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Self-similar inverse semigroups and Smale spaces

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Abstract

Self-similar inverse semigroups are defined using automata theory. Adjacency semigroups of s -resolved Markov partitions of Smale spaces are introduced. It is proved that a Smale space can be reconstructed from the adjacency semigroup of its Markov partition, using the notion of the limit solenoid of a contracting self-similar semigroup. The notions of the limit solenoid and a contracting semigroup is described.

1 Introduction

The aim of our paper is to show how self-similar inverse semigroups of automatic transformations appear naturally in dynamical system theory, in particular in connection with Smale spaces, minimal dynamical systems on Cantor sets and substitution tilings (the Penrose tilings, for instance).

We show, for example, that semigroups associated with tilings and Markov partitions of Smale spaces are generated by finite automata. This can be used to perform explicit computations and establishes an interesting connection between symbolic dynamics and semigroup theory. We also associate, in the spirit of the J. Kellendonk's construction, finitely generated semigroups (the *adjacency semigroups*) to Smale spaces of a special type. We show also that the Smale spaces are uniquely determined by the associated adjacency semigroups.

These results connect two different directions of investigation, which were developed independently before.

One direction investigates substitution tilings, Smale spaces and algebraic object related with them. Inverse semigroups were applied to the study of substitutional tilings by J. Kellendonk, who defined the *tiling semigroups* (see the articles [Kel97, Kel95, KL00] and [Pat99]). Inverse semigroups are essential in this situation, since substitutional tilings have typically a rich semigroup of partial symmetries, but are aperiodic, and thus have a small (trivial in the most cases) group of symmetries.

Interesting algebraic objects related with the Smale spaces are the *Ruelle groupoids* and the associated C^* -algebras, studied by I. F. Putnam and other authors (see [Put96, KPS97]). The

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space of all (pointed) tilings, obtained from a substitution rule is a Smale space in a natural way [AP98, KP00].

The other direction, the theory of self-similar groups, was developed during the study of groups of automata for purely group-theoretical reasons. For the first definitions of a self-similar (*state-closed*) group (or a group, *generated by an automaton*), see the works [Gri85, Sid98].

The class of self-similar groups contains many examples of groups with such interesting properties as intermediate growth, just-infiniteness, being an infinite finitely generated torsion group, etc. These examples have helped to solve many interesting problems in group theory. For examples of self-similar groups and their properties, see the papers [Gri80, Gri83, GS83, Gri98, BSV99, GLSŻ00]. See also the surveys [Gri00, GNS00, BGN03], and the book [Nek05] for a systematic study of self-similar groups and their relations to dynamical systems.

The study of self-similar groups had a dynamical flavor from the very beginning. Even the definitions of such groups were in a sense dynamical, since the groups were defined via their actions on graphs, topological spaces, or measure spaces.

The connections became more apparent during the subsequent investigations. For instance, there exist interesting applications of iterations of rational mappings to spectra of random walks on self-similar groups and their Schreier graphs (see [GŻ01, BG00]).

It was shown in [Nek05], that a naturally defined topological *limit space* \mathcal{J}_G and a dynamical system $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$ on it is associated with every self-similar group G belonging to an important class of *contracting* groups. Another object, associated to a contracting self-similar group, is the *limit solenoid* \mathcal{S}_G , which can be constructed as the inverse limit of the limit dynamical system (\mathcal{J}_G, s) . The limit solenoid has many properties, similar to the properties of the spaces of substitutional tilings. It is a union of *leaves*, where each leaf is a union of naturally defined *tiles*. The map s acts on the tiles as a substitution law (an *inflation*). In particular, the spaces of self-affine substitutional *digit* tilings (see definitions in [Vin00]) can be all obtained as limit solenoids of self-similar actions of the free abelian (or, in a more general setting, crystallographic) groups.

Self-similar groups can be naturally associated with branched self-coverings of topological spaces (for example, with the mappings of the complex sphere defined by rational functions) via the construction of the *iterated monodromy group* (see [Nek05, BGN03]). If the self-covering is expanding (if the rational function is sub-hyperbolic), then the iterated monodromy group G is contracting and its limit space \mathcal{J}_G is homeomorphic to the Julia set of the self-covering.

The notion of a self-similar inverse semigroup was introduced as a natural generalization of the notion of a self-similar group, see [Nek02, BGN03]. One of the first examples of a self-similar inverse semigroup was a variation of the tiling semigroup of the Penrose tiling (see its description in [BGN03] and in this article).

The current article is an attempt to generalize the mentioned results on limit spaces and iterated monodromy groups to the semigroup case.

The structure of the paper is the following. In Section 2 we define the notion of a self-similar inverse semigroup, its groupoid of germs and present some typical examples. Self-similar inverse semigroups act by local homeomorphisms on Markov shift spaces. We give the basic definitions of shift spaces in the first subsection of Section 2. The notion of self-similar inverse semigroup is defined in the second subsection. We use here the language of (non-deterministic) automata, similar to the group case. A previous definition given in [BGN03] did not use explicitly the notion of an automaton. The groupoid of germs is a natural object associated with every inverse semigroup of local homeomorphisms (see [BH99]) and is a convenient notion for self-similar semigroups. Self-similarity of the inverse semigroup can be, for example, formulated in terms of the groupoid of germs as the requirement that the shift induces an endomorphism of the groupoid.

Among the examples (Subsection 2.5) we present the self-similar inverse semigroups related to the “adic” transformations of the Cantor space defined by stationary Vershik-Bratelli diagrams and a variant of the tiling semigroup of the Penrose tiling.

The next section “Contracting self-similar semigroups” generalizes the notion of a contracting group and its limit solenoid to self-similar inverse semigroups. A self-similar inverse semigroup is called *contracting* if the iteration of the shift attracts every element of the groupoid of germs to a

compact subset. We associate with every contracting self-similar inverse semigroup H a metrizable finite-dimensional topological space \mathcal{S}_H together with a homeomorphism $s : \mathcal{S}_H \rightarrow \mathcal{S}_H$.

We apply in Section 4 the introduced notions and results to Smale spaces. Its first subsection contains the standard definitions of the Smale spaces (following [Rue78, Put96]) and the definition of an s -resolved Markov partition. A Markov partition is called s -resolved if every rectangle is a direct product (with respect to the canonical coordinates) of a clopen stable set and closed unstable set. In particular, if a Smale space has an s -resolved Markov partition, then its stable manifolds are 0-dimensional.

Subsection 4.2 contains the main results of our paper. We introduce a notion of the *adjacency semigroup* of an s -resolved Markov partition (Definition 4.2) and prove (Theorem 4.6) that it is self-similar contracting and that the limit dynamical system (\mathcal{S}_H, s) of the adjacency semigroup H is topologically conjugate with the Smale space. In particular, the action of the inverse semigroup on the shift space determines the Smale space uniquely.

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2 Self-similar inverse semigroups

2.1 Preliminary definitions

Let X , $|X| > 1$ be a finite set, called *alphabet*. We associate with every alphabet two spaces of sequences. One is the space $X^{\mathbb{N}}$ of infinite one-sided sequences of the form $x_1x_2\dots$, the other is the space $X^{\mathbb{Z}}$ of bi-infinite sequences of the form $\dots x_{-2}x_{-1} \cdot x_0x_1\dots$, where the dot marks the place between the coordinates number -1 and number 0 . We introduce on the spaces $X^{\mathbb{N}}$ and $X^{\mathbb{Z}}$ the topology of the direct products of the discrete spaces X . Then both spaces are compact metrizable spaces without isolated points, i.e., homeomorphic to the Cantor space.

We denote for every subset $A \subseteq X^{\mathbb{Z}}$ and a finite sequence $u = a_1a_2\dots a_n$ by uA the set of sequences of the form $a_1a_2\dots$ such that $a_{n+1}a_{n+2}\dots \in A$.

We introduce also the following maps on the spaces $X^{\mathbb{N}}$ and $X^{\mathbb{Z}}$, called *shifts*:

$$\begin{aligned} \sigma(x_1x_2\dots) &= x_2x_3\dots \\ \sigma_{\mathbb{Z}}(\dots x_{-2}x_{-1} \cdot x_0x_1\dots) &= \dots x_{-3}x_{-2} \cdot x_{-1}x_0\dots \end{aligned}$$

The map $\sigma : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ is an $|X|$ -to-one continuous surjection and the map $\sigma_{\mathbb{Z}} : X^{\mathbb{Z}} \rightarrow X^{\mathbb{Z}}$ is a homeomorphism. Note that σ and $\sigma_{\mathbb{Z}}$ are shifts in different directions.

A subspace $\mathcal{F} \subseteq X^{\mathbb{N}}$ or $\mathcal{F}_{\mathbb{Z}} \subseteq X^{\mathbb{Z}}$ is called *shift space* (or just *shift*) if it is closed and shift-invariant, i.e., if $\sigma(\mathcal{F}) \subseteq \mathcal{F}$, or $\sigma_{\mathbb{Z}}(\mathcal{F}_{\mathbb{Z}}) = \mathcal{F}_{\mathbb{Z}}$, respectively.

Important examples of shift spaces are the *Markov shifts*. Every Markov shift is defined by a set of *admissible pairs* $A \subseteq X \times X$. If A is a set of admissible pairs, then the respective one-sided Markov shift $\mathcal{F} = \mathcal{F}(A)$ is defined as the set of all sequences $x_1x_2\dots \in X^{\mathbb{N}}$ such that $(x_n; x_{n+1}) \in A$ for all $n \geq 1$. Similarly, the two-sided Markov shift $\mathcal{F}_{\mathbb{Z}} = \mathcal{F}_{\mathbb{Z}}(A)$ is defined as the set of sequences $(\dots x_{-2}x_{-1} \cdot x_0x_1\dots) \in X^{\mathbb{Z}}$ such that $(x_n; x_{n+1}) \in A$ for all $n \in \mathbb{Z}$.

If \mathcal{F} is a one-sided Markov shift, then its *natural extension* is the two-sided Markov shift $\mathcal{F}_{\mathbb{Z}}$ defined by the same set of admissible pairs. For more on shift spaces, see the books [LM95, Kit98].

