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Self–Triggered Linear Quadratic Networked Control

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SUMMARY

This paper deals with Networked Control Systems (NCS) design, under the constraint of limited bandwidth on the communication channel. A linear quadratic problem for a fixed sampling period is solved and this result is used for the development of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes, yielding to the statement and solution of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems. Finally, a self–triggered controller is designed with a switched system approach in order to improve performance. Several examples are presented in order to illustrate the validity of the developed theory. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Networks have become widespread in present days and this fact causes the integration of various areas of engineering to a networked environment. This is due to the flexibility in the data traffic between components of the system and to the reduced cost required to its implementation [1]. In this context, a Networked Control System (NCS) is a spatially distributed system in which the communication between sensors, actuators and controllers is done through a communication network [2]. Notice that the structure of a NCS differs from the classical control system one, in which the communication channels are idealized and are not shared. NCSs exhibit some limitations [1], [2], including limited bandwidth, sampling, coding, delay and packet dropout which may compromise the overall performance. In this paper, we focus our study on the so called direct structure, which requires more careful design techniques, see [1].

Several results may be found involving networked control systems analysis. In [3] we can find meaningful results on stabilization of linear systems with minimum *bit–rate* and in [4] some models of networked control systems with limited bandwidth are shown and necessary and sufficient

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stability conditions are discussed for state feedback and output feedback design. Another state feedback design approach is proposed in [5], where network delay and data packet dropout issues are taken into account. In some applications, data packet disorder may occur, due to the network-induced delay and this situation is discussed in [6]. Furthermore, [7], [8] provide stability conditions on NCSs involving minimum information rate, data delay and packet dropout.

Limited bandwidth NCSs are closely related to sampled-data systems. Indeed, a networked system of this class may be modeled as a sampled-data one with a maximum sampling rate constraint, defined by the main characteristics of the communication channel [9]. The classical approach when dealing with sampled-data systems consists in ideal sampling of the discrete-time measurements and in the use of zero-order holds to convert discrete-time control signals to continuous-time ones. In [10], the classical LQR problem is solved for a periodically sampled control signal through the definition of a specific discrete-time system, which will be used afterwards. Some analysis on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance for the sampled-data control system has also been done. More general results may be derived with the use of lifting techniques, in which the sampler and the holder may also be designed in order to optimize the closed-loop performance. A very comprehensive framework of lifted systems is given in [11], where $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller designs are developed for single rate sampled data systems. Another important contribution in the area is presented in [12] and provides $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design for synchronous multirate systems, which has been done by converting that problem to an equivalent one with a single sampling rate.

Another approach to NCSs design satisfying some performance index is done with *self–triggering* and *event–triggering* strategies [13], [14], [15], [16]. Both techniques try to reduce network resource consumption by the control system, but they differ in some aspects. The first one uses the current sampled state to determine the control signal and the next sampling time, involving a *scheduling* procedure. The second one uses an *event detector*, that is located next to the plant and uses a function of the current state to determine when the next sampling time must occur. An important comparison between both techniques is provided by [14] and [15], where the authors discuss some performance specifications and protocols that implement both control strategies via state feedback. The update time instants in the self-triggered case are designed in order to guarantee stability and to enforce the desired performance, which is measured by the decay of a Lyapunov function. An output feedback self-triggered design is developed in [13], where the authors generalize two strategies conceived for state feedback: *flow based scheduling* and *bound based scheduling* for the sampling instants.

Therefore, one may conclude that there are several results on stability and analysis of NCSs, but one may notice that there are few results on design of NCSs that satisfy some performance index. Optimal control results are obtained for fixed sampling time and this may not be well adapted for shared networks. Furthermore, we will show that better results are obtained if we allow a dynamically chosen sampling period. Another important remark is that event–triggering may consume network resources excessively in order to detect when the expected event will occur. Hence, we conclude that a self–triggering approach to the design of a NCS is well adapted to a networked environment, since it allows a sampled data control that does not demand network resources excessively. In this context, our goal is to provide results on NCSs optimal design for a fixed sampling period, based on [10], considering the well established $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes. Later on we relax this constraint and we design a switched control system, where the switching function determines the next sampling time, yielding a self–triggering controller. In

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this paper, stability results and guaranteed cost determination for discrete-time switched systems obtained by [17] will be extensively used.

The notation throughout is standard. For square matrices $\mathrm{Tr}(\cdot)$ denotes the trace function and for a symmetric matrix $\sigma_{max}(\cdot)$ is its maximum singular value. For real matrices or vectors ($'$) indicates transpose. For symmetric matrices, the symbol ($\bullet$) denotes each of its symmetric blocks. The set $\mathcal{M}$ is composed by all Metzler matrices with nonnegative elements $\pi_{ji} \geq 0, \forall i, j$ satisfying the normalization constraint $\sum_j \pi_{ji} = 1, \forall i$. The set of real and nonnegative integer numbers are denoted as $\mathbb{R}$ and $\mathbb{N}$ whereas the set of the N first natural numbers is denoted as $\mathbb{K} = \{1, 2, \cdots, N\}$. The squared norm of a trajectory $\xi(t)$ defined for all $t \geq 0$, denoted by $\|\xi\|_2^2$, is equal to $\|\xi\|_2^2 = \int_0^\infty \xi(t)'\xi(t)dt$. All trajectories with finite norm, that is $\|\xi\|_2 < \infty$, constitute the set $\mathcal{L}_2$. With a little abuse of notation, the symbol $\mathcal{L}_2$ also denotes the set of all trajectories with finite norm in discrete-time.

