Self-similar solutions of semilinear wave equations with a focusing nonlinearity

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Abstract
We prove that in three space dimensions a nonlinear wave equation $u_{tt} - \Delta u = u^p$, with $p \geqslant 7$ being an odd integer, has a countable family of regular spherically symmetric self-similar solutions.

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1. Introduction

An important feature of many nonlinear wave equations is that their solutions, corresponding to smooth initial data, may form singularities after a finite time. Such a phenomenon, usually referred to as blowup, has been a subject of intensive studies beginning with the pioneering works by Keller [1], John [2] and Glassey [3] (we refer the interested reader to the excellent online review of this subject with the complete bibliography [4]).

In this paper we consider the semilinear wave equation with a power nonlinearity

$$\Phi_{tt} - \Delta\Phi - \Phi^p = 0, \qquad \Phi = \Phi(t, x), \qquad x \in R^3, \tag{1}$$

where $p \geqslant 7$ is an odd integer. The sign of the nonlinear term corresponds to focusing which means that it tends to magnify the amplitude of the wave. If Φ is small this term is negligible and the evolution is essentially linear, leading to dispersion. However, if Φ is large the dispersive effect of the Laplacian may be overcome by the focusing effect of the nonlinearity and a singularity can form. Actually, neglecting the Laplacian altogether and solving the ordinary differential equation $\Phi_{tt} = \Phi^p$ one gets the exact, homogeneous in space, solution

$$\Phi_0 = \frac{b_0}{(T-t)^\alpha}, \qquad b_0 = \left[\frac{2(p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}}, \quad \alpha = \frac{2}{p-1}, \quad T > 0, \tag{2}$$

which blows up as $t \to T$. By the finite speed of propagation one can truncate this solution in space to get a solution with compactly supported initial data which blows up in finite time.

There is theoretical [5] and numerical [6] evidence that the solution Φ_0 determines the leading order asymptotics of blowup for generic large initial data. However, for specially prepared initial data, in particular for data fine-tuned to the threshold for blowup, singularities may have a different form which is given by self-similar solutions of equation (1). Such solutions were found numerically in [6]. The aim of this paper is to give a rigorous proof of their existence and discuss their properties.

By definition, self-similar solutions are invariant under rescaling

$$\Phi(t, x) \to \Phi_\lambda(t, x) = \lambda^{-\alpha} \Phi(t/\lambda, x/\lambda), \tag{3}$$

hence in the spherically symmetric case they have the form

$$\Phi(t, r) = (T - t)^{-\alpha} u(\rho), \qquad \rho = \frac{r}{T - t}, \tag{4}$$

where T is a positive constant. Note that each self-similar solution, if it is regular for $t < T$, provides an explicit example of regular initial data developing a singularity in a finite time. Substituting the ansatz (4) into equation (1) we obtain the ordinary differential equation for the similarity profile $u(\rho)$

$$(1 - \rho^2)u'' + \left(\frac{2}{\rho} - (2 + 2\alpha)\rho\right) u' - \alpha(\alpha + 1)u + u^p = 0. \tag{5}$$

It is easy to see that this equation has the constant solution $u_0(\rho) = b_0$ which, of course, corresponds to the homogeneous solution Φ_0 of equation (1). As mentioned above, numerical results indicate that, besides u_0, there also exist nontrivial regular solutions of equation (5). We remark that equation (5) has been studied by Kavian and Weissler [7] who made many interesting observations about the behaviour of solutions but unfortunately they imposed very restrictive fall-off conditions at infinity which excluded nontrivial solutions. However, as long as one wants to have an example of blowup, due to the finite speed of propagation, only the behaviour of solutions inside the past light cone of the blowup point $(t = T, r = 0)$ is relevant, which corresponds to the interval $0 \leqslant \rho \leqslant 1$. As we shall see below regular self-similar solutions are parametrized by their values at the endpoints $c = u(0)$ and $b = u(1)$. Due to the singular nature of equation (5) at $\rho = 0$ and $\rho = 1$, the generic solution obeying the regularity condition at $\rho = 0$ becomes singular at $\rho = 1$ and vice versa; hence globally regular solutions can exist only for discrete values of the parameters b and c. The goal of this paper is to show that there exists an infinite sequence of pairs (b_n, c_n) which give rise to globally regular solutions. To this end, in section 2 we first prove the existence of local solutions near the endpoints. Then, in section 3, we show, using a shooting argument, that for a discrete set of values of the parameters these local solutions match smoothly at a midpoint. In section 4 we show that the solutions constructed in section 3 can be extended beyond the past light cone to infinity. Finally, in section 5 we derive some scaling properties of the shooting parameters. In the appendix we present an alternative rigorous proof of existence.

2. Local existence

In the first step we will analyse the behaviour of solutions near the boundary points $\rho = 0$ and $\rho = 1$ and prove the local existence of one-parameter families of regular solutions there.

Near $\rho = 0$ we introduce a new variable $v = u'$ and rewrite equation (5) as the first order system:

$$\rho u' = \rho v, \tag{6a}$$

$$\rho v' = -2v + \frac{\rho}{1 - \rho^2} \left(\frac{4}{p - 1}\rho v + \frac{2(p + 1)}{(p - 1)^2}u - u^p\right). \tag{6b}$$

This system has the form required by proposition 1 of [8] guaranteeing the existence of a one-parameter family of regular solutions near $\rho = 0$ such that

$$u(\rho) = c + \frac{1}{3}\left(\frac{p+1}{(p-1)^2}c - \frac{1}{2}c^p\right)\rho^2 + O(\rho^4). \tag{7}$$

Similarly, near $\rho = 1$ we put $s = 1 - \rho$, $\bar{u}(s) = u(\rho)$, $\bar{v}(s) = \bar{u}'(s)$ to get

$$s\bar{u}' = s\bar{v}, \tag{8a}$$

$$s\bar{v}' = \frac{1}{(2-s)}\left(\left(\frac{2}{1-s} - \frac{2(p+1)(1-s)}{p-1}\right)\bar{v} + \frac{2(p+1)}{(p-1)^2}\bar{u} - \bar{u}^p\right). \tag{8b}$$