2.2 Automata

Definition 2.1. A (*non-deterministic*) *automaton* over an alphabet X is a directed graph Γ (loops and multiple edges are allowed) with the arrows labeled by pairs $(x; y)$ of letters from X . The vertices of the graph are called the *states* of the automaton.

We say that a *transition* $(u; v)$, where $u, v \in X^{\mathbb{N}}$, is *accepted by a state* $q \in \Gamma$ if there is an infinite directed path starting in q whose consecutive edges are labeled by the labels $(x_1; y_1), (x_2; y_2), \dots$ so that $u = x_1x_2\dots, v = y_1y_2\dots$

If Γ is an automaton, then its *inverse* is the automaton, obtained from Γ by changing every label $(x; y)$ to $(y; x)$.

If a transition $(u; v)$ is accepted by a state q then we say also that the state q *defines* the transition $(u; v)$.

Our automata may be infinite (i.e., may have infinitely many states), but we always suppose that they are *locally finite*, i.e., that at every state only a finite number of arrows start.

If Γ is an automaton and q is its state, then a sequence $v \in X^{\mathbb{N}}$ is *accepted* by the state q if there exists $u \in X^{\mathbb{N}}$ such that the transition $(v; u)$ is accepted by q . The set of all sequences, accepted by q is denoted $\text{Dom } q$. It is easy to see that $\text{Dom } q$ is a closed subset of the space $X^{\mathbb{N}}$.

The union $\text{Dom } \Gamma = \bigcup_{q \in \Gamma} \text{Dom } q$ is called the set of the sequences *accepted by the automaton* Γ . It is easy to see that the set $\text{Dom } \Gamma$ is invariant under the shift σ . If the automaton Γ is finite then the set $\text{Dom } \Gamma$ is closed and is a shift space.

An automaton Γ is said to be ω -*deterministic* if for every states q and for every $u \in X^{\mathbb{N}}$ there exists at most one path in Γ starting in q and labeled by $(x_1; y_1), (x_2; y_2), \dots$, where $u = x_1x_2\dots$. An ω -deterministic automaton is said to be ω -*invertible* if its inverse is ω -deterministic.

If Γ is an ω -deterministic automaton and q is its state, then the transformation *defined by the state* q is the partial map, denoted Γ_q , with the domain $\text{Dom } q$, mapping every sequence $u \in \text{Dom } q$ to the unique sequence $v \in X^{\mathbb{N}}$ such that $(u; v)$ is accepted by q .

Definition 2.2. Let Γ be an automaton over the alphabet X with the set of states Q . Then the *dual automaton* Γ^* is the automaton over the alphabet Q with the set of states X such that we have an arrow from $x \in X$ to $y \in X$ labeled by a pair $(p; q) \in Q^2$ if and only if there is an arrow from p to q labeled by the pair $(x; y)$ in the automaton Γ .

It is easy to see that $(\Gamma^*)^* = \Gamma$. We need the notion of a dual automaton, since in many cases the dual automaton is easier to draw.

2.3 Self-similar inverse semigroups

Remark. We use left actions of semigroups, so that the image of a point w under the action of a transformation g is denoted $g(w)$ and in a product g_1g_2 the transformation g_2 acts first.

All inverse semigroups appearing in our article come together with a fixed action by partial permutation. Therefore, being absolutely accurate, one should actually change every instance of the term “*inverse semigroup*” in our article to the term “*action of the inverse semigroup*”.

Definition 2.3. Let H be an inverse semigroup acting by local homeomorphisms on a Markov shift $\mathcal{F} \subseteq X^{\mathbb{N}}$. We say that the action is *self-similar* if there exists an ω -deterministic automaton $\Gamma(H)$ with the set of states H such that $\text{Dom } \Gamma(H) = \mathcal{F}$ and for every $g \in H$ the partial permutation g coincides with the partial permutation Γ_g defined by the state g of the automaton $\Gamma(H)$.

Here by a *local homeomorphism* of a topological space we mean a homeomorphism between two open subsets. In our case these open sets have to be also closed, since the set of sequences accepted by a state is closed. Note that in conditions of the last definition the union of the domains of the elements of H is the space \mathcal{F} and the automaton $\Gamma(H)$ is ω -invertible. When talking about an action of an inverse semigroup H on a space \mathcal{F} , we always assume that the union of the domains of H is the whole space \mathcal{F} .

We consider here only self-similar semigroups acting on a Markov space \mathcal{F} by local homeomorphisms, though Definition 2.3 has sense in a more general situation (see, for example the paper [BGN03]).

The automaton $\Gamma(H)$ from Definition 2.3 is called the *full automaton* of the self-similar action.

We have the following easy criterion. (It shows also that the notion of a self-similar semigroup, adopted here coincides with the notion, introduced in [BGN03].)

Proposition 2.1. *An action of an inverse semigroup H on a Markov shift space $\mathcal{F} \subseteq X^{\mathbb{N}}$ by local homeomorphisms is self-similar if and only if for every $g \in H$ the domain $\text{Dom } g$ is closed and for every $g \in H$ there exist a number $n \in \mathbb{N}$, a sequence $x_1, x_2, \dots, x_n \in X$, a sequence $y_1, y_2, \dots, y_n \in X$ and a sequence $h_1, h_2, \dots, h_n \in H$ such that $\text{Dom } g = \bigsqcup_{i=1}^n x_i \text{Dom } h_i$ and for every $w \in \text{Dom } h_i$ we have $g(x_i w) = y_i h_i(w)$.*

Here \bigsqcup denotes disjoint union.

Proof. If the action is self-similar, then for every $g \in H$ let e_1, e_2, \dots, e_n be the set of all arrows of the automaton $\Gamma(H)$, starting in g . If h_i is the end of the arrow e_i and the arrow is labeled by $(x_i; y_i)$, then the sequences (x_1, x_2, \dots, x_n) , (y_1, y_2, \dots, y_n) and (h_1, h_2, \dots, h_n) satisfy the condition of the proposition. The sets $x_i \text{Dom } h_i$ are disjoint by the definition of an ω -deterministic automaton.

On the other hand, if for every $g \in H$ such sequences (x_1, x_2, \dots, x_n) , (y_1, y_2, \dots, y_n) and (h_1, h_2, \dots, h_n) exist, then we can construct the automaton $\Gamma(H)$. We just have to draw n arrows e_i from g to h_i labeled by $(x_i; y_i)$. Let us prove that then the state g of the constructed automaton $\Gamma(H)$ defines the transformation g .

If $a_1 a_2 \dots \in \mathcal{F}$ is a sequence belonging to the domain of g , then the sequence $a_2 a_3 \dots$ belongs to the domain of the transformation $f_1 = h_1$, where $a_1 = x_i$ and $g(a_1 a_2 \dots) = y_i f_1(a_2 a_3 \dots)$. Let us denote $y_i = b_1$. Repeating the same arguments for $a_2 a_3 \dots$, then for $a_3 a_4 \dots$, etc., we see that if $b_1 b_2 \dots = g(a_1 a_2 \dots)$ then there exists a sequence $f_0 = g, f_1, f_2, \dots$ such that the vertex f_k is connected by an arrow labeled by $(a_{k+1}; b_{k+1})$ with the vertex f_{k+1} for all $k \geq 0$. This gives us the necessary path, showing that the state g of the automaton $\Gamma(H)$ admits the transition $(a_1 a_2 \dots; g(a_1 a_2 \dots))$.

On the other hand, suppose that the state g admits the transition $(a_1 a_2 \dots; b_1 b_2 \dots)$. Then there exists a path in the automaton $\Gamma(H)$ starting in g , labeled by the consecutive labels $(a_k; b_k)$, $k = 1, 2, \dots$, and passing through the consecutive vertices $g = f_0, f_1, \dots$. Then we have, by construction of the automaton $\Gamma(H)$, that $a_1 a_2 \dots a_k \text{Dom } f_k \subset \text{Dom } g$ and

$$g(a_1 a_2 \dots a_k x_{k+1} x_{k+2} \dots) = b_1 b_2 \dots b_k f_k(x_{k+1} x_{k+2} \dots)$$

for all $k \geq 1$ and $x_{k+1} x_{k+2} \dots \in \text{Dom } f_k$. But

$$\text{Dom } g \supset \bigcap_{k \geq 1} a_1 a_2 \dots a_k \text{Dom } f_k = \{a_1 a_2 a_3 \dots\},$$

since the sets $a_1 a_2 \dots a_k \text{Dom } f_k$ are compact and form a descending sequence, so that the intersection is not empty. Similarly,

$$\bigcap_{k \geq 1} b_1 b_2 \dots b_k \text{Ran } f_k = \{b_1 b_2 b_3 \dots\},$$

and consequently, $g(a_1 a_2 \dots) = b_1 b_2 \dots$

Let us show that the constructed automaton is ω -deterministic. Let $a_1 a_2 \dots \in X^{\mathbb{N}}$ and $g \in H$ be arbitrary. Then it follows from the previous consideration that $a_1 a_2 \dots \in \text{Dom } g$ if and only if there exists a path $e_1 e_2 \dots$ in the automaton, starting in the state g and labeled by $(a_1; b_1), (a_2; b_2), \dots$. It also follows that the first arrow e_1 is necessary the arrow corresponding to the unique set $a_1 \text{Dom } h$ in the decomposition $\text{Dom } g = \bigsqcup x_i \text{Dom } h_i$ to which belongs the sequence $a_1 a_2 \dots$. Hence the first arrow e_1 and similarly all the subsequent arrows e_i of the path $e_1 e_2 \dots$ are uniquely determined by the sequence $a_1 a_1 \dots$. \square

The full automaton $\Gamma(H)$ is infinite if the semigroup H is infinite. On the other hand, finite automata *generate* self-similar inverse semigroups in the following sense.

Definition 2.4. If Γ is a finite ω -invertible automaton, then the inverse semigroup *generated* by Γ is the inverse semigroup acting on the space $\text{Dom } \Gamma$ generated by the partial permutations of the form Γ_q , where q runs through the set of all states of the automaton Γ .

The following proposition is an easy corollary of Proposition 2.1.