2. PRELIMINARIES

We begin by the calculation of two performance indexes that we think are specially well adapted to NCS, where the effect of limited bandwidth in measurement and control channels are taken into account. This is the basis for the solution of what we call Linear Quadratic Networked Control Problem - LQN to be defined afterwards, see [10] for similar results. For the moment, we point out that optimal control problems of this class are formulated in continuous-time and are solved, without introducing any conservatism, from the positive definite stabilizing solution of a discrete-time Riccati equation. Hence, from the numerical viewpoint these problems are adequately handled by the methods available in the literature to date even for systems with large dimensions.

2.1. *Sampled data system*

Let us consider a continuous-time linear system with minimal state space realization

$$\dot{x}(t) = Ax(t) + Bu_k(t) \; , \; x(0) = \xi \tag{1}$$

$$z(t) = Cx(t) + Du_k(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $z(t) \in \mathbb{R}^q$ is the controlled output and $u_k(t) \in \mathbb{R}^m$ is the control signal transmitted through a limited bandwidth channel. Following [9], let us suppose that the communication channel is a perfect noiseless channel, with no time delay, and that $\mu > 0$ is the maximum allowed bit-rate. Then, for a given $r > 0$, we consider the class $\mathcal{S}_r$ of such channels with any sampling period $T > 0$ satisfying

$$\frac{\mu}{T} \leq r, \tag{3}$$

which models bandwidth limitations. Hence, this constraint is equivalent to $T \geq T_\star \equiv \mu/r > 0$, where $T_\star$ is the minimum sampling period allowed for transmission.

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With this in mind, supposing that quantization effects can be ignored, the control input is modeled as a piecewise constant signal of the form

$$u_k(t) = u_k \ , \ \forall t \in [t_k, t_{k+1}) \tag{4}$$

for all $k \in \mathbb{N}$. For the moment, the time duration between two successive sampling instants is constant, that is

$$t_{k+1} - t_k = T \geq T_\star \ , \ \forall k \in \mathbb{N} \tag{5}$$

This assumption will be relaxed afterwards to cope with non-uniform samplings $T_k = t_{k+1} - t_k, \forall k \in \mathbb{N}$, defined as an additional decision variable. This is a control system of data-rate-limited type with limited bandwidth equal to $1/T_\star$. We start by introducing the following notation associated to the continuous-time system (1)-(2), namely $\mathcal{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ and $\mathcal{C} \in \mathbb{R}^{q\times(n+m)}$ given by

$$\mathcal{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} C & D \end{bmatrix} \tag{6}$$

and stating a central result to be extensively used in the sequel.

Lemma 1

Consider the system (1)-(2), define matrices (A_d, B_d, C_d, D_d) of compatible dimensions such that

$$e^{\mathcal{A}T} = \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} \tag{7}$$

$$\int_0^T e^{\mathcal{A}'t}\mathcal{C}'\mathcal{C}e^{\mathcal{A}t}dt = \begin{bmatrix} C_d' \\ D_d' \end{bmatrix} \begin{bmatrix} C_d' \\ D_d' \end{bmatrix}' \tag{8}$$

and the discrete-time invariant linear system

$$x_{k+1} = A_d x_k + B_d u_k \ , \ x_0 = \xi \tag{9}$$

$$z_k = C_d x_k + D_d u_k \tag{10}$$

then the following equality holds

$$\int_0^\infty z(t)'z(t)dt = \sum_{k=0}^\infty z_k' z_k \tag{11}$$

The proof of this lemma, see [10], makes clear that, generally, the dimensions of the output vectors $z(t)$ and z_k are not the same. For instance, it may occur that the left hand side of (8) be a positive definite matrix which imposes $z_k \in \mathbb{R}^{n+m}$ even though $z(t) \in \mathbb{R}^q$ with $q \leq n < n+m$. Moreover, given the state space representation of the continuous-time system (A, B, C, D) and $T > 0$, it is a simple matter to calculate the state space minimal representation of the discrete-time system (A_d, B_d, C_d, D_d) such that equality (11) is preserved. This result is important in the context of networked control systems since it provides a way to calculate a discrete-time linear system such that the $\mathcal{L}_2$ norm of the output trajectory $z_k, \forall k \in \mathbb{N}$, equals the $\mathcal{L}_2$ norm of the output

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trajectory $z(t), \forall t \geq 0$, of the continuous-time system. This fact, expressed by the equality (11), does not involve any kind of approximation. Furthermore, it is important to stress that for any $T > 0$ the result of Lemma 1 can be used to determine the exact value of squared norm of the output trajectory, whenever the control signal $u_k(t)$ is transmitted to the limited bandwidth channel satisfying (4). With this result, problems formulated in continuous-time are exactly solved in discrete-time.

2.2. *Performance indexes*

Based on the result of Lemma 1, in this section, we develop two performance indexes associated to the networked system under consideration. They are closely related to the concept of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of LTI systems, respectively. Their importance stems from the fact that the $\mathcal{H}_2$ norm follows from an impulsive input which implies that all frequencies are equally excited. On the other hand, the $\mathcal{H}_\infty$ norm, whenever bounded, imposes to the system certain robustness properties against parameter perturbations, for details see [18].

