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Selfdecomposability and Semi-selfdecomposability in Subordination of Cone-parameter Convolution Semigroups

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Abstract. Extension of two known facts concerning subordination is made. The first fact is that, in subordination of 1-dimensional Brownian motion with drift, selfdecomposability is inherited from subordinator to subordinated. This is extended to subordination of cone-parameter convolution semigroups. The second fact is that, in subordination of strictly stable cone-parameter convolution semigroups on $\mathbf{R}^d$, selfdecomposability is inherited from subordinator to subordinated. This is extended to semi-selfdecomposability.

1. Introduction

A subset K of $\mathbf{R}^N$ is called a cone if it is a non-empty closed convex set which is closed under multiplication by nonnegative reals and contains no straight line through 0 and if $K \neq \{0\}$. Given a cone K, we call $\{\mu_s : s \in K\}$ a K-parameter convolution semigroup on $\mathbf{R}^d$ if it is a family of probability measures on $\mathbf{R}^d$ satisfying

$$\mu_{s_1} * \mu_{s_2} = \mu_{s_1+s_2} \quad \text{for } s_1, s_2 \in K, \tag{1.1}$$

$$\mu_{ts} \to \delta_0 \quad \text{as } t \downarrow 0, \quad \text{for } s \in K, \tag{1.2}$$

where δ_0 is delta distribution located at $0 \in \mathbf{R}^d$. Convergence of probability measures is understood as weak convergence. It follows from (1.1) and (1.2) that $\mu_0 = \delta_0$.

Subordination of a cone-parameter convolution semigroup is defined as follows. Let K_1 and K_2 be cones in $\mathbf{R}^{N_1}$ and $\mathbf{R}^{N_2}$, respectively. Let $\{\mu_u : u \in K_2\}$ be a K_2-parameter convolution semigroup on $\mathbf{R}^d$ and $\{\rho_s : s \in K_1\}$ a K_1-parameter convolution semigroup on $\mathbf{R}^{N_2}$ supported on K_2 (that is, $\mathrm{Supp}(\rho_s) \subseteq K_2$). Define a probability measure σ_s on $\mathbf{R}^d$ by

$$\sigma_s(B) = \int_{K_2} \mu_u(B)\rho_s(du) \quad \text{for } B \in \mathcal{B}(\mathbf{R}^d), \tag{1.3}$$

where $\mathcal{B}(\mathbf{R}^d)$ is the class of Borel sets in $\mathbf{R}^d$. Then $\{\sigma_s : s \in K_1\}$ is a K_1-parameter convolution semigroup on $\mathbf{R}^d$. This procedure to get $\{\sigma_s : s \in K_1\}$ is called subordination of $\{\mu_u : u \in K_2\}$ by $\{\rho_s : s \in K_1\}$. Convolution semigroups $\{\mu_u : u \in K_2\}$, $\{\rho_s : s \in K_1\}$,

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and $\{\sigma_s : s \in K_1\}$ are respectively called subordinand, subordinating (or subordinator), and subordinated.

Cone-parameter convolution semigroups on $\mathbf{R}^d$ and their subordination are introduced in Pedersen and Sato [11]. Their basic properties are proved in Theorems 2.8, 2.11, and 4.4 of [11]. A number of examples are given there. In Barndorff-Nielsen, Pedersen, and Sato [1], several models leading to $\mathbf{R}_+$-parameter convolution semigroups supported on $\mathbf{R}_+^N$ are discussed, including some financial models. Here $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_+^N = (\mathbf{R}_+)^N$.