Proposition 2.2. *An inverse semigroup generated by a finite automaton is self-similar (if it acts on $\text{Dom } \Gamma$ by local homeomorphisms).*

2.4 Groupoid of H -germs

An important notion related to self-similar inverse semigroups is the associated groupoid. Recall that *groupoid* is a small category of isomorphisms (see a more explicit definition in [Pat99, Ren80]).

Let H be a self-similar inverse semigroup acting on a Markov space \mathcal{F} . The *groupoid of H -germs* (denoted $\mathcal{G}(H)$) is the set of the equivalence classes of pairs (h, w) , where $h \in H$ and $w \in \text{Dom } h$. Two pairs (h_1, w_1) and (h_2, w_2) are considered to be equivalent if and only if $w_1 = w_2$, $h_1(w_1) = h_2(w_2)$ and the respective paths in the full automaton $\Gamma(H)$ starting in h_1 and h_2 and defining the transitions $(w_1; h(w_1)) = (w_2; h_2(w_2))$ are cofinal, i.e., there exists a number n_0 such that the edge number n of the first path coincides with the edge number n of the second path for every $n > n_0$.

We will identify the germs with the respective pairs (h, w) and with the paths in $\mathcal{G}(H)$ starting in h and defining the transition $(w; h(w))$.

We define the multiplication of the elements of the groupoid $\mathcal{G}(H)$ by the rules

$$(h_1, w_1) \cdot (h_2, w_2) = (h_1 h_2, w_2),$$

where the product is defined if and only if $h_2(w_2) = w_1$. Then the inverse to the pair (h, w) is the pair $(h^{-1}, h(w))$.

It is easy to see that the associated groupoid $\mathcal{G}(H)$ is well defined. Our construction is a particular case of a general notion of the groupoid of germs of an inverse semigroup of local homeomorphisms. The groupoid of germs comes with a naturally defined *germ topology* (see [BH99]) and is an *étale (r -discrete)* locally compact groupoid. Namely, a neighborhood of a germ (h, w) is determined by a number $n \in \mathbb{N}$ and consists of all the germs (h, w') such that the corresponding paths in $\Gamma(H)$ have a common beginning of length at least n .

The shift $\sigma : \mathcal{F} \rightarrow \mathcal{F}$ induces an endomorphism (functor) $\sigma_* : \mathcal{G}(H) \rightarrow \mathcal{G}(H)$ defined by the map deleting the first edge of a path in the full automaton $\Gamma(H)$. It is easy to see that the shift σ_* agrees with the cofinality relation, thus is a well defined map of the groupoid. It follows also directly from the definition that σ_* is an étale functor.

Definition 2.5. A self-similar inverse semigroup is said to be *recurrent* if the shift endomorphism σ_* of the associated groupoid is onto.

Note that if the inverse semigroup acting on a space \mathcal{F} is recurrent then $\sigma(\mathcal{F}) = \mathcal{F}$, since the space \mathcal{F} is identified with the space of units (with the set of objects) of the groupoid $\mathcal{G}(H)$.

2.5 Examples

Self-similar groups. A group G acting by permutations on the space $X^{\mathbb{N}}$ is said to be self-similar (see [Nek05]) if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that $g(xw) = yh(w)$ for every $w \in X^{\mathbb{N}}$.

There exist many interesting examples of self-similar group actions. For example, many branch groups [Gri00] are self-similar. Branch groups often have interesting properties, rare among other classes of groups (many of them are just-infinite, have intermediate growth, are infinite torsion groups, etc.). An example of a self-similar branch group is the Grigorchuk group (see [Gri80, Har00]).

Another class of self-similar groups are the iterated monodromy groups of (branched) self-coverings of topological spaces (for example, of rational functions on the complex sphere). For the definition of the iterated monodromy groups and their basic properties see [Nek05] and [BGN03].

The iterated monodromy group of a self-covering map f contains much of the information about the dynamics of the map. In particular, in many cases the Julia set of f can be reconstructed from its iterated monodromy group.

Let us give one of important examples of self-similar group actions. It is the action of the group \mathbb{Z} on the space $\{0, 1\}^{\mathbb{N}}$ generated by the transformation a , called the *adding machine* and defined by the rules:

$$a(0w) = 1w, \quad a(1w) = 0a(w),$$

where $w \in \{0, 1\}^{\mathbb{N}}$ is arbitrary. It is easy to see that the transformation a acts on the sequences from $\{0, 1\}^{\mathbb{N}}$ in the same way as addition of 1 to a dyadic integer. More precisely, $a(x_0x_1\dots) = y_0y_1\dots$ is equivalent to the equality

$$1 + \sum_{k=0}^{\infty} x_k \cdot 2^k = \sum_{k=0}^{\infty} y_k \cdot 2^k$$

in the ring \mathbb{Z}_2 of dyadics.

The adding machine action of \mathbb{Z} is generated by the automaton shown on the left-hand side of Figure 1. On the right-hand side of the Figure the dual automaton is shown.

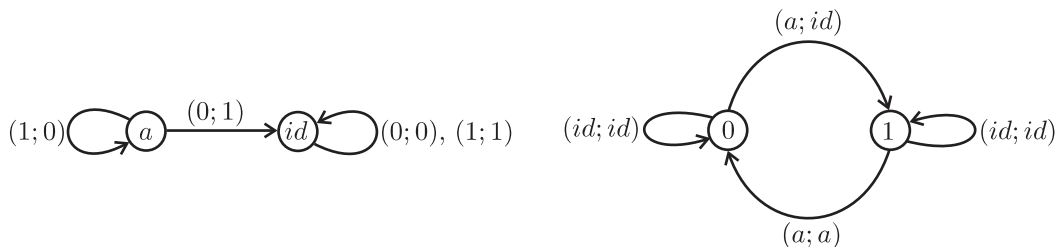


Figure 1: Adding machine.

“Adic” transformations. An interesting class of transformations defined by finite automata are “adic” transformations, introduced by Vershik.

Recall that a *Bratelli diagram* \mathcal{D} is defined by a sequence of finite sets V_0, V_1, \dots (sets of the *vertices* of the diagram) and a sequence of finite sets E_1, E_2, \dots (sets of the *edges* of the diagram), together with maps $s_i : E_i \rightarrow V_{i-1}$ and $r_i : E_i \rightarrow V_i$. If $e \in E_i$ is an edge, then the vertex $s_i(e) \in V_{i-1}$ is the *source* and the vertex $r_i(e) \in V_i$ is the *range* of e .

A *Vershik-Bratelli diagram* is a Bratelli diagram with partial orders on every set E_i . Two edges must be comparable in this order if and only if they have equal ranges.

If we have a Vershik-Bratelli diagram \mathcal{D} , then the order on the edges induces the *lexicographical order* on the paths of \mathcal{D} . Namely a (finite or infinite) path (e_1, e_2, \dots) (i.e., such a sequence $e_i \in E_i$ that $r_i(e_i) = s_{i+1}(e_{i+1})$) is less than a path of the same length (f_1, f_2, \dots) if and only if for some index i_0 we have that $e_i = f_i$ for all $i > i_0$ and the edge e_{i_0} is less than f_{i_0} . It is easy to prove that for every infinite path w , which is not maximal in the lexicographical order there exists exactly one infinite path $\tau(w)$ (the *next path to w*) minimal among the paths which are greater than w .

The transformation τ defined on the set of non-maximal paths is called the *adic transformation*, defined by the Vershik-Bratelli diagram.

If the maximal path w_1 is unique and the minimal path w_0 is unique, then one can define $\tau(w_1) = w_0$ and the transformation τ will be a minimal homeomorphism of the space of infinite paths in the diagram \mathcal{D} . In fact, every minimal homeomorphism of the Cantor space is of this form, i.e., is conjugate to an adic transformation (see [GPS95]).

A *stationary* Vershik-Bratelli diagram \mathcal{D} is a Vershik-Bratelli diagram defined by constant sequences $V_i = V$, $E_i = E$, and $s_i = s : E \rightarrow V$, $r_i = r : E \rightarrow V$ such that all the orderings on $E_i = E$ are equal. The space of all infinite paths in a stationary diagram is obviously a Markov shift over the alphabet $X = E$.

For every $n \in \mathbb{N}$ the n th *telescoping* $\mathcal{D}^{(n)}$ of a stationary Vershik-Bratelli diagram \mathcal{D} is the diagram defined by the same set of vertices V , by the set of edges $E^{(n)}$ equal to the set of paths $e = (e_1, e_2, \dots, e_n)$ of length n in the original diagram, the maps $s^{(n)}(e) = s(e_1)$, $r^{(n)}(e) = r(e_n)$ and the lexicographic order on $E^{(n)}$ induced by the order on E . The telescoping changes the adic transformation to a conjugate one. See more on Vershik-Bratelli diagrams and adic transformations in the papers [HPS92, GPS95].

Suppose that we have a stationary diagram \mathcal{D} , which has a unique minimal and a unique maximal path. For every edge a of the first level of the diagram, denote by τ_a the restriction of the minimal homeomorphism τ onto the set of paths with the first edge a . Denote by $H_{\mathcal{D}}$ the inverse semigroup generated by the transformations τ_a , $a \in E$. We have the following result (see [Nek02]).

Theorem 2.3. *Let \mathcal{D} be a stationary Vershik-Bratelli diagram with a single maximal and a single minimal path. Then there exists $n \in \mathbb{N}$ such that the semigroup $H_{\mathcal{D}^{(n)}}$ is self-similar.*

For example, consider the stationary Vershik-Bratelli diagram shown on the upper part of Figure 2. We introduce the ordering of the edges as follows:

$$a_0 < a_1 < a_2, \quad b_0 < b_1.$$

Then the adic transformations τ_x are defined by the dual automaton, shown on the lower part of Figure 2. Here a label $(a; b_1, b_2, \dots, b_n)$ means that we have at this place n arrows with the labels $(a; b_i)$. The transformations 1_x are the identical transformations, with the domain equal to the set of paths with the first edge x . We have omitted the loops, defining the identical transformations 1_x , since it is obvious how to draw them (one must only not forget that some of transitions $(1_x; 1_y)$ are prohibited by the Vershik-Bratelli diagram).

