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Self-similar solutions for conic cusps formation at the surface of dielectric liquids in electric field

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The nonlinear dynamics is studied for the free surface of an ideal dielectric fluid in an electric field. Self-similar solutions of electrohydrodynamic equations describing the formation of surface conic cusps are revealed. The behavior of physical quantities (field strength, fluid velocity, and surface curvature) near the singularity is established. The threshold value of the fluid dielectric constant required for the proposed mechanism of a cusp development is found.

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It is well known [1–5] that free surfaces of both conducting fluids and liquid dielectrics, which conductivity is assumed to be negligibly small, are unstable in a strong external electric field. In the former case the interaction of the electric field and the free-surface charges induced by this field causes conic cusps to form in a finite time (see, for example, Refs. [6–10]). What is the fluid behavior at the stage of a singularity formation in the latter case, when the surface contains only bound polarization charges? Despite different physical nature of the instabilities, equations of motion for an ideal dielectric liquid in a vertical electric field, and the equations of perfectly conducting liquid motion have similar structures. Furthermore, they completely coincide in the formal limit of infinite permittivity of a dielectric, $\varepsilon\to\infty$, when the field does not penetrate into the medium. Then there are good grounds to believe that instabilities of both conducting liquids and deionized dielectric liquids with high ε values will develop through a common scenario. So, the aim of the present paper is to study the transient problem of the cusp formation at the surface of a perfect dielectric, and, in particular, to find allowable values of dielectric constant ε required for conic structures development.

The essential progress in understanding of nature of conic cusps at a charged fluid surface is associated with Taylor's work [7], where it was demonstrated that the surface electrostatic pressure P_E for an equipotential cone with angle 98.6° depends on the distance from its axis as r^{-1} and, hence, can be counterbalanced by the surface pressure $P_S\propto r^{-1}$. This result was extended by Ramos and Castellanos [11,12] to the case of dielectric liquids with arbitrary permittivity. They have shown that the pressures P_S and P_E cancellation is possible if the following condition is valid:

$$\frac{P'_{1/2}(\cos\theta_0)P_{1/2}(-\cos\theta_0)}{P'_{1/2}(-\cos\theta_0)P_{1/2}(\cos\theta_0)}=-\varepsilon, \tag{1}$$

where $\theta_0=\pi-\beta/2$ (β is the cone angle), $P_{1/2}$ is the Legendre function of order 1/2, and $P'_{1/2}$ is its derivative with respect to the argument. It follows from the relation (1) that the stationary conic structures can exist only for $\varepsilon>17.6$. The corresponding cone angle β falls in the range

$$0°<\beta<98.6°. \tag{2}$$

As for the nonstationary problem, a model for the Taylor cone formation on the conducting fluid boundary was proposed in my recent paper [13]. There it was shown that the surface evolution near the singularity could be adequately described by the self-similar solutions of the electrohydrodynamic equations. It might be supposed for pure dielectric liquids that, as for conducting liquids, the self-similar solutions are responsible for cusps formation.

Let us check the validity of this hypothesis. Consider the potential motion of an ideal dielectric fluid occupying the region bounded by free surface $z=\eta(x,y,t)$. We will assume that the vector of an external electric field is directed along the z axis. The velocity potential Φ and the electric-field potentials φ and φ' in and above the liquid obey the Laplace equations

$$\nabla^2\Phi=0,\quad \nabla^2\varphi=0,\quad \nabla^2\varphi'=0. \tag{3}$$

The evolution of the free surface is determined by the dynamic and kinematic boundary conditions

$$\Phi_t+\frac{|\nabla\Phi|^2}{2}=\frac{(\varepsilon-1)^2(\partial_n\varphi)^2+(\varepsilon-1)|\nabla\varphi|^2}{8\pi\rho}+\frac{\alpha}{\rho}\nabla_\perp\cdot\frac{\nabla_\perp\eta}{\sqrt{1+(\nabla_\perp\eta)^2}},\quad z=\eta(x,y,t), \tag{4}$$

