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Semi Analytical Technique For the Solution of Fractional Maxwell Fluid

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Abstract

In this article, the flow of a fractional Maxwell fluid is discussed. The velocity function and time dependent shear stress of a Maxwell fluid with fractional derivatives are calculated. It is considered that the fluid in the infinitely long circular cylinder is moving with a velocity ft. The fluid in the infinitely long circular cylinder of radius R is initially at rest and at $t = 0^+$, because of shear, it instantly starts to move longitudinally. To obtain the solutions, we have employed Laplace transformation and modified Bessel equation. The solutions are in series form which are expressed in terms of modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$, satisfy all given conditions. In this paper, Laplace inverse transformation has been calculated numerically by using MATLAB. The behavior of the following physical parameters on the flow are investigated: relaxation time, dynamic viscosity, kinematics viscosity, similarity parameters of fractional derivatives and radius of the circular cylinder. Finally, the impact of the fractional parameter and material elements is shown by graphical demonstration.

Keywords: Maxwell fluid; Velocity field; Linear shear stress; Laplace transformation; Modified Bessel function.

1 Introduction

The phenomenon in which fluid is moving in circular pipes, annulus and concentric rotating cylinders has many applications in polymer industry, food processing industry, chemical reactors and bio engineering. The non-Newtonian fluids as compared to Newtonian fluids are considered more essential and vital in technological applications. The Navier-Stokes equations, which are non-linear differential equations, are considered fundamental equations in fluid mechanics. The best way to describe the rheological properties of non-Newtonian fluids is through Navier-Stokes equations. These rheological properties includes, shear thickening, shear thinning, normal stress and stress relaxation. Many models of non-Newtonian fluids have been proposed due to their implementations in industry and engineering.

The books written by Chandrasehkar [1], Drazin and Reid [2] are considered standard for the exact solutions of non-Newtonian fluids in cylindrical domains. After the publication of these books, many papers associated to non-Newtonian fluids in cylindrical geometry have been proclaimed [3-10]. To study the rheological flow assumptions of differential type fluids by Rajagopal [11] and rate type fluids by Dunn and Rajagopal [12] have acquired acceptance of both experimentalists and theoreticians. In cylindrical geometry, the first analytical solutions corresponding to the motion of non-Newtonian fluids have been presented by Waters and King [13], Srivastava [14] and Ting [15]. The exact solutions of unsteady flow of non-Newtonian fluids in circular cylinders in the presence of shear stress on boundary is given by Bandelli [16].

Furthermore, mathematical model of Maxwell fluid with fractional derivatives has been found quite handy in polymer industry to develope transition in amorphous materials and glass states, like quartz and polystyrene. It is worthy to mention here that Palade *et al.* [17] constructed constitutive equation for an unsteady incompressible fluid which reduces to generalized Maxwell fluid model by deformations hypothesis. Friedrich [18] showed that the molecular theory of non-Newtonian fluids has been well described by converting ordinary derivatives into fractional derivatives. For the flow of incompressible viscoelastic fluids, the manipulation of fractional derivatives has been firstly presented by Germant [19]. After that, Bagley and Torvik [20] demonstrated the applications of fractional calculus in the study of viscoelastic fluids. The transient flow of a viscoelastic fluid in annulus and cylindrical pipes is examined by Wood [21]. The solution for the unidirectional flow of a viscoelastic fluid with fractional Maxwell model is analyzed by Hayat *et al.* [22]. Fetecau *et al.* [23] discussed the exact solutions for the flow of a viscoelastic fluid due to time dependent shear stress in cylindrical domain. The unsteady flow of a Maxwell fluid with fractional derivatives between two parallel plates has been presented by Makris *et al.* [24] and the flow of non-Newtonian fluids with fractional derivatives over a plane surface is given by Tan and Xu [25]. Yang and Zhu [26] produced solutions of velocity and shear stress of a fractional Maxwell model in a pipe. The flow of a Maxwell fluid model with fractional derivatives corresponding to two parallel side walls has been studied by Vieru *et al.* [27]. The exact solution of fractional Maxwell fluid related to Stokes' first problem is solved by Jamil *et al.* [28]. The movement of fractional Maxwell fluid lying inside two rotating circular cylinders is presented by Fetecau *et al.* [29].

