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SELF-ORTHOGONAL LATIN SQUARES OF ALL ORDERS $n \neq 2, 3, 6$

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1. **Introduction.** E. Nemeth **[15]** has coined the term "self-orthogonal latin square" to denote a latin square orthogonal to its transpose. We have proved that there exists a self-orthogonal latin square of order n if and only if $n \neq 2, 3, 6$. Previous literature on this problem ([**3**], [**6**]–[**10**], [**12**]–[**15**], [**17**]–[**19**]) had constructed such squares for certain infinite classes of n and certain isolated values. We only became aware of this work after having constructed s.o.l.s. for all but approximately a dozen values of n, for we were motivated to consider the question by a problem in tennis. This we describe in §2. In §3 we give a rough summary of the proof of the theorem, details of which will appear elsewhere.

2. **Spouse-avoiding mixed doubles round robins.** In tennis, a mixed doubles match consists of two teams, each of which consist of one man and one woman. In informal play at a tennis club, it is common for a husband and wife to be a team, but this is not always so. One of us was asked by John Melian **[11]**, director of the Briarcliff Racquet Club in Briarcliff, New York, to arrange a schedule of matches for n couples in which husband and wife did not necessarily play together. We interpreted this as follows:

(i) Husband and wife would *never* appear in the same match either as partners or opponents.

(ii) Each pair of players of the same sex would oppose each other exactly once.

(iii) Each pair of players of opposite sex, if not married to each other, would play in exactly one match as partners, and exactly one match as opponents.

Suppose we are given a s.o.l.s. A of order n. It follows that the diagonal entries are a rearrangement of $\{1, \cdots, n\}$, and without loss of generality we may assume $a_{ii} = i$. We now determine the $\binom{n}{2}$ matches in a spouse-

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avoiding mixed doubles round robin as follows. In the unique match in which Mr. i opposes Mr. j ($i \neq j$), the partner of Mr i is Mrs. a_{ij} (and the partner of Mr. j is Mrs. a_{ji}). It is easy to verify that (i), (ii), (iii) are satisfied. Conversely, given a spouse-avoiding mixed doubles round robin for n couples, the matrix A given by $a_{ii}=i$, and a_{ij} is the family name of the woman who is the partner of Mr. i in his unique match opposing Mr. j, is clearly a s.o.l.s. of order n.

3. Summary of proof. We first note that the impossibility of 2, 3, 6 is obvious. Further, a remarkable theorem of Wilson [**20**] yields instantly that s.o.l.s. of all sufficiently large orders exist, but we have not used this. In [**3**], an outline of a method intended to cover all large n is also given. Our proof is based on (i) imitation of techniques of Bose, Parker and Shrikhande ([**1**], [**2**], [**16**]) in their disproof of the Euler conjecture, and those of Hanani and Wilson on the existence conjecture ([**5**], [**20**]), (ii) results of Sade [**17**] on s.o.l.s., and of Dulmage, Johnson and Mendolsohn [**4**] on the number of mutually orthogonal latin squares of order 12, (iii) the case $n=10$, due to Hedayat [**6**], which helped us to do 14 and 18, (iv) special constructions for 12 and 15, (v) remembering the isomorphism with tennis.

Let $B=\{n \mid \text{there exists a s.o.l.s. of order } n\}$. If there exists a pseudo-geometry on v points $\pi(v)$ (certain distinguished subsets are called lines, and two points are contained in exactly one line) such that every line cardinality is in B, then $v \in B$. From [**17**], $4, 5 \in B$. This is used to show that if there are at least 3 mutually orthogonal latin squares of order n, and if $n, m \in B$, $m \leqq n$, then a $\pi(4n+m)$ exists in which each line cardinality is 4, 5, m, or n, so $4n+m \in B$.

From $12 \in B$, and from [**4**] and [**16**], we are able to conclude that, for each $k \geqq 1$, $4k \in B$ and there are at least 3 mutually orthogonal latin squares of order $4k$. It follows that, if one can find a representative in B of each residue class mod 16 (which we can), then one only needs to construct a modest number of squares of certain specified orders to complete the proof (which we do).