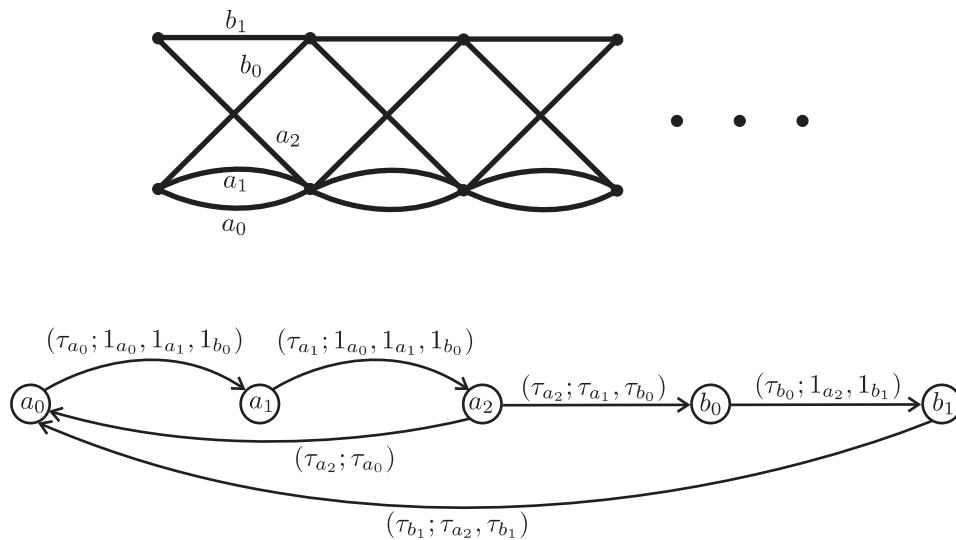


Figure 2: Adic transformations.

Another example of a homeomorphism defined by a stationary Vershik-Bratelli diagram is the binary adding machine described before. It is defined by the set V consisting of a single point and the set $E = \{0, 1\}$ with the ordering $0 < 1$.

Penrose tiling. The following example of a self-similar semigroup is defined in [BGN03]. It is an example similar to the *tiling semigroup* of J. Kellendonk, different from it only in some technical details.

Recall, that a Penrose tiling (see [Pen84, Gar77, GS87]) is a tiling of the plane by the triangles of two types shown on Figure 3, which has to satisfy the matching rules.

The angles of the triangles are equal to $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{2\pi}{5}$ and to $\frac{3\pi}{5}$, $\frac{\pi}{5}$, $\frac{\pi}{5}$. They can be cut from a regular pentagon by the diagonals issued from a common vertex.

The *matching rules* require that the common vertices of two tiles are marked by the same color (black or white) and that the arrows on the adjacent sides of the triangles point at the same direction (see Figure 3 for the marking of the vertices and the arrows).

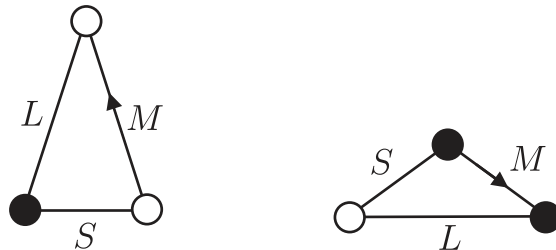


Figure 3: Tiles and matching rules.

It follows from the matching rules that if we mark the sides of the tiles by the letters S, L and M in the way it is done on Figure 3, then in every Penrose tiling the common sides of any two adjacent tiles will be marked by the same letter.

One can check, just considering all possible patches of Penrose tilings of sufficient size, that the tiles of any Penrose tiling can be grouped into blocks of two or three tiles, as it is shown on Figure 4 (the mirror images are also allowed).

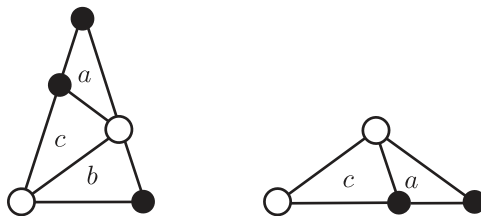


Figure 4: Inflation

The grouping into blocks is unique [GS87] and it is easy to see that the blocks are similar to the original tiles and that the blocks also form a Penrose tiling. This tiling is called the *inflation* of the original one.

The inflation procedure defines an encoding of the tiles of Penrose tilings by infinite sequences from the Markov shift \mathcal{F} over the alphabet $X = \{a, b, c\}$ defined by the set of admissible pairs $A = X \times X \setminus \{(b; a)\}$. Namely, let T be a tile of a Penrose tiling. Then T is a subset of a tile T_n of the Penrose tiling obtained by the n th iteration of the inflation procedure. Let $a_n \in X$ be the label of the tile T_n on Figure 4, according to its participation in the inflation. Then $a_0 a_1 \dots \in \mathcal{F}$ is the label of T .

This encoding is one-to-one, except for the case when a tiling has a non-trivial symmetry group (this group is always finite and isomorphic either to $\mathbb{Z}/2\mathbb{Z}$ or to the dihedral group \mathbb{D}_5 of order 10). If the tiling has a non-trivial symmetry group, then the tiles belonging to one orbit of the symmetry group correspond to one sequence (see analysis of this situation in [BGN03]).

Define transformations L, M, S of the space \mathcal{F} by the formulae:

$$\begin{aligned}
S(aw) &= cw & L(aaw) &= b \cdot S(aw) \\
S(bw) &= b \cdot M(w) & L(abw) &= a \cdot M(bw) \\
S(cw) &= aw & L(acw) &= a \cdot M(cw) \\
M(aw) &= a \cdot L(w) & L bbw) &= b \cdot S(bw) \\
M(bw) &= cw & L(bcw) &= a \cdot S(cw) \\
M(caw) &= c \cdot M(aw) & L(cw) &= c \cdot L(w) \\
M(cbw) &= bbw \\
M(ccw) &= bcw
\end{aligned}$$

If $w \in \mathcal{F}$ corresponds to a tile T of the Penrose tiling, then $S(w), M(w)$ and $L(w)$ correspond to the tiles having a common side with T , marked on Figure 3 by the letters S, M and L respectively. This is also proved just by considering all possible patches of sufficient size in the Penrose tilings (actually, by considering one sufficiently large patch, since all Penrose tilings are locally isomorphic).

If we denote by S_x, M_x, L_x the restrictions of S, M and L onto the sets of sequences starting with x , for $x \in \{a, b, c\}$, then the transformations $S_x, M_x, L_x, x \in \{a, b, c\}$ are defined by the dual automaton shown on Figure 5. We have not written the loops defining the idempotents 1_x .

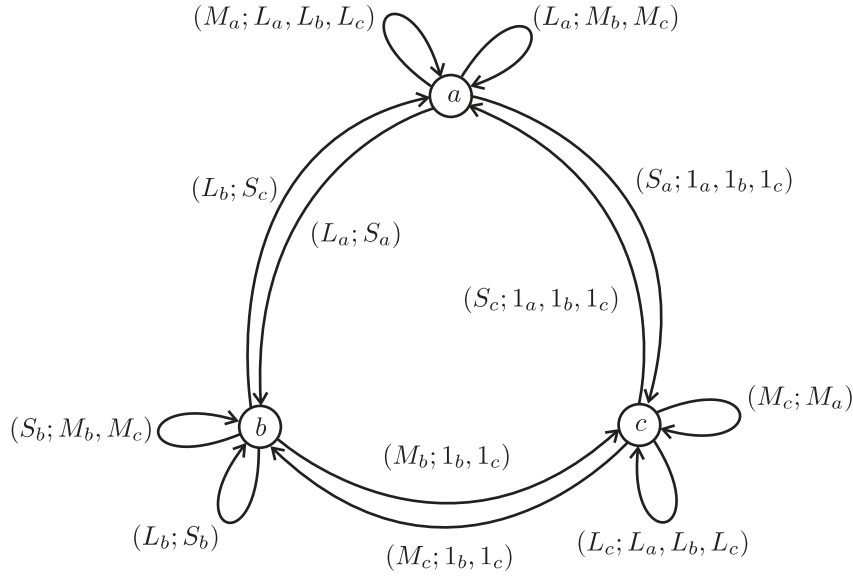


Figure 5: Adjacency semigroup of the Penrose tilings.

3 Contracting self-similar semigroups

3.1 Definitions

Definition 3.1. A *subautomaton* of an automaton Γ is a subgraph Γ_1 of Γ such that every arrow of Γ starting in a vertex of Γ_1 also belongs to Γ_1 .

So, a sub-automaton Γ_1 is uniquely determined by the corresponding subset of states of Γ . If Γ is ω -deterministic and Γ_1 is a sub-automaton of Γ , then every state q of Γ_1 defines the same transformation as the state q of Γ .

Definition 3.2. Let H be a self-similar semigroup and let $\Gamma(H)$ be its full automaton. We say that the semigroup H is *contracting* if there exists a finite subautomaton $\mathcal{N} \subset \Gamma(H)$ such that every infinite path e_1, e_2, \dots in the automaton $\Gamma(H)$ has only a finite number of arrows e_i outside \mathcal{N} .

The minimal set \mathcal{N} satisfying the conditions of the definition is called the *nucleus* of the action.

Proposition 3.1. *The nucleus exists if the semigroup is contracting. It consists of all the states g of $\Gamma(H)$ such that there exists a directed path to g from a directed cycle in $\Gamma(H)$.*

Proof. Suppose that the semigroup is contracting. Let N be any finite subautomaton satisfying the conditions of Definition 3.2. Every cycle of $\Gamma(H)$ must belong to N , since there exists a path going infinitely many times through each of its arrows. Consequently, every end of a path starting in a vertex of a cycle must belong to N .

Let us prove that the set C of the states g which are ends of the directed paths starting in elements of the cycles is the nucleus. We have shown that it is a sub-automaton of any finite automaton, satisfying the conditions of Definition 3.2. It remains to show that C itself satisfies the conditions of Definition 3.2.