In fluid mechanics, the system of cylindrical coordinates is adequate to solve many physical problems, including geophysical and meteorological problems, flows in rotating cavities or flows in pipes. The flows of biological fluids through arteries or veins are intensively studied in the last period. The physiological fluids, having a complex rheology, are described by the non-Newtonian models, including the Maxwell fluid model.

Many other materials used in industrial processes, such as oils and greases, polymer melts, emulsions are treated as non-Newtonian fluids. The motion of a fluid in a cylindrical domain is of interest to both theoretical and practical purposes. These types of motions of the visco-elastic fluid have applications in many industrial fields, such as in oil exploitation, chemical and food industry or bio-engineering.

In the all mentioned problems, the fluids have a complex rheology, and the ordinary models do not respond satisfactorily. In many cases, the models described by the fractional differential equations lead to more adequate results. For these reasons, the fluid models with fractional derivatives are intensively studied.

The goal of this monograph is to examined the flow of a fractional Maxwell fluid in an infinite circular cylinder. The semi analytical solutions for the velocity function and the adequate shear stress associated to the flow of a fractional Maxwell fluid in an infinitely long circular cylinder of radius R has been established. Initially the fluid in the circular cylinder is at rest and at $t = 0^+$, because of the shear stress, it starts to move longitudinally. The semi analytical solutions are procured with the tool of Laplace transformation and modified Bessel equation. The inverse Laplace transformation has been achieved through MATLAB. These solutions are presented in series form and gratify all initial and boundary conditions. At the end, profiles of velocity and adequate shear stress for the translational flow of fractional Maxwell fluid are plotted and discussed in detail.

2 Formulation of the Problem

The velocity function $\mathbf{U}$ and the extra-stress $\mathbf{E}$ for the movement of fluid [9] are considered as

$$\mathbf{U} = \mathbf{U}(r,t) = u(r,t)\mathbf{e}_z, \qquad \mathbf{E} = \mathbf{E}(r,t), \tag{1}$$

where $\mathbf{e}_z$ is the unit vector of the cylindrical coordinate system (r,θ,z) along the z-direction. Moreover, when the fluid starts to move, we have

$$\mathbf{U}(r,0) = \mathbf{0}, \qquad \mathbf{E}(r,0) = \mathbf{0}. \tag{2}$$

The balance of linear momentum, in the absence of body forces and a pressure gradient in the flow direction, lead to $E_{rr} = E_{\theta\theta} = E_{rz} = E_{\theta z} = 0$ and the governing equations related to incompressible Maxwell fluid [9]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu\frac{\partial u(r,t)}{\partial r}, \tag{3}$$

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial u(r,t)}{\partial t} = \nu\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial u(r,t)}{\partial r}, \tag{4}$$

where $\tau(r,t) = E_{r\theta}(r,t)$ is the only nontrivial shear stress, $\nu = \frac{\mu}{\rho}$ is the kinematics viscosity, μ is the dynamic viscosity and λ is the relaxation time. The Riemann-Liouville fractional differential operator [9] is defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_0^t \frac{f(\zeta)}{(t-\zeta)^\alpha}d\zeta, & 0 \le \alpha < 1; \\ \frac{df(t)}{dt}, & \alpha = 1, \end{cases} \tag{5}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

The equations governing the motion of a Maxwell fluid with fractional derivatives can be obtained. For this we replace the ordinary time derivatives in Eqs. (3) and (4) by fractional differential operator D_t^α.

$$(1 + \lambda^\alpha D_t^\alpha)\,\tau(r,t) = \mu\frac{\partial u(r,t)}{\partial r}, \tag{6}$$

$$(1 + \lambda^\alpha D_t^\alpha)\frac{\partial u(r,t)}{\partial t} = \nu\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial u(r,t)}{\partial r}. \tag{7}$$

Consider a fractional Maxwell fluid which is initially at rest in an infinitely long circular cylinder of radius R. In the presence of shear stress, after some time the cylinder starts to move. Due to this movement, the fluid is gently stirred with the velocity ft of the form shown in Fig. 1.

