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SEMI-CLASSICAL ANALYSIS ON H-TYPE GROUPS

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ABSTRACT. In this paper, we develop a semi-classical analysis on H-type groups. We define semi-classical pseudodifferential operators, prove the boundedness of their action on square integrable functions and develop a symbolic calculus. Then, we define the semi-classical measures of bounded families of square integrable functions which consists of a pair formed by a measure defined on the product of the group and its unitary dual, and by a field of trace class positive operators acting on the Hilbert spaces of the representations. We illustrate the theory by analyzing examples, which show in particular that this semi-classical analysis takes into account the finite-dimensioned representations of the group, even though they are negligible with respect to the Plancherel measure.

1. INTRODUCTION

This article is dedicated to Jean-Yves Chemin for his 60-th birthday. With this article devoted to the Heisenberg-type groups (H-type groups), the authors wish to thank Jean-Yves Chemin for his inspiring works on the Heisenberg groups (see [9, 10, 11, 12]), especially the recent articles [6, 7] and the monograph [8]. The first author will not forget her first discovery of the cotangent space, with Jean-Yves Chemin's explanations, when she was a second-year bachelor, together with numerous stimulating discussions, not always about mathematics. This article is an attempt to celebrate these friendly moments.

The H-type groups are connected simply connected Lie group G which are stratified of step 2 as the Heisenberg group. There are simple relationships between the dual of the centrum and the first strata which makes them very good toys-model for testing ideas and building examples. Our aim here is to develop semi-classical analysis on such groups by introducing semi-classical pseudo-differential operators and semi-classical Wigner measures in the spirit of previous works [13, 14, 28, 26].

Semi-classical pseudodifferential operators are basic tools for semi-classical analysis, and their development is related with this of microlocal analysis. Microlocal analysis (which uses classical pseudo-differential operators, without small parameter) was mainly developed in the 70's. The funding idea is to study phenomenon simultaneously in standard and Fourier variables, which correspond to position and impulsion variables, the phase space variables of quantum mechanics. Therefore, with such a view point, it is crucial to be equipped with localization operators in position and impulsion, and that is what makes possible the pseudodiffrential operators introduced by Shubin [45] and systematically studied by Hörmander [36]. In his annotated bibliography, Bernard Helffer [34] dates semi-classical earliest results to André Voros thesis where one sees the first use of pseudodifferential calculus in a semi-classical context. The question is then to use microlocal technics on problems where a small parameter is present and to analyze microlocal properties when this small parameter goes to 0.

The transposition of the aforementioned point of view in the setting of Lie groups has been a subject of investigation where microlocal approach or phase-space analysis, has been developed in the context of the Heisenberg group (see [15, 10, 9] for example). Such strategies are faced to the difficulty of the operator structure of the Fourier transform and require to use a non-commutative setting. Various attempts have been made to construct a pseudodifferential calculus since the 80's. One can cite the pioneer works of [48, 16, 30, 18]. We also refer to [44] on compact Lie groups and to the introductions of [13, 28] for an overview on the subject. More recently, one of the authors and

her colleagues have developed such a calculus in the context of the Heisenberg group in [13]. The full program has been achieved by the second author in [28] where pseudodifferential operators are defined on graded Lie groups. We define here semi-classical operators in the framework of H-type groups, using the general theory developed in the preceding references and we develop semi-classical measures in this very context, as we shall see in the next sections.

Semi-classical measures, also called Wigner measures, were used throughout the 90's, in particular in articles [40] and [32] (see also [31] and [33]). They have opened an elegant path to study the compactness defect of sequences of functions which have a special size of oscillations. For that reason, they have contributed to prove important results on the density of families of eigenfunctions of the Laplacian on the torus (cf. [4] and [5]); it is indeed possible to link semi-classical measures with weak limits of densities of wave functions. The authors think that, once suitably generalized in the framework of Lie groups, these technics should be useful for studying quantum ergodicity in sub-Riemaninan geometries, in the spirit of the recent result [20].

Of course, the approach developed here adapts to more general graded Lie groups. However, we choose the simple case of H-type groups in order to illustrate that, in this simple example, the microlocal approach developed here takes into account tricky questions linked for example with representations. Indeed, despite the fact that finite dimensional representations are of null measure with respect to the Plancherel measure, they are taken into account by the averaging process implemented in the computation of semi-classical measures. We illustrate this fact with examples which should answer to the questions of interest to Fulvio Ricci and asked by Bernard Helffer to the authors. These examples also illustrate the large complexity of the Fourier space in H-type groups, which has been the subject of [7] and of numerous recent conversations of the second author with Jean-Yves Chemin late in the eve of board meetings at the CIRM cafeteria. We hope he will appreciate the examples we develop in Section 5, where we see the representations "above $\lambda = 0$" emerging from a semi-classical averaging process.

In the next section of this paper, we recall basic facts about H-type groups. Then, we introduce the algebra of semi-classical pseudo-differential operators and describe its main properties, we define semi-classical measures and introduce a notion of Wigner distributions. Finally, we analyze the semi-classical measures of several families of functions, emphasizing concentration effects in space variables, and oscillations properties that we interpret as concentration in Fourier variables, including concentration on finite-dimensional representations.