Consider an open-loop continuous-time system

$$\dot{x}(t) = Ax(t) + Ew(t)\ ,\ x(0) = 0 \tag{12}$$

$$z(t) = Cx(t) + Fw(t) \tag{13}$$

where, as before, $x(t) \in \mathbb{R}^n$ is the state, $z(t) \in \mathbb{R}^q$ is the output and $w(t) \in \mathbb{R}^r$ is the exogenous perturbation. In this section, it is assumed that matrix A is Hurwitz which implies that matrix A_d is Schur for all $T > 0$. First, as usually done in the definition of $\mathcal{H}_2$ norm we assume that $F = 0$ and the perturbation is such that $w(t) = e_i\delta(t)$ where $e_i \in \mathbb{R}^r, i = 1, \cdots, r$, are the columns of the identity matrix. Hence, denoting by $z^i(t)$ the output trajectory, the performance index is expressed simply as

$$J_2 = \sum_{i=1}^{r} \int_0^\infty z^i(t)' z^i(t) dt \tag{14}$$

and, remembering that the impulse at $t = 0$ induces a discontinuity on the initial condition corresponding to move it instantaneously from $x(0) = 0$ to $x(0^+) = Ee_i, i = 1, \cdots, r$, Lemma 1 applied to $\dot{x}(t) = Ax(t)$, $z(t) = Cx(t)$ for each initial condition yields

$$J_2 = \sum_{i=1}^{r} \sum_{k=0}^{\infty} (z_k^i)'(z_k^i) \tag{15}$$

where $x_{k+1} = A_d x_k$, $x_0 = Ee_i$ and $z_k^i = C_d x_k$ for each $i = 1, \cdots, r$ and

$$e^{AT} = A_d\ ,\ \int_0^T e^{A't} C' C e^{At} dt = C_d' C_d \tag{16}$$

From this calculation it follows that

$$\begin{aligned} J_2 &= \mathrm{Tr}\left(E' \sum_{k=0}^{\infty} A_d'^k C_d' C_d A_d^k E \right) \\ &= \left\| C_d (zI - A_d)^{-1} E \right\|_2^2 \end{aligned} \tag{17}$$

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It is interesting to observe that the pair of matrices (A_d, C_d) depends on $T > 0$ but the index J_2 does not. We conclude that

$$\left\|C(sI-A)^{-1}E\right\|_2^2 = \left\|C_d(zI-A_d)^{-1}E\right\|_2^2, \forall T > 0 \tag{18}$$

which means that this is just an alternate and equivalent way to calculate the performance index in terms of the $\mathcal{H}_2$ norm of the associated discrete-time system transfer function. As it will be clear in the sequel, this is an useful result to dealing with networked control systems.

Now, let us move our attention to the $\mathcal{H}_\infty$ performance index. To this end, we assume that $w(t) \neq 0$ is an arbitrary perturbation in $\mathcal{L}_2$ with finite norm transmitted through the limited bandwidth channel. In other words, it is modeled as an additive perturbation that acts in the control channel, which is, in our opinion, an adequate performance index to be considered in the NCS framework. After transmission, denoting again $z(t)$ the corresponding output signal given by (12)-(13), we define the performance index

$$J_\infty = \sup_{w_k(t)\neq 0 \in \mathcal{L}_2} \frac{\int_0^\infty z(t)'z(t)dt}{\int_0^\infty w_k(t)'w_k(t)dt} \tag{19}$$

where $w_k(t) = w_k,\ \forall t \in [t_k, t_{k+1})$ for all $k \in \mathbb{N}$. It is important to give some interpretation about the index we have just introduced that is strongly related to the $\mathcal{H}_\infty$ norm of the continuous-time. Indeed, we immediately have

$$\begin{aligned} J_\infty &\leq \sup_{w(t)\neq 0 \in \mathcal{L}_2} \frac{\int_0^\infty z(t)'z(t)dt}{\int_0^\infty w(t)'w(t)dt} \\ &\leq \left\|C(sI-A)^{-1}E + F\right\|_\infty^2 \end{aligned} \tag{20}$$

Although the upper bound does not depend on $T > 0$, the value of the index J_∞ does. However, it can be calculated by applying once again the result of Lemma 1 which yields

$$J_\infty = \sup_{w_k \neq 0 \in \mathcal{L}_2} \frac{\sum_{k=0}^\infty z_k' z_k}{T\sum_{k=0}^\infty w_k' w_k} \tag{21}$$

where $x_{k+1} = A_d x_k + E_d w_k$, $z_k = C_d x_k + F_d w_k$. This is now a classical $\mathcal{H}_\infty$ norm evaluation in the discrete-time domain, which immediately gives

$$J_\infty = \left\| \frac{C_d(zI-A_d)^{-1}E_d + F_d}{\sqrt{T}} \right\|_\infty^2 \tag{22}$$

which means that, for $T > 0$ given, the proposed index is proportional to the $\mathcal{H}_\infty$ squared norm of the transfer function from the input w_k to the output z_k of the previously given discrete-time system.

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Finally, denoting $H(s) = C(sI - A)^{-1}E + F$ and taking $T > 0$ arbitrarily small we have

$$
\begin{aligned}
J_\infty &= (1/T)\left\|C_d(zI - A_d)^{-1}E_d + F_d\right\|_\infty^2 \\
&\approx \left\|C((zI - I)/T - A)^{-1}E + F\right\|_\infty^2 \\
&\approx \sup_{\omega \in [0,\ \pi/T]} \sigma_{max}^2\left(H((e^{j\omega T} - 1)/T)\right) \\
&\approx \|H(s)\|_\infty^2
\end{aligned} \tag{23}
$$

where the last approximation follows from the relation

$$
j\omega \approx \frac{e^{j\omega T} - 1}{T} \tag{24}
$$

valid for all $\omega \in \mathbb{R}$ and $T > 0$ arbitrarily small. As expected, the index J_∞ recovers the square $\mathcal{H}_\infty$ norm of the continuous-time system under consideration, without any bandwidth limitation. In our opinion, both indexes are valid for all values of $T > 0$ whenever the performance deterioration due to bandwidth limitation has to be measured. The next section is devoted to state and solve the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems for networked systems taking into account the limited bandwidth constraint. The rationale to be adopted is to convert the continuous-time model to the discrete-time one as indicated in Lemma 1.