Let e_1, e_2, \dots be an arbitrary infinite directed path. Its all but finite number of arrows belong to a finite set satisfying the conditions of Definition 3.2. Hence, some arrow e_i appears in this path twice. All the states between these two appearances belong to a cycle in $\Gamma(H)$. Consequently, all the states appearing after them belong to C . \square

Lemma 3.2. *Let H be a contracting self-similar inverse semigroup acting on a Markov shift \mathcal{F} . Then for every $g \in H$ there exists $n_0 \in \mathbb{N}$ such that every path in $\Gamma(H)$ of length $\geq n_0$ which starts in g ends in an element of the nucleus.*

Proof. Let $w \in \text{Dom } g$ be an arbitrary sequence. Then the path starting in g and defining the transition $(w; g(w))$ has all edges inside the nucleus \mathcal{N} , except for some finite beginning. Let n_w be the length of this beginning, and let $h_w \in \mathcal{N}$ be its end. Then we have for every $u \in \text{Dom } h_w$ that

$$g(w_0u) = v_0h_w(u),$$

where w_0 and v_0 are the beginnings of length n_w of the sequences w and $g(w)$, respectively.

The sets of the form $w_0 \text{Dom } h_w$ are open and cover the set $\text{Dom } g$. Thus there exists a finite subcover $\{w_1 \text{Dom } h_1, w_2 \text{Dom } h_2, \dots, w_m \text{Dom } h_m\}$, where w_i are some finite sequences and h_i are elements of the nucleus. Then the maximal length n_0 of the sequences w_i satisfies the conditions of the lemma. \square

Corollary 3.3. *Suppose that a recurrent self-similar semigroup H acting on \mathcal{F} is contracting and let \mathcal{N} be its nucleus. Then for every $w \in \mathcal{F}$ there exists an idempotent $1_U \in H$ defined by an element of \mathcal{N} such that w belongs to the domain U of 1_U .*

Proof. The space \mathcal{F} is covered, by definition, by the domains of the idempotents from H . The domains are open, thus the space \mathcal{F} is covered by domains U_i of a finite number of idempotents $\{1_{U_1}, 1_{U_2}, \dots, 1_{U_m}\} \subset H$.

There exists, by Lemma 3.2, for every $1_{U_i} \in H$ a number n_i such that every path of length n_i starting in $1_{U_i} \in \Gamma(H)$ ends in an element of the nucleus. Every such an end is obviously an idempotent. Let $n = \max n_i$. Then the sets $\sigma^n(U_i)$ and, consequently, the space $\mathcal{F} = \sigma^n(\mathcal{F})$ are covered by the domains of idempotents belonging to \mathcal{N} . \square

Denote by $\mathcal{G}(\mathcal{N})$ the set of germs $(h, w) \in \mathcal{G}(H)$ such that $h \in \mathcal{N}$. Then the condition in Definition 3.2 can be formulated as the requirement that $\sigma_*^n((h, w)) \in \mathcal{G}(\mathcal{N})$ for all (h, w) and for all $n = n(h, w)$ big enough. Note also that the set $\mathcal{G}(\mathcal{N})$ is a compact σ_* -invariant subset of $\mathcal{G}(H)$.

3.2 Limit solenoid

Definition 3.3. Let H be a recurrent contracting self-similar semigroup, acting on a shift space \mathcal{F} . Let \mathcal{N} be its nucleus and let $\mathcal{F}_{\mathbb{Z}}$ be the natural extension of \mathcal{F} . We say that two bi-infinite sequences $(\dots x_{-2}x_{-1} \cdot x_0x_1 \dots), (\dots y_{-2}y_{-1} \cdot y_0y_1 \dots) \in \mathcal{F}_{\mathbb{Z}}$ are *asymptotically equivalent* if there exists a bi-infinite path in the full automaton $\Gamma(H)$ labeled by the consecutive labels

$$\dots (x_{-2}; y_{-2}), (x_{-1}; y_{-1}), (x_0; y_0), (x_1; y_1) \dots$$

with all its vertices belonging to the nucleus \mathcal{N} of the semigroup.

Lemma 3.4. *Two sequences $(\dots x_{-2}x_{-1} \cdot x_0x_1 \dots)$ and $(\dots y_{-2}y_{-1} \cdot y_0y_1 \dots) \in \mathcal{F}_{\mathbb{Z}}$ are asymptotically equivalent if and only if for every $n \in \mathbb{Z}$ there exists $g_n \in \mathcal{N}$ such that*

$$g_n(x_n x_{n+1} \dots) = y_n y_{n+1} \dots$$

Proof. The “only if” part of the lemma is trivial.

Let $A_n \subset \mathcal{N}$ be the set of all elements g_n such that $g_n(x_n x_{n+1} \dots) = y_n y_{n+1} \dots$. It is a non-empty finite set, and we have a natural map $A_{n-1} \rightarrow A_n$, mapping a state g_n to the next state g_{n-1} in the path defining the respective transition. In this way we get an inverse sequence of finite sets. It is well known that the inverse limit of this sequence is not empty. Any point of the inverse limit defines an infinite path labeled by the labels $\dots (x_{-1}; y_{-1}), (x_0; y_0), (x_1; y_1), \dots$ \square

Proposition 3.5. *The asymptotic equivalence is an equivalence relation on $\mathcal{F}_{\mathbb{Z}}$ and the quotient of the space $\mathcal{F}_{\mathbb{Z}}$ by this equivalence relation is a metrizable topological space of topological dimension less than $|\mathcal{N}|$.*

Proof. It is easy to prove that if $g \in \mathcal{N}$, then $g^{-1} \in \mathcal{N}$ and thus the asymptotic equivalence is symmetric. The fact that it is reflexive follows directly from Corollary 3.3 and Lemma 3.4. Let us prove that it is transitive.

Suppose that a sequence $\xi = \dots x_{-2}x_{-1} \cdot x_0x_1 \dots \in \mathcal{F}_{\mathbb{Z}}$ is equivalent to a sequence $\eta = \dots y_{-2}y_{-1} \cdot y_0y_1 \dots \in \mathcal{F}_{\mathbb{Z}}$ and η is equivalent to $\zeta = \dots z_{-2}z_{-1} \cdot z_0z_1 \dots \in \mathcal{F}_{\mathbb{Z}}$. Let $(\dots g_{-1}, g_0, g_1 \dots)$ be such a sequence of elements of \mathcal{N} that $g_n(x_n x_{n+1} \dots) = y_n y_{n+1} \dots$ and let $(\dots h_{-1}, h_0, h_1 \dots)$ be such a sequence of elements of \mathcal{N} that $h_n(y_n y_{n+1} \dots) = z_n z_{n+1} \dots$. Then

$$h_n g_n(x_n x_{n+1} \dots) = z_n z_{n+1} \dots$$

There exists, by Lemma 3.2, a number $n_0 \in \mathbb{N}$ such that every path of the length $\geq n_0$ in $\Gamma(H)$, starting in a product of two elements of \mathcal{N} , ends in \mathcal{N} . This implies that the transition $(x_n x_{n+1} \dots; z_n z_{n+1} \dots)$ is defined by an element of the nucleus for every $n \in \mathbb{Z}$ and the sequences ξ and ζ are equivalent by Lemma 3.4.

It follows directly from the definitions that the asymptotic equivalence relation is closed (since even its graph is closed). It follows also that for every sequence $\xi = \dots x_{-2}x_{-1} \cdot x_0x_1 \dots \in \mathcal{F}_{\mathbb{Z}}$ and for every $n \in \mathbb{Z}$ the number of possible tails $y_n y_{n+1} \dots$ of a sequence $\dots y_{-2}y_{-1} \cdot y_0y_1 \dots$ asymptotically equivalent to ξ is not greater than $|\mathcal{N}|$. This implies that every asymptotic equivalence class on $\mathcal{F}_{\mathbb{Z}}$ has not more than $|\mathcal{N}|$ elements.

Now by Theorem 4.2.13 from [Eng77], the quotient space is metrizable, since it is a quotient of a compact separable metrizable space $\mathcal{F}_{\mathbb{Z}}$ by a closed equivalence relation with compact equivalence classes. The assertion about the dimension follows from Hurewicz formula (see [Kur61] page 52). \square

Definition 3.4. The quotient of the space $\mathcal{F}_{\mathbb{Z}}$ by the asymptotic equivalence relation is called the *limit solenoid* of the self-similar semigroup and is denoted \mathcal{S}_H .

The shift $\sigma_{\mathbb{Z}} : \mathcal{F}_{\mathbb{Z}} \rightarrow \mathcal{F}_{\mathbb{Z}}$ induces a homeomorphism of the limit solenoid \mathcal{S}_H , since the asymptotic equivalence relation agrees with the shift. Let us denote this homeomorphism by s . We have associated in this way a dynamical system (\mathcal{S}_H, s) with every contracting recurrent action of an inverse semigroup.

Definition 3.5. A *tile* \mathcal{T}_w corresponding to a sequence $w \in \mathcal{F}$ is the image of the set of the sequences of the form $\dots x_{-2}x_{-1} \cdot w \in \mathcal{F}_{\mathbb{Z}}$ in the solenoid \mathcal{S}_H . A *leaf* of the solenoid is the union of all the tiles \mathcal{T}_w , for which w belongs to a given H -orbit.

Every tile \mathcal{T}_w is a closed subset of \mathcal{S}_H , and the solenoid \mathcal{S}_H is a disjoint union of the leaves. It follows directly from the definition that

$$s(\mathcal{T}_w) = \bigcup_{x \in X} \mathcal{T}_{xw},$$

where \mathcal{T}_{xw} is considered to be empty if $xw \notin \mathcal{F}$. This relation is similar to the *inflation* of substitution tilings, see [AP98].

3.3 Examples

Adding machine. The nucleus of the adding machine action consists of the states (vertices) $a, 1, a^{-1}$ (see [BGN03]). It follows directly from the diagram of the nucleus that two sequences $(\dots x_{-2}x_{-1} \cdot x_0x_1 \dots), (\dots y_{-2}y_{-1} \cdot y_0y_1 \dots) \in \{0, 1\}^{\mathbb{Z}}$ are asymptotically equivalent if and only if they are either equal or are of the form $(\dots 0001a_n a_{n+1} \dots), (\dots 1110a_n a_{n+1} \dots)$. But this is the usual identification of the reals written in the binary numeration system. More precisely, the sequences are asymptotically equivalent if and only if

$$\sum_{n=0}^{+\infty} x_n \cdot 2^n - \sum_{n=0}^{+\infty} y_n \cdot 2^n = \sum_{n=-\infty}^{-1} y_n \cdot 2^n - \sum_{n=-\infty}^{-1} x_n \cdot 2^n, \quad (1)$$

where the left-hand side of the equality is computed in the ring \mathbb{Z}_2 of dyadics, the right-hand side in the field \mathbb{R} of real numbers, but both sides must belong to \mathbb{Z} and be equal.

