2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Self-Triggered Model Predictive Control for Nonholonomic Systems

Alina Eqtami, Shahab Heshmati-alamdari, Dimos V. Dimarogonas and Kostas J. Kyriakopoulos

Abstract— **This paper proposes a Model Predictive Control (MPC) framework combined with a self-triggering mechanism for constrained uncertain systems. Under the proposed scheme, the control input as well as the next control update time are provided at each triggering instant. Between two consecutive triggering instants, the control trajectory given by the MPC is applied to the plant in an open-loop fashion. This results to less frequent computations while preserving stability and convergence of the closed-loop system. A scenario for the stabilization of a nonholonomic robot subject to constraints and disturbances is considered, with the aim of reaching a specific triggering mechanism. The robot under the proposed control framework is driven to a compact set where it is ultimately bounded. The efficiency of the proposed approach is illustrated through a simulated example.**

I. Introduction

The formulation of control schemes in event-based rather than traditional periodic frameworks is a recent development that leads to the alleviation of energy consumption and it can result to the mitigation of the network traffic in network control systems. The key attribute of these approaches is that the decision for the control update is based on a certain condition of the state of the system. The event-based schemes can lead not only to a more flexible aperiodic sampling but also can preserve necessary properties of the system such as stability and convergence. Event and self-triggered control are two particular event-based approaches that however have some similarities. Both have a controller that provides the control input and both have a triggering mechanism that determines when the new control update should be. Nevertheless, the event-triggered techniques require a constant measurement of the actual state of the plant in order to decide when the control execution must be triggered while in the case of self-triggered control only the latest state measurement needs to be known for determining the next triggering instant. Related works on event-triggered control can be found in [6], [7], [17], [20]. Some relevant results for the self-triggered set-up can be found in [1], [9], [10], [19], [21].

Nonlinear Model Predictive controllers have the capability to deal with nonlinearities and constraints. This is particularly desired in real applications where constraints on the inputs as well as the states must be taken explicitly into consideration. However, most NMPC schemes are computationally demanding which gives us the motivation to design the NMPC law along with a triggering mechanism in order to compute the control law only when is needed. Note, that in [22] a NMPC was applied to a nonholonomic vehicle under a discrete-time framework. However, the control horizon was decided ad-hoc and no triggering condition was given.

In this paper a self-triggered MPC strategy is presented. We treat the case of constrained nonholonomic systems with additive disturbances under a NMPC law. The contribution relies in finding a framework that will provide control trajectories that lead to stable closed-loop responses and a mechanism that decides when the control updates should occur. In [4], a similar analysis was proposed for an event-based MPC framework. In the event-based set-up there is the need for continuously taking state measurements, in contrast to the proposed self-triggered set-up where this need is relaxed.

Even though event-based control have been considered extensively in recent years, the case of event-based MPC controllers has just started to gain attention whereas very few results have been presented for the self-triggered MPC set-up. For event-triggered MPC the reader is referred to [3], [4], [5], [15], [18]. In the context of self-triggered MPC, an analysis was presented in [8] for Network Scheduling. The authors focus on discrete-time LTI systems and they propose a cost function of the MPC that depends on the control performance and the cost for sampling. In [2], a self-triggered MPC framework was presented for constrained discrete-time linear systems. The MPC controller is designed to maintain some specific optimality levels while the control input that is sent to the actuators is the current control value and not the trajectory of the optimal inputs as is the case in the current paper. An approach for network control systems which is extended to continuous time systems, but not in the area of MPC, proposes a self-triggered selection based on quadratic programming, [12]. There, the authors present an analysis that leads to an optimization problem for maximizing the intersampling period.

The remainder of the paper is organized as follows. The scenario for the control of the nonholonomic robot as well as the problem statement are presented in Section II. Section III accommodates the robust stability analysis for the NMPC scheme which leads to the self-triggered framework. In Section IV, some simulated examples are presented which show the efficiency of the proposed scheme, along with some comparative results. Finally, Section V summarizes the results of this paper and indicates further research goals.

Alina Eqtami, Shahab Heshmati-alamdari and Kostas J. Kyriakopoulos are with the Control Systems Lab, Department of Mechanical Engineering, National Technical University of Athens, 9 Heroon Polytechniou Street, Zografou 15780, Greece {alina,shahab,kkyria@mail.ntua.gr}. Their work was supported by the EU funded project PANDORA: "Persistent Autonomy through learNing aDaptation, Observation and ReplAnning", FP7-288273, 2012-2014. Dimos V. Dimarogonas is with the KTH ACCESS Linnaeus Center and the KTH Centre for Autonomous Systems, School of Electrical Engineering, Royal Institute of Technology (KTH), Stockholm, Sweden {dimos@ee.kth.se}. His work was supported by the Swedish Research Council and the EU HYCON2 NoE.

II. Problem Formulation

In this section the specifics for the stabilization scenario are presented. First, the mathematical model of the nonholonomic system is given along with the constraints that must be fulfilled. Next, the design and analysis of the proposed controller is provided along with some assumptions that are necessary in order to achieve stability of the closed-loop system.