In $\mathbf{R}_+$-parameter case, any convolution semigroup on $\mathbf{R}^d$ corresponds to a unique (in law) Lévy process. For a general cone K, any K-parameter Lévy process $\{X_s : s \in K\}$ on $\mathbf{R}^d$ defined in Pedersen and Sato [12] induces a K-parameter convolution semigroup $\{\mu_s\}$ on $\mathbf{R}^d$ as $\mu_s = \mathcal{L}(X_s)$, the law of X_s. But, for a given K-parameter convolution semigroup on $\mathbf{R}^d$, neither existence nor uniqueness (in law) of a K-parameter Lévy process which induces the semigroup can be proved in general, as is shown in [12]. The existence is proved when $d = 1$, when K is isomorphic to $\mathbf{R}_+^N$, or when μ_s does not have Gaussian part for any s. The non-existence is proved for the canonical (d-dimensional Gaussian) $\mathbf{S}_d^+$-parameter convolution semigroup defined in [12] for $d \geqslant 2$, where $\mathbf{S}_d^+$ is the cone of $d \times d$ symmetric nonnegative-definite matrices. Concerning the uniqueness, some sufficient conditions for the uniqueness and for the non-uniqueness are given in [12]. For example, if $\{\mu_s\}$ is an $\mathbf{R}_+^2$-parameter convolution semigroup on $\mathbf{R}$ such that the Gaussian part of μ_s is nonzero for any $s \neq 0$, then the corresponding $\mathbf{R}_+^2$-parameter Lévy process on $\mathbf{R}$ is not unique in law. Subordination of a K_2-parameter Lévy process on $\mathbf{R}^d$ by a K_1-parameter Lévy process on K_2 results in a new K_1-parameter Lévy process on $\mathbf{R}^d$, as is shown in Pedersen and Sato [12] and earlier, in the case $K_2 = \mathbf{R}_+^N$ and $K_1 = \mathbf{R}_+$, in Barndorff-Nielsen, Pedersen, and Sato [1]. It induces subordination of a cone-parameter convolution semigroup. But subordination of a cone-parameter convolution semigroup is not always accompanied by subordination of a cone-parameter Lévy process.

In this paper we give some results on inheritance of selfdecomposability, semi-selfdecomposability, and some related properties from subordinating to subordinated in subordination of cone-parameter convolution semigroups. Applications to distributions of type mult$G$ are given.

Semi-selfdecomposable distributions were introduced by Maejima and Naito [8]. Their probabilistic representations were given by Maejima and Sato [9]. Their remarkable continuity properties were discovered by Watanabe [19]. Recent papers of Kondo, Maejima, and Sato [5] and Lindner and Sato [7] studied them in stationary distributions of some generalized Ornstein–Uhlenbeck processes.

2. One-dimensional Gaussian subordinands

Let $G_{a,\gamma}$ denote Gaussian distribution on $\mathbf{R}$ with variance $a \geqslant 0$ and mean $\gamma \in \mathbf{R}$, where $G_{0,\gamma} = \delta_\gamma$. A K-parameter convolution semigroup $\{\mu_u : u \in K\}$ is called 1-dimensional Gaussian if, for each $u \in K$, μ_u is $G_{a,\gamma}$ with some a and γ.

A distribution μ on $\mathbf{R}^d$ is said to be selfdecomposable if, for each $b > 1$, there is a distribution μ' on $\mathbf{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu'}(z), \quad z \in \mathbf{R}^d. \tag{2.1}$$

Here $\widehat{\mu}(z)$ and $\widehat{\mu'}(z)$ are the characteristic functions of μ and μ', respectively. If μ is selfdecomposable, then μ is infinitely divisible.

Noting that selfdecomposability is equivalent to semi-selfdecomposability with span b for all $b > 1$ (see Section 3 for the definition) and using Theorem 15.8 of [15], we see that an infinitely divisible distribution μ on $\mathbf{R}^d$ with Lévy measure ν is selfdecomposable if and only if

$$\nu(b^{-1}B) \geqslant \nu(B) \quad \text{for } b > 1 \text{ and } B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}). \tag{2.2}$$

The condition (2.2) holds if and only if ν has a polar representation

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-1} k_\xi(r) dr \quad \text{for } B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\}), \tag{2.3}$$

where $S = \{\xi : |\xi| = 1\}$, the unit sphere in $\mathbf{R}^d$, λ is a measure on S, and $k_\xi(r)$ is a nonnegative function measurable in ξ and decreasing in $r > 0$ (Theorem 15.10 of [15]). We are using the word *decrease* in the wide sense allowing flatness.