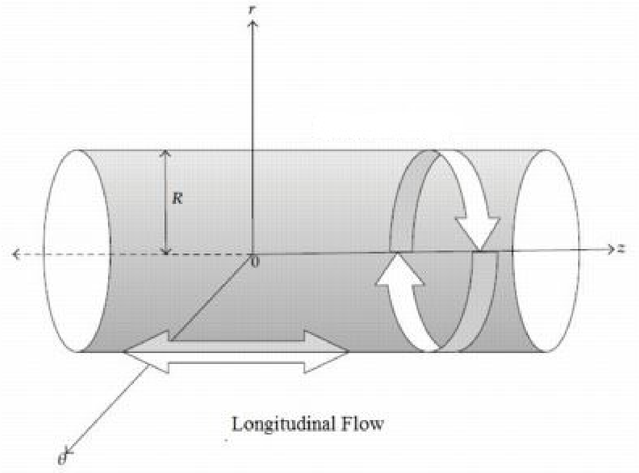

Figure 1: Geometry of the problem

The governing equations are stated by Eqs. (6) and (7) and initial and boundary conditions are,

$$u(r,0)=\left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0}=0,\qquad \tau(r,0)=0;\qquad r\in[0,R], \tag{8}$$

$$\left(1+\lambda^{\alpha}D_t^{\alpha}\right)\tau(R,t)=\mu\left.\frac{\partial u(r,t)}{\partial r}\right|_{r=R}=ft. \tag{9}$$

Eqs. (6) and (7) involves fractional derivatives, which can be solved with different methods. But the integral transforms method is organized, logical and sturdy tool to solve such equations. Here governing equations are solved by using the tool of Laplace transformation and modified Bessel equation.

3 Calculation of the Velocity Field

Taking the Laplace transformation of Eqs. (7) and (9), we have

$$\frac{\partial^2\overline{u}(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}(r,s)}{\partial r}-\frac{(s+\lambda^{\alpha}s^{\alpha+1})}{\nu}\overline{u}(r,s)=0, \tag{10}$$

$$\left.\frac{\partial\overline{u}(r,s)}{\partial r}\right|_{r=R}=\frac{f}{\mu s^2}. \tag{11}$$

Eqs. (10) and (11) can be written in suitable form as

$$\frac{\partial^2\overline{u}(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}(r,s)}{\partial r}-a(s)\overline{u}(r,s)=0, \tag{12}$$

$$\left.\frac{\partial \overline{u}(r,s)}{\partial r}\right|_{r=R} = b(s), \tag{13}$$

where

$$a(s) = \frac{(s + \lambda^{\alpha} s^{\alpha+1})}{\nu} \quad \text{and} \quad b(s) = \frac{f}{\mu s^2}.$$

By using variable transformation $m = r\sqrt{a(s)}$ in Eq. (12), we get

$$m^2 \frac{d^2\overline{u}}{dm^2} + m\frac{d\overline{u}}{dm} - (m^2 - 0^2)\overline{u} = 0. \tag{14}$$

Eq. (14) represents the modified Bessel equation and the general solution of this equation is given by the following equation

$$\overline{u}(m,s) = C_1 I_0(m) + C_2 K_0(m), \tag{15}$$

where C_1 and C_2 are constants and $I_0(m)$, $K_0(m)$ are the Modified Bessel functions of the first and second kind respectively. In order to have a finite solution at $m = 0 (r = 0)$, C_2 must be zero. Then Eq. (15) becomes

$$\overline{u}(r,s) = C_1 I_0(m), \tag{16}$$

by solving Eqs. (13) and (16) we get, the following value of C_1

$$C_1 = \frac{b(s)}{\sqrt{a(s)} I_1(R\sqrt{a})}, \tag{17}$$

introducing the value of C_1 into Eq. (16), we have

$$\overline{u}(r,s) = \frac{b(s) I_0(r\sqrt{a(s)})}{\sqrt{a(s)} I_1(R\sqrt{a(s)})}. \tag{18}$$

The expression of Eq. (18) is in complex form of modified Bessel functions of first and second kind of order zero. For the solution of velocity function, it is very difficult to obtain the inverse Laplace transform traditionally. So, we have found the inverse Laplace transform numerically through MATLAB.