A. Mathematical Modeling

Consider that the motion of the robot is governed by unicycle kinematics with respect to a global cartesian coordinate frame G. The kinematic model is given by

$$\dot{x} = f(x,u) \Rightarrow \begin{bmatrix} \dot{\chi} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \upsilon \\ \omega \end{bmatrix} \tag{1}$$

where $x = [\chi, y, \theta]^\top$ is the state vector comprised by the position of the robot (χ, y) and the orientation θ with respect to G. The vector $u = [\upsilon, \omega]^\top$ denotes the control inputs, and υ, ω are the linear and angular velocity of the robot, respectively, expressed in the body-fixed frame B.

The requirements of the robotic system along with a saturation bound in the velocity impose the constraints of the problem. Particularly, the requirements are captured by the connected state constraint set X, given by

$$x(t) \in X \subset \mathbb{R}^3 \tag{2}$$

Note that the whole state $x(t)$ is assumed to be available, for all $t \in \mathbb{R}_{\geq 0}$. The control constraint set U is assumed to be compact and it is given by:

$$u(t) \triangleq [\upsilon(t), \omega(t)]^\top \in U \subset \mathbb{R}^2 \tag{3}$$

The constraints of the input are of the form $|\upsilon| \leq \bar{\upsilon}$ and $|\omega| \leq \bar{\omega}$. Therefor we get $||u|| \leq \bar{u}$, where $\bar{u} = \sqrt{\bar{\upsilon}^2 + \bar{\omega}^2}$, where we have $\bar{u}, \bar{\upsilon}, \bar{\omega} \in \mathbb{R}_{\geq 0}$. The nominal system (1) is Lipschitz continuous with Lipschitz constant $0 < L_f < \infty$. More specifically,

Lemma 1: The nominal model $f(x,u)$, given the constraints (2) and (3), is locally Lipschitz in x for all $x \in X$, with a Lipschitz constant $L_f \triangleq \sqrt{2}\bar{\upsilon}$.

Proof: The Euclidean norm is used for the sake of simplicity. We have

$$||f(x_1,u) - f(x_2,u)||^2 = || \begin{pmatrix} \upsilon\cos\theta_1 - \upsilon\cos\theta_2 \\ \upsilon\sin\theta_1 - \upsilon\sin\theta_2 \\ \omega - \omega \end{pmatrix} ||^2$$
$$= |\upsilon|^2 |\cos\theta_1 - \cos\theta_2|^2 + |\upsilon|^2 |\sin\theta_1 - \sin\theta_2|^2$$
$$\leq 2|\upsilon|^2 |\theta_1 - \theta_2|^2$$

where the mean value theorem is used. Thus, it can be concluded that $||f(x_1,u) - f(x_2,u)|| \leq \sqrt{2}\bar{\upsilon}||x_1 - x_2||$ for all $x_1, x_2 \in X$. ■

We assume that the robot moves under the influence of a current w with respect to the global frame. Therefor we consider a perturbed system of the form:

$$\dot{x} = f(x,u) + w \tag{4}$$

with $w(t) \in W \subset \mathbb{R}^3$ and W to be a compact set. Since the uncertainty is assumed to be bounded we set $||w|| \leq \bar{w}$.

B. Control Design and Objective

The goal is to control the actual system (4) subject to $x(t) \in X$ and $u(t) \in U$, to a desired compact set that includes the desired state $x_d \triangleq [\chi_d, y_d, \theta_d]^T \in X$. A predictive controller is employed in order to achieve this task. With the NMPC law the state of the system is proven to converge to the desired set. Inside this set, an auxiliary terminal controller is used to drive the system to the desired point. The design of an ISS stable controller for system (4) is presented next.

The NMPC consists in solving a finite-horizon, open-loop optimal control problem, based on the actual state of the plant $x(t_i)$, at time t_i. The solution is a control trajectory $u(t)$, for $t \in [t_i, t_i + T_p]$, where T_p is the prediction horizon. The Optimal Control Problem (OCP) of the NMPC is given as

$$\min_{u(\cdot)} J(u(\cdot), x(t_i)) =$$

$$\min_{u(\cdot)} \int_{t_i}^{t_i+T_p} F(\hat{x}(\tau), u(\tau)) \, \mathrm{d}\tau + E(\hat{x}(t_i + T_p)), \tag{5a}$$

subject to

$$\dot{\hat{x}} = f(\hat{x}(t), u(t)), \qquad \hat{x}(t_i) = x(t_i), \tag{5b}$$

$$u(t) \in U, \tag{5c}$$

$$\hat{x}(t) \in X_{t-t_i} \qquad t \in [t_i, t_i + T_p], \tag{5d}$$

$$\hat{x}(t_i + T_p) \in \mathscr{E}_f, \tag{5e}$$

where $\hat{\cdot}$ denotes the controller internal variables, corresponding to the nominal dynamics of the system. F and E are the running and terminal costs functions, respectively. The design parameters F and E, as well as the sets X_{t-t_i} and $\mathscr{E}_f$ are defined later in the text.

In order to proceed to the subsequent analysis a few definitions and some preliminary results are presented first.