THEOREM 2.1. *Let K_1 and K_2 be cones in $\mathbf{R}^{N_1}$ and $\mathbf{R}^{N_2}$, respectively. Let $\{\mu_u : u \in K_2\}$ be a 1-dimensional Gaussian K_2-parameter convolution semigroup (subordinand), $\{\rho_s : s \in K_1\}$ a K_1-parameter convolution semigroup supported on K_2 (subordinating), and $\{\sigma_s : s \in K_1\}$ the subordinated K_1-parameter convolution semigroup on $\mathbf{R}$. Fix $s \in K_1$. If ρ_s is selfdecomposable, then σ_s is selfdecomposable.*

We stress that the Gaussian distribution μ_u is not necessarily centered. For the centered Gaussian (that is strictly 2-stable), the result is largely extended in Theorem 3.1 in Section 3. Historically, Halgreen [4] raised a question equivalent to asking whether the statement of Theorem 2.1 for $K_1 = K_2 = \mathbf{R}_+$ is true. After 22 years, Theorem 1.1 of Sato [16] answered this question affirmatively. The theorem above is an extension of it. In order to prove the theorem, we prepare a lemma.

LEMMA 2.2. *Let $f(r)$ be a nonnegative decreasing function of $r > 0$ satisfying $\int_0^\infty (r \wedge 1) r^{-1} f(r) dr < \infty$. Let $a \geqslant 0$ and $\gamma \in \mathbf{R}$. Then, for every $b > 1$ and $B \in \mathcal{B}(\mathbf{R} \setminus \{0\})$,*

$$\int_0^\infty G_{ra,r\gamma}(b^{-1}B) r^{-1} f(r) dr \geqslant \int_0^\infty G_{ra,r\gamma}(B) r^{-1} f(r) dr. \tag{2.4}$$

PROOF. Let $\{X_t : t \in \mathbf{R}_+\}$ be the Lévy process with distribution $G_{a,\gamma}$ at time 1. Let $\{Z_t : t \in \mathbf{R}_+\}$ be a selfdecomposable subordinator with Lévy measure $r^{-1}f(r)dr$ and drift 0. Let $\{Y_t : t \in \mathbf{R}_+\}$ be the Lévy process on $\mathbf{R}$ obtained by subordination of $\{X_t\}$ by $\{Z_t\}$. Then Theorem 30.1 of [15] tells us that the Lévy measure ν^Y of $\{Y_t\}$ is expressed as

$$\nu^Y(B) = \int_0^\infty G_{ra,r\gamma}(B)r^{-1}f(r)dr\,, \quad B \in \mathcal{B}(\mathbf{R}\setminus\{0\})\,.$$

If $a > 0$, then Theorem 1.1 of [16] establishes that Y_t has a selfdecomposable distribution for any $t \geqslant 0$. If $a = 0$, then $\{X_t\}$ is a trivial Lévy process (that is, $X_t = \gamma t$, nonrandom) and $Y_t = \gamma Z_t$, which has a selfdecomposable distribution. In any case, $\{Y_t\}$ is selfdecomposable. Hence $\nu^Y(b^{-1}B) \geqslant \nu^Y(B)$, which is exactly (2.4). □

PROOF OF THEOREM 2.1. Let ν^{μ_u}, ν^{ρ_s}, and ν^{σ_s} denote the Lévy measures of μ_u, ρ_s, and σ_s, respectively. We have $\mu_u = G_{a_u,\gamma_u}$ with some $a_u \geqslant 0$ and $\gamma_u \in \mathbf{R}$. These a_u and γ_u are continuous functions of u (Theorem 2.8 of [11]). Since μ_u has Lévy measure 0, Theorem 4.4 of [11] says that

$$\nu^{\sigma_s}(B) = \int_{K_2} G_{a_u,\gamma_u}(B)\nu^{\rho_s}(du)\,, \quad B \in \mathcal{B}(\mathbf{R}\setminus\{0\})\,.$$