The limit solenoid of the adding machine action is then the well known *Smale solenoid*, homeomorphic to the inverse limit of the circle \mathbb{R}/\mathbb{Z} with respect to the iterations of its double self-covering $f : t \mapsto 2t$ (see, for example [Put96] and [BGN03]). The respective shift is induced by f on the inverse limit.

This solenoid $\mathcal{S}_{\mathbb{Z}}$ can be also constructed in the following way. Take the direct product $[0, 1] \times X^{\mathbb{N}}$ of the real segment by the Cantor space $X^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ and identify every point $(1, w)$ with the point $(0, a(w))$, where a is the adding machine. The obtained space is homeomorphic to the limit solenoid $\mathcal{S}_{\mathbb{Z}}$ of the adding machine action. This follows easily from (1).

Penrose tilings. The self-similar semigroup associated with the Penrose tiling is contracting. The tiled leaves of its limit solenoid are exactly the Penrose tilings. (Except for the tilings having a non-trivial symmetry group; then the respective leaf of the solenoid is the quotient of the tiling by its symmetry group.)

The limit solenoid is homeomorphic to the space of Penrose tilings with marked points. Two marked Penrose tilings are close in this space if the tilings can be superimposed by an affine transformation in such a way that the marked points will be close to each other and the tilings agree on a large ball around one of the marked point. For an exact definition of this space, see the paper [AP98]. Note, however, that we do not care about the orientation of the tiles. Respectively, our space is a finite-to-one quotient of the space defined in [AP98].

The fact that the semigroup is contracting and the above description of the limit space follow easily from Theorem 4.6 below.

4 Smale spaces

4.1 Definitions

Let us define the notion of a *Smale space*, following [Rue78] and [Put96]. Let (Ω, d) be a compact metric space together with a homeomorphism $f : \Omega \rightarrow \Omega$. We require that there exists a number

$\epsilon_0 > 0$ and a continuous function (*canonical coordinates*):

$$[\cdot, \cdot] : \{(\xi, \eta) \in \Omega \times \Omega : d(\xi, \eta) < \epsilon_0\} \longrightarrow \Omega$$

such that

$$[\xi, \xi] = \xi, \quad [[\xi, \eta], \zeta] = [\xi, \zeta], \quad [\xi, [\eta, \zeta]] = [\xi, \zeta]$$

for all $\xi, \eta, \zeta \in \Omega$ for which the respective expressions are defined.

Define for every $0 < \epsilon \leq \epsilon_0$

$$W_\epsilon^s(\xi) = \{\eta \in \Omega : [\xi, \eta] = \eta, \quad d(\xi, \eta) < \epsilon\} \quad (2)$$

$$W_\epsilon^u(\xi) = \{\eta \in \Omega : [\eta, \xi] = \eta, \quad d(\xi, \eta) < \epsilon\}. \quad (3)$$

We require that

$$f([\xi, \eta]) = [f(\xi), f(\eta)],$$

whenever both sides of the equality are defined and that

$$\begin{aligned} d(f(\eta), f(\zeta)) &\leq \lambda_0 d(\eta, \zeta), & \eta, \zeta \in W_\epsilon^s(\xi) \\ d(f^{-1}(\eta), f^{-1}(\zeta)) &\leq \lambda_0 d(\eta, \zeta), & \eta, \zeta \in W_\epsilon^u(\xi) \end{aligned}$$

for some fixed $0 < \lambda_0 < 1$.

Then we have for every $\xi \in \Omega$ a homeomorphism

$$[\cdot, \cdot] : W_{\epsilon_0/2}^u(\xi) \times W_{\epsilon_0/2}^s(\xi) \longrightarrow \Omega$$

onto a neighborhood of ξ , and

$$\{[\xi, \eta]\} = W_{\epsilon_0}^s(\xi) \cap W_{\epsilon_0}^u(\eta).$$

In general, a subset $R \subset \Omega$ is called *rectangle*, if its diameter is less than ϵ_0 and for every point $\xi \in R$ the map

$$[\cdot, \cdot] : W_{\epsilon_0}^u(\xi, R) \times W_{\epsilon_0}^s(\xi, R) \longrightarrow R$$

is a homeomorphism, where $W_{\epsilon_0}^u(\xi, R) = W_{\epsilon_0}^u(\xi) \cap R$ and $W_{\epsilon_0}^s(\xi, R) = W_{\epsilon_0}^s(\xi) \cap R$.

The next lemma follows directly from the definition.

Lemma 4.1. *If R is a rectangle and $\xi_1, \xi_2 \in R$, then the map*

$$W_{\epsilon_0}^s(\xi_1, R) \longrightarrow W_{\epsilon_0}^s(\xi_2, R) : \eta \mapsto [\xi_2, \eta].$$

is a homeomorphism. Similarly, the map $\eta \mapsto [\eta, \xi_2]$ is a homeomorphism from $W_{\epsilon_0}^u(\xi_1, R)$ to $W_{\epsilon_0}^u(\xi_2, R)$.

A *Markov partition* of a Smale space (see [Bow75]) is a finite cover \mathcal{M} of the space Ω by closed proper rectangles with disjoint interiors (a rectangle R is said to be *proper* if $R = \overline{\text{int } R}$), such that if $\xi \in \text{int } R$ and $f(\xi) \in \text{int } R'$ for $R, R' \in \mathcal{M}$, then

$$f(W_{\epsilon_0}^s(\xi, R)) \subseteq R' \quad \text{and} \quad f^{-1}(W_{\epsilon_0}^u(f(\xi), R')) \subseteq R.$$

If R is a closed rectangle, then (see [Bow75]) its boundary ∂R is a union $\partial^s R \cup \partial^u R$, where

$$\begin{aligned} \partial^s R &= \{\xi \in R : \xi \notin \text{int } W_{\epsilon_0}^u(\xi, R)\}, \\ \partial^u R &= \{\xi \in R : \xi \notin \text{int } W_{\epsilon_0}^s(\xi, R)\}, \end{aligned}$$

where the interiors of $W_{\epsilon_0}^u(\xi, R)$ and $W_{\epsilon_0}^s(\xi, R)$ are taken as in subsets of the topological spaces $W_{\epsilon_0}^u(\xi)$ and $W_{\epsilon_0}^s(\xi)$, respectively.

Definition 4.1. We say that a Markov partition \mathcal{M} is *s-resolved* if for every rectangle $R \in \mathcal{M}$ and $\xi \in R$ the set $W_{\epsilon_0}^s(\xi, R)$ is open in $W_{\epsilon_0}^s(\xi)$, i.e., if we have

$$\partial R = \partial^s R.$$

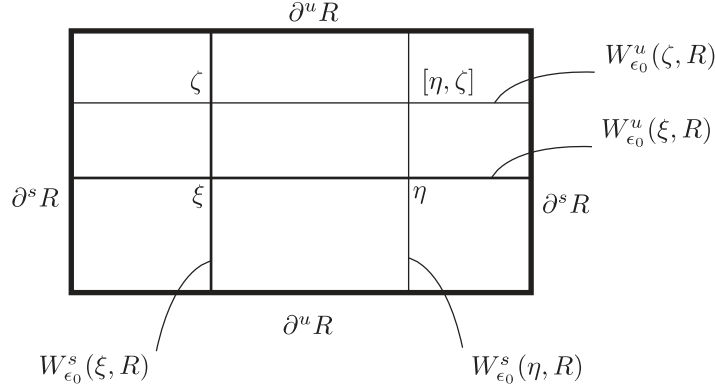


Figure 6: A rectangle.

The set $W_{\epsilon_0}^s(\xi, R)$ is always closed, since R is closed, $W_{\epsilon_0}^s(\xi, R) = \{\eta \in R : [\xi, \eta] = \eta\}$ and the map $[\cdot, \cdot]$ is continuous. Hence, if the Markov partition is s -resolved, then the sets $W_{\epsilon_0}^s(\xi, R)$ are clopen in $W_{\epsilon_0}^s(\xi)$ (and totally disconnected, as we will see later).

Fix an alphabet X together with a one-to-one correspondence $a \mapsto R_a$ with the elements of a Markov partition \mathcal{M} of a Smale space (Ω, d, f) . Then for every point $\xi \in \Omega$ its *code* is a sequence $(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots) \in X^{\mathbb{Z}}$ such that for every $n \in \mathbb{N}$ we have

$$\xi \in \bigcap_{-n \leq k \leq n} f^k(R_{a_k}) = C_n$$

and the set C_n has a non-empty interior.

Note that the diameter of the set C_n is less than $\lambda_0^n \epsilon_0$. Consequently, there is not more than one point having a given code. On the other hand, one point can have different codes, since the elements of the Markov partition can have non-empty intersections.

The following result is classical.

Lemma 4.2. *The set $\mathcal{F}_{\mathbb{Z}}$ of all codes of points $\xi \in \Omega$ is equal to the set of all sequences $(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots) \in X^{\mathbb{Z}}$ such that the set $\text{int } R_{a_k} \cap \text{int } f(R_{a_{k+1}})$ is not empty for all $k \in \mathbb{Z}$.*

The map $\pi : \mathcal{F}_{\mathbb{Z}} \rightarrow \Omega$, putting into correspondence to a code the respective point of Ω is a continuous surjection such that

$$f(\pi(w)) = \pi(\sigma_{\mathbb{Z}}(w)).$$

Consequently, the set $\mathcal{F}_{\mathbb{Z}}$ is a Markov shift space. We denote by \mathcal{F} the respective one-sided shift.

The following lemma follows directly from the definitions.

Lemma 4.3. *If $\pi(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots) = \xi$ and $\pi(\dots b_{-2}b_{-1} \cdot b_0b_1 \dots) = \eta$, where $a_0 = b_0$ and $\xi, \eta \in R_{a_0}$, then $\pi(\dots a_{-2}a_{-1} \cdot b_0b_1 \dots) = [\xi, \eta]$. In particular, the set $W_{\epsilon_0}^s(R_{a_0}, \xi)$ coincides with the set of points having codes of the form $(\dots a_{-2}a_{-1} \cdot a_0x_1x_2 \dots)$.*

Let us prove the following fact about s -resolved Markov partitions.