The predicted state of the nominal system (1) at time $t_i+\tau$ with $\tau \geq 0$, is denoted as $\hat{x}(t_i+\tau, u(\cdot), x(t_i))$ and it is based on the measurement of the actual state $x(t_i)$ at time t_i, when a control trajectory $u(\cdot; x(t_i))$ is applied to the system for time period t_i until $t_i + \tau$. It holds that $\hat{x}(t_i, u(\cdot), x(t_i)) \equiv x(t_i)$. Moreover the following result is given:

Lemma 2: The difference between the actual state $x(t_i+t)$ at time $t_i + t$ and the predicted state at the same time under the same control law $u(t_i + t, x(t_i))$, with $0 \leq t \leq T_p$, starting at the same initial state $x(t_i)$, can be shown to be upper bounded by

$$||x(t_i + t) - \hat{x}(t_i + t, u(\cdot), x(t_i))|| \leq \gamma(t) \tag{6}$$

where $\gamma(t) \triangleq (2\sqrt{2}\bar{\upsilon} + \bar{w})t$ for all $t \in [0, T_p]$.

Proof: Set the control trajectory $u(\cdot) \triangleq u(t_i + t, x(t_i))$ and $x(t) \triangleq x(t, u(\cdot), x(t_i))$ to be the state trajectory for system (4). Also we denote for the sake of simplicity, $\hat{x}(t) \triangleq \hat{x}(t, u(\cdot), x(t_i))$ for all $t \in \mathbb{R}_{\geq 0}$. Using the Euclidian norm and the triangular inequality for system (1) and system (4), we get

$$||x(t_i+t)-\hat{x}(t_i+t)|| = ||x(t_i) + \int_{t_i}^{t_i+t} f(x(\tau),u(\cdot))\,\mathrm{d}\,\tau$$
$$+ \int_{t_i}^{t_i+t} w(\tau)\,\mathrm{d}\,\tau - x(t_i) - \int_{t_i}^{t_i+t} f(\hat{x}(\tau),u(\cdot))\,\mathrm{d}\,\tau||$$
$$\leq || \int_{t_i}^{t_i+t} f((x(\tau),u(\cdot)) - f(\hat{x}(\tau),u(\cdot)))\,\mathrm{d}\,\tau||$$
$$+ || \int_{t_i}^{t_i+t} w(\tau)\,\mathrm{d}\,\tau|| \leq \int_{t_i}^{t_i+t} ||[\upsilon(\cos\theta(\tau)-\cos\hat{\theta}(\tau)),$$
$$\upsilon(\sin\theta(\tau)-\sin\hat{\theta}(\tau)),0]^\top||\,\mathrm{d}\,\tau + \int_{t_i}^{t_i+t} ||w(\tau)||\,\mathrm{d}\,\tau$$
$$\leq (2\sqrt{2}\bar{\upsilon}+\bar{w})t$$

■

To address to the divergence between the actual state trajectory of system (4) and the predicted state trajectory of the nominal system as given in *Lemma* 2, we replace the state constraint set X with the restricted constraint set X_{t-t_i} into (5d), with $X_{t-t_i} \subseteq X$. We resort to this constraint' tightening technique presented in [11] and [13] since the control trajectory that results from (5a)-(5e) when it is applied to the system (4), results to a state trajectory that does not violate the state constraint set X. In particular, the restricted constraint set is defined as $\mathcal{X}_{t-t_i} = \mathcal{X} \sim \mathscr{B}_{t-t_i}$ where $\mathscr{B}_{t-t_i} = \{x \in \mathbb{R}^n : ||x|| \leq \gamma(t-t_i)\}$, with $t \in [t_i, t_i+T_p]$. The set operator "$\sim$" denotes the Pontryagin difference, i.e., given two sets $A, B \subseteq \mathbb{R}^n$ the Pontryagin difference set C is defined as $C = A \sim B \triangleq \{x \in \mathbb{R}^n : x+\xi \in A, \forall \xi \in B\}$.

Assume now that the terminal cost $E(x)$ as well as the cost function $F(x,u)$, are quadratic of the form $E(x) = x^\top Px$ and $F(x,u) = x^\top Qx + u^\top Ru$, respectively, with P, Q and R being positive definite matrices. More specifically we set $P = diag\{p_1,p_2,p_3\}$, $Q = diag\{q_1,q_2,q_3\}$ and $R = diag\{r_1,r_2\}$. Moreover it can be shown that $F(0,0) = 0$ and that $F(x,u) \geq \min\{q_1,q_2,q_3,r_1,r_2\}||[x,u]^\top||^2 \geq \min\{q_1,q_2,q_3,r_1,r_2\}||x||^2$. Since X and U are bounded, it can be concluded that:

Lemma 3: The stage cost $F(x,u)$ is Lipschitz continuous in $X \times U$, with a Lipschitz constant $L_F \triangleq 2(R_{\max}^2 + (\frac{\pi}{2})^2)^{1/2}\sigma_{\max}(Q)$, where $\sigma_{\max}(Q)$ is the largest singular value of matrix Q and R_{max} is the largest distance of the robot with respect to the target.

Proof: We have

$$||F(x_1,u)-F(x_2,u)|| = ||x_1^\top Qx_1 - x_2^\top Qx_2|| = ||x_1^\top Qx_1$$
$$- x_1^\top Qx_2 + x_1^\top Qx_2 - x_2^\top Qx_2|| = ||x_1^\top Q(x_1-x_2)$$
$$+ (x_1-x_2)^\top Qx_2|| \leq (||x_1||+||x_2||)\sigma_{\max}(Q)||x_1-x_2||$$

Notice though that $\forall x \in X$ we have $||x||^2 \leq |\chi|^2 + |y|^2 + |\theta|^2 \leq R_{\max}^2 + (\frac{\pi}{2})^2$, which concludes the proof. ■

In order to assert that the NMPC strategy results in a robust stabilizing controller, some stability conditions are stated in the following:

Assumption 1: Assume that a set $\mathscr{E} \subset X$ is an admissible positively invariant set for the nominal system (1), and that $\mathscr{E}$ is such that $\mathscr{E} \triangleq \{x \in X : ||x|| \leq \varepsilon_0\}$, with ε_0 being a positive parameter.

