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SELFCOMMUTATORS OF MULTICYCLIC HYPONORMAL OPERATORS ARE ALWAYS TRACE CLASS

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1. For A, B operators on the Hilbert space H, $[A, B] = AB - BA$. The selfcommutator of A is $[A^*, A]$. If E is a closed proper subset of the plane, $R(E)$ will be the rational functions analytic on E. The operator A is said to be n-multicyclic if there are n vectors $g_1, \ldots, g_n \in H$, called generating vectors, such that $\{r(A)g_i : r \in R(\mathrm{sp}(A)), 1 \leqq i \leqq n\}$ has span dense in H. This paper will outline a circle of ideas culminating in the following result.

MAIN THEOREM. *If A is an n-multicyclic hyponormal operator, then $[A^*, A]$ is in trace class, and* $\mathrm{tr}[A^*, A] \leqq (n/\pi)\omega(\mathrm{sp}(A))$, *where ω is planar Lebesgue measure.*

This result is especially interesting because of the scarcity of known conditions insuring that the selfcommutator lie in trace class. The above result is new even when A is subnormal and has a cyclic vector in the usual sense. The best previous result in this direction is due to T. Kato [**1**], and states that if $\mathrm{Re}(A)$ has finite spectral multiplicity n, then $[A^*, A]$ is in trace class. Kato provides a trace estimate which Putnam [**4**] is able to use to prove the above estimate, where n is an upper bound for the spectral multiplicity of $\mathrm{Re}(A)$.

The Kato-Putnam estimate and the main theorem above are independent. For example, using a result of J. W. Helton and R. Howe, unpublished as yet, which provides a lower bound for the spectral multiplicity of the real part of a hyponormal operator, one can see that the real part of the 1-multicyclic operator given by multiplication by z on R^2 of a Swiss cheese has infinite spectral multiplicity almost everywhere.

Throughout the following, a space and the orthogonal projection onto that space will be denoted by the same symbol. All spaces are Hilbert spaces.

2. The following lemma is central.

STRUCTURE LEMMA. *Let T and A be hyponormal operators on H and K*

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respectively, and let $W: H \to K$ *be a trace class operator with dense range, such that* $WT = AW$. *Then* $\operatorname{tr}[A^*, A] \leqq \operatorname{tr}[T^*, T]$.

PROOF. It may be assumed that $\operatorname{tr}[T^*, T] < \infty$. Let N be the null space of W. Since N is an invariant space for T, TN is also hyponormal. It will be shown that $\operatorname{tr}[A^*, A] + \operatorname{tr}[NT^*, TN] \leqq \operatorname{tr}[T^*, T]$.

Let $\{\varphi_n\}_n$ be a complete orthonormal system of eigenvectors for W^*W, with $W^*W\varphi_n = \lambda_n^2\varphi_n$, $\lambda_n \geqq 0$. Then the vectors $\{\psi_n : \lambda_n > 0\}$ given by $W\varphi_n = \lambda_n\psi_n$ are a complete orthonormal basis for K. Let $L_t = H \oplus K$ have the norm $\|h \oplus k\|_t^2 = t^2\|h\|^2 + \|k\|^2$, for $t > 0$, and let J be the closed subspace spanned by the vectors $\{h \oplus Wh : h \in H\}$.

$$\{(t^2 + \lambda_n^2)^{-1/2}(\varphi_n \oplus \lambda_n\psi_n)\}_n$$

is a complete orthonormal basis for J. Note that J is an invariant space for $T \oplus A$, so $(T \oplus A)J$ is hyponormal. $(T \oplus A)H = (T \oplus 0)$, which, when restricted to $H \oplus 0$, is unitarily equivalent to T, so if it can be shown that $H - J$ is in trace class, $[J(T \oplus A)^*, (T \oplus A)J]$ will lie in trace class, and

$$\operatorname{tr}[J(T \oplus A)^*, (T \oplus A)J] = \operatorname{tr}[H(T \oplus A)^*, (T \oplus A)H] = \operatorname{tr}[T^*, T].$$