3. LINEAR QUADRATIC NETWORKED CONTROL

Let the control system with bandwidth limitation be given as

$$
\dot{x}(t) = Ax(t) + Bu_k(t) + Ew(t) \tag{25}
$$

$$
z(t) = Cx(t) + Du_k(t) + Fw(t) \tag{26}
$$

with zero initial condition and where $w(t)$ is the exogenous input. The main goal is to determine a state feedback gain $L \in \mathbb{R}^{m\times n}$ such that the closed-loop system with $u(t) = u_k = Lx_k, \forall t \in [t_k, t_{k+1})$, minimizes one of the indexes J_2 or J_∞ that we have discussed in the previous section.

3.1. $\mathcal{H}_2$ norm optimization

Applying Lemma 1, we convert the above problem in the one that consists in finding $u_k(t)$ in order to minimize the index J_2. As usual, it is assumed that $F = 0$. Based on the results of the previous section, this problem can be recast in the equivalent form:

$$
\inf_L \|(C_d + D_d L)(zI - (A_d + B_d L))^{-1}E\|_2^2 \tag{27}
$$

which is nothing else than a classical LQ problem in discrete-time. Hence, the optimal gain can be determined from the positive definite stabilizing solution of the algebraic Riccati equation obtained from

$$
(A_d + B_d L)'P(A_d + B_d L) - P + (C_d + D_d L)'(C_d + D_d L) = 0 \tag{28}
$$

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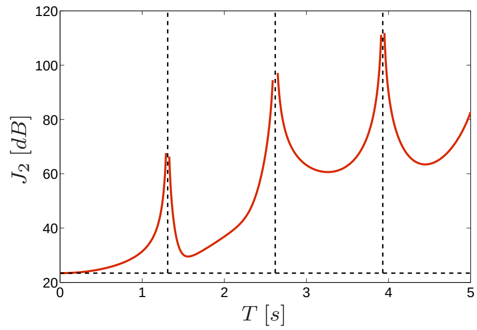

Figure 1. Values of the performance index J_2 for $T > 0$.

together with

$$L = -(B_d'PB_d + D_d'D_d)^{-1}(B_d'PA_d + D_d'C_d). \tag{29}$$

This state feedback gain is unique and the best one as far as the control constraint (4) is taken into account. Moreover, whenever it admits a positive definite stabilizing solution, the closed-loop continuous-time system becomes asymptotically stable as a consequence of the equality (11).

As before, it is interesting to see that for $T > 0$ arbitrarily small, we can adopt the approximations $A_d \approx I + TA$, $B_d \approx TB$, $C_d \approx \sqrt{T}C$ and $D_d \approx \sqrt{T}D$ and the previous Lyapunov equation (28) collapses to

$$\Big(I + T(A + BL)\Big)' P \Big(I + T(A + BL)\Big) - P + T(C + DL)'(C + DL) = 0$$

that is

$$(A + BL)'P + P(A + BL) + (C + DL)'(C + DL) + \mathcal{O}(T) = 0$$

where $\mathcal{O}(T) \geq 0$ and, as expected, it goes to zero when $T > 0$ goes to zero. We recover the optimal continuous-time state feedback gain for the classical problem without bandwidth constraint. Of course for $T > 0$ arbitrary the optimal solution does not necessarily coincide with this one, which is valid only for $T > 0$ sufficiently small.

Example 1

Consider a networked system (25)-(26) defined by matrices

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Figure 1 shows the index J_2 against the sampling period $T > 0$. For $T \to 0$, it illustrates the fact that the optimal continuous-time solution is generated.

Notice the vertical asymptotes for the discrete-time system clearly indicated in the same figure. We have observed that these lines occur periodically with period approximately equal to π/ω_n, where ω_n is the natural frequency of the open-loop system. These singularities put in evidence the existence of values of the sampling period for which the closed-loop system is unstable. In this case,

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the discrete-time algebraic Riccati equation does not admit a stabilizing positive definite solution. This phenomenon is caused by the *pathological sampling* [19] of the state variables, which creates uncontrollable (*hidden*) unstable poles in the connected system. If we use $T = 0.5$ $[s]$, the optimal state feedback gain is given by $L = [2.3758 \;\; -1.3907]$ yielding the minimum cost $J_2^* = 17.5661$. This figure makes clear that the increasing of $T > 0$ reflects an increasing on the optimal $\mathcal{H}_2$ cost is not necessarily true, showing that the performance index $J_2(\cdot)$ is not a monotonic function of the sampling period T.