Assumption 2: Assume that for the terminal set $\mathscr{E}_f$, there exists a local stabilizing controller $u_T(x(t)) \in U$, $\forall x \in \mathscr{E}$. The associated Lyapunov function $E(\cdot)$ has the following properties

$$\frac{\partial E}{\partial x} f(x(\tau),u_T(x(\tau))) + F(x(\tau),u_T(x(\tau))) \leq 0 \qquad \forall x \in \mathscr{E}$$

and is Lipschitz in $\mathscr{E}$, with Lipschitz constant $L_E = 2\varepsilon_0\sigma_{\max}\{P\}$ for all $x \in \mathscr{E}$. The proof for finding the Lipschitz constant L_E is the same as the proof of *Lemma* 3.

Assumption 3: For the set $\mathscr{E}$ we have $E(x) = x^\top Px \leq \alpha_{\mathscr{E}}$ where $\alpha_{\mathscr{E}} = \max\{p_1,p_2,p_3\}{\varepsilon_0}^2 > 0$ and we assume that $\mathscr{E} = \{x \in X_{T_P} : u_T(x) \in U\}$. Take $\alpha_{\mathscr{E}_f} \in (0, \alpha_{\mathscr{E}})$ and assume that $\mathscr{E}_f = \{x \in \mathbb{R}^3 : E(x) \leq \alpha_{\mathscr{E}_f}\}$ is such that $\forall x \in \mathscr{E}, f(x,u_T) \in \mathscr{E}_f$.

C. Problem Statement

The solution of the OCP (5a)-(5e) at time t_i provides an optimal control trajectory denoted as $u^*(t;x(t_i))$, for $t \in [t_i, t_i+T_p]$. A portion this control trajectory, is then applied to the plant, i.e.,

$$u(t) = u^*(t;x(t_i)), \qquad t \in [t_i, t_{i+1}) \tag{7}$$

During the time interval $[t_i, t_{i+1})$ the control law is applied to the plant in an open-loop fashion. A question that naturally arises is how large this time interval can be? The self-triggered strategy that will be presented later in this paper, provides sufficient conditions for finding the recalculation periods, or in other words sufficient conditions for triggering the computation of the NMPC law. In particular, the presented framework not only provides the control law to be applied to the actual system (4), but also provides the time of the next triggering instant, t_{i+1}. This leads us to the statement of the problem treated in this paper:

Problem Statement 1: Consider the system (4) that is subject to constraints (2) and (3). The objective is (i) to design a feedback control law provided by (5a)-(5e) such that the system (4) converges to the terminal constraint set and (ii) to find a mechanism to decide when the next control update should be.

III. Stability Analysis of NMPC

In this section a stability analysis for the closed-loop system (4)-(7) is presented. Due to the fact that the system in consideration is perturbed, we only require "ultimate boundedness" results. Accordingly, it can be proven that the closed-loop scheme is Input to State stable (ISS) with respect to the disturbances, [16]. Moreover, through the ISS analysis it is possible to reach to a self-triggering mechanism which provides the triggering instants.

The proof of stability of a system under a predictive controller consists in guaranteeing (i) the feasibility property and (ii) the convergence property of the closed-loop system.

We begin by showing that initial feasibility implies feasibility afterwards. Consider two successive triggering events t_i and t_{i+1}. A feasible control trajectory $\bar{u}(\cdot, x(t_{i+1}))$, at t_{i+1}, may be the following:

$$\bar{u}(\tau, x(t_{i+1})) = \begin{cases} u^*(\tau, x(t_i)) & \forall \tau \in [t_{i+1}, t_i + T_p] \\ u_T(\hat{x}(t_i + T_p, u^*(\cdot), x(t_i))) & \forall \tau \in [t_i + T_p, t_{i+1} + T_p] \end{cases} \tag{8}$$

where $u^*(\cdot, x(t_i))$ is the optimal solution of the OCP at t_i.

From feasibility of $u^*(\cdot, x(t_i))$ and the fact that $u_T(x) \in U$ for all $x \in \mathscr{E}$, it follows that $\bar{u}(\tau, x(t_{i+1})) \in \mathcal{U}$ for all $\tau \in [t_{i+1}, t_{i+1} + T_p]$. We continue by showing that $\hat{x}(t_{i+1} + T_p, \bar{u}(\tau, x(t_{i+1})), x(t_{i+1})) \in \mathscr{E}_f$. We have

$$E(\hat{x}(t_i + T_p, u(\cdot), x(t_{i+1})) \leq E(\hat{x}(t_i + T_p, u(\cdot), x(t_i)) + L_E \gamma(T_p) \leq \alpha_{\mathscr{E}_f} + L_E(2\sqrt{2}\bar{v} + \bar{w})T_p \leq \alpha_{\mathscr{E}}$$

The uncertainties must then be bounded by $\bar{w} + 2\sqrt{2}\bar{v} \leq \frac{\alpha_{\mathscr{E}} - \alpha_{\mathscr{E}_f}}{L_E T_p}$. Moreover, the state constraints must be fulfilled: according to [11] and [14] and considering that $||x(t) - \hat{x}(t, u(\cdot), x(t_i))|| \leq \gamma(t)$, for all $t \geq t_i$, it can be verified that since $\hat{x}(t, u^*(\cdot), x(t_i)) \in \mathcal{X}_{t-t_i}$, then $\hat{x}(t, \bar{u}(\cdot), x(t_{i+1})) \in \mathcal{X}_{t-t_{i+1}}$.