But the space spanned by the vectors $\{\varphi_n, \psi_n\}$ reduces $H - J$, and on this space $H - J$ has trace norm $2\lambda_n(t^2 + \lambda_n^2)^{-1/2}$. Thus, $H - J$ has trace norm $\sum_n 2\lambda_n(t^2 + \lambda_n^2)^{-1/2} \leqq 2t^{-1}\sum_n \lambda_n$. Now consider

$$\begin{aligned}
&\operatorname{tr}[J(T \oplus A)^*, (T \oplus A)J] \\
&\qquad = \sum_{\lambda_n > 0} \{\|(T \oplus A)(t^2 + \lambda_n^2)^{-1/2}(\varphi_n \oplus \lambda_n\psi_n)\|_t^2 \\
&\qquad\quad - \|J(T^* \oplus A^*)(t^2 + \lambda_n^2)^{-1/2}(\varphi_n \oplus \lambda_n\psi_n)\|_t^2\} \\
&\qquad\quad + \sum_{\lambda_n = 0} \{\|(T \oplus A)(t^{-1}\varphi_n \oplus 0)\|_t^2 - J(T^* \oplus A^*)(t^{-1}\varphi_n \oplus 0)\|_t^2\}.
\end{aligned}$$

The diligent reader will discover that the summand in the first sum approaches $\|A\psi_n\|^2 - \|A^*\psi_n\|^2$ as $t \to 0$. (To show that $\|J(0 \oplus u)\|_t^2 \to \|u\|^2$, he will evaluate the norm of the projection using the orthonormal basis for J, and apply the Lebesgue monotone convergence theorem to the resulting sum.) A similar technique, applied to the summands of the second sum, and now invoking Lebesgue dominated convergence, shows that they approach

$$\|T\varphi_n\|^2 - \sum_{\lambda_m = 0} \{|\langle T^*\varphi_n, \varphi_m\rangle|^2\} = \{\|TN\varphi_n\|^2 - \|NT^*\varphi_n\|^2\}.$$

Thus, by Fatou's theorem, $\operatorname{tr}[A^*, A] + \operatorname{tr}[NT^*, TN] \leqq \operatorname{tr}[T^*, T]$.

In light of the Structure Lemma, it is obviously desirable to produce a supple family of hyponormal operators T with trace class selfcommutators.

DEFINITION. For μ a finite measure with compact support E contained in the compact set F, $R^2(F, \mu)$ will be the closure of $R(F)$ in $L^2(\mu)$. $R^2(E, \mu)$ will be written $R^2(\mu)$. If F does not divide the plane, $R^2(F, \mu) = H^2(\mu)$. T_f on $R^2(F, \mu)$ will be the operator PL_fP, where P is the orthogonal projection on $L^2(\mu)$ with range $R^2(F, \mu)$.

COMPUTATIONAL LEMMA. *Let* $D = \{z : |z| < 1\}$, *and let* $H = H^2(\chi_D\omega)$. *For* $f \in H^\infty(\chi_D\omega)$, *let* $T_f = L_f$ *on* H, *where* L_f *is the Laurent operator. If* $f = \sum_{n=0}^\infty a_n z^n$, *then*

$$\operatorname{tr}[T_f^*, T_f] = \sum_{n=1}^{\infty} n\,|a_n|^2 = \frac{1}{\pi}\int |f'|^2\, d\omega$$

$$= \pi^{-1}\,\{\textit{Area of } f(D), \textit{counting the multiplicity of the covering}\}.$$

PROOF. The first equality may be computed directly, using the basis $\{(n+1)^{1/2}z^n\}_{n=0}^\infty$. The others are well known.