$$\eta_t=\Phi_z-\nabla_\perp\eta\cdot\nabla_\perp\Phi,\quad z=\eta(x,y,t), \tag{5}$$

where α is the surface-tension coefficient, ρ is the mass density of a medium, and ∂_n denotes the derivative along the normal to the fluid surface. Since the electric field potential and normal component of the displacement vector have to be continuous at the interface, we should add the following conditions at the boundary:

$$\varphi=\varphi',\quad z=\eta(x,y,t), \tag{6}$$

$$\varepsilon\partial_n\varphi=\partial_n\varphi',\quad z=\eta(x,y,t). \tag{7}$$

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The system of equations is closed by the conditions of the velocity field decay and the electric-field uniformity at infinite distance from the surface

$$|\nabla\Phi|\to 0, \quad z\to-\infty, \tag{8}$$

$$\varphi\to -Ez/\varepsilon, \quad z\to-\infty, \tag{9}$$

$$\varphi'\to -Ez, \quad z\to\infty, \tag{10}$$

where E is the magnitude of the external electric-field strength.

We are interested in the dynamics of formation of a singular profile for the fluid surface. It is natural to assume that the electric field near the cusp appreciably exceeds the external field; i.e., $|\nabla\varphi|\gg E/\varepsilon$ and $|\nabla\varphi'|\gg E$. In this case, the interface evolution is fully determined by the induced field, which decreases with distance from the singularity. One can thus use the conditions

$$|\nabla\varphi|\to 0, \quad z\to-\infty, \tag{11}$$

$$|\nabla\varphi'|\to 0, \quad z\to+\infty, \tag{12}$$

instead of the field uniformity conditions (9) and (10). This agrees with the assumption about the universal behavior of a fluid in the formation of a singular surface profile, because it allows the fluid motion near the singular point to be analyzed without regard for the particular geometry of the problem (the fluid "forgets" the boundary conditions at infinity at the final stages of the instability). The applicability of these conditions will be discussed below in more detail after establishing some regularities for the dynamics of a dielectric fluid near the singularity.

Note that the equations of motion with conditions (11) and (12) allow the self-similar substitution

$$\Phi(x,y,z,t)=2\alpha(\pi/\rho)^{1/2}E^{-1}\tilde{\Phi}(\tilde{r},\tilde{z})\tau^{1/3},$$

$$\varphi(x,y,z,t)=4\pi\alpha E^{-1}\tilde{\varphi}(\tilde{r},\tilde{z})\tau^{1/3},$$

$$\varphi'(x,y,z,t)=4\pi\alpha E^{-1}\tilde{\varphi}'(\tilde{r},\tilde{z})\tau^{1/3},$$

$$\eta(x,y,t)=4\pi\alpha E^{-2}\tilde{\eta}(\tilde{r})\tau^{2/3},$$

$$\tilde{r}=E^2(4\pi\alpha)^{-1}r\tau^{-2/3},$$

$$\tilde{z}=E^2(4\pi\alpha)^{-1}z\tau^{-2/3},$$

$$\tau=E^3(64\pi^3\alpha^2\rho)^{-1/2}(t_c-t),$$

which corresponds to the most important case of the axially symmetric protrusion on the surface (here, $r=\sqrt{x^2+y^2}$ is the distance from the symmetry axis and t_c is the blow-up time). Substituting these expressions in Eqs. (3)–(8), (11), and (12), one finds that the dimensionless functions $\tilde{\Phi}$, $\tilde{\varphi}$, $\tilde{\varphi}'$, and $\tilde{\eta}$ obey the following set of partial differential equations:

$$\tilde{\Phi}_{\tilde{r}\tilde{r}}+\tilde{r}^{-1}\tilde{\Phi}_{\tilde{r}}+\tilde{\Phi}_{\tilde{z}\tilde{z}}=0, \quad \tilde{z}<\tilde{\eta}(\tilde{r}), \tag{13}$$