The convergence of the state is discussed now. A proper value function must be shown to be decreasing in order to prove stability of the closed-loop system. Consider the optimal cost $J^*(u^*(\cdot; x(t_i)), x(t_i)) \triangleq J^*(t_i)$ from (5a) as a Lyapunov function candidate. Then, consider the cost of the feasible trajectory, indicated by $\bar{J}(\bar{u}(\cdot; x(t_{i+1})), x(t_{i+1})) \triangleq \bar{J}(t_{i+1})$. Note that t_i, t_{i+1} are two successive triggering instants. Also, we introduce the "feasible" state $\bar{x}(\tau, \bar{u}(\tau; x(t_{i+1})), x(t_{i+1}))$ which accounts for the predicted state at time τ, with $\tau \geq t_{i+1}$, based on the measurement of the real state at time t_{i+1}, while using the feasible control trajectory $\bar{u}(\tau; x(t_{i+1}))$ from (8).

Set $x_1(\tau) = \bar{x}(\tau, \bar{u}(\tau; x(t_{i+1})), x(t_{i+1}))$, $u_1(\tau) = \bar{u}(\tau; x(t_{i+1}))$, $x_2(\tau) = \hat{x}(\tau, u^*(\tau; x(t_i)), x(t_i))$ and $u_2(\tau) = u^*(\tau; x(t_i))$.

The difference between the optimal cost and the feasible cost is:

$$\begin{aligned} &\bar{J}(t_{i+1}) - J^*(t_i) = \\ &\int_{t_{i+1}}^{t_{i+1}+T_p} F(x_1(\tau), u_1(\tau))\, \mathrm{d}\tau + E(x_1(t_{i+1} + T_p)) \\ &- \int_{t_i}^{t_i+T_p} F(x_2(\tau), u_2(\tau))\, \mathrm{d}\tau - E(x_2(t_i + T_p)) \\ &= \int_{t_{i+1}}^{t_i+T_p} F(x_1(\tau), u_1(\tau))\, \mathrm{d}\tau + E(x_1(t_{i+1} + T_p)) \\ &+ \int_{t_i+T_p}^{t_{i+1}+T_p} F(x_1(\tau), u_1(\tau))\, \mathrm{d}\tau \\ &- \int_{t_i}^{t_{i+1}} F(x_2(\tau), u_2(\tau))\, \mathrm{d}\tau \\ &- \int_{t_{i+1}}^{t_i+T_p} F(x_2(\tau), u_2(\tau))\, \mathrm{d}\tau - E(x_2(t_i + T_p)) \end{aligned} \tag{9}$$

From (8), we have that $u_1(t) \equiv u_2(t) \equiv \bar{u}(t)$ for $t \in [t_{i+1}, t_i + T_p]$. Imposing this control law to the system (1) we get:

$$\begin{aligned} &||x_1(t) - x_2(t)|| = ||x(t_{i+1}) + \int_{t_{i+1}}^{t} f(\bar{x}(\tau), \bar{u}(\tau))\, \mathrm{d}\tau - \\ &x(t_i) - \int_{t_i}^{t_{i+1}} f(\hat{x}(\tau), u^*(\tau))\, \mathrm{d}\tau - \int_{t_{i+1}}^{t} f(\hat{x}(\tau), \bar{u}(\tau))\, \mathrm{d}\tau|| \\ &= ||x(t_{i+1}) - \hat{x}(t_{i+1}, u^*(\cdot), x(t_i))|| \leq \gamma(t_{i+1} - t_i) \end{aligned} \tag{10}$$

The difference between the running costs, with the help of (10), becomes:

$$\begin{aligned} &\int_{t_{i+1}}^{t_i+T_p} F(x_1(\tau), u_1(\tau))\, \mathrm{d}\tau - \int_{t_{i+1}}^{t_i+T_p} F(x_2(\tau), u_2(\tau))\, \mathrm{d}\tau \\ &\leq \int_{t_{i+1}}^{t_i+T_p} ||F(x_1(\tau), \bar{u}(\cdot)) - F(x_2(\tau), \bar{u}(\cdot))||\, \mathrm{d}\tau \\ &\leq L_F \int_{t_{i+1}}^{t_i+T_p} ||x_1(\tau) - x_2(\tau)||\, \mathrm{d}\tau \\ &\leq L_F \int_{t_{i+1}}^{t_i+T_p} \gamma(t_{i+1} - t_i)\, \mathrm{d}\tau \\ &= L_F(2\sqrt{2}\bar{v} + \bar{w})(t_{i+1} - t_i)(t_i + T_p - t_{i+1}) \geq 0 \end{aligned} \tag{11}$$