COROLLARY. *Let* U *be a simply connected open set with a smooth Jordan curve for its boundary. Let* g *be the Riemann map from* U *to* $\mathbf{D}$. *Then the map* T_z *on* $H^2(\chi_U\,|g'|^2\,\omega)$ *satisfies* $\operatorname{tr}[T_z^*, T_z] = \pi^{-1}\omega(U)$.

PROOF. Taking $g^{-1} = f$, T_z is unitarily equivalent to T_f above.

REMARK. If $A_1, \ldots, A_n$ are each T_z on the respective spaces $R^2(\mu_i)$, if their spectra are pairwise disjoint and if $\operatorname{tr}[A_i^*, A_i] = \rho_i < \infty$, then the operator T_z on $R^2(\mu_1 + \cdots + \mu_n)$ satisfies $\operatorname{tr}[T_z^*, T_z] = \rho_1 + \cdots + \rho_n$.

It is also necessary to produce trace class intertwining maps. Let $T \in B(H)$. Suppose there is a map $z \to k_z$, from the open set U to H, which is conjugate analytic as a map into H in the strong topology, and such that there is a vector $x \in H$ satisfying $\langle r(T)x, k_z\rangle = r(z)$, for all rational functions r with poles off $\operatorname{sp}(T)$, and all $z \in U$. Then the triple (U, k_z, x) will be called an analytic evaluation for T, if $T^*k_z = \bar{z}k_z$ for all $z \in U$.

INTERTWINING LEMMA. *Let* (U, k_z, x) *be an analytic evaluation for* $T \in B(H)$, *and suppose that* x *is a* 1-*multicyclic vector for* T. *If* $u \in H$, *let* $\hat{u}(z) = \langle u, k_z\rangle$, *for* $z \in U$. *Let* $A \in B(K)$ *such that* $\operatorname{sp}(A) \subset U$, *and let* $y \in K$. *Define* $W: H \to K$, $Wu = \hat{u}(A)y$. *Then* $WT = AW$, *and* W *lies in trace class.*

PROOF. $\hat{u}$ is analytic on an open neighborhood of $\operatorname{sp}(A)$, and so $\hat{u}(A)$ is well defined, say by the Riesz integral. Since k_z is an eigenvector for T^* with eigenvalue $\bar{z}$, $(Tu)^\wedge = z\hat{u}$. Thus $WT = AW$. That W lies in trace class results from the fact that the map $z \to k_z$ is strongly conjugate analytic on an open neighborhood of $\mathrm{sp}(A)$. Let Γ_1 be a finite set of smooth Jordan curves bounding $\mathrm{sp}(A)$ from U^c, and let Γ_2 be another such set bounding Γ_1 from U^c, and Γ_3 a third, bounding Γ_2 from U^c. Let λ_i be arc length on Γ_i. Let H_i be the closure of the functions $\{\hat{u}: u \in H\}$ in $L^2(\lambda_i)$. Let $W_3: H \to H_3$ by $W_3 u = \hat{u}|_{\Gamma_3}$. H_3, H_2, and H_1 admit analytic evaluations. Define $W_i u = \hat{u}|_{\Gamma_i}$ for $u \in H_{i+1}$ for $i = 2, 1$ and $W_0 u = \hat{u}(A)y$ for $u \in H_1$. $W = W_0 W_1 W_2 W_3$, each W_i is bounded and it is easy to represent W_2 and W_1 as integral operators with square-summable kernels. Thus W_2 and W_1 are Hilbert-Schmidt operators, and so $W_2 W_1$ is in trace class [**2**].

COROLLARY. *Let μ be a finite measure with compact support. Let $K = H^2(\mu)$ and let E be the complement of the unbounded component of the complement of* $\mathrm{sp}(T_z)$. $[T_z^*, T_z]$ *is in trace class and* $\mathrm{tr}[T_z^*, T_z] \leqq \pi^{-1}\omega(E)$.