In order to bring this into the form required by proposition 1 of [8] we introduce $\tilde{v} = \bar{v} - \frac{p+1}{2(p-1)}\bar{u} + \frac{p-1}{4}\bar{u}^p$ and rewrite the system (8a) and (8b) as

$$s\bar{u}' = s\left(\tilde{v} + \frac{p+1}{2(p-1)}\bar{u} - \frac{p-1}{4}\bar{u}^p\right), \tag{9a}$$

$$s\tilde{v}' = -\frac{2}{p-1}\tilde{v} + sf(\bar{u}, \tilde{v}, s), \tag{9b}$$

where the function $f(\bar{u}, \tilde{v}, s)$ is analytic in $\bar{u}$, $\tilde{v}$ and s near $s = 0$. We conclude that there is a one-parameter family of regular solutions near $\rho = 1$ such that

$$u(\rho) = b + \frac{1}{2}\left(\frac{1}{2}(p-1)b^p - \frac{p+1}{p-1}b\right)(\rho - 1) + O((\rho - 1)^2). \tag{10}$$

The singular solutions behave like $u \sim (1-\rho)^{\frac{p-3}{p-1}}$ near $\rho = 1$ leading to a singularity of u'. For later use it is convenient to introduce the variables $\sigma = (1-\rho)^{2/(p-1)}$, $\psi(\sigma) = u(\rho)$ and $\theta(\sigma) = \sigma u'$. System (8a) and (8b) may then be written in the form

$$\psi' = -\frac{p-1}{2}\sigma^{\frac{p-5}{2}}\theta, \tag{11a}$$

$$\theta' = g(\psi, \theta, \sigma), \tag{11b}$$

where the function $g(\psi, \theta, \sigma)$ is analytic in ψ, θ and σ near $\sigma = 0$. This implies that there is a two-parameter family of regular solutions parametrized by the values $b = \psi(0) = u(1)$ and $d = \theta(0)$. These solutions are analytic as functions of b, d and σ in a neighbourhood of $\sigma = 0$. Thus, the general solution is in fact regular at $\rho = 1$ as a function of $\sigma = (1-\rho)^{2/(p-1)}$ with a convergent Taylor series at $\sigma = 0$.

3. Global behaviour and proof of existence

The main result of this paper is as follows.

Theorem 1. *For any odd $p \geqslant 7$ and any nonnegative integer n, equation (5) on the interval $0 \leqslant \rho \leqslant 1$ has an analytic solution $u_n(\rho)$ which satisfies the boundary conditions (7) and (10) and for which the function $w_n = u_n/u_\infty - 1$, with u_∞ defined by (18), has exactly $n + 1$ zeros.*

Remark. In this section we give a 'physicist' proof based on scaling arguments. We believe that by sacrificing slightly the mathematical rigour we gain better understandability of the basic mechanism which is responsible for the existence and structure of solutions. An alternative rigorous proof is presented in the appendix.

Proof. First, we need to show that solutions which are regular at $\rho = 0$ or $\rho = 1$ remain bounded on the whole interval $0 < \rho < 1$. This is conveniently proven using a suitable Lyapunov function [9] defined as

$$H = (1 - \rho^2)\frac{u'^2}{2} + \frac{u^{p+1}}{p + 1} - \frac{(p + 1)u^2}{(p - 1)^2}, \tag{12}$$

obeying

$$H(\rho) \geqslant -\frac{1}{p - 1}\left(\frac{2(p + 1)}{(p - 1)^2}\right)^{\frac{2}{p-1}} \quad \text{and} \quad H' = \left(\frac{p + 3}{p - 1}\rho - \frac{2}{\rho}\right)u'^2 \leqslant 0. \tag{13}$$

Hence, for solutions regular at $\rho = 0$ we get

$$H(\rho) \leqslant H(0) = \frac{c^{p+1}}{p + 1} - \frac{(p + 1)c^2}{(p - 1)^2}, \tag{14}$$

which implies

$$\sqrt{1 - \rho^2}\,|u'(\rho)| \leqslant c^{\frac{p+1}{2}} \tag{15}$$

and

$$|u(\rho)| \leqslant c \qquad \text{for} \quad c \geqslant (2(p + 1)/(p - 1)^2)^{1/(p-1)}. \tag{16}$$

Furthermore, $H(\rho)$ and thus $u(\rho)$ have a finite limit at $\rho = 1$.

On the other hand, for solutions regular at $\rho = 1$ we may integrate the function H to the left. It is easy to see that

$$-(H + 1)^{-1}H' \leqslant 2/\rho, \tag{17}$$

hence H stays finite on $0 < \rho \leqslant 1$ and with it u and u'. We emphasize that this argument does not exclude solutions which are regular at $\rho = 1$ and singular at $\rho = 0$. Actually, such an explicit singular solution

$$u_\infty(\rho) = b_\infty\rho^{-\frac{2}{p-1}}, \qquad b_\infty = \left(\frac{2(p - 3)}{(p - 1)^2}\right)^{\frac{1}{p-1}}, \tag{18}$$

will play an important role in our analysis. Note that u_∞ corresponds to the singular static solution $\Phi = b_\infty r^{-\frac{2}{p-1}}$ of equation (1).

Next, we consider the behaviour of solutions regular at $\rho = 0$, i.e. solutions satisfying the initial condition (7), with large values of $c = u(0)$. To this order we rescale the variables

$$\rho = \frac{x}{c^{(p-1)/2}}, \qquad u(\rho) = cU(x). \tag{19}$$

From equation (5) we get

$$\frac{d^2U}{dx^2} + \frac{2}{x}\frac{dU}{dx} + U^p = \frac{1}{c^{p-1}}\left(x^2\frac{d^2U}{dx^2} + \frac{2(p + 1)}{p - 1}x\frac{dU}{dx} + \frac{2(p + 1)}{(p - 1)^2}U\right). \tag{20}$$

In the limit $c \to \infty$ we obtain the simple equation

$$\frac{d^2U}{dx^2} + \frac{2}{x}\frac{dU}{dx} + U^p = 0 \tag{21}$$

on the interval $0 \leqslant x < \infty$ together with the boundary condition at $x = 0$

$$U(x) = 1 - x^2/6 + O(x^4). \tag{22}$$

Note that the special solution u_∞ is invariant under the performed rescaling and thus $U_\infty(x) = b_\infty x^{-2/(p-1)}$ solves equation (21). From our numerical analysis we know that U oscillates around U_∞ suggesting the change in variables

$$\bar{U}(\tau) = x^{2/(p-1)} U(x), \qquad \tau = \ln(x). \tag{23}$$

Substituting (23) into equation (21) we get the autonomous equation

$$\frac{\mathrm{d}^2 \bar{U}}{\mathrm{d}\tau^2} + \frac{p-5}{p-1}\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau} + \bar{U}^p - \frac{2(p-3)}{(p-1)^2}\bar{U} = 0. \tag{24}$$