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ADDED IN PROOF. In the manuscript "Research on Sade's disproof of the Euler conjecture with an application to latin squares orthogonal to their transposes", the authors of [**3**] show that elaboration of the ideas of [**17**] proves the existence of a s.o.l.s. of order n with at most 217 exceptions.

D. Knuth has kindly informed us that a s.o.l.s. of order 10 was constructed in Louis Weisner, *Special orthogonal latin squares of order* 10, Canad. Math. Bull. **6** (1963), 61–63. MR **26**, 3621.

REFERENCES

1. R. C. Bose, E. T. Parker and S. Shrikhande, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, Canad. J. Math. **12** (1960), 189–203. MR **23** #A69.

2. R. C. Bose and S. Shrikhande, *On the construction of sets of mutually orthogonal Latin squares and the falsity of a conjecture of Euler*, Trans. Amer. Math. Soc. **95** (1960), 191–209. MR **22** #2557.

3. D. J. Crampin and A. J. W. Hilton, *The spectrum of latin squares orthogonal to their transposes* (manuscript).

4. A. L. Dulmage, D. M. Johnson, and N. S. Mendelsohn, *Orthomorphisms of groups and orthogonal Latin squares* I, Canad. J. Math. **13** (1961), 356–372.

5. H. Hanani, *The existence and construction of balanced incomplete block designs*, Ann. Math. Stat. **32** (1961), 361–386. MR **29** #4161.

6. A. Hedayat, *An application of sum composition: a self orthogonal latin square of order ten*, J. Combinatorial Theory, Series A **14** (1973), 256–260.

7. J. D. Horton, *Variations on a theme by Moore*, Proceedings of the Louisiana Conference on Graph Theory, Combinatorics and Computing, Louisiana State University, Baton Rouge, March 1–5, 1970.

8. C. C. Lindner, *The generalized singular direct product for quasigroups*, Canad. Math. Bull. **14** (1971), 61–63. MR **45** #435.

9. ———, *Construction of quasigroups satisfying the identity $x(xy)=yx$*, Canad. Math. Bull. **14** (1971), 57–59. MR **45** #434.

10. ———, *Application of the singular direct product to constructing various types of orthogonal latin squares*, Memphis State University Combinatorial Conference, 1972.

11. John Melian (oral communications).

12. N. S. Mendelsohn, *Combinatorial designs as models of universal algebras*, Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968), Academic Press, New York, 1969, pp. 123–132. MR **41** #85.

13. ———, *Latin squares orthogonal to their transposes*, J. Combinatorial Theory Ser. A. **11** (1971), 187–189. MR **45** #88.

14. R. C. Mullin and E. Nemeth, *A construction for self orthogonal latin squares form certain Room squares*, Proc. Louisiana Conf. on Graph Theory, Combinatorics and Computing (Louisiana State Univ., Baton Rouge, La., 1970), Louisiana State Univ., Baton Rouge, La., 1970, pp. 213–226. MR **42** #2957.

15. E. Nemeth, *Study of Room squares*, Ph.D. Thesis, University of Waterloo, Ontario

16. E. Parker, *Construction of some sets of mutually orthogonal Latin squares*, Proc. Amer. Math. Soc. **10** (1959), 946–949. MR **22** #674.

17. A. Sade, *Produit direct-singulier de quasigroupes orthogonaux et anti-abéliens*, Ann. Soc. Sci. Bruxelles, Sér. I, **74** (1960), 91–99. MR **25** #4017.

18. ———, *Une nouvelle construction des quasigroupes orthogonaux à leur conjoint*, Notices Amer. Math. Soc. **19** (1972), 434. Abstract # 72T-A105.

19. S. K. Stein, *On the foundations of quasigroups*, Trans. Amer. Math. Soc. **85** (1957), 228–256. MR **20** #922.

20. R. M. Wilson, *An existence theory for pairwise balanced designs*, I, II, J. Combinatorial Theory Ser. A. **13** (1972), 220–273.

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