PROOF. Let $A = T_z$ on K. Let U be a simply connected open set with smooth Jordan boundary such that $E \subseteq U$ and $\omega(U) - \omega(E)$ is small. Let T be T_z on $H = H^2(\chi_U |g'|^2\, \omega)$, where g is as in the corollary to the Computational Lemma. Then $\mathrm{tr}[T^*, T] = \pi^{-1}\omega(U)$. Since $|g'|^2$ is bounded away from zero on compact sets in U, there exist vectors $k_z \in H$ such that $(U, k_z, 1)$ is an analytic evaluation for T. Thus the Intertwining Lemma applies. $W1 = 1$ is a cyclic vector for T_z on K, so W has dense range. Thus, the Structure Lemma applies, and so $\mathrm{tr}[A^*, A] \leqq \pi^{-1}\omega(U)$. Thus $\mathrm{tr}[A^*, A] \leqq \pi^{-1}\omega(E)$.

SUBSPACE DOMINANCE LEMMA. *Let the hyponormal operator $A \in B(H)$ be n-multicyclic, with generating vectors $g_1, \ldots, g_n$. Let E be a compact set containing* $\mathrm{sp}(A)$. *Let V be the closure of the space spanned by $\{r(A)g_i: r \in R(E)$, and $1 \leqq i \leqq n\}$. Then V is an invariant space for A, AV is hyponormal,* $\mathrm{sp}(A|_V) \subseteq E$, *$AV$ is n-multicyclic with generating vectors $g_1, \ldots, g_n$ and* $\mathrm{tr}[A^*, A] \leqq \mathrm{tr}[VA^*, AV]$.

PROOF. Unless $\mathrm{tr}[VA^*, AV] < \infty$, there is nothing to prove. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of points in $E \sim \mathrm{sp}(A)$ which land densely in each component of $\mathrm{sp}(A)^c$ which lies entirely in E. Let $r_m(z) = \prod_{i=1}^{m} (z - a_i)^{-1}$. Let $V_m = r_m(A)V$, $V_0 = V$. Then $V_{m+1} \supset V_m$, rank $(V_{m+1} - V_m) \leqq n$, and $V_m \nearrow H$ strongly. Thus $\mathrm{tr}[V_m A^*, AV_m] = \mathrm{tr}[VA^*, AV]$. Let $\{e_k\}_k$ be an orthonormal basis for H.

$$\mathrm{tr}[V_m A^*, AV_m] = \sum_k [\|AV_m e_k\|^2 - \|V_m A^* e_k\|^2].$$

Thus, since the summands are all nonnegative and approach the corresponding terms for $\mathrm{tr}[A^*, A]$, Fatou's lemma guarantees the desired inequality.

SECOND COMPUTATIONAL LEMMA. *Let $U_1, \ldots, U_n$ be open sets with* *disjoint closures, each bounded by finitely many disjoint smooth Jordan curves. Let* $U = \bigcup_i^n U_i$ *and* $H = R^2(\chi_{U^-}\omega)$. *Then* T_z *on* H *satisfies* $\operatorname{tr}[T_z^*, T_z] \leqq \pi^{-1}\omega(U)$.

PROOF. Let $\{G_i\}_{i=1}^m$ be simply connected open sets with smooth Jordan curves as boundaries such that each G_i^- lies in a separate bounded component of U^{-c}, and such that $\sum_i \omega(G_i)$ is close to the total area of the bounded components of U^{-c}. Choose g_i so that T_z on $H^2(|g_i'|^2\, \chi_{G_i}\omega)$ satisfies $\operatorname{tr}[T_z^*, T_z] = \pi^{-1}\omega(G_i)$. Let T be T_z on H, S be T_z on

$$R^2\left(\chi_{U^-}\omega + \sum_i |g_i'|^2\, \chi_{G_i}\omega\right),$$

T_i be T_z on $H^2(|g_i'|^2\, X_{G_i}\omega)$, and let S' be T_z on $H^2(\chi_{U^-}\omega + \sum_i |g_i'|^2\, \chi_{G_i}\omega)$. Let $\tilde{U}$ be the complement of the unbounded component of U^c. Then

$$\begin{aligned}\operatorname{tr}[T^*, T] + \pi^{-1}\sum_i \omega(G_i) &= \operatorname{tr}[T^*, T] + \sum_{i=1}^n \operatorname{tr}[T_i^*, T_i] \\ &= \operatorname{tr}[S^*, S] \leqq \operatorname{tr}[S'^*, S'] \leqq \pi^{-1}\omega(\tilde{U}).\end{aligned}$$