4 Calculation of the Shear Stress

Taking Laplace transform of Eq. (6), we have

$$\overline{\tau}(r,s) = \frac{\mu}{(1 + \lambda^{\alpha} s^{\alpha})} \frac{\partial \overline{u}(r,s)}{\partial r}. \tag{19}$$

Differentiating Eq. (18) w.r.t 'r', we obtain

$$\frac{\partial \overline{u}(r,s)}{\partial r} = b(s)\frac{I_1(r\sqrt{a(s)})}{I_1(R\sqrt{a(s)})}, \tag{20}$$

by substituting Eq. (20) into Eq. (19), we get

$$\overline{\tau}(r,s) = b(s)\frac{\mu}{(1+\lambda^{\alpha}s^{\alpha})}\frac{I_1(r\sqrt{a(s)})}{I_1(R\sqrt{a(s)})}. \tag{21}$$

The expression of Eq. (21) is also in complex form of modified Bessel functions of order first and second kind. For the solution of shear stress, it is very difficult to obtain the inverse Laplace transform traditionally. So, we have found the inverse Laplace transform with some numerical package through MATLAB.

5 Results and Discussion

In this article, semi analytical solutions of velocity function and tangential stress of a fractional Maxwell fluid in an infinitely long circular cylinder are fixed. The tool of Laplace transformation and Modified Bessel equation are used for determining these solutions. The solutions are presented in series form in terms of Modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$. Our general solutions presented by Eqs. (18) and (21) satisfy both the governing equations and all imposed initial and boundary conditions. In the end, the reaction of different physical parameters is presented by graphs. In order to provide the validation of results, we presented in Table 1, numerical results of fluid velocity, obtained with MATLAB program and with other two numerical algorithms, namely the Stehfest's algorithm [31] and Tzou's algorithm [32]. According with Stehfest's algorithm, the inverse Laplace transform is given by

$$u(r,t) = \frac{ln2}{t}\sum_{k=1}^{N} V_k \overline{u}\left(r, \frac{ln2}{t}\right),$$

$$V_k = (-1)^{k+\frac{N}{2}} \sum_{j=\left[\frac{k+1}{2}\right]}^{min(k,\frac{N}{2})} \frac{j^{\frac{N}{2}}(2j)!}{(\frac{N}{2}-j)!j!(j-1)!(k-j)!(2j-k)!}, \tag{22}$$

where N is the number of the expansion terms and must be an even number (N=16 leads to a very good precision).

The Tzou's algorithm is based on the Riemann-sum approximation. In this method the inverse Laplace is given by

$$u(r,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\overline{u}\left(r, \frac{4.7}{t}\right) + Re\left(\sum_{k=1}^{N_1}(-1)^k \overline{u}\left(r, \frac{4.7 + k\pi i}{t}\right)\right)\right], \tag{23}$$

where Re$(\cdot)$ is the real part, i is the imaginary unit and N_1 is a natural number. The values of inverse Laplace transform obtained with MATLAB, Eqs. (22) and (23) are given in the following table. It is

Table 1: Comparison Between Different Numerical Algorithms

r	u(r,t) (MATLAB)[30]	u(r,t) (Stehfest's)[31]	u(r,t) (Tzou's)[32]
0	0.287953	0.287516	0.288339
0.05	0.299452	0.299009	0.299839
0.1	0.334179	0.333718	0.334571
0.15	0.392822	0.392336	0.393222
0.2	0.476514	0.476001	0.476926
0.25	0.586815	0.586275	0.587241
0.3	0.725680	0.725118	0.726124
0.35	0.895421	0.894843	0.895886
0.4	1.098664	1.098073	1.099153
0.45	1.338305	1.337696	1.338823
0.5	1.617478	1.616838	1.619028

observed that the results obtained with different numerical algorithms has a good agreement between them.

Finally, we have plotted some graphs for velocity function and tangential stress of the fluid by using Eqs. (18) and (21) respectively, to see the effect of various material parameters on our results. Figs. 2(a) and 2(b) depict the velocity $u(r,t)$ and shear stress $\tau(r,t)$ are directly proportional to t. Others graph have been plotted against the values of t. Figs. 3(a) and 3(b) depict the velocity $u(r,t)$ and tangential stress $\tau(r,t)$ are directly proportional to r. The effect of time required for a fluid to attain its initial position is shown in Figs. 4(a), 4(b). From 4(a) and 4(b) Figures, we conclude that velocity and tangential stress in absolute value, are opposite to it. The influence of the kinematic viscosity

ν is shown in Figs. 5(a) and 5(b), which shows that kinematics viscosity is directly proportional to both velocity field and tangential stress. Figs. 6(a) and 6(b) represents the velocity $u(r,t)$ and shear stress $\tau(r,t)$ are opposite to R. Figs. 7(a) and 7(b) shows that the velocity $u(r,t)$ and tangential stress $\tau(r,t)$ are inversely proportional to α. The impact of the dynamics viscosity ν is shown in Fig. 8(a), which shows that dynamics viscosity is also inversely proportional to velocity function.