Using the Lyapunov function

$$h = \frac{1}{2}\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau}^2 + \frac{\bar{U}^{p+1}}{p+1} - \frac{(p-3)\bar{U}^2}{(p-1)^2} \qquad \text{with} \quad \frac{\mathrm{d}h}{\mathrm{d}\tau} = -\frac{p-5}{p-1}\left(\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau}\right)^2, \tag{25}$$

we conclude that for $\tau \to \infty$ the solution $\bar{U}(\tau)$ tends to the fixed point $\bar{U}_\infty = b_\infty$ of equation (24) and $\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau} \to 0$ for $\tau \to \infty$. Putting $\bar{U} = b_\infty + y$ and $\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau} = z$ we find

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = z, \tag{26a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = -\frac{p-5}{p-1}z - \frac{2(p-3)}{p-1}y + N(y), \tag{26b}$$

where $N(y)$ denotes the nonlinear terms in y. Neglecting these nonlinear terms we get a linear system with the eigenvalues $\lambda = (-p + 5 \pm \mathrm{i}\sqrt{7p^2 - 22p - 1})/2(p-1)$. This implies that the solution of equation (21) which is regular at $x = 0$ has for $x \to \infty$ the asymptotic form

$$U(x) = b_\infty x^{-\frac{2}{p-1}}(1 + A_0 x^{-\frac{p-5}{2(p-1)}}\sin(\omega \ln x + \delta_0)) \tag{27}$$

with some constants A_0 and δ_0 and $\omega = \sqrt{7p^2 - 22p - 1}/2(p-1)$.

In order to prove the existence of regular solutions for sufficiently large c we choose some x_0, such that for $x > x_0$ we can safely use the asymptotic expression (27) for the solution of equation (21) with the boundary condition (22). Furthermore, we choose c_0 so large that we can neglect the rhs of equation (20) on the interval $0 \leqslant x < x_0$ for $c > c_0$. Integrating solutions regular at $\rho = 0$ to $\rho_0 = x_0/c_0^{(p-1)/2}$ and varying the parameter $c = u(0)$ we get a smooth curve C_0 in the $(u(\rho_0), u'(\rho_0))$ plane. Using the asymptotic expression (27) for the part of C_0 corresponding to $c > c_0$ we get

$$u(\rho) \approx b_\infty \rho^{-\frac{2}{p-1}}(1 + A_0 c^{\frac{5-p}{4}} \rho^{-\frac{p-5}{2(p-1)}} \sin(\omega \ln(c^{\frac{p-1}{2}}\rho) + \delta_0)). \tag{28}$$

Keeping $\rho = \rho_0 \ll 1$ fixed and letting $c \to \infty$ (and with it x) we find that C_0 spirals down (in the negative direction) to the limit point

$$P(\rho_0) = \left(b_\infty \rho_0^{-2/(p-1)}, -\frac{2}{p-1}b_\infty \rho_0^{-(p+1)/(p-1)}\right). \tag{29}$$

We are now ready to show the existence of regular solutions for large c. Integrating the solutions regular at $\rho = 1$ back to ρ_0 we obtain a smooth curve C_1 parametrized by b. For $0 \leqslant b \leqslant b_0$ this curve goes from the point $(0, 0)$ to $(b_0, 0)$ passing through $P(\rho_0)$ for $b = b_\infty$. Thus, curve C_0 intersects curve C_1 again and again as c goes to infinity (note that curve C_1 has positive rotation around $P(\rho_0)$). This is illustrated in figure 1 in the case $p = 7$. Obviously each such intersection yields a solution which is regular on the whole interval $0 \leqslant \rho \leqslant 1$. As c and thus the domain for x increase the corresponding solutions $U(x)$ perform more and more oscillations around $U_\infty(x)$. In fact, each further oscillation of $\bar{U}(x)$ about zero leads
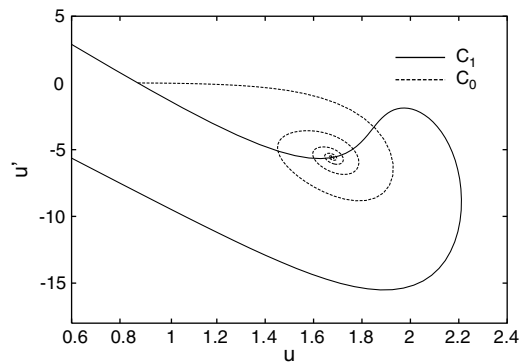
Figure 1. The intersection of curves C_0 and C_1 for $\rho_0 = 0.1$ ($p = 7$).

to another cycle in the (u, u') plane at ρ_0. Thus, we arrive at the conclusion that for any sufficiently large n there exist regular solutions of equation (5) on the interval $0 \leqslant \rho \leqslant 1$ with n of such oscillations.

In the second step of our existence proof we shall show that we can decrease n and thus prove the existence of regular solutions for any positive integer n. The constant solution u_0 with $c = c_0 = b_0$ has exactly one such intersection. Thus we must be able to decrease the number of intersections to the one decreasing c from some large value $c \gg 1$ corresponding to a regular solution with $n \gg 1$ intersections. Obviously the number of intersections can only change at $\rho = 1$, because, by the uniqueness theorem for solutions of differential equations, a double zero of $u(\rho, c) - u_\infty(\rho)$ is not possible at a regular point $0 < \rho < 1$. On the other hand, no solution regular on the whole interval $0 \leqslant \rho \leqslant 1$ can have $b = u(1) = b_\infty$. Thus, a change of n can only happen for a value of c corresponding to a solution which is singular at $\rho = 1$. We showed in section 2 that such solutions are parametrized by the numbers $b = u(1)$ and $d = V(1)$, where the latter is the finite coefficient of the singular mode. For solutions regular at $\rho = 0$ the coefficient d depends smoothly on c. Suppose $b(c) = b_\infty$ for some $c = c_s$. Then, we get a decrease of n exactly by one if either $b - b_\infty$ moves from positive to negative values as c decreases through c_s and $d(c_s) < 0$ or $b - b_\infty$ moves from negative to positive values as c decreases through c_s and $d(c_s) > 0$. This shows that as we decrease c from the value corresponding to a regular solution with $k \gg 1$ intersections to c_0 we necessarily encounter regular solutions with any value of n between one and k. This concludes the proof of theorem 1.