$$\tilde{\varphi}_{\tilde{r}\tilde{r}}+\tilde{r}^{-1}\tilde{\varphi}_{\tilde{r}}+\tilde{\varphi}_{\tilde{z}\tilde{z}}=0, \quad \tilde{z}<\tilde{\eta}(\tilde{r}), \tag{14}$$

$$\tilde{\varphi}'_{\tilde{r}\tilde{r}}+\tilde{r}^{-1}\tilde{\varphi}'_{\tilde{r}}+\tilde{\varphi}'_{\tilde{z}\tilde{z}}=0, \quad \tilde{z}>\tilde{\eta}(\tilde{r}), \tag{15}$$

$$\frac{2\tilde{\Phi}_{\tilde{r}}\tilde{r}+2\tilde{\Phi}_{\tilde{z}}\tilde{\eta}-\tilde{\Phi}}{3}+\frac{\tilde{\Phi}_{\tilde{r}}^2+\tilde{\Phi}_{\tilde{z}}^2}{2} = \frac{1}{\sqrt{1+\tilde{\eta}_{\tilde{r}}^2}}\left(\frac{\tilde{\eta}_{\tilde{r}\tilde{r}}}{1+\tilde{\eta}_{\tilde{r}}^2}+\frac{\tilde{\eta}_{\tilde{r}}}{\tilde{r}}\right) + \frac{(\varepsilon-1)^2(\partial_n\tilde{\varphi})^2+(\varepsilon-1)(\tilde{\varphi}_{\tilde{r}}^2+\tilde{\varphi}_{\tilde{z}}^2)}{2}, \quad \tilde{z}=\tilde{\eta}(\tilde{r}), \tag{16}$$

$$2\tilde{\eta}_{\tilde{r}}\tilde{r}-2\tilde{\eta}=3\tilde{\Phi}_{\tilde{z}}-3\tilde{\eta}_{\tilde{r}}\tilde{\Phi}_{\tilde{r}}, \quad \tilde{z}=\tilde{\eta}(\tilde{r}), \tag{17}$$

$$\tilde{\varphi}=\tilde{\varphi}', \quad \varepsilon\partial_n\tilde{\varphi}=\partial_n\tilde{\varphi}', \quad \tilde{z}=\tilde{\eta}(\tilde{r}), \tag{18}$$

$$\tilde{\Phi}_{\tilde{r}}^2+\tilde{\Phi}_{\tilde{z}}^2\to 0, \quad \tilde{r}^2+\tilde{z}^2\to\infty, \tag{19}$$

$$\tilde{\varphi}_{\tilde{r}}^2+\tilde{\varphi}_{\tilde{z}}^2\to 0, \quad \tilde{\varphi}_{\tilde{r}}'^2+\tilde{\varphi}_{\tilde{z}}'^2\to 0, \quad \tilde{r}^2+\tilde{z}^2\to\infty, \tag{20}$$

$$\tilde{\Phi}_{\tilde{r}}=0, \quad \tilde{\varphi}_{\tilde{r}}=0, \quad \tilde{\varphi}'_{\tilde{r}}=0, \quad \tilde{\eta}_{\tilde{r}}=0, \quad \tilde{r}=0. \tag{21}$$

For the self-similar solutions concerned, the surface profile forms first at the periphery and then extends to the center $r=z=0$ (the spatial scale decreases as $\tau^{2/3}$). This implies that the formation of conic cusps at $t=t_c$ is described by those solutions to the set of Eqs. (13)–(21) which provide conic asymptotic shape of the surface. In such a situation, the presence of asymptotic solutions for which $\tilde{\eta}\propto\tilde{r}$ at $\tilde{r}\to\infty$ is the necessary condition for the validity of our assumption about the self-similar nature of conic points.