Integrating the inequality from *Assumption* 2 for $t \in [t_i + T_p, t_{i+1} + T_p]$ results in the following:

$$\begin{aligned} &\int_{t_i+T_p}^{t_{i+1}+T_p} F(x_1(\tau), u_1(\tau))\, \mathrm{d}\tau + E(x_1(t_{i+1} + T_p)) \\ &- E(x_2(t_i + T_p)) - E(x_1(t_i + T_p)) + E(x_1(t_i + T_p)) \\ &\leq E(x_1(t_i + T_p)) - E(x_2(t_i + T_p)) \\ &\leq L_E||x_1(t_i + T_p) - x_2(t_i + T_p)|| \\ &\leq L_E(2\sqrt{2}\bar{v} + \bar{w})(t_{i+1} - t_i) \geq 0 \end{aligned} \tag{12}$$

Since function F is positive definite, it can be concluded that

$$\int_{t_i}^{t_{i+1}} F(x_2(\tau), u_2(\tau))\, \mathrm{d}\tau \geq L_Q(t_{i+1}) \geq 0 \tag{13}$$

with $L_Q(t) \triangleq \min\{q_1, q_2, q_3, r_1, r_2\} \cdot \int_{t_i}^{t} ||\hat{x}(\tau, u^*(\tau; x(t_i)), x(t_i))||^2\, \mathrm{d}\tau$ for $t \geq t_i$. Substituting (11), (12), (13) to (9), the following is derived

$$\begin{aligned} &\bar{J}(t_{i+1}) - J^*(t_i) \\ &\leq L_F(2\sqrt{2}\bar{v} + \bar{w})(t_{i+1} - t_i)(t_i + T_p - t_{i+1}) \\ &+ L_E(2\sqrt{2}\bar{v} + \bar{w})(t_{i+1} - t_i) - L_Q(t_{i+1}) \end{aligned} \tag{14}$$

The optimality of the solution yields

$$J^*(t_{i+1}) - J^*(t_i) \leq \bar{J}(t_{i+1}) - J^*(t_i) \tag{15}$$

The Lyapunov function $J^*(\cdot)$ has been proven to be decreasing, thus the closed-loop system converges to a compact set $\mathscr{E}_f$, where it is ultimately bounded, due to Assumption 3.

A. Self-triggered Framework

In this section the self-triggering mechanism is going to be presented. Consider that at time t_i an event is triggered. The ISS of the NMPC was proven considering that the time t_{i+1}, i.e, the next triggering instant, was known. Here, the next control update time t_{i+1} is considered to be unknown and should be found. The next control update time t_{i+1} should be such that the closed-loop system does not lose any of its desired properties. Thus, we still need the Lyapunov function $J^*(\cdot)$ to be decreasing, which will preserve the convergence of the closed-loop system.

Given (14) and (15), then for some triggering instant t_i and some time t with $t \in [t_i, t_i + T_p]$ we get

$$\begin{aligned} &J^*(t) - J^*(t_i) \\ &\leq L_F(2\sqrt{2}\bar{v} + \bar{w})(t - t_i)(t_i + T_p - t) \\ &+ L_E(2\sqrt{2}\bar{v} + \bar{w})(t - t_i) - L_Q(t) \end{aligned} \quad (16)$$

The time instant t should be such that

$$\begin{aligned} &L_F(2\sqrt{2}\bar{v} + \bar{w})(t - t_i)(t_i + T_p - t) \\ &+ L_E(2\sqrt{2}\bar{v} + \bar{w})(t - t_i) \leq \sigma L_Q(t) \end{aligned} \quad (17)$$

with $0 < \sigma < 1$. Plugging in (17) to (16) we get

$$J^*(t) - J^*(t_i) \leq (\sigma - 1) \cdot L_Q(t) \quad (18)$$

This suggests that provided $\sigma < 1$, the convergence property is still guaranteed. Thus, the next control update time should be triggered when (17) is violated. This provides the triggering mechanism. Notice that the time t_{i+1} can be found beforehand at time t_i, i.e, this is a self-triggering mechanism. Moreover, it should be pointed out that the term $L_Q(t)$ includes only predictions of the nominal system that is forming a trajectory and that it can be found by forward integration of (1) for time $t \in [t_i, t_i + T_p]$.

Next we describe the self-triggering mechanism. At time t_i a control update is triggered and a control trajectory for $[t_i, t_i + T_p]$ is provided. With the help of (17) we get

$$(2\sqrt{2}\bar{v} + \bar{w})[L_F(t_i + T_p - t) + L_E](t - t_i) = \sigma L_Q(t) \quad (19)$$

The solution of (19) will provide the next update time t_{i+1}. During the time interval $t \in [t_i, t_{i+1})$ the control trajectory $u(t) = u^*(t, x(t_i))$ is applied to the plant in an open-loop fashion. Next, at time t_{i+1} the OCP is solved again using the current measure of the state $x(t_{i+1})$ as the initial state. The controller follows this procedure until the system converges to the terminal constraint set.

We are now ready to state the stability result for this self-triggered MPC framework:

Theorem 1: Consider the system (4) that is subject to constraints (2) and (3) under the NMPC strategy and assume that *Assumptions* 1-4 hold. The control update times that are provided by (19) and the NMPC law provided by (5a)-(5e) which is applied to the system in an open-loop fashion during the inter-sampling periods, drive the closed-loop system towards a compact set $\mathscr{E}_f$ where it is ultimately bounded.