Clearly, there could be more than one regular solution with the same number of intersections, because $b - b_\infty$ can have zeros of even order. However, our numerical analysis shows that there is exactly one regular solution for any n. The first few solutions generated numerically are shown in figure 2.

4. Extension beyond the past light cone

In this section we show that the solutions constructed above can be smoothly extended beyond $\rho = 1$ to infinity.

Consider the function

$$Q(\rho) = \frac{1}{2}(1 - \rho^2)\rho^3 u'^2 + \frac{1}{2}\rho^2(1 - \rho^2)uu' - \frac{3p^2 - 18p + 23}{4(p - 1)^2}\rho^3 u^2 + \frac{1}{p + 1}\rho^3 u^{p+1}. \quad (30)$$
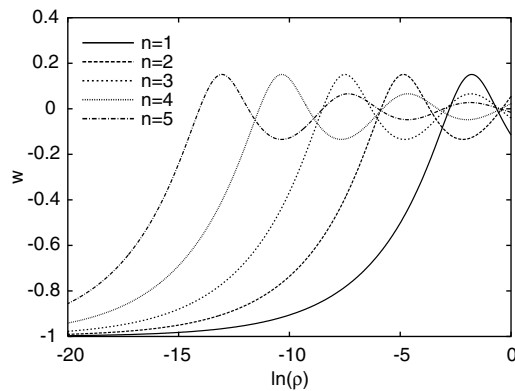
Figure 2. The first five self-similar solutions for $p = 7$. To show oscillations around u_∞ we plot $w_n = u_n/u_\infty - 1$.

This function was introduced by Kavian and Weissler [7] in their study of equation (5). By straightforward (but tedious) computation one gets

$$Q'(\rho) = \frac{5-p}{2(p-1)}\rho^2\left[2(3u/2 + \rho u')^2 + \frac{3p-7}{2(p-1)}u^2 + \frac{p-1}{p+1}u^{p+1}\right], \quad (31)$$

hence $Q(\rho)$ is monotone decreasing for $p > 5$. Since $Q(0) = 0$ for analytic solutions, it follows that $Q(\rho) \leqslant 0$ for $\rho > 0$ and thus $u(\rho)$ is positive for $0 < \rho \leqslant 1$.

Remark. We note in passing that for $p = 5$ it follows from (31) that the function $Q(\rho)$ is the first integral which implies that u_0 is the only regular self-similar solution.

It is convenient to rewrite equation (5) in a self-adjoint form

$$(\rho^2(1-\rho^2)^\alpha u')' = \rho^2(1-\rho^2)^{\alpha-1}u(b_0^{p-1} - u^{p-1}). \quad (32)$$

Proposition 1. If $u(\rho)$ is a regular solution of equation (32) defined for $0 \leqslant \rho \leqslant 1$ and $u(0) > b_0$, then $u(1) < b_0$.

Proof. First, note that if u is analytic at $\rho = 1$ and $u(1) = b_0$, then $u(\rho) \equiv b_0$, which contradicts the assumption that $u(0) > b_0$. Thus, the case $u(1) = b_0$ is impossible. Next, suppose that $u(1) > b_0$. If $u(\rho) \geqslant b_0$ on the whole interval $0 \leqslant \rho \leqslant 1$ then the rhs of equation (32) is negative and hence by integrating it we get that $u'(1) = \infty$. Thus, $u(\rho) < b_0$ for some $0 < \rho < 1$ and therefore u must have a minimum value at some ρ_0, i.e. $u'(\rho_0) = 0$ and $0 < u(\rho_0) < b_0$. If so, between ρ_0 and 1 there is a ρ_1 where $u(\rho_1) = b_0$ and $u(\rho) < b_0$ for $\rho_0 \leqslant \rho < \rho_1$. We now show that this is not possible. To this order we integrate equation (32) over the interval $\rho_0 \leqslant s < \rho < \rho_1$. Since $u'(\rho_0) = 0$ we get

$$\rho^2(1-\rho^2)^\alpha u'(\rho) = \int_{\rho_0}^{\rho} s^2(1-s^2)^{\alpha-1}u(b_0^{p-1} - u^{p-1})\,ds. \quad (33)$$

We have $u(s) < u(\rho) < b_0$; hence $u(b_0^{p-1} - u^{p-1}) = u(b_0^{p-2} + b_0^{p-3}u + \cdots + u^{p-2})(b_0 - u) < (p-1)b_0^{p-1}(b_0 - u)$. Thus,

$$\int_{\rho_0}^{\rho} s^2(1-s^2)^{\alpha-1}u(b_0^{p-1} - u^{p-1})\,ds < (p-1)b_0^{p-1}(b_0 - u(\rho))\int_{\rho_0}^{\rho} s^2(1-s^2)^{\alpha-1}\,ds$$

$$= f(\rho)(b_0 - u(\rho)) \quad (34)$$

for some bounded continuous function $f(\rho)$. It follows from (33) and (34) that

$$\frac{u'(\rho)}{b_0 - u(\rho)} < \frac{f(\rho)}{\rho^2 (1 - \rho^2)^\alpha} = g(\rho), \tag{35}$$

where $g(\rho)$ is a bounded continuous function on $\rho_0 < \rho \leqslant \rho_1 < 1$. Integrating (35) from ρ_0 to ρ, we get $\ln(b_0 - u(\rho_0)) - \ln(b_0 - u(\rho)) < \int_{\rho_0}^\rho g(s)\,\mathrm{d}s$; hence $\ln(b_0 - u(\rho))$ is bounded for $\rho \leqslant \rho_1$. This contradicts the assumption that $u(\rho_1) = b_0$ and concludes the proof.

Next, we consider the behaviour of solutions for $\rho > 1$.