3.2. $\mathcal{H}_\infty$ norm optimization

Following the same steps of the $\mathcal{H}_2$ case, our concern now is to determine the state feedback gain $L \in \mathbb{R}^{m\times n}$ that minimizes the cost J_∞. Imposing $u(t) = u_k = Lx_k, \forall t \in [t_k, t_{k+1})$, from Lemma 1 we obtain

$$\inf_L \frac{1}{T} \|(C_d + D_d L)(zI - (A_d + B_d L))^{-1} E_d + F_d\|_\infty^2 \tag{30}$$

Since $T > 0$ is fixed, the matrices $(A_d, B_d, C_d, D_d, E_d, F_d)$ are readily determined by Lemma 1 applied to the augmented system $(A, [B \;\; E], C, [D \;\; F])$, allowing us to see (30) as a standard $\mathcal{H}_\infty$ problem in discrete-time, which can be entirely expressed through LMIs [20]. Indeed, the global optimal solution of problem (30) is obtained from

$$\inf_{X>0, Z, \rho} \rho \tag{31}$$

subject to the LMI constraint

$$\begin{bmatrix} X & \bullet & \bullet & \bullet \\ XA_d' + Z'B_d' & X & \bullet & \bullet \\ E_d' & 0 & I & \bullet \\ 0 & C_d X + D_d Z & F_d & \rho I \end{bmatrix} > 0 \tag{32}$$

which provides the state feedback gain $L = ZX^{-1}$ and the cost $J_\infty = \rho/T$. The advantage of this approach is the linear dependence of all involved variables, including ρ, which avoids the adoption of an interactive method based on the Riccati equation calculation to get the minimum value of the norm. On the other hand, for $T > 0$ arbitrarily small, considering the approximations previously adopted together with $E_d \approx TE$ and $F_d \approx \sqrt{T}F$ the inequality (32) reduces to

$$\begin{bmatrix} AQ + QA' + BR + R'B' & \bullet & \bullet \\ E' & -I & \bullet \\ CQ + DR & F & -(\rho/T)I \end{bmatrix} < 0 \tag{33}$$

where $Q = X/T$ and $R = Z/T$, which is nothing else but the $\mathcal{H}_\infty$ condition in continuous-time. This inequality puts in evidence the scaling $1/T$ appearing in the ρ variable in order to compensate the same scaling in the cost (30).

Example 2

Consider the same dynamic system described in Example 1. Figure 2 shows the index J_∞ against the sampling period $T > 0$. Notice that periodic vertical asymptotes also occur for this index, due

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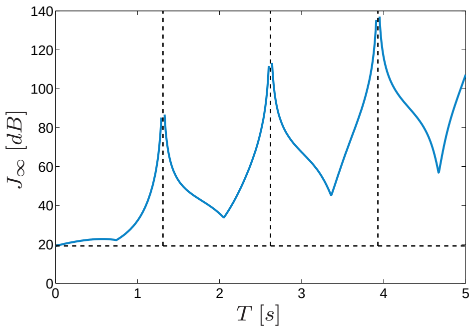

Figure 2. Values of the performance index J_∞ for $T > 0$.

to the pathological sampling of the state vector. These peaks put in evidence the existence of values of the sampling period for which the closed-loop system is unstable. In this case, the LMIs for the $\mathcal{H}_\infty$ control design do not admit a stabilizing feasible solution. If we use $T = 0.5\ [s]$, the optimal state feedback gain is given by $L = [1.1351\ \ -2.9486]$ yielding the minimum cost $J_\infty^* = 13.5919$. As before, the performance index $J_\infty(\cdot)$ is not a monotonic function of the sampling period T.

4. SELF-TRIGGERED DESIGN

In this section we introduce what we call a *discrete self-triggered control design* based on the following observation. Contrarily of what is adopted in the event-triggering strategy [15], where the sampling period is considered a continuous variable $T \in \mathbb{R}$ to be determined in real time, in order to improve stability and performance, we assume that, for a given $T_\star > 0$, the sampling period satisfies $T \in \{T_i \geq T_\star\ :\ i \in \mathbb{K}\}$ where the number N and the values of different time samplings $T_i,\ i \in \mathbb{K}$, are provided by the designer. They are chosen taking into account two important features of networked control systems: $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance and bandwidth limitations. The first criterion induces small T while the second one goes in the opposite direction. One possible choice for the set of $T_i, i \in \mathbb{K}$ is to equally divide the interval one would vary $T \in \mathbb{R}$ over N discrete values.

4.1. $\mathcal{H}_2$ Control Design

The main problem to be faced is the determination of a switching rule that orchestrates the use of a particular T_i for some $i \in \mathbb{K}$. The first step yields the strictly proper state space models $(A_{Li}, E, C_{Li}, 0)$ for all $i \in \mathbb{K}$ as follows. For each $T_i,\ i \in \mathbb{K}$, the optimal solution of problem (27) provides the closed-loop system matrices $A_{Li} = A_{di} + B_{di}L_i$ and $C_{Li} = C_{di} + D_{di}L_i$ which define the switched linear system

$$x_{k+1} = A_{L\sigma}x_k + Ew_k \quad (34)$$

$$z_k = C_{L\sigma}x_k \quad (35)$$

evolving from $x_0 = 0$, where $\sigma(k) : \mathbb{N} \to \mathbb{K}$ is the switching function to be designed and $w_k \in \mathbb{R}^r$ is an impulsive external input of the form $w_k = \delta(k)e_\ell$, being e_ℓ the ℓ^{th} column of the identity matrix