IV. Simulation Results

In this section, a simulated example of the proposed framework for a nonholonomic robot is presented. The objective is to control the robot through a NMPC law of the form (5a)-(5e) in order to reach a desired terminal constraint set. The nominal model of the nonholonomic system has the form (1). Furthermore we assume that disturbances exist and that they are bounded by $||w|| \leq 0.5$. Thus, the actual model is (4). The initial position of the robot is $x_{\text{initial}} = [-43, 11.5, -\pi/6]^\top$ and the desired position is $x_d = [0, 0, 0]^\top$.

In order to evaluate the proposed self-triggered approach we are going to present some comparison results. The traditional time-triggered, periodical, scheme is given and the event-triggered MPC framework that was proposed in our earlier work [4] is given as well. The simulation shows that the actual system (4) under all three schemes, i.e., time, event and self-triggered NMPC, converges to the terminal set around the desired state, see Fig. 1.

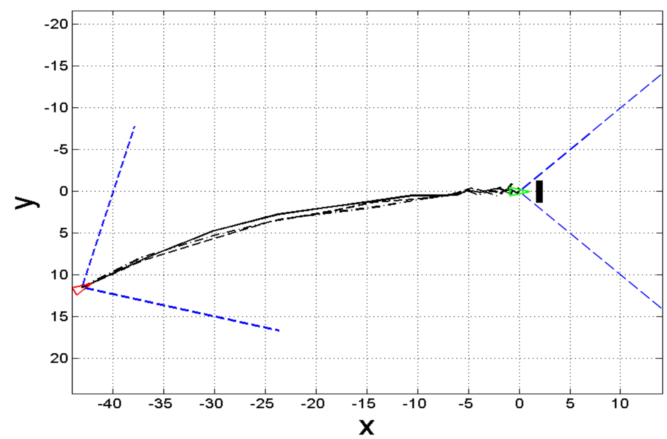

Fig. 1. State trajectories of the nonholonomic robot under robust MPC. The solid line represents the trajectory of the robot under periodic MPC. The dashed line as well as the dash-dotted line represents the trajectories of the robot under the event-triggered MPC and the self-triggered MPC, respectively. The red triangle is the initial position of the robot, while the green is the desired state.

In Fig. 2, the evolution of the system trajectories under all three schemes is depicted. It is apparent that all three schemes have comparable results. Finally, Fig. 3, is capturing the triggering instants on both the event-triggered and the self-triggered frameworks. The time-triggered framework is not depicted because it is trivially triggered at each sampling period, i.e., the smallest triggering period $\beta = 0.1$sec.

V. Summary and Future Work

We provided a self-triggered formulation for constrained nonholonomic systems under a model predictive controller. The main idea is to trigger the solution of the optimal control problem only when it is needed and not periodically as in the case of classic MPC schemes. This approach results to an improvement on the requirements on the computation resources. With the self-triggered approach both the control

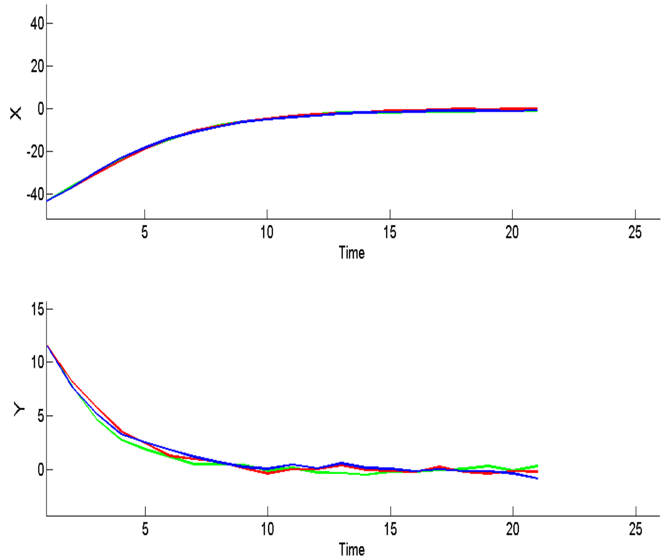

Fig. 2. The evolution of the system trajectories in time. The green line represents the time-triggered case. The red and blue represent the event-triggered and self-triggered, respectively.

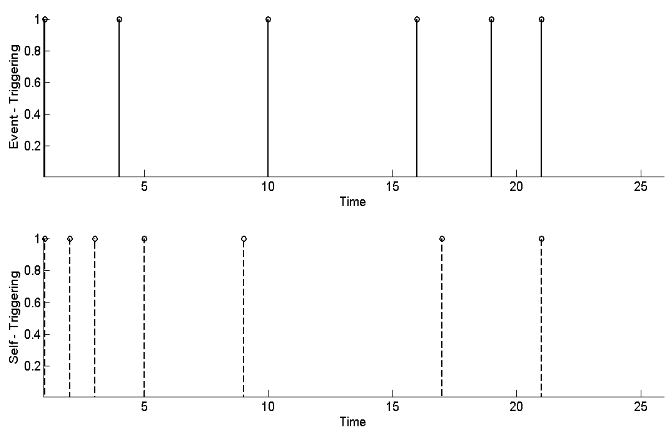

Fig. 3. The triggering instants. When the vertical axis has the value 1, the NMPC is triggered. For value 0 the control law is implemented on the system in an open-loop fashion. (a) The event-triggered set-up, (b) The self-triggered set-up.

input and the next control update time are evaluated in order to avoid continuous supervision of the actual state of the system. During the inter-sampling times the control trajectory from the NMPC is applied to the system in an open-loop fashion. In this paper sufficient conditions for triggering were presented along with some simulation results that depict the validity of the overall framework.