Proposition 2. If $u(\rho)$ is a solution to equation (32) that is analytic at $\rho = 1$ and $u(1) = b < b_0$, then $u(\rho)$ is defined for all $\rho > 1$ and $0 < u(\rho) < b_0$.

Proof. Integrating equation (32) from 1 to some $\rho > 1$ we get

$$\rho^2 (\rho^2 - 1)^\alpha u'(\rho) = \int_1^\rho s^2 (1 - s^2)^\alpha u(b_0^{p-1} - u^{p-1})\,\mathrm{d}s, \tag{36}$$

hence if u is bounded, so is u' and therefore the solution cannot become singular. Thus, it suffices to show that $0 < u(\rho) < b$. From equation (32) we have that $u'(\rho) = 0$ and $0 < u(\rho) < b_0$ for $\rho > 1$ implies that $u''(\rho) < 0$; thus u is monotone decreasing. To show that u cannot become negative we introduce the function $f(\rho) = \rho u'(\rho) + (\alpha + 1)u(\rho)/2$. Evaluating f' at $f = 0$ we obtain

$$f'|_{f=0} = \frac{1}{2}(1 - \alpha^2)u + \frac{2\rho^2}{\rho^2 - 1}u^p > 0, \tag{37}$$

which, together with $f(1) = b^p/2\alpha > 0$, implies that $f(\rho) > 0$ for $\rho > 1$. Since $u'(\rho) < 0$ (as noted above), we conclude that $u > 0$, as claimed. $\quad\blacksquare$

5. Large n asymptotics

Our numerical analysis shows that the shooting parameters b_n and c_n exhibit remarkable scaling properties in the large n limit. In this section we explain this phenomenon.

In order to derive the large n asymptotic behaviour of solutions which are regular at $\rho = 1$ we introduce a new variable w defined by $u = u_\infty(1 + w)$. Using equations (5) and (10) we find that w satisfies the equation

$$\rho^2(1 - \rho^2)w'' + \left(\frac{2(p-3)}{p-1}\rho - 2\rho^3\right)w' - \frac{2(p-3)}{(p-1)^2}(1 + w)((1 + w)^{p-1} - 1) = 0, \tag{38}$$

with the boundary condition at $\rho = 1$

$$w(\rho) = \frac{b - b_\infty}{b_\infty} + \frac{p-1}{4}\frac{b}{b_\infty}(b^{p-1} - b_\infty^{p-1})(\rho - 1) + O((\rho - 1)^2). \tag{39}$$

For any $0 < \rho_0 \ll 1$ and b sufficiently close to b_∞, the solution of equation (38) satisfying the initial condition (39) will stay arbitrarily close to $w = 0$ on the interval $\rho_0 \leqslant \rho \leqslant 1$; hence it can be approximated by $w = \frac{b - b_\infty}{b_\infty}w_L$, where w_L satisfies the linearized equation

$$\rho^2(1 - \rho^2)w_L'' + \left(\frac{2(p-3)}{p-1}\rho - 2\rho^3\right)w_L' - \frac{2(p-3)}{(p-1)}w_L = 0, \tag{40}$$

with the boundary condition $w_L(1) = 1$, $w_L'(1) = (p-3)/2$. We could express w_L in terms of a hypergeometric function but for our purposes it is sufficient to have its asymptotics for small values of ρ

$$w_L \approx \rho^{\frac{5-p}{2(p-1)}} A_1 \sin(\omega \ln \rho + \delta_1), \qquad \omega = \sqrt{7p^2 - 22p - 1}/2(p-1), \tag{41}$$

Table 1. Shooting parameters of solutions u_n and their quotients $\Delta c_n = c_{n+1}/c_n$ and $\Delta b_n = (b_{n+1} - b_\infty)/(b_\infty - b_n)$ for $p = 7$. The last row for $n = \infty$ corresponds to the analytic results (45) and (46).

n	c_n	b_n	Δc_n	Δb_n
1	2.054 390 385	0.688 698 572	2.8018	0.4713
2	5.756 037 116	0.820 493 408	2.4090	0.7428
3	13.866 556 15	0.746 908 360	2.5899	0.5670
4	35.913 433 30	0.796 055 093	2.4577	0.6752
5	88.266 611 66	0.766 263 419	2.5326	0.6059
6	223.550 738 1	0.785 548 198	2.4823	0.6493
7	554.921 549 5	0.773 546 658	2.5128	0.6217
8	1 394.439 242	0.781 209 584	2.4930	0.6391
9	3 476.402 010	0.776 393 971	2.5053	0.6281
10	8 709.676 250	0.779 451 184	2.4974	0.6351
11	21 752.408 61	0.777 522 645	2.5012	0.6307
12	54 434.147 14	0.778 744 142	2.4993	0.6334
13	136 047.675 9	0.777 972 446	2.5000	0.6317
14	340 293.102 2	0.778 460 765	2.5008	0.6328
15	850 746.135 8	0.778 152 079	2.5003	0.6321
∞	∞	0.778 271 716	2.5005	0.6324

where A_1 and δ_1 are constants. Thus, the solution regular at $\rho = 1$ with b near b_∞ has the following form for small ρ

$$u(\rho) \approx u_\infty \left(1 + \frac{b - b_\infty}{b_\infty} \rho^{\frac{5-p}{2(p-1)}} A_1 \sin(\omega \ln \rho + \delta_1) \right). \tag{42}$$

The key point is that the regions of validity of the approximations (42) and (28) overlap so we can match them. Matching the amplitudes of solutions (28) and (42) at ρ_0 we get the condition

$$A_0 c^{\frac{p-5}{4}} = \pm A_1 \frac{b - b_\infty}{b_\infty}. \tag{43}$$

On the other hand, using (28) and the fact that the phases of two adjacent solutions $u_n(\rho)$ and $u_{n+1}(\rho)$ differ by π we obtain

$$\frac{p-1}{2} \omega \ln \left(\frac{c_{n+1}}{c_n} \right) = \pi, \tag{44}$$

hence

$$\frac{c_{n+1}}{c_n} \approx e^{\frac{2\pi}{(p-1)\omega}}, \tag{45}$$

which, using (43), implies that

$$\frac{b_{n+1} - b_\infty}{b_\infty - b_n} \approx e^{-\frac{(p-5)\pi}{2(p-1)\omega}}. \tag{46}$$

The numerical verification of these scaling properties is shown in table 1 in the case $p = 7$.