Assume that ρ_s is selfdecomposable. Then ν^{ρ_s} is expressed as in the right-hand side of (2.3) with $d = N_2$. Since $\mathrm{Supp}(\rho_s) \subseteq K_2$, it follows from Skorohod's theorem [17] (or Lemma 4.1 of [11]) that the measure λ is supported on $S \cap K_2$ and that

$$\int_{S\cap K_2} \lambda(d\xi)\int_0^\infty (r\wedge 1)r^{-1}k_\xi(r)dr < \infty\,.$$

For any $b > 1$ and $B \in \mathcal{B}(\mathbf{R}\setminus\{0\})$ we have

$$\begin{aligned}\nu^{\sigma_s}(b^{-1}B) &= \int_{K_2} G_{a_u,\gamma_u}(b^{-1}B)\nu^{\rho_s}(du)\\ &= \int_{S\cap K_2}\lambda(d\xi)\int_0^\infty G_{a_{r\xi},\gamma_{r\xi}}(b^{-1}B)r^{-1}k_\xi(r)dr = I \quad \text{(say)}\,.\end{aligned}$$

Notice that $k_\xi(r)$ is decreasing in r and satisfies $\int_0^\infty (r\wedge 1)r^{-1}k_\xi(r)dr < \infty$ for λ-almost every ξ and that $a_{r\xi} = ra_\xi$ and $\gamma_{r\xi} = r\gamma_\xi$ (see Proposition 2.7 of [11]). Thus we can apply Lemma 2.2 to obtain

$$I \geqslant \int_{S\cap K_2}\lambda(d\xi)\int_0^\infty G_{a_{r\xi},\gamma_{r\xi}}(B)r^{-1}k_\xi(r)dr = \nu^{\rho_s}(B)\,.$$

This means that σ_s is selfdecomposable. □

REMARK 2.3. Let K be a cone and let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbf{R}^d$. Let $s_0 \in K\setminus\{0\}$. If μ_{s_0} is selfdecomposable, then μ_{ts_0} is selfdecomposable for all $t \geqslant 0$ since $\mu_{ts_0} = \mu_{s_0}^t$, the t th convolution power of μ_{s_0} (Proposition 2.7 of [11]), but

μ_{s_1} may not be selfdecomposable for some $s_1 \in K \setminus \{ts_0 : t \geqslant 0\}$. This follows from Sections 2 and 3 of [11].

REMARK 2.4. In Theorem 2.1 let $K_1 = K_2 = \mathbf{R}_+$ and replace "Gaussian" by "α-stable (not necessarily strictly α-stable)", where $\alpha \in (0, 2]$. Then the statement for $\alpha = 2$ is exactly Theorem 1.1 of [16]. The statement for $\alpha \in (1, 2)$ is not true, which is pointed out by Kozubowski [6] using Theorem 2.1(v) of Ramachandran [13]. It is not known whether the statement for $\alpha \in (0, 1]$ is true.

REMARK 2.5. If μ is selfdecomposable, then the distribution μ' in (2.1) is uniquely determined by μ and b, and μ' is also infinitely divisible. For nonnegative integers m we define $L_m(\mathbf{R}^d)$ as follows: $L_0(\mathbf{R}^d)$ is the class of selfdecomposable distributions on $\mathbf{R}^d$; for $m \geqslant 1$, $L_m(\mathbf{R}^d)$ is the class of $\mu \in L_0(\mathbf{R}^d)$ such that, for every $b > 1$, μ' in (2.1) belongs to $L_{m-1}(\mathbf{R}^d)$. Thus we get a strictly decreasing sequence of subclasses of the class $ID(\mathbf{R}^d)$ of infinitely divisible distributions on $\mathbf{R}^d$. We define $L_\infty(\mathbf{R}^d)$ as the intersection of $L_m(\mathbf{R}^d)$, $m = 0, 1, 2, \ldots$. It is not known even in the case $K_1 = K_2 = \mathbf{R}_+$ whether Theorem 2.1 is true with "selfdecomposable" replaced by "of class L_m" for $m \in \{1, 2, \ldots, \infty\}$.