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which is used to define the $\mathcal{H}_2$ performance

$$J_2(\sigma) = \sum_{\ell=1}^{r} \|z^\ell\|_2^2 \tag{36}$$

It is important to stress that whenever the switching function is enforced to be equal to $\sigma(k) = i \in \mathbb{K}$ for all $k \in \mathbb{N}$ then $J_2(\sigma)$ equals the optimal performance of the closed-loop system with the sampling T_i. Hence, our purpose is to determine a state dependent switching strategy of the form $\sigma(k) = g(x_k)$ in order to improve the final performance by an appropriate choice of the samplings $T_i, i \in \mathbb{K}$, dynamically, that is, at each instant of time t_k, for all $k \in \mathbb{N}$. Following [17], we consider

$$g(x) = \arg\min_{i\in\mathbb{K}} x' P_i x \tag{37}$$

where $P_i > 0,\ \forall i \in \mathbb{K}$, have to be adequately determined. Indeed, we define the set $\mathcal{X}_2$ composed by all matrices $\{P_1, \cdots, P_N\}$ and a Metzler matrix $\Pi \in \mathcal{M}$ that satisfy the so called Lyapunov-Metzler inequalities

$$A_{Li}' P_{pi} A_{Li} - P_i + C_{Li}' C_{Li} < 0,\ i \in \mathbb{K} \tag{38}$$

where $P_{pi} = \sum_{j\in\mathbb{K}} \pi_{ji} P_j$. It is proven in [21] that any feasible matrices $\{P_1, \cdots, P_N\}$ whenever plugged in (37) makes the switched linear system asymptotically stable and

$$J_2(\sigma) < \min_{i\in\mathbb{K}} \mathrm{Tr}(E' P_i E) \tag{39}$$

Hence, the minimum upper bound is given by the optimal solution of the problem

$$J_2^{so} = \inf_{\{P_1,\cdots,P_N,\Pi\}\in\mathcal{X}_2} \min_{i\in\mathbb{K}} \mathrm{Tr}(E' P_i E) \tag{40}$$

which provides the switching strategy σ_2^{so} that clearly satisfies $J_2(\sigma_2^{so}) \le J_2^{so}$. This problem is, in general, difficult to solve due to the product of variables $\pi_{ji} P_j$ which makes it nonconvex. To this end, some general purpose nonlinear programming method has to be adopted, see [22], unless N is small (typically 2 or 3) in which case it is solved by grid search by taking advantage to the fact that for $\Pi \in \mathcal{M}$ fixed, (38) reduces to N LMIs.

Problem (40) exhibits a remarkable property. By construction, all discrete-time subsystems matrices $A_{Li},\ i \in \mathbb{K}$, are Schur and consequently the matrix $\Pi = I \in \mathcal{M}$ is feasible. Indeed, plugging this matrix in (38) the LMIs become decoupled, that is

$$A_{Li}' P_i A_{Li} - P_i + C_{Li}' C_{Li} < 0,\ i \in \mathbb{K} \tag{41}$$

implying that

$$\begin{aligned} J_2(\sigma_2^{so}) &\le J_2^{so} \\ &\le \inf_{\{P_1,\cdots,P_N,\Pi=I\}\in\mathcal{X}_2} \min_{i\in\mathbb{K}} \mathrm{Tr}(E' P_i E) \\ &\le \min_{i\in\mathbb{K}} \inf_{P_i>0} \{\mathrm{Tr}(E' P_i E)\ :\ (41)\} \\ &\le \min_{i\in\mathbb{K}} \|C_{Li}(zI - A_{Li})^{-1} E\|_2^2 \end{aligned} \tag{42}$$

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Based on this last relation we can drawn the conclusion that the procedure proposed in this paper promotes, in general, a gain of the performance due to the dynamic allocation of time sampling periods. It is also important to state that the obtained $\mathcal{H}_2$ performance is preserved for the original system (25)-(26). Furthermore, the switching strategy dynamically chooses the sampling period $T_{\sigma_2^{so}}$ so that (25)-(26) has the same $\mathcal{H}_2$ cost as (34)-(35) with $\sigma = \sigma_2^{so}$. In fact, without loss of generality, we suppose $n_w = 1$ and we denote $\sigma_k = \sigma_2^{so}(x_k)$, for simplicity. The general case is straightforward. Then, the $\mathcal{H}_2$ cost for (34)-(35) is

$$\begin{aligned}
\sum_{k=0}^{\infty} z_k' z_k &= \sum_{k=0}^{\infty} x_k' C_{L\sigma_k}' C_{L\sigma_k} x_k \\
&= \sum_{k=0}^{\infty} x_k' \int_0^{T_{\sigma_k}} e^{(A+BL_{\sigma_k})' t} (C + DL_{\sigma_k})' \times \\
&\quad \times (C + DL_{\sigma_k}) e^{(A+BL_{\sigma_k}) t} \, dt \, x_k \\
&= \sum_{k=0}^{\infty} x_k' \int_{t_k}^{t_{k+1}} e^{(A+BL_{\sigma_k})' (t-t_k)} (C + DL_{\sigma_k})' \times \\
&\quad \times (C + DL_{\sigma_k}) e^{(A+BL_{\sigma_k})(t-t_k)} \, dt \, x_k \\
&= \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} x(t)' (C + DL_{\sigma_k})' (C + DL_{\sigma_k}) x(t) \, dt \\
&= \int_0^{\infty} z(t)' z(t) \, dt, \qquad (43)
\end{aligned}$$

which is the $\mathcal{H}_2$ cost for (25)-(26). This result is a natural consequence of our $\mathcal{H}_2$ performance index and it ensures the stability of the closed-loop system, since (34)-(35) is clearly stable by the Lyapunov–Metzler conditions (38). The next example illustrates the more important features of the procedure.