Note added in proof. P Breitenlohner (personal communication) pointed out that a minor modification of our proof applies also to even values of p; hence the conclusion of theorem 1 holds for all integers $p \geqslant 6$.

Acknowledgments

Appendix

We present here an alternative version of the proof of theorem 1. Throughout the appendix the range of ρ is $0 \leqslant \rho \leqslant 1$.

We know from section 2 and equation (16) that given any c there is a unique solution $u(\rho, c)$ of equation (5) satisfying $u(0, c) = c$ defined for all $0 \leqslant \rho \leqslant 1$. For $\rho > 0$ we define $v(\rho, c) = u(\rho, c)/u_\infty(\rho)$ and $w(\rho, c) = v(\rho, c) - 1$. We also define the function

$$H_v = \frac{1}{2}\rho^2(1 - \rho^2)v'^2 - \alpha(1 - \alpha)\left(\frac{v^2}{2} - \frac{v^{p+1}}{p+1}\right). \tag{47}$$

Using equation (5) we get

$$H_v' = \frac{5 - p}{p - 1}\rho v'^2, \tag{48}$$

hence H_v is decreasing along solutions. Furthermore, H_v has a minimum equal to $-(p - 3)/(p^2 - 1)$ for $v' = 0$ and $v = 1$.

Lemma 1. Let $d = (p - 1)/p$. If $c > b_0/d$, then there is $\rho_1 = \rho_1(c) < (b_\infty/dc)^{1/\alpha}$ such that $w(\rho_1, c) = 0$ and $0 < \rho_1 w'(\rho_1, c) < \alpha$.

Proof. It follows from equation (32) that $u(\rho, c)$ is a decreasing function of ρ for $c > b_0$ as long as $u > b_0$. Integrating equation (32) and noting that $0 < u(\rho, c) \leqslant c$ we get

$$\rho^2(1 - \rho^2)^\alpha u'(\rho) = -\int_0^\rho s^2(1 - s^2)^{\alpha-1}[u(s)^p - \alpha(1 + \alpha)u(s)]\,\mathrm{d}s >$$

$$-\int_0^\rho s^2(1 - s^2)^{\alpha-1}u(s)^p\,\mathrm{d}s > -\int_0^\rho s^2(1 - s^2)^{\alpha-1}c^p\,\mathrm{d}s. \tag{49}$$

Thus

$$\rho^2(1 - \rho^2)^\alpha u'(\rho) > -\int_0^\rho \frac{s^2(1 - s^2)^{\alpha-1}}{(1 - \rho^2)^{\alpha-1}}c^p\,\mathrm{d}s > -\int_0^\rho s^2c^p\,\mathrm{d}s = -\frac{\rho^3}{3}c^p, \tag{50}$$

hence

$$u'(\rho) > -\frac{\rho}{3(1 - \rho^2)}c^p. \tag{51}$$

Integrating (51) from 0 to x where $u(x, c) = dc$ gives

$$-c(1 - d) > \frac{c^p}{6}\ln(1 - x^2), \qquad \text{so } x^2 > 1 - \exp\left(-\frac{6}{pc^{p-1}}\right) > \frac{6}{pc^{p-1}}. \tag{52}$$

Let y be a point where $u_\infty(y) = b_\infty y^{-\alpha} = dc$, that is, $y = (b_\infty/dc)^{1/\alpha}$. We have

$$\left(\frac{b_\infty}{d}\right)^{2/\alpha} = \left(\frac{b_\infty}{d}\right)^{p-1} = \left(\frac{p}{p - 1}\right)^{p-1}\frac{2(p - 3)}{(p - 1)^2} < \frac{6}{p}, \tag{53}$$

hence $x > y$, and therefore $u(y, c) > u(x, c) = dc = u_\infty(y)$ or $w(y, c) > 0$. Since $w(\rho, c) < 0$ for small ρ, we conclude that there is a ρ such that $w(\rho, c) = 0$ and if ρ_1 is the smallest such ρ, then $w'(\rho_1, c) > 0$. Note that $\rho_1 < y$. This concludes the proof of lemma 1.

Corollary 1. $w(\rho) > -2\alpha$ for $\rho > \rho_1$.

Proof. Using the fact that the function $H_v(\rho)$ is monotone decreasing we obtain

$$H_v(\rho) \leqslant H_v(\rho_1) \leqslant \frac{\alpha^2}{2} - \alpha(1 - \alpha)\left(\frac{1}{2} - \frac{1}{p + 1}\right), \tag{54}$$

for $\rho \geqslant \rho_1$. On the other hand, for $\rho \geqslant \rho_1$

$$H_v(\rho) = \frac{1}{2}\rho^2(1 - \rho^2)v'^2 - \alpha(1 - \alpha)\left(\frac{v^2}{2} - \frac{v^{p+1}}{p - 1}\right) \geqslant -\alpha(1 - \alpha)\left(\frac{v^2}{2} - \frac{v^{p+1}}{p - 1}\right). \tag{55}$$

Combining equations (54) and (55) we have

$$f(v) \equiv \frac{\alpha^2}{2} - \alpha(1 - \alpha)\left(\frac{1}{2} - \frac{1}{p + 1}\right) + \alpha(1 - \alpha)\left(\frac{v^2}{2} - \frac{v^{p+1}}{p - 1}\right) \geqslant 0, \tag{56}$$

for $\rho \geqslant \rho_1$. Note that $f'(v) > 0$ for $v < 1$. A straightforward calculation yields $f(1 - 2\alpha) < 0$ if $p \geqslant 7$; hence $v > 1 - 2\alpha$ (or $w > -2\alpha$) for $\rho \geqslant \rho_1$, as claimed.