6 Conclusions

In this paper, the flows of Maxwell fluid governed by the fractional differential equations with Caputo derivative were studied. The flow domain is the inner of a circular cylinder and flow is generated by the longitudinal stress-force given on the cylinder surface. By applying the Laplace transform with respect to the time variable t, the exact solutions for the fluid velocity and longitudinal shear stress are obtained in terms of the modified Bessel functions of first kind I_0 and I_1. Since the Laplace transforms of the velocity and of the shear stress are enough complicated, we have obtained the inverse Laplace transforms by means of the numerical procedures. Firstly, we used a MATLAB numerical code. In order to provide a validation of results, we have used other two numerical algorithms, namely the Stehfest's algorithm and Tzou's algorithm.

- As shown in Table 1, we found a good agreement between the results obtained for inverse Laplace transform with three numerical methods.
- It is important to observe that the fluid layers situated close cylinder surface have a significant motion, while the fluid situated in the central area of the cylinder has a very slow motion.
- The fluids modeled with fractional derivatives flow faster than the ordinary fluid. When the fractional parameter decreases, the fluid velocity increases.
- The shear stress has the behavior similar with velocity; therefore, it is increasing when the fractional parameter decreases.
- The physical parameters, t, r and ν are directly proportional to both velocity function and shear stress.
- The physical parameters, R, α, μ and λ are opposite to both velocity function and shear stress.

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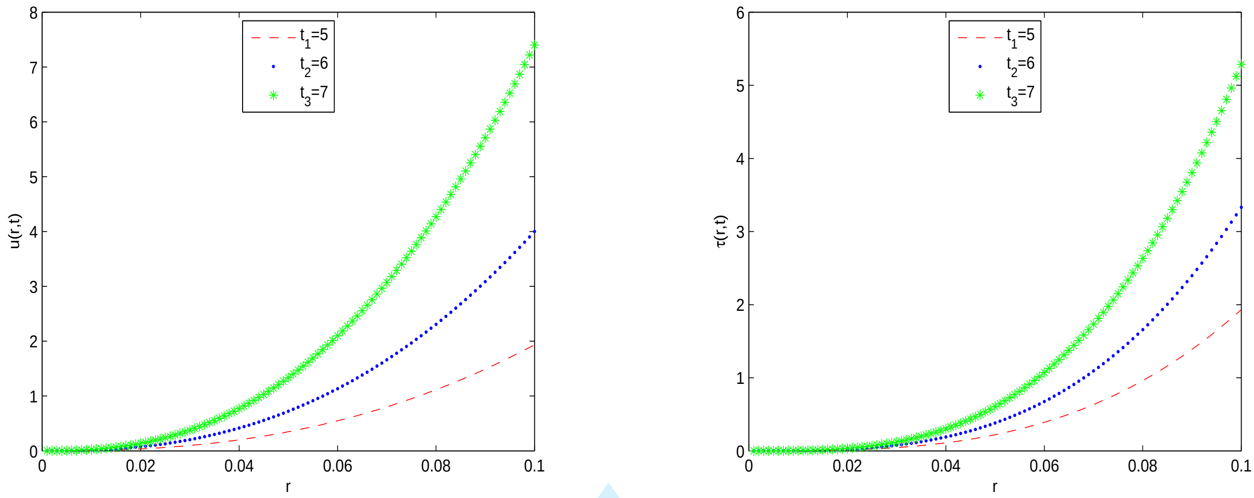
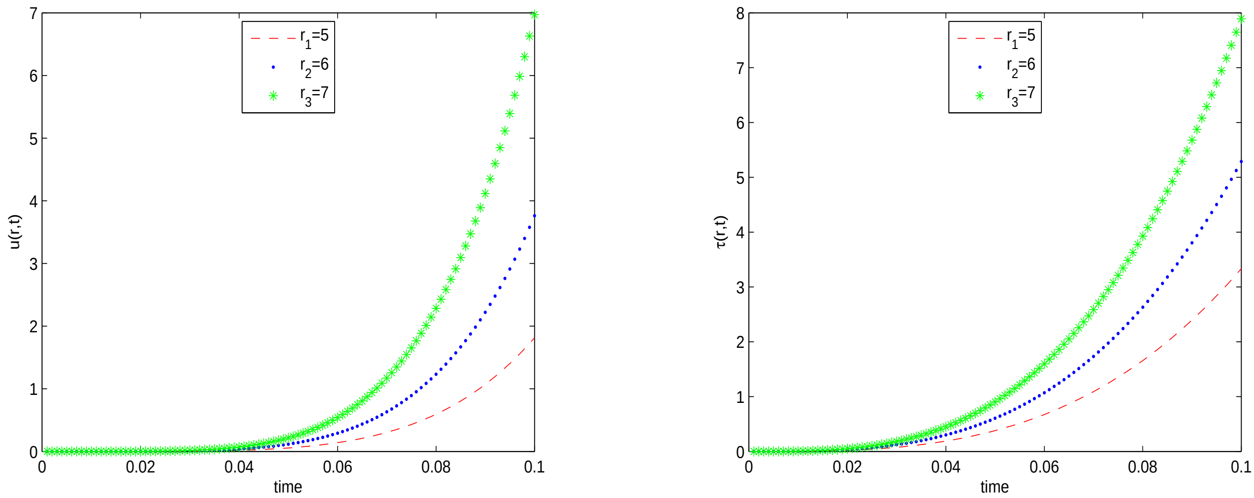