REMARK 2.6. Let $d \geqslant 2$. Theorem 2.1 cannot be generalized to d-dimensional Gaussian. If $\{\mu_u : u \in \mathbf{R}_+\}$ is an $\mathbf{R}_+$-parameter convolution semigroup (subordinand) induced by d-dimensional Brownian motion with nonzero drift and $\{\rho_t : t \in \mathbf{R}_+\}$ is an $\mathbf{R}_+$-parameter convolution semigroup supported on $\mathbf{R}_+$ (subordinating) of Thorin class (of generalized gamma convolutions, in other words) satisfying some additional condition, then the subordinated $\mathbf{R}_+$-parameter convolution semigroup $\{\sigma_t : t \in \mathbf{R}_+\}$ on $\mathbf{R}^d$ is not selfdecomposable for any $t > 0$. This fact was noticed by Takano [18] and Grigelionis [3]. Recall that the Thorin class is a subclass of the class of selfdecomposable distributions. This σ_t supplies an example of an infinitely divisible non-selfdecomposable distribution whose one-dimensional projections are selfdecomposable, since we can apply Theorem 1.1 of [16] to one-dimensional projections of $\{\mu_u : u \in \mathbf{R}_+\}$. The first example of a distribution with this projection property was constructed in Sato [14].

REMARK 2.7. It is not known even in the case $K_1 = K_2 = \mathbf{R}_+$ whether Theorem 2.1 is true with "selfdecomposable" replaced by "semi-selfdecomposable", which will be defined in the next section.

3. Inheritance of semi-selfdecomposability

A distribution on $\mathbf{R}^d$ is called semi-selfdecomposable if there are $b > 1$ and $\mu' \in ID(\mathbf{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu'}(z), \quad z \in \mathbf{R}^d . \tag{3.1}$$

The b in this definition is called a span of μ; it is not uniquely determined by μ. The class of semi-selfdecomposable distributions on $\mathbf{R}^d$ having b as a span is denoted by $L_0(b^{-1}, \mathbf{R}^d)$. If $\mu \in L_0(b^{-1}, \mathbf{R}^d)$, then μ is infinitely divisible and the distribution μ' is uniquely determined by μ and b. For any positive integer m we inductively define

$$L_m(b^{-1}, \mathbf{R}^d) = \{\mu \in L_0(b^{-1}, \mathbf{R}^d) \colon \mu' \in L_{m-1}(b^{-1}, \mathbf{R}^d)\}.$$

Then $L_m(b^{-1}, \mathbf{R}^d)$ is a subclass of $L_{m-1}(b^{-1}, \mathbf{R}^d)$. In fact we can prove that the former is a strict subclass of the latter (see Remark 3.1 of [10]). Further we define $L_\infty(b^{-1}, \mathbf{R}^d)$ as the intersection of $L_m(b^{-1}, \mathbf{R}^d)$ for $m = 0, 1, \ldots$.