Let us define

$$R(\rho, c) = \sqrt{w(\rho, c)^2 + w'(\rho, c)^2}, \qquad 0 < \rho \leqslant 1. \tag{57}$$

By uniqueness of solutions of ODEs, a solution starting at $(u, u') = (c, 0)$ at $\rho = 0$ cannot have $(u(\rho, c), u'(\rho, c) = (u_\infty(\rho), u'_\infty(\rho))$, which implies that $R(\rho, c) > 0$. Thus, we may also define

$$\Theta(\rho, c) = \arctan\left(\frac{\rho w'(\rho, c)}{w(\rho, c)}\right) \qquad \text{up to a multiple of } \pi. \tag{58}$$

Since the region $\{(\rho, c) | 0 < \rho \leqslant 1, c \geqslant 0\}$ is simply connected we may unambiguously define a function $\Theta(\rho, c)$ once we specify its value at any point in the domain. Since $w(\rho, 0) \equiv -1, w'(\rho, 0) \equiv 0$, we set $\Theta(1/2, 0) = \pi$. Note that $\lim_{\rho \to 0+} w(\rho, c) = -1$ and $\lim_{\rho \to 0+} \rho w'(\rho, c) = 0$; hence $\lim_{\rho \to 0+} \Theta(\rho, c) = \pi$.

Lemma 2. Let $\rho_2 \ll 1$. We have $\lim_{c \to \infty} \Theta(\rho_2, c) = -\infty$.

Proof. Consider the equation satisfied by the function $\Theta(\rho) = \Theta(\rho, c)$:

$$\Theta'(\rho) = -\frac{1}{\rho}\left[\sin^2 \Theta + \frac{(p - 5) - \rho^2(p - 1)}{(p - 1)(1 - \rho^2)} \sin \Theta \cos \Theta \right.$$

$$\left. + \frac{2(p - 3)v}{(p - 1)^2(1 - \rho^2)}(1 + v + \cdots + v^{p-2}) \cos^2 \Theta. \right] \tag{59}$$

We will show that the quantity in brackets is bounded from below by a positive constant η for $\rho \geqslant \rho_1$. It follows that

$$\lim_{c \to \infty} \Theta(\rho_2, c) = \lim_{c \to \infty}\left(\Theta(\rho_1(c), c) + \int_{\rho_1(c)}^{\rho_2} \Theta'(\rho)d\rho\right) \leqslant \frac{\pi}{2}$$

$$+ \lim_{c \to \infty} \int_{\rho_1(c)}^{\rho_2}\left(-\frac{\eta}{\rho}\right) = \frac{\pi}{2} - \eta \lim_{c \to \infty} \ln \frac{\rho_2}{\rho_1(c)} = -\infty, \tag{60}$$

and the lemma is proven.

To get a bound on the quantity in brackets we regard it as a (formal) quadratic form in $\sin\Theta$ and $\cos\Theta$: $[\ ] = A\sin^2\Theta + B\sin\Theta\cos\Theta + C\cos^2\Theta$, where

$$A = 1, \qquad B = \frac{(p-5) - \rho^2(p-1)}{(p-1)(1-\rho^2)}, \qquad C = \frac{2(p-3)v}{(p-1)^2(1-\rho^2)}(1 + v + \cdots + v^{p-2}).$$
(61)

Since $\rho < \rho_2$, we may replace these coefficients by

$$A = 1, \qquad \tilde{B} = \frac{(p-5)}{(p-1)}, \qquad \tilde{C} = \frac{2(p-3)v}{(p-1)^2}(1 + v + \cdots + v^{p-2}).$$
(62)

We need to show that the discriminant $\Delta = \tilde{B}^2 - 4A\tilde{C}$ is negative for $v > 1 - 2\alpha$. We have

$$\Delta = \frac{(p-5)^2}{(p-1)^2} - \frac{8(p-3)}{(p-1)^2}(1 + v + \cdots + v^{p-2}).$$
(63)

Clearly, $(p-1)^2\Delta$ is a decreasing function of v. When $v = 1 - 2\alpha$ we get

$$(p-1)^2\Delta = (2p^2 - 8p + 6)\left(\frac{p-5}{p-1}\right)^p - p^2 + 6p - 5,$$
(64)

which is negative for $p \geqslant 7$; hence $\Delta < 0$ which concludes the proof of lemma 2.

Next, we consider solutions $U(\rho, b)$ which start from initial value (10). As we showed in section 3 these solutions are defined for $0 < \rho \leqslant 1$; hence we may define

$$\tilde{R}(\rho, b) = \sqrt{W(\rho, b)^2 + W'(\rho, b)^2}, \qquad 0 < \rho \leqslant 1,$$
(65)

where $W(\rho, b) = U(\rho, b)/u_\infty(\rho) - 1$. By uniqueness of solutions of ODEs, a solution starting at $\rho = 1$ cannot have $(U(\rho, b), U'(\rho, b) = (u_\infty(\rho), u'_\infty(\rho))$, unless $b = b_\infty$, which implies that $\tilde{R}(\rho, b) > 0$ if $b \neq b_\infty$. Thus, we may also define

$$\tilde{\Theta}(\rho, b) = \arctan\left(\frac{\rho W'(\rho, b)}{W(\rho, b)}\right) \qquad \text{up to a multiple of } \pi.$$
(66)

Since the region $\{(\rho, b)|0 < \rho \leqslant 1, 0 \leqslant b \leqslant b_\infty\}$ is simply connected we may unambiguously define a function $\tilde{\Theta}(\rho, b)$ once we specify its value at any point in the domain. We set $\tilde{\Theta}(1, 0) = \pi$.

We define a map

$$\Phi : R_+ = \{c|c \geqslant 0\} \to R_+^2 = \{(x, y) \in R^2|y > 0\}, \qquad \Phi(c) = (\Theta(\rho_0, c), R(\rho_0, c))$$
(67)

and a map

$$\Psi_n : t \in (-\infty, b_\infty) \to R_+^2, \qquad \Psi_n(t) = \begin{cases} (\tilde{\Theta}(\rho_0, b) - 2n\pi & \text{if } 0 \leqslant t < b_\infty, \\ \tilde{R}(\rho_0, b)) & \text{(segment 1)}, \\ (\pi - 2n\pi, 1 - t) & \text{if } t < 0 \\ & \text{(segment 2)}. \end{cases}$$
(68)

Note that if $\Psi_n(b) = \Phi(c)$ for some $b \in (0, b_\infty)$ and some c, then $\Phi(c)$ cannot lie on segment 2 because $w(\rho_0, c) \geqslant -1$ and $y\cos x = -y < -1$ on segment 2. Thus, if $\Psi_n(b) = \Phi(c)$ for some $b \in (0, b_\infty)$ and some c, then the functions $u(\rho, c)$ and $U(\rho, b)$ and their derivatives match at ρ_0; hence we have a solution defined on the whole interval $0 \leqslant \rho \leqslant 1$ in the nth

nodal class. The integer part of $(\tilde{\Theta}(1, b) - \Theta(0, c))/\pi$ counts the number of zeros of w. In particular, $\Psi_0(0) = \Phi(0)$ since $u(\rho, 0) \equiv U(\rho, 0) \equiv 0$ is a solution in the 0th nodal class.