Lemma 4.4. *Let \mathcal{M} be an s -resolved Markov partition of a Smale space (Ω, d, f) . For every $R_{a_0} \in \mathcal{M}$ and every point $\xi \in R_{a_0}$ having a code $(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots)$, the restriction of π onto the set $\mathcal{F}^s(\xi)$ of sequences of the form $\dots a_{-2}a_{-1} \cdot a_0x_1x_2 \dots \in \mathcal{F}_{\mathbb{Z}}$ is a homeomorphism onto the space $W_{\epsilon_0}^s(\xi, R_{a_0})$.*

Proof. It is sufficient to prove that the restriction of the map π onto the set $\mathcal{F}^s(\xi)$ is injective, since it is surjective by Lemma 4.3 and the spaces $\mathcal{F}^s(\xi)$ and $W_{\epsilon_0}^s(\xi, R_{a_0})$ are compact.

Suppose at first that $\xi \in \text{int } R_{a_0}$. Then $W_{\varepsilon_0}^s(\xi, R_{a_0}) \subset \text{int } R_{a_0}$, by definition of an s -resolved Markov partition. Let $\eta \in W_{\varepsilon_0}^s(\xi, R_{a_0})$ be an arbitrary point and let $(\dots a_{-2}a_{-1} \cdot a_0b_0b_1 \dots)$ be its code. The point η is an internal point of R_{a_0} and thus it is an internal point of $W_{\varepsilon_0}^u(\eta, R_{a_0})$. Then for every $n \in \mathbb{N}$ the point $f^{-n}(\eta)$ is an internal point of the set $f^{-n}(W_{\varepsilon_0}^u(\eta, R_{a_0})) \subset R_{b_n}$, and thus is an internal point of R_{b_n} , by definition of an s -resolved Markov partition. Consequently, the point η belongs only to the rectangle R_{b_n} and the sequence $b_1b_2 \dots$ is uniquely determined by $(\dots a_{-2}a_{-1}a_0)$ and η , i.e., the restriction of the map π onto $\mathcal{F}^s(\xi)$ is injective.

For other points of R_{a_0} one needs just to apply Lemma 4.1 and Lemma 4.3. \square

4.2 Adjacency semigroup

Let us fix now some s -resolved Markov partition $\mathcal{M} = \{R_a\}_{a \in X}$ of a Smale space (Ω, d, f) .

Let $a_0, b_0 \in X$ be arbitrary letters such that the intersection $R_{a_0} \cap R_{b_0}$ is not empty, and let $\xi \in R_{a_0} \cap R_{b_0}$. Then the point ξ has codes of the form $(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots)$ and $(\dots b_{-2}b_{-1} \cdot b_0b_1 \dots)$.

Define now a partial transformation $T_{(a_0; b_0)}$ of the space \mathcal{F} , which maps a sequence $a_0x_1x_2 \dots \in \mathcal{F}$ to a sequence $b_0y_1y_2 \dots \in \mathcal{F}$ if and only if

$$\pi(\dots a_{-2}a_{-1} \cdot a_0x_1x_2 \dots) = \pi(\dots b_{-2}b_{-1} \cdot b_0y_1y_2 \dots).$$

Proposition 4.5. *The transformation $T_{(a_0; b_0)}$ is a well defined local homeomorphism of the space \mathcal{F} depending only on the letters a_0 and b_0 .*

Proof. Let us prove that the transformation $T_{(a_0; b_0)}$ does not depend on the choice of the point ξ and its codes $w_1 = (\dots a_{-2}a_{-1} \cdot a_0a_1 \dots)$, $w_2 = (\dots b_{-2}b_{-1} \cdot b_0b_1 \dots)$. Let

$$w'_1 = (\dots a'_{-2}a'_{-1} \cdot a'_0a'_1 \dots), \quad w'_2 = (\dots b'_{-2}b'_{-1} \cdot b'_0b'_1 \dots) \in \mathcal{F}$$

be codes of a point ξ' such that $a'_0 = a_0$ and $b'_0 = b_0$.

Let $(\dots a_{-2}a_{-1} \cdot a_0x_1x_2 \dots)$ and $(\dots b_{-2}b_{-1} \cdot b_0y_1y_2 \dots)$ be codes of a point ζ . Then, by Lemma 4.3, the sequences $(\dots a'_{-2}a'_{-1} \cdot a'_0x_1x_2 \dots)$ and $(\dots b'_{-2}b'_{-1} \cdot b'_0y_1y_2 \dots)$ are codes of the point $[\xi', \zeta]$. Hence, the transformation $T_{(a_0; b_0)}$ defined by ξ , w_1 and w_2 is the same as the transformation defined by ξ' , w'_1 and w'_2 .

Let us prove that the transformation $T_{(a_0; b_0)}$ is a local homeomorphism (in particular, that it is a well defined map). Let $\xi \in R_{a_0} \cap R_{b_0}$ be a point, and let $(\dots a_{-2}a_{-1} \cdot a_0a_1 \dots)$ and $(\dots b_{-2}b_{-1} \cdot b_0b_1 \dots)$ be codes of the points ξ . Recall that $\mathcal{F}^s(\xi)$ denotes the set of sequences of the form $(\dots a_{-2}a_{-1} \cdot a_0x_1x_2 \dots) \in \mathcal{F}$. The map

$$\rho_\xi : a_0x_1x_2 \dots \mapsto \dots a_{-2}a_{-1} \cdot a_0x_1x_2 \dots$$

is a homeomorphism from the space \mathcal{F}_{a_0} of all the sequences belonging to \mathcal{F} and starting with a_0 to the set $\mathcal{F}^s(\xi)$. Let us denote by $\pi_{a_0, \xi}$ the homeomorphism $\pi|_{\mathcal{F}^s(\xi)} \circ \rho_\xi : \mathcal{F}_{a_0} \longrightarrow W_{\varepsilon_0}^s(\xi, R_{a_0})$. It is a homeomorphism by Lemma 4.4.

Denote

$$\begin{aligned} D_{a_0} &= \pi_{a_0, \xi}^{-1} (W_{\varepsilon_0}^s(\xi, R_{a_0}) \cap W_{\varepsilon_0}^s(\xi, R_{b_0})) \\ D_{b_0} &= \pi_{b_0, \xi}^{-1} (W_{\varepsilon_0}^s(\xi, R_{a_0}) \cap W_{\varepsilon_0}^s(\xi, R_{b_0})). \end{aligned}$$

Then, by Lemma 4.4 the set D_{a_0} is the domain of the map $T_{(a_0; b_0)}$, and

$$T_{(a_0; b_0)} = (\pi|_{D_{a_0}})^{-1} \circ \pi|_{D_{b_0}},$$

so that $T_{(a_0; b_0)}$ is a local homeomorphism. \square

Note that the transformation $T_{(a_0; a_0)}$ is the idempotent with domain \mathcal{F}_{a_0} .

Definition 4.2. The *adjacency semigroup* of an s -resolved Markov partition is the inverse semigroup generated by the set of partial homeomorphisms $\mathcal{T} = \{T_{(a_0; b_0)} : R_{a_0} \cap R_{b_0} \neq \emptyset\}$ of the space \mathcal{F} .

The groupoid of germs of the adjacency semigroup can be interpreted as the *holonomy groupoid* of the unstable foliation coming from the local structure of a direct product on the Smale space. See [Hae01] for the definition of the holonomy groupoid of a foliation.

Theorem 4.6. *The adjacency semigroup H of a Smale space (Ω, f) is self-similar and contracting. Its limit dynamical system (\mathcal{S}_H, s) is topologically conjugate to the system (Ω, f) . In particular, the limit solenoid \mathcal{S}_H is homeomorphic to Ω .*

Proof. Let us prove that the semigroup H is self-similar. We shall construct the automaton generating the semigroup H and use then Proposition 2.2.

Let us construct a directed graph Γ with the vertices \mathcal{T} and connect the vertex $T_{(a_0; b_0)}$ to the vertex $T_{(a_1; b_1)}$ by an arrow labeled by $(a_0; b_0)$ if and only if there exists a pair of sequences

$$(\dots a_{-2} a_{-1} \cdot a_0 a_1 a_2 \dots), \quad (\dots b_{-2} b_{-1} \cdot b_0 b_1 b_2 \dots) \in \mathcal{F}_{\mathbb{Z}},$$

representing the same point of the Smale space Ω .

Let us prove that the constructed automaton defines the transformations $T_{(a; b)}$ and thus generates the semigroup H .

Let $T_{(a_0; b_0)}$ be an element of \mathcal{T} and let $T_{(a_1^{(k)}; b_1^{(k)})}$, $k = 1, \dots, m$, be all the ends of the arrows starting in $T_{(a_0; b_0)}$. Then for every k there exists a pair of codes

$$(\dots a_{-2}^{(k)} a_{-1}^{(k)} \cdot a_0^{(k)} a_1^{(k)} a_2^{(k)} \dots), \quad (\dots b_{-2}^{(k)} b_{-1}^{(k)} \cdot b_0^{(k)} b_1^{(k)} b_2^{(k)} \dots) \in \mathcal{F}_{\mathbb{Z}},$$

of a point $\xi_k \in \Omega$ such that $a_0^{(k)} = a_0$ and $b_0^{(k)} = b_0$. It follows directly from the definition of the transformation $T_{(a_1^{(k)}; b_1^{(k)})}$ (and the fact that it is well defined) that the condition $a_1^{(k)} x_2 x_3 \dots \in \text{Dom } T_{(a_1^{(k)}; b_1^{(k)})}$ is equivalent to the condition $a_0 a_1^{(k)} x_2 x_3 \dots \in \text{Dom } T_{(a_0; b_0)}$ and that the equality

$$T_{(a_0; b_0)} \left(a_0 a_1^{(k)} x_2 x_3 \dots \right) = b_0 T_{(a_1^{(k)}; b_1^{(k)})} \left(a_1^{(k)} x_2 x_3 \dots \right) \quad (4)$$

holds. In particular, we have

$$\text{Dom } T_{(a_0; b_0)} = \bigsqcup_{k=1, \dots, m} a_0 \text{Dom } T_{(a_1^{(k)}; b_1^{(k)})}. \quad (5)$$

The union is disjoint, since every sequence $a_0 a_1 \dots \in \text{Dom } T_{(a_0; b_0)}$ belongs only to $a_0 \text{Dom } T_{(a_1; b_1)}$, where b_1 is the second letter of the sequence $T_{(a_0; b_0)}(a_0 a_1 \dots)$.