Let $0 < \alpha \leqslant 2$. A distribution μ on $\mathbf{R}^d$ is called strictly α-semistable if $\mu \in ID(\mathbf{R}^d)$ and if there is a real number $b > 1$ such that

$$\widehat{\mu}(z)^{b^\alpha} = \widehat{\mu}(bz), \quad z \in \mathbf{R}^d, \tag{3.2}$$

or, equivalently, $\widehat{\mu}(z)^{b^{-\alpha}} = \widehat{\mu}(b^{-1}z)$, $z \in \mathbf{R}^d$. In this case we say that the α-semistable distribution μ has a span b, which is not uniquely determined by μ. If μ is strictly α-semistable on $\mathbf{R}^d$ with a span b, then it is easy to see that $\mu \in L_\infty(b^{-1}, \mathbf{R}^d)$, since we have

$$\widehat{\mu}(z) = \widehat{\mu}(z)^{b^{-\alpha}} \widehat{\mu}(z)^{1-b^{-\alpha}} = \widehat{\mu}(b^{-1}z)\widehat{\mu}(z)^{1-b^{-\alpha}}.$$

For description and examples of Lévy measures of semi-selfdecomposable and semistable distributions, see Sections 14 and 15 of [15].

The statement of Remark 2.3 is true also for "semi-selfdecomposable with a span b" and "strictly α-semistable with a span b" in place of "selfdecomposable".

THEOREM 3.1. *Let K_1 and K_2 be cones in $\mathbf{R}^{N_1}$ and $\mathbf{R}^{N_2}$, respectively. Let $\{\mu_u \colon u \in K_2\}$ be a K_2-parameter convolution semigroup on $\mathbf{R}^d$ (subordinand), $\{\rho_s \colon s \in K_1\}$ a K_1-parameter convolution semigroup supported on K_2 (subordinating), and $\{\sigma_s \colon s \in K_1\}$ the subordinated K_1-parameter convolution semigroup on $\mathbf{R}^d$. Suppose that there are $0 < \alpha \leqslant 2$ and $b > 1$ such that, for every $u \in K_2$, μ_u is strictly α-semistable with a span $b^{1/\alpha}$. Fix $s \in K_1$. Then the following statements are true.*

(i) *Let $m \in \{0, 1, \ldots, \infty\}$. If*

$$\rho_s \in L_m(b^{-1}, \mathbf{R}^{N_2}), \tag{3.3}$$

then

$$\sigma_s \in L_m(b^{-1/\alpha}, \mathbf{R}^d). \tag{3.4}$$

(ii) *Let $0 < \alpha' \leqslant 1$. If*

$$\rho_s \text{ is strictly } \alpha'\text{-semistable with a span } b, \tag{3.5}$$

then

$$\sigma_s \text{ is strictly } \alpha\alpha'\text{-semistable with a span } b^{1/\alpha}. \tag{3.6}$$

Note that strictly 1-semistable distributions supported on a cone are delta distributions. This theorem is an extension of Theorem 4.10 of Pedersen and Sato [11] to the "semi" case. We prepare a lemma. This is an analogue of Lemma 4.11 of [11] and the proof is almost the same.

LEMMA 3.2. *Let K_2 be a cone in $\mathbf{R}^{N_2}$. Suppose that ρ is in $L_0(b^{-1}, \mathbf{R}^{N_2})$ and that* $\mathrm{Supp}(\rho) \subseteq K_2$. *Let ρ' be defined by $\widehat{\rho}(z) = \widehat{\rho}(b^{-1}z)\widehat{\rho'}(z)$, $z \in \mathbf{R}^{N_2}$. Then* $\mathrm{Supp}(\rho') \subseteq K_2$.

PROOF OF THEOREM 3.1. Let us prove assertion (i) for $m = 0$. Assume that $\rho_s \in L_0(b^{-1}, \mathbf{R}^{N_2})$. Define ρ_s'' as $\widehat{\rho_s''}(z) = \widehat{\rho_s}(b^{-1}z)$. Then

$$\widehat{\rho_s}(z) = \widehat{\rho_s''}(z)\widehat{\rho_s'}(z)$$

and thus $\rho_s = \rho_s'' * \rho_s'$. Lemma 3.2 tells us that ρ_s' is supported on K_2. Clearly ρ_s'' is also supported on K_2. Hence