Future work involves experimental results for the proposed framework with a real nonholonomic underwater vehicle. Moreover, the robust MPC controller will be combined with vision-based techniques that should provide the estimate of the state vector with respect to the target frame.

References

[1] A. Anta and P. Tabuada. To sample or not to sample: self-triggered control for nonlinear systems. *IEEE Transactions on Automatic Control*, 55(9):2030–2042, 2010.

[2] J.D.J. Barradas Berglind, T.M.P. Gommans, and W.P.M.H. Heemels. Self-triggered mpc for constrained linear systems and quadratic costs. *4th IFAC Nonlinear Model Predictive Control Conference*, 2012.

[3] D. Bernardini and A. Bemporad. Energy-aware robust model predictive control based on wireless sensor feedback. *47th IEEE Conference on Decision and Control*, pages 3342 – 3347, 2008.

[4] A. Eqtami, D.V. Dimarogonas, and K.J. Kyriakopoulos. Novel event-triggered strategies for model predictive controllers. *50th IEEE Conf. Decision and Control & Eur. Control Conf.*, pages 3392–3397, 2011.

[5] R. Findeisen and P. Varutti. Stabilizing nonlinear predictive control over nondeterministic communication networks. In *Nonlinear Model Predictive Control, Lecture Notes in Control and Information Sciences*, pages 167–179. Springer Berlin / Heidelberg, 2009.

[6] E. Garcia and P.J. Antsaklis. Model-based event-triggered control with time-varying network delays. *50th IEEE Conf. Decision and Control & Eur. Control Conf.*, pages 1650 – 1655, 2011.

[7] W.P.M.H. Heemels, J.H. Sandee, and P.P.J. Van Den Bosch. Analysis of event-driven controllers for linear systems. *International Journal of Control*, 81(4):571–590, 2007.

[8] E. Henriksson, D. E. Quevedo, H. Sandberg, and K. Henrik Johansson. Self-triggered model predictive control for network scheduling and control. *8th IFAC Symposium on Advanced Control of Chemical Processes*, 2012.

[9] M. Mazo Jr., A. Anta, and P. Tabuada. On self-triggered control for linear systems: Guarantees and complexity. *European Control Conference*, 2009.

[10] D. Lehmann and J. Lunze. Event-based control: A state feedback approach. *European Control Conference*, pages 1716–1721, 2009.

[11] D. Limon Marruedo, T. Alamo, and E.F. Camacho. Input-to-state stable mpc for constrained discrete-time nonlinear systems with bounded additive uncertainties. *41st IEEE Conf. Decision and Control*, pages 4619 – 4624, 2002.

[12] P. Millán, L. Orihuela, D. Mu noz de la Peña, C. Vivas, and F.R. Rubio. Self-triggered sampling selection based on quadratic programming. *Proceedings of the 18th IFAC World Congress*, pages 8896–8901, 2011.

[13] G. Pin, D.M. Raimondo, L. Magni, and T. Parisini. Robust model predictive control of nonlinear systems with bounded and state-dependent uncertainties. *IEEE Transactions on Automatic Control*, 54(7):1681 – 1687, 2009.

[14] M. Rubagotti, D. M. Raimondo, A. Ferrara, and L. Magni. Robust model predictive control with integral sliding mode in continuous-time sampled-data nonlinear systems. *IEEE Transactions on Automatic Control*, 56(3):556 –570, 2011.

[15] J. Sijs, M. Lazar, and W.P.M.H. Heemels. On integration of event-based estimation and robust mpc in a feedback loop. *Proceedings of the 13th ACM international conference on Hybrid systems: computation and control*, pages 31–40, 2010.

[16] E.D. Sontag. Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, pages 163–220. Springer Berlin / Heidelberg, 2008.

[17] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

[18] P. Varutti, B. Kern, T. Faulwasser, and R. Findeisen. Event-based model predictive control for networked control systems. *48th IEEE Conf. Decision and Control*, pages 567 – 572, 2009.

[19] M. Velasco, J. Fuertes, and P. Marti. The self-triggered task model for real-time control systems. *In Work in Progress Proceedings of the 24th IEEE Real-Time Systems Symposium*, pages 67–70, 2005.

[20] X. Wang and M.D. Lemmon. Event design in event-triggered feedback control systems. *47th IEEE Conf. Decision and Control*, pages 2105–2110, 2008.

[21] X. Wang and M.D. Lemmon. Self-triggered feedback control systems with finite-gain L2 stability. *IEEE Transactions on Automatic Control*, 45(3):452–467, 2009.

[22] Z. Yongjie and Ü. Özgüner. Robustness analysis on constrained model predictive control for nonholonomic vehicle regulation. *American Control Conference*, pages 3896 – 3901, 2009.