Thus $\operatorname{tr}[T^*, T] \leqq \pi^{-1}\omega(U)$.

It is now possible to prove the Main Theorem.

THEOREM 1. *Let* $A \in B(K)$ *be hyponormal, with n-multicyclic generating vectors* $g_1, \ldots, g_n$. *Then* $\operatorname{tr}[A^*, A] \leqq (n/\pi)\omega(\operatorname{sp}(A))$.

PROOF. Let U be an open set bounded by a finite number of disjoint smooth Jordan curves, such that $\operatorname{sp}(A) \subset U$, and $\omega(U) - \omega(\operatorname{sp}(A))$ is small. Let K' be the space spanned by $\{r(A)g_i : r \in R(U^-), \text{and } 1 \leqq i \leqq n\}$. Let A' be the restriction of A to K'. A' is hyponormal, and $\operatorname{sp}(A') \subseteq U$. $\{g_1, \ldots, g_n\}$ is a set of n-multicyclic vectors for A'. By the Subspace Dominance Lemma, $\operatorname{tr}[A^*, A] \leqq \operatorname{tr}[A'^*, A']$.

Let $T = \oplus \sum_{i=1}^n T_z$ acting on $H = \oplus \sum_{i=1}^n R^2(\chi_U\omega)$.

By the Second Computational Lemma, $\operatorname{tr}[T^*, T] \leqq (n/\pi)\omega(U)$. Thus, it only remains to produce an intertwining map between T and A' satisfying the conditions of the Structure Lemma.

$R^2(\chi_{U^-}\omega)$ has reproducing kernel k_z at each $z \in U$. The map $z \to k_z$ is strongly conjugate analytic, and the triple $(U, k_z, 1)$ is an analytic evaluation. Thus by the Intertwining Lemma, the map $W_i : R^2(\chi_{U^-}\omega) \to K'$ defined by $Wf = \hat{f}(A')g_i$ lies in trace class, and $W_iT_z = A'W_i$. Let $W : \oplus \sum_{i=1}^n R^2(\chi_{U^-}\omega) \to K'$ by $W = \sum_{i=1}^n W_i$. W lies in trace class, and $WT = A'W$. Clearly, the range of W is dense in K'. Thus

$$\operatorname{tr}[A^*, A] \leqq \operatorname{tr}[A'^*, A'] \leqq \operatorname{tr}[T^*, T] \leqq (n/\pi)\omega(U).$$

COROLLARY (PUTNAM'S THEOREM [3]). *If $A \in B(H)$ is hyponormal, then* $\|[A^*, A]\| \leqq \pi^{-1}\omega(\mathrm{sp}(A))$.

PROOF. Let $x \in H$, $\|x\| = 1$, and let V be the closure of the set of vectors $\{r(A)x : r \in R(\mathrm{sp}(A))\}$. V is an invariant space for A. Let A' be the restriction of A to V. A' is hyponormal.