Example 3

Consider the following marginally stable dynamic system of the form (25)-(26), given by its state space realization

$$A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To build the discrete-time switched linear system we suppose the transmission is allowed only with sampling periods equal to $T_1 = 0.80$ [s] and $T_2 = 1.1$ [s]. Using the procedure developed in this section, we obtain the associated costs $J_2(\sigma = 1) = 8.8501$ and $J_2(\sigma = 2) = 9.9731$. In addition, solving problem (40) by performing a grid search in the box $[0, 1] \times [0, 1]$ we have determined the minimum guaranteed $\mathcal{H}_2$ cost $J_2^{so} = 8.6059$.

We also solved this problem using the event–triggering strategy proposed by [15]. Through simulation, the event–triggered regulator achieves an $\mathcal{H}_2$ gain of 7.80 and our approach provides a cost $J_2(\sigma_2^{so}) = 8.31 < J_2^{so} < \min_{i=1,2}\{J_2(\sigma = i)\}$, which confirms (42). However, to accomplish this performance, in various intervals of time, the event–triggered compensator works with very

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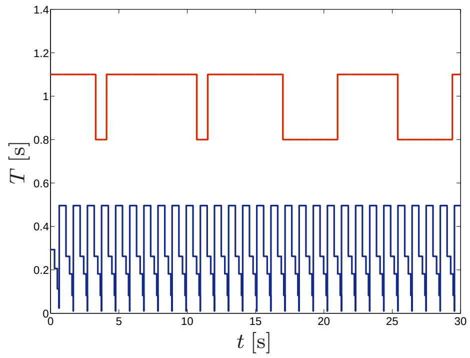

Figure 3. Sampling period for both techniques.

small sampling periods, which may not be possible on limited bandwidth communication channels. This behavior is shown in Figure 3, which displays the sampling period for both strategies. As it can be seen in this figure, the upper curve describes the sampling period with respect to time for our switching approach and the second one represents the sampling period for the event–triggered controller. It is interesting to observe that, from Figure 3, the self-triggered technique works with an average sampling period $T_{avg} = 1.02$ [s]. If a periodic sampling strategy with $T = T_{avg}$ was adopted, the associated optimal cost, obtained from (27), would be $J_{avg} = 12.0622$, which is clearly outperformed by the self-triggered controller designed in this example.

With this in mind, in order to compare the two strategies under the same limited bandwidth, we impose a constraint of the form (5) with $T_\star = 0.80$ [s]. In this situation the event–triggering technique provides a cost of 11.39, which shows that our technique enables a gain of 27% in performance, approximately. It is also important to state that, if we use our approach and allow sampling with a period less than or equal to 0.55 $[s]$, we obtain better $\mathcal{H}_2$ costs than the event–triggered strategy and may demand less network resources, depending on the chosen period.

The previous example puts in evidence that our approach is well adapted to NCS, when bandwidth limitations are imposed. However, the event–triggering technique does not provide a trustable controller when communication constraints are additionally imposed, in which case instability may occur. In fact, the event–triggered controller demands a large amount of network resources in some time intervals, which may not be possible in shared sensors/actuators networks.

4.2. $\mathcal{H}_\infty$ Control Design

Following the same reasoning that we have developed for the $\mathcal{H}_2$ problem, the main goal of this section is to determine a switching rule that imposes a particular sampling period T_i, $i \in \mathbb{K}$. To this end, we define the state space models $(A_{Li}, E_{di}, C_{Li}, F_{di})$, for each $i \in \mathbb{K}$, as follows. For each sampling period T_i, we obtain the $\mathcal{H}_\infty$ discrete-time equivalent system described in Subsection 3.2 and define the closed-loop matrices $A_{Li} = A_{di} + B_{di}L_i$ and $C_{Li} = C_{di} + D_{di}L_i$, where the feedback gains L_i are given from the solution of problem (31). Thus, we define the switched system

$$x_{k+1} = A_{L\sigma}x_k + E_{d\sigma}w_k \tag{44}$$

$$z_k = \frac{C_{L\sigma}}{\sqrt{T_\sigma}}x_k + \frac{F_{d\sigma}}{\sqrt{T_\sigma}}w_k \tag{45}$$

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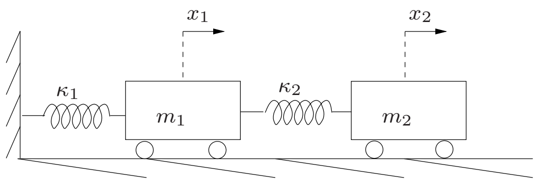

Figure 4. Mass–spring system.

evolving from $x_0 = 0$. Notice that the definition of the $\mathcal{H}_\infty$ performance index assures that the exogenous disturbance $w_k \in \mathcal{L}_2$ is a piecewise constant function and, due to (22), the output matrices in (45) are divided by the active period T_σ. Then, let us consider the $\mathcal{H}_\infty$ performance

$$J_\infty(\sigma) = \sup_{w_k \neq 0 \in \mathcal{L}_2} \frac{\sum_{k=0}^{\infty} z_k' z_k}{\sum_{k=0}^{\infty} w_k' w_k} \tag{46}$$

which, for $\sigma(k) = i$, constant for all $k \in \mathbb{N}$, coincides with the $\mathcal{H}_\infty$ cost for the closed-loop system with a fixed sampling period $T_i, i \in \mathbb{K}$.