$$\begin{aligned}\widehat{\sigma}_s(z) &= \int_{K_2} \widehat{\mu}_u(z)\rho_s(du) = \iint_{K_2\times K_2} \widehat{\mu}_{u_1+u_2}(z)\rho_s''(du_1)\rho_s'(du_2) \\ &= \iint_{K_2\times K_2} \widehat{\mu}_{u_1}(z)\widehat{\mu}_{u_2}(z)\rho_s''(du_1)\rho_s'(du_2) \\ &= \int_{K_2} \widehat{\mu}_{b^{-1}u_1}(z)\rho_s(du_1)\int_{K_2} \widehat{\mu}_{u_2}(z)\rho_s'(du_2)\,.\end{aligned}$$

Using Proposition 2.7 of [11] and the assumption that μ_u is strictly α-semistable with a span $b^{1/\alpha}$, we have

$$\widehat{\mu}_{b^{-1}u}(z) = \widehat{\mu}_u(z)^{b^{-1}} = \widehat{\mu}_u(b^{-1/\alpha}z)\,.$$

It follows that

$$\widehat{\sigma}_s(z) = \widehat{\sigma}_s(b^{-1/\alpha}z)\int_{K_2} \widehat{\mu}_u(z)\rho_s'(du)\,. \tag{3.7}$$

Since $\int_{K_2} \widehat{\mu}_u(z)(\rho_s')^t(du)$ is subordination of $\{\mu_u\}$ by $\{(\rho_s')^t : t \in \mathbf{R}_+\}$, we see that $\int_{K_2} \widehat{\mu}_u(z)\rho_s'(du)$ is infinitely divisible. This shows that $\sigma_s \in L_0(b^{-1/\alpha}, \mathbf{R}^d)$.

Next, we assume that (i) is true for a fixed $m \in \{0, 1, \ldots\}$. We claim that (i) is true for $m + 1$. Suppose that $\rho_s \in L_{m+1}(b^{-1}, \mathbf{R}^{N_2})$. Then $\widehat{\rho_s}(z) = \widehat{\rho_s}(b^{-1}z)\widehat{\rho_s'}(z)$ with $\rho_s' \in L_m(b^{-1}, \mathbf{R}^{N_2})$. We have (3.7) since $L_{m+1}(b^{-1}, \mathbf{R}^{N_2}) \subseteq L_0(b^{-1}, \mathbf{R}^{N_2})$. Now $\int_{K_2} \widehat{\mu}_u(z)(\rho_s')^t(du)$ is subordination such that $(\rho_s')^t$ is in $L_m(b^{-1}, \mathbf{R}^{N_2})$. Hence $\int_{K_2} \widehat{\mu}_u(z)\rho_s'(du)$ is the the characteristic function of a distribution in $L_m(b^{-1/\alpha}, \mathbf{R}^d)$. It follows that $\sigma_s \in L_{m+1}(b^{-1/\alpha}, \mathbf{R}^d)$, which shows (i) for $m + 1$.

Assertion (i) for $m = \infty$ is a consequence of that for finite m.

To prove (ii), assume (3.5). Let us show (3.6), that is,

$$\widehat{\sigma}_s(z)^{b^{\alpha'}} = \widehat{\sigma}_s(b^{1/\alpha}z)\,. \tag{3.8}$$

Using

$$\widehat{\rho}_{b^{\alpha'}s}(z) = \widehat{\rho}_s(z)^{b^{\alpha'}} = \widehat{\rho}_s(bz)$$

and

$$\widehat{\mu}_{bu}(z) = \widehat{\mu}_u(z)^b = \widehat{\mu}_u(b^{1/\alpha}z)\,,$$

we obtain

$$\begin{aligned}\widehat{\sigma}_s(z)^{b^{\alpha'}} &= \widehat{\sigma}_{b^{\alpha'}s}(z) = \int_{K_2} \widehat{\mu}_u(z)\rho_{b^{\alpha'}s}(du) = \int_{K_2} \widehat{\mu}_{bu}(z)\rho_s(du) \\ &= \int_{K_2} \widehat{\mu}_u(b^{1/\alpha}z)\rho_s(du) = \widehat{\sigma}_s(b^{1/\alpha}z)\,,\end{aligned}$$