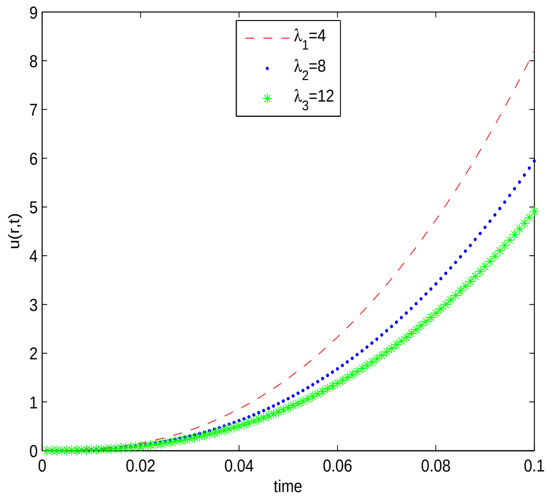
(a) Velocity (b) Shear Stress

Figure 2: Velocity $u(r,t)$ and tangential stress $\tau(r,t)$ graphs of the fluid by using material parameters $R = 0.5$, $\nu = 0.035754$, $\mu = 15$, $f = 90$, $\lambda = 5$, $\alpha = 0.5$ and various values of t.

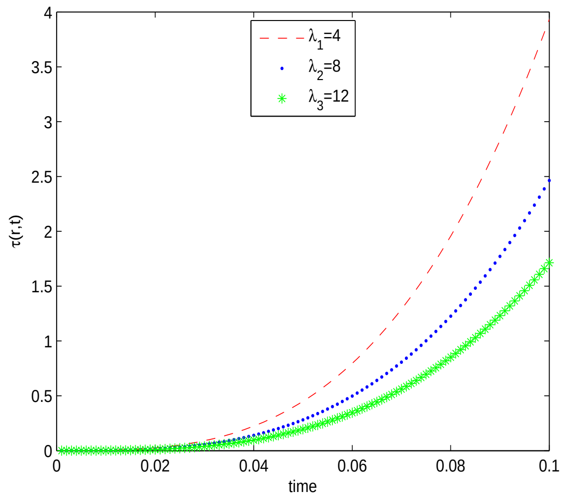


(c) Velocity (d) Shear Stress

Figure 3: Velocity $u(r,t)$ and tangential stress $\tau(r,t)$ graphs of the fluid by using material parameters $R = 0.5$, $\nu = 0.035754$, $\mu = 15$, $f = 90$, $\lambda = 5$, $\alpha = 0.5$ and various values of r.