As before, we consider the state dependent switching function $\sigma(k) = g(x_k)$, where $g(\cdot)$ is given by (37), and the set $\mathcal{X}_\infty$ composed by all positive definite matrices $\{P_1, \ldots, P_N\}$ and a Metzler matrix $\Pi \in \mathcal{M}$ that satisfy the Riccati-Metzler inequalities

$$\begin{bmatrix} P_i & \bullet & \bullet & \bullet \\ 0 & \rho I & \bullet & \bullet \\ P_{pi}A_{Li} & P_{pi}E_{di} & P_{pi} & \bullet \\ C_{Li} & F_{di} & 0 & T_i I \end{bmatrix} > 0, \quad i \in \mathbb{K} \tag{47}$$

It is proven in [23] that any feasible solution in $\mathcal{X}_\infty$ assures the closed-loop system is globally asymptotically stable. Moreover, the minimum upper bound is given by the solution of the optimization problem

$$J_\infty^{so} = \inf_{\{P_1,\ldots,P_N,\Pi\} \in \mathcal{X}_\infty} \rho \tag{48}$$

which provides a switching strategy σ_∞^{so} such that $J(\sigma_\infty^{so}) \leq J_\infty^{so}$. Unfortunately, in the $\mathcal{H}_\infty$ case, we cannot guarantee that a similar inequality as (42) holds for the designed switching function associated to the matrix $\Pi = I \in \mathcal{M}$. The adoption of a more general Metzler matrix must be investigated.

5. PRACTICAL APPLICATION

In this section, we consider the following example given in [24]. It consists of two cars with masses m_1 and m_2, with positions $x_1(t)$ and $x_2(t)$, coupled by two springs, as described in Figure 4. Let us consider that the elastic constants of the springs are $\kappa_1 = \kappa_2 = 1.0$ [N/m] and the masses are $m_1 = 1.0$ [kg] and $m_2 = 0.5$ [kg]. Our goal is to control the position of the car with mass m_2 by applying a force u_k, of the form (4), to the other car.

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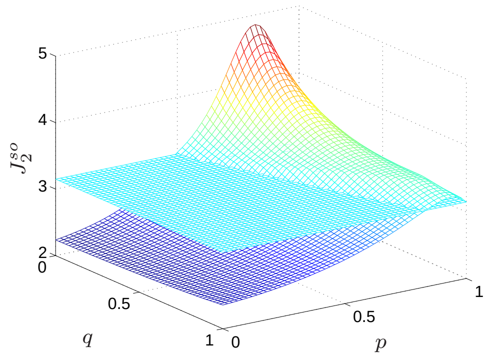

Figure 5. J_2^{so} as a function of (p, q).

Define the state vector $x_p(t) = [x_1(t)\ x_2(t)\ \dot{x}_1(t)\ \dot{x}_2(t)]'$. Following the same notation used in the previous section, the state space realization of the system is given by (25)-(26), with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{49}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$

In this self–triggered control design, it is supposed that the allowed transmission periods are $T_1 = 1.65$ [s] and $T_2 = 2.10$[s], due to bandwidth limitations. Applying the results developed in this section, we obtain the associated costs $J_2(\sigma = 1) = 3.1501$ and $J_2(\sigma = 2) = 3.3468$. Hence, performing a grid search in $(p, q) \in [0, 1] \times [0, 1]$, considering the Metzler matrix of the form

$$\Pi = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} \in \mathcal{M}, \tag{51}$$

we obtain the minimum guaranteed cost $J_2^{so} = 2.2356$, for $p = q = 0$, as shown in Figure 5.

We simulate the closed-loop system with the derived switching function σ_2^{so}, obtaining the actual $\mathcal{H}_2$ cost $J_2(\sigma_2^{so}) = 2.0338$, which is 35% better than the minimum $\mathcal{H}_2$ cost related to the subsystems, which confirms the inequality (42). The dynamic sampling period is shown in Figure 6. It is important to state that the optimal $\mathcal{H}_2$ cost for the continuous-time system, without any communication constraint, is $J_2^c = 1.1609$. Hence, it is clear that, even with large sampling periods, our approach has a comparable $\mathcal{H}_2$ performance. The quality of our results is due to the precise definition of the performance indexes, to the optimal control problems solved in Section 3 and to the self-triggered controller designed in Section 4. As before, it is possible to verify that the average sampling period is $T_{avg} = 1.84$ [s] with the associated cost $J_{avg} = 2.2368$, which is very close to J_2^{so} but greater than $J_2(\sigma_2^{so})$, as expected.

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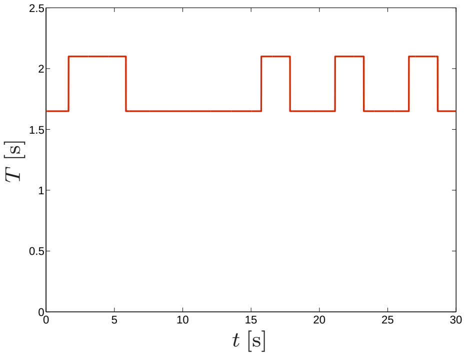

Figure 6. Sampling period of the closed-loop system using self–triggering.

6. CONCLUSION

In this paper, we have presented results in Networked Control Systems design subject to limited bandwidth communication constraints. To this end, our first approach consisted in obtaining an optimal state feedback gain that stabilizes the sampled time system through the solution of a linear quadratic problem of a specific discrete-time system. This important result has been used afterwards in order to develop $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes for the limited bandwidth problem. With these indexes, we state and solve the optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems for the NCS.

Finally, a self–triggering control system has been designed using a discrete-time switched control approach. It has been shown that this strategy is well adapted to the networked environment and it may improve the overall performance of the closed-loop system. It is important to mention that the conditions are given by LMIs or Riccati equations, which can be efficiently solved numerically, using methods available in the literature to date.

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