Analysis of Eqs. (13)–(21) in the limit $R=\sqrt{\tilde{r}^2+\tilde{z}^2}\to\infty$ showed that they have an asymptotic solution of the form

$$\tilde{\Phi}=\tilde{\Phi}_0(\tilde{r},\tilde{z})=sR^{-1}, \tag{22}$$

$$\tilde{\varphi}=\tilde{\varphi}_0(\tilde{r},\tilde{z})=\left[\frac{2R(s_0-s)}{q(\varepsilon-1)}\right]^{1/2}P_{1/2}(-\cos\theta), \tag{23}$$

$$\tilde{\varphi}'=\tilde{\varphi}_0'(\tilde{r},\tilde{z})=-\left[\frac{2R(s_0-s)}{p(\varepsilon-1)}\right]^{1/2}P_{1/2}(\cos\theta), \tag{24}$$

$$\tilde{\eta}=\tilde{\eta}_0(\tilde{r})=-s_0\tilde{r}, \tag{25}$$

$$p=P_{1/2}^2(\cos\theta_0)/4+\sin^2\theta_0 P_{1/2}'^{\,2}(\cos\theta_0)/\varepsilon,$$

$$q=P_{1/2}^2(-\cos\theta_0)/4+\varepsilon\sin^2\theta_0 P_{1/2}'^{\,2}(-\cos\theta_0),$$

$$s_0=-\cot\theta_0,$$

where $\theta=\arctan(\tilde{r}/\tilde{z})$ and s is a constant satisfying inequality $0<s<s_0$ (the quantities R and θ are the radial and angular

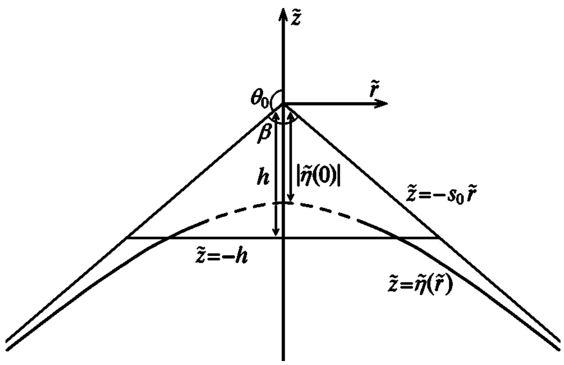

FIG. 1. Schematic drawing of a surface of dielectric fluid $\tilde z=\tilde\eta(\tilde r)$ corresponding to the self-similar solutions of the equations of motion.

spherical polar coordinates). This solution describes a conic surface of the angle $\beta=2\pi-2\theta_0$ whose ε dependence is given by the relation (1). It may be considered as a dynamic generalization for the static solutions obtained in Refs. [11,12].

According to Eq. (22), the fluid motion is spherically symmetric, and fluid moves to the sink point $R=0$ along the tangent to the surface (25). Since the surface shape at $\tau\to 0$ is determined by the asymptotic solutions of Eqs. (13)–(21), a conic cusp forms at time t_c in accordance with the expression (25). The electric field at the cusp increases as $\tau^{-1/3}$, the cusp growth velocity increases as $\tau^{-1/3}$, and the cusp curvature increases as $\tau^{-2/3}$. Returning to dimensional quantities, we get at the protrusion apex

$$\partial_n\varphi|_{r=0}\sim\alpha^{1/3}\rho^{1/6}(t_c-t)^{-1/3},$$

$$\partial_n\varphi'|_{r=0}\sim\alpha^{1/3}\rho^{1/6}(t_c-t)^{-1/3},$$

$$\partial_n\Phi|_{r=0}\sim\alpha^{1/3}\rho^{-1/3}(t_c-t)^{-1/3},$$

$$\eta_{rr}|_{r=0}\sim\alpha^{-1/3}\rho^{1/3}(t_c-t)^{-2/3}.$$

One can see that all these quantities become infinite in a finite time.