If $y \in V$ and $a \in \mathrm{sp}(A)^c$, $(A - aI)^{-1}y \in V$. Thus $\mathrm{sp}(A) \supseteq \mathrm{sp}(A')$. It is clear that A' is 1-multicyclic. Thus

$$\begin{aligned}\langle[A^*, A]x, x\rangle &= \|Ax\|^2 - \|A^*x\|^2 \leqq \|Ax\|^2 - \|VA^*x\|^2\\ &= \|A'x\|^2 - \|A'^*x\|^2\\ &= \langle[A'^*, A']x, x\rangle \leqq \mathrm{tr}[A'^*, A']\\ &\leqq \pi^{-1}\omega(\mathrm{sp}(A')) \leqq \pi^{-1}\omega(\mathrm{sp}(A)).\end{aligned}$$

3. The techniques used above suffice to yield the following results.

THEOREM 2. *If the hyponormal operator A has analytic evaluation (U, k_z, x), then* $\mathrm{tr}[A^*, A] \geqq \pi^{-1}\omega(U)$.

THEOREM 3. *If A is a 1-multicyclic hyponormal operator with generating vector x, if V is an invariant space for A containing x, and if A' is the restriction of A to V, then*

$$\mathrm{tr}[A^*, A] + \pi^{-1}\omega(\mathrm{sp}(A') \sim \mathrm{sp}(A)) \leqq \mathrm{tr}[A'^*, A'].$$

The corresponding result for n-multicyclic hyponormal operators is rather more complicated, and requires a fairly lengthy explanation.

THEOREM 4. *For $r \in R(E)$, T_r on $R^2(E, \mu)$ satisfies*

$$[T_r^*, T_r] \leqq \frac{1}{\pi}\int_{\mathrm{sp}(T_z)} |r'|^2 \, d\omega.$$

Note that the quantity $[T_r^*, T_r]$ is a quadratic norm on $R(E)$. The above theorem may be generalized to all functions in the Hilbert space so determined. The following is unknown.

CONJECTURE. There is a measurable function g defined on $\mathrm{sp}(T_z)$ such that $0 \leqq g \leqq 1$, and $\mathrm{tr}[T_r^*, T_r] = \pi^{-1}\int_{\mathrm{sp}(T_z)} |r'|^2 g \, d\omega$ for all $r \in R(E)$.

THEOREM 5. *If $R^2(E, \mu)$ has analytic evaluation $(U, k_z, 1)$, F is a compact subset of U, ν is a finite measure supported on F, and $r \in R(E)$, then $\mathrm{tr}[T_r^*, T_r]$ is the same, whether computed on $R^2(E, \mu)$ or on $R^2(E, \chi_{F^c}\mu + \nu)$.*

THEOREM 6. *If $R^2(E, \mu)$ has analytic evaluation $(U, k_z, 1)$, and $0 \leqq g \leqq 1$ is a measurable function such that $g^{-1}([0, 1)) \subset U$, then for all $r \in R(E)$, $\mathrm{tr}[\mathrm{Tr}^*, \mathrm{Tr}]$ is not increased when it is computed on $R^2(E, g\mu)$ rather than on $R^2(E, \mu)$.*

THEOREM 7. *Let $A^2(U)$ be the Hilbert space of all functions analytic on the open set U, and square summable with respect to $\chi_U \omega$. Let f be bounded and analytic on U. Then* $\mathrm{tr}[T_f^*, T_f] = \pi^{-1} \int_{f(U)} \eta(z, f)\, d\omega$, *where $\eta(z, f)$ is the cardinality of $f^{-1}(z)$.*

This theorem may be generalized to the setting of complex manifolds.

For μ a finite measure with compact support E, and F a compact set containing E, let $R = R^2(F, \mu) \subseteq L^2(\mu)$, and for $f \in L^\infty(\mu)$, define the "Hankel operator" H_f by $H_f = (I - R)L_f R$. Let $\mathscr{H} = \{f \in L^\infty(\mu) : H_f$ is compact$\}$.

THEOREM 8. *If $f \in R(F)$, then H_f is a Hilbert-Schmidt operator. $\mathscr{H}$ is a closed subalgebra of $L^\infty(\mu)$, and $\mathscr{H}$ contains $L^\infty(\mu) \cap R^2(F, \mu) + C(E)$.*

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