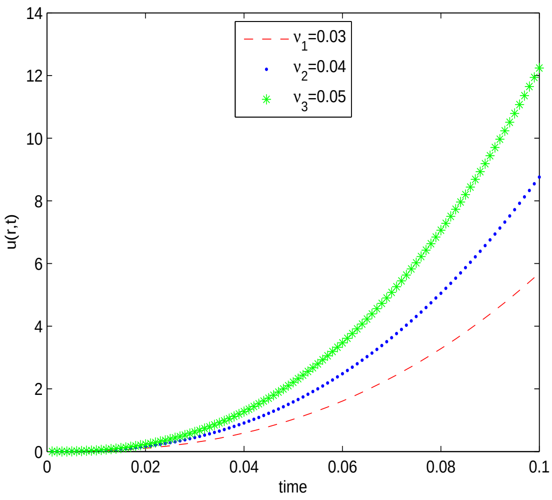


(a) Velocity

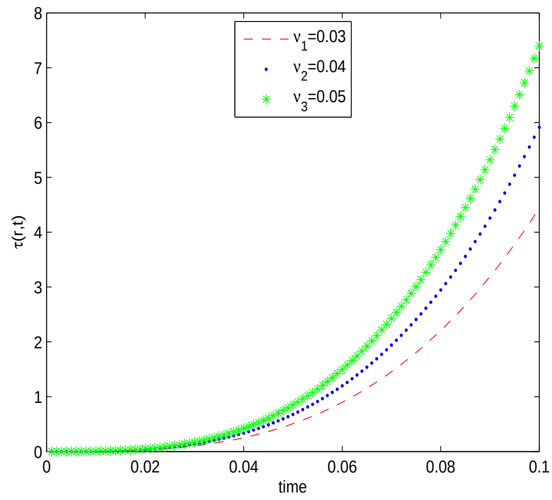


(b) Shear Stress

Figure 4: Velocity $u(r, t)$ and tangential stress $\tau(r, t)$ graphs of the fluid by using material parameters $R = 0.5$, $\nu = 0.035754$, $\mu = 15$, $f = 90$, $\alpha = 0.5$, $r = 5$ and various values of λ.