This analysis is valid only if the solutions of the set of partial differential Eqs. (13)–(21) with the asymptotics (22)–(25) satisfy the condition $\tilde\eta<-s_0\tilde r$, i.e., if the fluid surface $\tilde z=\tilde\eta(\tilde r)$ is positioned below the asymptotic cone (see Fig. 1). Otherwise, the surface velocity would be directed in opposition to the z axis that certainly contradicts our notion of the fluid behavior at the blow-up stage. Let us check how this condition is fulfilled in the limit of large $\tilde r$. We will seek the solutions of the system (13)–(21) at $R\to\infty$ as the asymptotic expansion with leading terms given by the expressions (22)–(25),

$$\tilde\varphi(\tilde r,\tilde z)=\tilde\varphi_0(\tilde r,\tilde z)+\sum_{n=1}^{\infty}a_n\frac{\partial^{3n}}{\partial\tilde z^{3n}}[R^{1/2}P_{1/2}(-\cos\theta)],$$

$$\tilde\varphi'(\tilde r,\tilde z)=\tilde\varphi_0'(\tilde r,\tilde z)+\sum_{n=1}^{\infty}a_n'\frac{\partial^{3n}}{\partial\tilde z^{3n}}[R^{1/2}P_{1/2}(\cos\theta)],$$

$$\tilde\Phi(\tilde r,\tilde z)=\tilde\Phi_0(\tilde r,\tilde z)+\sum_{n=1}^{\infty}b_n\frac{\partial^{3n}}{\partial\tilde z^{3n}}[R^{-1}],$$

$$\tilde\eta(\tilde r)=\tilde\eta_0(\tilde r)+\sum_{n=1}^{\infty}c_n\tilde r^{1-3n}.$$

It turns out that, to the first order of the expansion, the surface is conic

$$a_1=0,\quad a_1'=0,\quad b_1=-\frac{s^2(1+s_0^2)^{3/2}}{18s_0(3-2s_0^2)},\quad c_1=0.$$

The correction to Eq. (25) for the surface shape appears in the next order. One finds from kinematic boundary condition (17) that

$$c_2=-\frac{s^2(4s_0^2-1)}{8s_0(1+s_0^2)^2(3-2s_0^2)}.$$

It is clear that the surface shape deviates from the conic one in the direction specified by the sign of the c_2 coefficient. Indeed, the evolution of the fluid boundary away from the singularity is determined by the leading terms of the expansion in small (t_c-t) value,

$$\eta=-s_0r+c_2(\alpha/\rho)^2(t_c-t)^4r^{-5},$$

from whence it follows that, when forming a conic cusp, the fluid moves upwards only at $c_2<0$. As is seen from the expression for the coefficient c_2, its value is negative only if $1/4<s_0^2<3/2$ or, what is the same, if

$$\sqrt{2/3}<-\tan\theta_0<2.$$

This condition restricts the applicability of our approach to the description of the process of the conic cusps formation, which is based on the analysis of self-similar solutions. It suggests that the cone angle $\beta=2\pi-2\theta_0$ should be in the range $78.5°<\beta<126.9°$. Comparing the allowable values of β with the inequality (2), one can see that the following condition must hold:

$$78.5°<\beta<98.6°.$$

Taking into account the relation (1), we find that the corresponding value of the fluid dielectric constant ε must be more than $\varepsilon_c\approx 22.2$ (note that $\varepsilon\approx 26$ for ethyl alcohol and $\varepsilon\approx 81$ for water). For $\varepsilon<\varepsilon_c$ it holds $c_2>0$ and our solutions are physically meaningless.

Let us now define the velocity of the fluid boundary at the protrusion apex. According to the proposed model, we have

$$\eta_t|_{r=0}=\frac{2(\alpha/\rho)^{1/3}}{3(t_c-t)^{1/3}}|\tilde\eta(0)|.$$

It is possible to estimate the involved in this expression distance $|\tilde{\eta}(0)|$ from the cone apex to the fluid surface (our asymptotic expansion diverges for small $\tilde{r}$). Multiplying kinematic boundary condition (17) by $\pi\tilde{r}/3$ and integrating it over $\tilde{r}$, one obtains after simple mathematics