Repeating now the arguments from the proof of Proposition 2.1, we deduce from (4) and (5) that the state $T_{(a_0; b_0)}$ of the automaton Γ defines the transformation $T_{(a_0; b_0)}$.

Let us prove that the semigroup H is contracting. We need at first some auxiliary definitions and facts.

Let R_1, R_2, R_3 be three rectangles from the Markov partition, and denote by Ξ_2 the set all $\xi_2 \in R_2$ such that there exist $\xi_1 \in R_1$ and $\xi_3 \in R_3$ such that

$$W_{\varepsilon_0}^u(\xi_1, R_1) \cap W_{\varepsilon_0}^u(\xi_2, R_2) \neq \emptyset, \quad W_{\varepsilon_0}^u(\xi_2, R_2) \cap W_{\varepsilon_0}^u(\xi_3, R_3) \neq \emptyset. \quad (6)$$

Note that the sets $W_{\varepsilon_0}^u(\xi_1, R_1)$ and $W_{\varepsilon_0}^u(\xi_3, R_3)$ depend only on $W_{\varepsilon_0}^u(\xi_2, R_2), R_1, R_2, R_3$.

It is easy to see that the set Ξ_2 is a closed rectangle and is the image of the set $W_{\varepsilon_0}^u(\xi_2, R_2) \times S$ under the homeomorphism $[\cdot, \cdot]$, where S is a closed subset of $W_{\varepsilon_0}^s(\xi_2, R_2)$ (see Figure 7).

The set

$$(\Xi_2 \cap R_1) \cap (\Xi_2 \cap R_3)$$

is also a rectangle and is equal to the image of the set $U(\xi_2) \times S$ under the homeomorphism $[\cdot, \cdot]$, where

$$U(\xi_2) = W_{\varepsilon_0}^u(\xi_1, R_1) \cap W_{\varepsilon_0}^u(\xi_2, R_2) \cap W_{\varepsilon_0}^u(\xi_3, R_3),$$

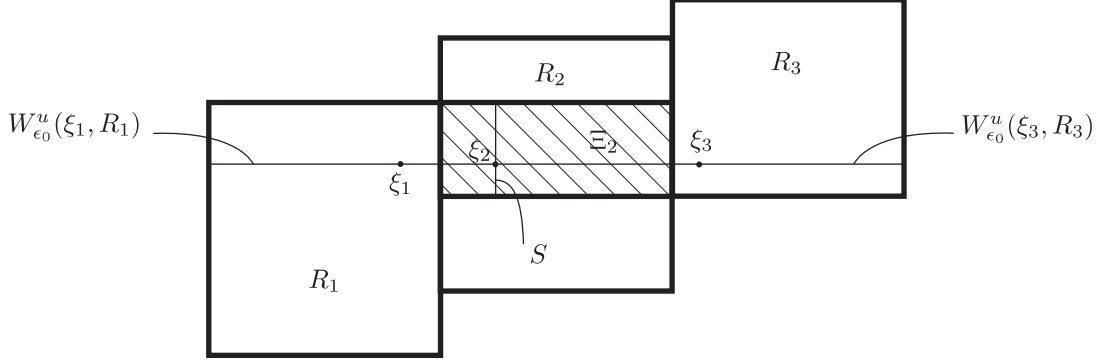


Figure 7: The set Ξ_2 .

and ξ_1, ξ_3 are some points satisfying conditions (6). In particular, the set $U(\xi_2)$ does not depend, up to a homeomorphism on $\xi_2 \in \Xi_2$.

Define $c(R_1, R_2, R_3)$ to be equal to the distance between the compact sets $\Xi_2 \cap R_1$ and $\Xi_2 \cap R_3$, if they are disjoint, i.e., if the set $U(\xi_2)$ (for some, and thus for all $\xi_2 \in \Xi_2$) is empty. Let c be the minimal among the defined numbers $c(R_1, R_2, R_3)$ for all possible triples of rectangles $R_1, R_2, R_3 \in \mathcal{M}$. If the set of the defined numbers $c(R_1, R_2, R_3)$ is empty, then we choose c to be any positive number.

The positive number c is so defined that if $R_1, R_2, R_3 \in \mathcal{M}$ and $\xi_i \in R_i$ are such that $\xi_1, \xi_2 \in R_2$, conditions (6) hold and $d(\xi_1, \xi_3) < c$, then $W_{\epsilon_0}^u(\xi_1, R_1) \cap W_{\epsilon_0}^s(\xi_2, R_2) \cap W_{\epsilon_0}^s(\xi_3, R_3) \neq \emptyset$.

Let now $T_{(b_0; c_0)}, T_{(a_0; b_0)}$ be two generators of the adjacency semigroup H , and sequences $a_0 a_1 \dots, b_0 b_1 \dots, c_0 c_1 \dots \in \mathcal{F}$ are such that $a_0 a_1 \dots \in \text{Dom}(T_{(b_0; c_0)} \cdot T_{(a_0; b_0)})$ and

$$\begin{aligned} b_0 b_1 \dots &= T_{(a_0; b_0)}(a_0 a_1 \dots), \\ c_0 c_1 \dots &= T_{(b_0; c_0)}(b_0 b_1 \dots). \end{aligned}$$

Then there exist sequences from $\mathcal{F}_{\mathbb{Z}}$ and points of Ω such that

$$\begin{aligned} \xi &= \pi(\dots a_{-2} a_{-1} \cdot a_0 a_1 a_2 \dots) = \pi(\dots b_{-2} b_{-1} \cdot b_0 b_1 b_2 \dots) \\ \eta &= \pi(\dots b'_{-2} b'_{-1} \cdot b_0 b_1 b_2 \dots) = \pi(\dots c_{-2} c_{-1} \cdot c_0 c_1 c_2 \dots). \end{aligned}$$

The definition of a Smale space implies that for every $n > \frac{\log(c/\text{Diam } R_{b_0})}{-\log \lambda_0}$ the inequality $d(f^{-n}(\xi), f^{-n}(\eta)) < c$ holds. We have $f^{-n}(\xi) \in R_{a_n} \cap R_{b_n}$ and $f^{-n}(\eta) \in R_{b_n} \cap R_{c_n}$. We also have $W_{\epsilon_0}^u(\xi, R_{b_n}) = W_{\epsilon_0}^u(\eta, R_{b_n})$. Then it follows from the choice of the constant c that the intersection $W_{\epsilon_0}^u(\xi, R_{a_n}) \cap W_{\epsilon_0}^u(\eta, R_{c_n})$ is not empty. This implies that $T_{(a_n; c_n)}(a_n a_{n+1} \dots) = c_n c_{n+1} \dots$ and that the beginning of length n of the path starting in $T_{(b_0; c_0)} \cdot T_{(a_0; b_0)}$ and defining the transition $(a_0 a_1 \dots; c_0 c_1 \dots)$ ends in $T_{(a_n; c_n)}$.

We have proved that a path in the full automaton of the adjacency semigroup which starts in a product of two generators is eventually trapped in the set $\mathcal{T} = \{T_{(a; b)} : a, b \in X\}$. But it follows then by induction that the path starting in a product of any number of generators is also eventually trapped in the set \mathcal{T} . This shows that the semigroup H is contracting with the nucleus a subset of \mathcal{T} .

Let us prove that the dynamical systems (Ω, f) and (\mathcal{S}_H, s) are topologically conjugate. Both Ω and \mathcal{S}_H are quotients of the topological space $\mathcal{F}_{\mathbb{Z}}$. It is sufficient to prove that the respective equivalence relations are equal. Then the conjugacy of the dynamical systems will follow automatically, due to the definition of the map s and Lemma 4.2.

Let $w_1 = (\dots a_{-2} a_{-1} \cdot a_0 a_1 \dots)$ and $w_2 = (\dots b_{-2} b_{-1} \cdot b_0 b_1 \dots)$ be two points, which are asymptotically equivalent in $\mathcal{F}_{\mathbb{Z}}$ with respect to the action of the semigroup H . Then for every

$n \in \mathbb{Z}$ there exists a pair of sequences

$$w_1^{(n)} = (\dots a_{-2}^{(n)} a_{-1}^{(n)} \cdot a_0^{(n)} a_1^{(n)} \dots), \quad w_2^{(n)} = (\dots b_{-2}^{(n)} b_{-1}^{(n)} \cdot b_0^{(n)} b_1^{(n)} \dots) \in \mathcal{F}_{\mathbb{Z}}$$

such that $\pi(w_1^{(n)}) = \pi(w_2^{(n)})$ and $a_k = a_k^{(n)}$, $b_k = b_k^{(n)}$ for all $k \geq n$ (we used here the fact that the nucleus of H is a subset of \mathcal{T}). Then

$$\pi(w_1) = \pi\left(\lim_{n \rightarrow -\infty} w_1^{(n)}\right) = \lim_{n \rightarrow -\infty} \pi\left(w_1^{(n)}\right) = \lim_{n \rightarrow -\infty} \pi\left(w_2^{(n)}\right) = \pi(w_2).$$

In the other direction, suppose that $\pi(w_1) = \pi(w_2)$. Then we have also that

$$\pi(\sigma_{\mathbb{Z}}^n(w_1)) = f^n(\pi(w_1)) = f^n(\pi(w_2)) = \pi(\sigma_{\mathbb{Z}}^n(w_2))$$

for all $n \in \mathbb{Z}$. But this means, by definition of the transformations $T_{(a;b)}$, that

$$T_{(a_{-n}; b_{-n})}(a_{-n} a_{-n+1} \dots) = b_{-n} b_{-n+1} \dots,$$

so that the sequences w_1 and w_2 are asymptotically equivalent. \square

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