completing the proof. □

Application to distributions of type mult$G$. Following Barndorff-Nielsen and Pérez-Abreu [2], we say that a probability measure σ on $\mathbf{R}^d$ is of type mult$G$ if $\sigma = \mathcal{L}(Z^{1/2}X)$, where X is a standard Gaussian on $\mathbf{R}^d$, Z is an $\mathbf{S}_d^+$-valued infinitely divisible random variable, $Z^{1/2}$ is the nonnegative-definite symmetric square root of Z, and X and Z are independent. Here, as in Section 1, $\mathbf{S}_d^+$ is the class of $d \times d$ symmetric nonnegative-definite matrices and elements of $\mathbf{R}^d$ are considered as column d-vectors. Regarding the lower triangle $(s_{jk})_{k\leqslant j}$ of $s = (s_{jk})_{j,k=1}^d \in \mathbf{S}_d^+$ as a $d(d+1)/2$-vector, $\mathbf{S}_d^+$ is identified with a cone in $\mathbf{R}^{d(d+1)/2}$. The $\mathbf{S}_d^+$-parameter convolution semigroup $\{\mu_s : s \in \mathbf{S}_d^+\}$ on $\mathbf{R}^d$ where μ_s is d-dimensional Gaussian with mean vector 0 and covariance matrix s is called the canonical $\mathbf{S}_d^+$-parameter convolution semigroup ([11]). The following fact is known (Theorem 4.7 of [11] and its proof).

PROPOSITION 3.3. *Let $\{\mu_u : u \in \mathbf{S}_d^+\}$ be the canonical $\mathbf{S}_d^+$-parameter convolution semigroup (subordinand), $\{\rho_t : t \in \mathbf{R}_+\}$ an $\mathbf{R}_+$-parameter convolution semigroup on $\mathbf{R}^{d(d+1)/2}$ supported on $\mathbf{S}_d^+$ (subordinating), and $\{\sigma_t : t \in \mathbf{R}_+\}$ the subordinated $\mathbf{R}_+$-parameter convolution semigroup on $\mathbf{R}^d$. Then σ_1 (or, more generally, σ_t) is of type* mult$G$. *Conversely, any distribution on $\mathbf{R}^d$ of type* mult$G$ *is expressible as σ_1 of such an $\mathbf{R}_+$-parameter convolution semigroup $\{\sigma_t : t \in \mathbf{R}_+\}$. The correspondence of the two representations of a distribution of type* mult$G$ *is that $\rho_1 = \mathcal{L}(Z)$.*

We can show the following.

PROPOSITION 3.4. *Let σ be a distribution of type* mult$G$, *that is, let $\sigma = \mathcal{L}(Z^{1/2}X)$, where X is a standard Gaussian on $\mathbf{R}^d$, $Z^{1/2}$ is the nonnegative-definite symmetric square root of $\mathbf{S}_d^+$-valued infinitely divisible random variable Z, and X and Z are independent.*

(i) *Let $m \in \{0, 1, \ldots, \infty\}$ and $b > 1$. If $\mathcal{L}(Z) \in L_m(b^{-1}, \mathbf{R}^{d(d+1)/2})$, then $\sigma \in L_m(b^{-1/2}, \mathbf{R}^d)$.*

(ii) *Let* $0 < \alpha' \leqslant 1$ *and* $b > 1$. *If* $\mathcal{L}(Z)$ *is strictly* α'*-semistable with a span* b, *then* σ *is strictly* $2\alpha'$*-semistable with a span* $b^{1/2}$.

PROOF. Recall that a distribution μ is strictly α-stable if and only if it is strictly α-semistable with a span b for all $b > 1$. Apply Theorem 3.1 combined with Proposition 3.3. □

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