$$V=\frac{E^3\rho^{1/2}}{16\tau\alpha^2\pi^{3/2}}\int_S \partial_n\Phi\, dS,$$

where S stands for the fluid surface $z=\eta(x,y,t)$, and V is the volume of a region bounded from above by the conic surface $\tilde{z}=-s_0\tilde{r}$ and from below by the surface $\tilde{z}=\tilde{\eta}(\tilde{r})$. The integral on the right-hand side of this expression is the fluid velocity flux through the surface S. Since the function Φ is harmonic, the flux of the vector-field $\nabla\Phi$ through any closed surface is zero. This fact allows the flux through the surface S to be determined using the asymptotic form of velocity potential at $r^2+z^2\to\infty$. Taking into account that the fluid flows into a solid angle $2\pi(1+\cos\theta_0)$ at infinity, one can get from Eq. (22)

$$V=\pi s(1+\cos\theta_0).$$

Notice that the volume of a region bounded by the conic surface $\tilde{z}=-s_0\tilde{r}$ and the plane $\tilde{z}=-h$ (a circular right cone of height h) equals V at

$$h=h(s,s_0)=[3s\cot^2\theta_0(1+\cos\theta_0)]^{1/3}.$$

Clearly, if the volume V is fixed and the condition $-s_0<\tilde{\eta}_{\tilde{r}}(\tilde{r})\leqslant 0$ is fulfilled for any $\tilde{r}$ (for the conditions (21) to be valid, the surface near the cone apex must be "rounded off," the quantity $|\tilde{\eta}(0)|$ cannot exceed the cone height (see Fig. 1). That is, the inequality

$$|\tilde{\eta}(0)|\leqslant h(s,s_0)$$

connecting the characteristic spatial dimension at small R with the asymptotic parameter s is satisfied. Since the maximum possible value of h for the fixed permittivity ε and, hence, for the fixed angle θ_0 corresponds to the maximum allowed value s_0 of the s constant, the following estimate is also valid:

$$|\tilde{\eta}(0)|\leqslant h(s_0,s_0)=-\cot\theta_0(3+3\cos\theta_0)^{1/3},$$

which does not involve the free parameter s.

Let us return to the question of the applicability of the conditions (11) and (12). As was pointed out above, they can be used instead of Eqs. (9) and (10) only if the external electric field is much weaker than the cusp field. After the transition to the self-similar variables, this requirement is recast as

$$\tilde{\varphi}_{\tilde{r}}^2+\tilde{\varphi}_{\tilde{z}}^2\gg\varepsilon^{-2}\tau^{2/3},\qquad \tilde{\varphi}_{\tilde{r}}'^2+\tilde{\varphi}_{\tilde{z}}'^2\gg\tau^{2/3}.$$

It is clear that for small τ (i.e., immediately before the collapse) this condition is fulfilled near the singularity in a natural way. In this case, R_0 and T values exist for which models (13)–(21) with $0\leqslant r^2+z^2<R_0^2$ and $t_c-T<t\leqslant t_c$ adequately describes the strongly nonlinear stages of electrohydrodynamic instability development for the surface of a dielectric fluid in an external electric field. One can find that the following inequalities must hold:

$$R_0\ll\lambda\sim\alpha E^{-2},\qquad T\ll T_0\sim\alpha\rho^{1/2}E^{-3},$$

where λ and T_0 are respectively the characteristic spatial and temporal scales at the linear stage of the instability.

Note also that, together with the asymptotic solution for the velocity potential (22), the system of the Eqs. (13)–(21) admits the more general solution,

$$\tilde{\Phi}=s_1R^{-1}+s_2R^{1/2}P_{1/2}(-\cos\theta),$$

which demands a separate consideration. This expression coincides with Eq. (22) for $s_2=0$. In any case (i.e., for arbitrary s_1 and s_2), the self-similar solutions describe the formation of conic cusps with ε-dependent angles at the free surface of dielectric liquids in an applied electric field.

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