Remark. The choices of $\Theta(1/2, 0) = \pi$ and $\tilde{\Theta}(1, 0) = \pi$ were made so that the indexing of solutions is correct.

Now we are ready to prove the following proposition.

Proposition 3. For any positive integer n there is $B_n < b_\infty$ such that the solution $U(\rho, B_n)$ is in the nth nodal class.

Proof. Since $b_\infty < b_0$, there is a $\rho_0 < 1$ such that $U(\rho_0, b) < b_0$ for all $b \in [0, b_\infty]$. Note that $U'(1, b) < 0$ for $b \leqslant b_\infty$ and thus $U'(\rho, b) < 0$ on $\rho_0 < \rho < 1$ because $U''(\rho, b) > 0$ if $U'(\rho, b) = 0$ and $U(\rho, b) < b_0$. It follows that $U(\rho, b) \geqslant 0$ and hence $W(\rho, b) \geqslant -1$ for $\rho_0 < \rho \leqslant 1$ and $0 \leqslant b \leqslant b_\infty$. From this conclude that the curve $\Psi_n(b)$ is simple, i.e. $\Psi_n(b) = \Psi_n(\bar{b})$ implies that $b = \bar{b}$. For nonnegative b and $\bar{b}$ this follows from the uniqueness of solutions of ODEs, while for $\bar{b} < 0 \leqslant b$, $\Psi_n(\bar{b}) = \Psi_n(b)$ is impossible because $W(\rho_0, b) > -1$ by choice of ρ_0 and $y \cos x = -y < -1$ on segment 2 of the curve. Since $\lim_{b \to b_\infty} \tilde{R}(\rho_0, b) = \tilde{R}(\rho_0, b_\infty) = 0$ and $\lim_{b \to -\infty} \tilde{R}(\rho_0, b) = \infty$, it follows from the Jordan curve theorem that the curve Ψ_n separates the half-plane R_+^2 into two open regions: $R_+^2 \setminus \text{image}(\Psi_n) = A \cup B$. Moreover, points p and q are in the different components A and B iff there is a curve from p to q crossing $\text{image}(\Psi_n)$ transversally at exactly one point.

Note that $\tilde{\Theta}(1, b) > \pi/2$ if $b < b_\infty$ and $\tilde{\Theta}'(\rho, b) > 0$ when $\tilde{\Theta}(\rho, b) = \pi/2$; hence $\tilde{\Theta}(\rho, b) \geqslant \pi/2$ for $b < b_\infty$. The region

$$Z = \{(x, y) \in R_+^2 | x < \pi/2 - 2n\pi\} \tag{69}$$

is connected and does not meet $\text{image}(\Psi_n)$ because $\tilde{\Theta}(\rho, b) \geqslant \pi/2$; hence Z must be contained in A or B; suppose that $Z \subset A$. Curve Φ meets A because $\lim_{c \to \infty} \Theta(\rho, c) = -\infty$ by lemma 2 and hence curve Φ meets $Z \subset A$. Note that $\text{image}(\Psi_n) \cap \text{image}(\Psi_m) = \emptyset$ if $n \neq m$ by uniqueness of solutions of ODEs; hence $\Phi(0) \notin \text{image}(\Psi_n)$ if $n > 0$.

We will now show that $\Phi(0) \in B$. This will complete the proof because the curve Φ is connected; hence it must cross $\text{image}(\Psi_n)$, which means that there is a solution in the nth nodal class. To show this we construct a curve in R_+^2 from $\Phi(0)$ to Z that crosses $\text{image}(\Psi_n)$ transversally at exactly one point. We denote by M an upper bound of $\tilde{R}(\rho_0, b)$ on $0 \leqslant b \leqslant b_\infty$ and define a curve (see figure 3)

$$\Gamma(t) = \begin{cases} (\pi, 1 + t) & \text{for } 0 \leqslant t \leqslant M \text{ (segment 1)}, \\ (\pi + M - t, M + 1) & \text{for } M \leqslant t \leqslant (2n + 1)\pi \text{ (segment 2)}. \end{cases} \tag{70}$$

Note that segment 1 of Γ cannot meet segment 1 of Ψ_n because segment 1 of Ψ_n has $w > -1$ and segment 1 of Γ has $y \cos x < -1$. Also, segment 1 of Γ cannot meet segment 2 of Ψ_n because they have different first coordinates $\pi - 2\pi < \pi$. Finally, segment 2 of Γ cannot meet segment 1 of Ψ_n because $\tilde{R}(\rho_0, b) \leqslant M < M = 1$. Thus, the only intersection point of Γ and Ψ_n is at $(\pi - 2n\pi, M + 1)$. This completes the proof of proposition 3.

Proposition 4. For any positive integer n there is $\tilde{B}_n > b_\infty$ such that the solution $U(\rho, b_n)$ is in the nth nodal class.
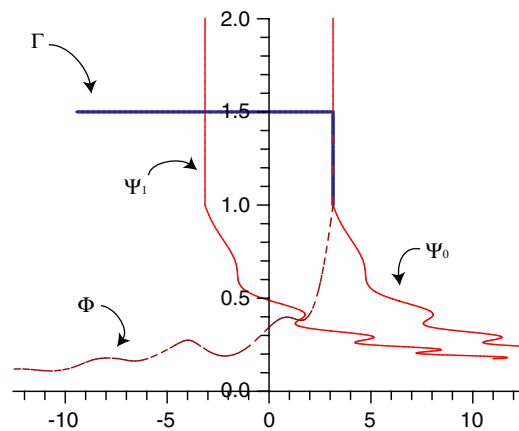
Figure 3. Illustration to the proof of proposition 3.

(This figure is in colour only in the electronic version)

Proof. The proof proceeds along the same lines as the proof of proposition 3 except for some technical modifications which we leave to the reader.

Theorem 1 is the immediate consequence of propositions 3 and 4 if we set $b_{2n-1} = B_n$ and $b_{2n} = \tilde{B}_n$.

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