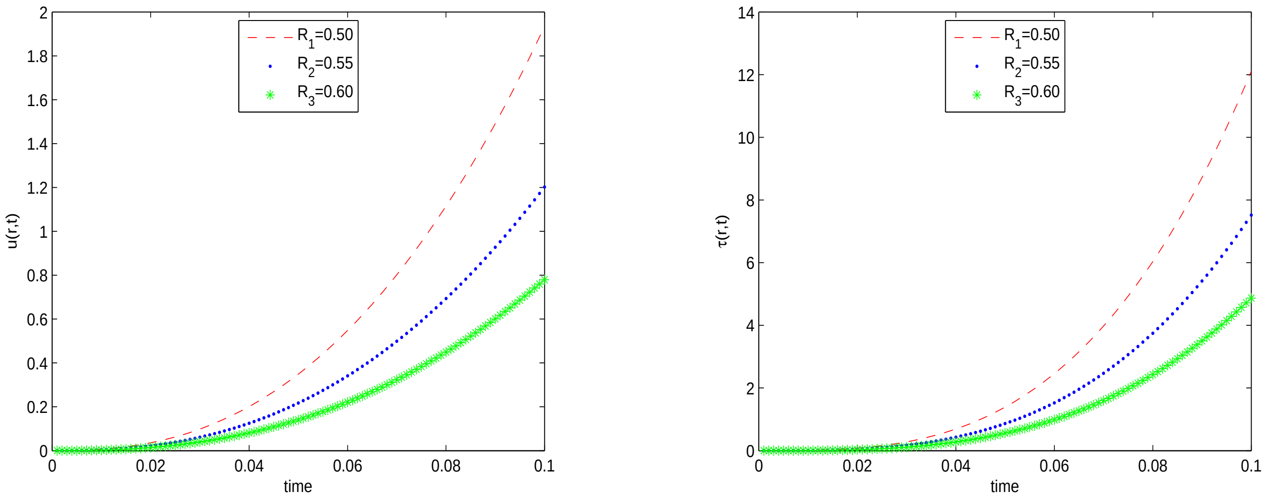


(a) Velocity



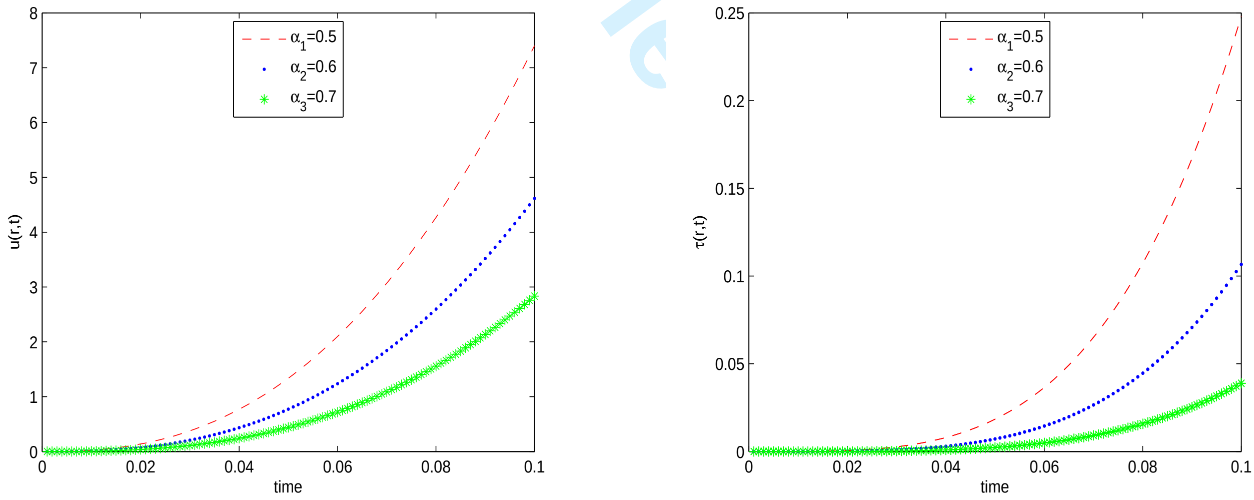
(b) Shear Stress

Figure 5: Velocity $u(r, t)$ and tangential stress $\tau(r, t)$ graphs of the fluid by using material parameters $R = 0.5$, $\mu = 15$, $f = 90$, $\lambda = 5$, $\alpha = 0.5$, $r = 5$ and various values of ν.



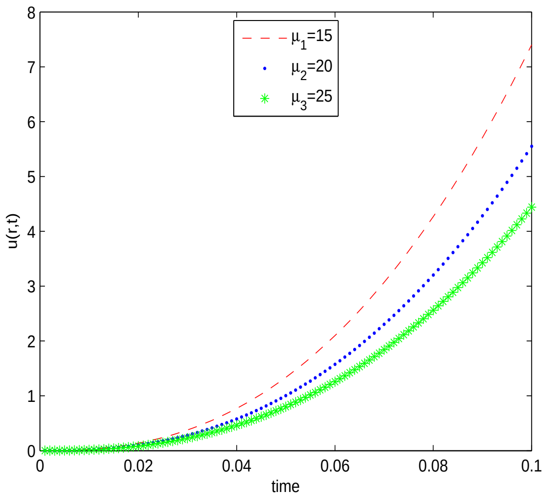
(a) Velocity (b) Shear Stress

Figure 6: Velocity $u(r,t)$ and tangential stress $\tau(r,t)$ graphs of the fluid by using material parameters $\nu = 0.035754$, $\mu = 15$, $f = 90$, $\lambda = 5$, $\alpha = 0.5$, $r = 5$ and various values of R.



(a) Velocity (b) Shear Stress

Figure 7: Velocity $u(r,t)$ and tangential stress $\tau(r,t)$ graphs of the fluid by using material parameters $R = 0.5$, $\nu = 0.035754$, $\mu = 15$, $f = 90$, $\lambda = 5$, $r = 5$ and various values of α.



(a) Velocity

Figure 8: Velocity $u(r,t)$ graph of the fluid by using material parameters $R = 0.5$, $\nu = 0.035754$, $f = 90$, $\lambda = 5$, $\alpha = 0.5$, $r = 5$ and various values of μ.

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