2. H-TYPE GROUPS

A connected simply connected Lie group G is said to be stratified of step 2 if its left-invariant Lie algebra $\mathfrak{g}$ (assumed real-valued and of finite dimension n) is endowed with a vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z} \,,$$

such that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Via the exponential map

$$\mathrm{Exp} : \mathfrak{g} \to G$$

which is in that case a diffeomorphism from $\mathfrak{g}$ to G, one identifies the sets of G and $\mathfrak{g}$ with the underlying vector space. It turns out that under this identification, the group law on G (which is generally not commutative) provided by the Campbell-Baker-Hausdorff formula, $(x, y) \mapsto xy$ is a polynomial map. More precisely, if $x = \mathrm{Exp}(v_x + z_x)$ and $y = \mathrm{Exp}(v_y + z_y)$ then

$$xy = \mathrm{Exp}(v + z), \ \ v = v_x + v_y \in \mathfrak{v}, \ \ z = z_x + z_y + \frac{1}{2}[v_x, v_y] \in \mathfrak{z}.$$

If $x = \mathrm{Exp}(v)$ then $x^{-1} = \mathrm{Exp}(-v)$.

For any $\lambda \in \mathfrak{z}^\star$ (the dual of the center $\mathfrak{z}$) we define a skew-symmetric bilinear form on $\mathfrak{v}$ by

(2.1)
$$\forall U, V \in \mathfrak{v}, \quad B(\lambda)(U,V) := \lambda([U,V]).$$

Following [37], we say that G is of H-type (or Heisenberg type) if, once fixed an inner product on $\mathfrak{v}$ and on $\mathfrak{z}$, and in any orthonormal basis of $\mathfrak{v}$, the endomorphism of this skew symmetric form (that we still denote by $B(\lambda)$) satisfies

$$\forall \lambda \in \mathfrak{z}^*, \quad B(\lambda)^2 = -|\lambda|^2 \mathrm{Id}_{\mathfrak{v}}.$$

This implies in particular that the dimension of $\mathfrak{v}$ is even. We set

$$\dim \mathfrak{v} = 2d, \quad \dim \mathfrak{z} = p.$$

Naturally, a Heisenberg group is an H-type group; more precisely, an H-type group is Heisenberg if and only if the dimension of the centre p is equal to 1. Recall that the Heisenberg group $\mathbb{H}_d$ is the set $\mathbb{R}^{2d+1}$ equipped with the following product

$$hh' = (p + p', q + q', z + z' + \frac{1}{2}(pq' - p'q), \quad \text{where} \quad h = (p, q, z), h' = (p', q', z') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R};$$

here pq' denotes the standard inner product of the two vectors $p, q' \in \mathbb{R}^d$. Its Lie algebra is the vector space $\mathbb{R}^{2d+1}$ equipped with the following Lie bracket

$$[(p,q,z), (p',q',z')] = (0, 0, pq' - p'q).$$

The first stratum and the centre are $\mathfrak{v} = \mathbb{R}^d \times \mathbb{R}^d \times \{0\} \sim \mathbb{R}^d \times \mathbb{R}^d$ and $\mathfrak{z} = \{0\} \times \{0\} \times \mathbb{R}$. For $\lambda \in \mathfrak{z}^*$, the skew-symmetric bilinear form on $\mathfrak{v}$ is

(2.2)
$$B(\lambda)((p,q),(p',q')) = \lambda \begin{pmatrix} p \\ q \end{pmatrix}^t J \begin{pmatrix} p' \\ q' \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

We fix an inner product on $\mathfrak{z}$, this allows us to consider the Lebesgue measure $dv\,dz$ on $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$. Via the identification of G with $\mathfrak{g}$ via the exponential map, this induces a Haar measure dx on G. This measure is invariant under left and right translations:

$$\forall f \in L^1(G, d\mu), \quad \forall x \in G, \quad \int_G f(y)dy = \int_G f(xy)dy = \int_G f(yx)dy.$$

Note that the convolution of two functions f and g on G is given by

(2.3)
$$f * g(x) := \int_G f(xy^{-1})g(y)dy = \int_G f(y)g(y^{-1}x)dy,$$

and as in the Euclidean case we define Lebesgue spaces by

$$\|f\|_{L^q(G)} := \left(\int_G |f(y)|^q \, dy \right)^{\frac{1}{q}},$$

for $q \in [1, \infty)$, with the standard modification when $q = \infty$.

Since G is stratified, there is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we can decompose X as $X = V + Z$ with $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, then

$$\delta_t X := tV + t^2 Z.$$

This allows us to define the dilation on the Lie group G via the identification by the exponential map:

$$
\begin{array}{ccc}
\mathfrak{g} & \overset{\delta_t}{\to} & \mathfrak{g} \\
\mathrm{Exp} \downarrow & & \downarrow \ \mathrm{Exp} \\
G & \underset{\mathrm{Exp} \circ \delta_t \circ \mathrm{Exp}^{-1}}{\to} & G
\end{array}
$$

To avoid heavy notations, we shall still denote by δ_t the map $\mathrm{Exp} \circ \delta_t \circ \mathrm{Exp}^{-1}$. The dilations δ_t, $t > 0$, on $\mathfrak{g}$ and G form a one-parameter group of automorphisms of the Lie algebra $\mathfrak{g}$ and of the group G. The Jacobian of the dilation δ_t is t^Q where

$$Q := \dim \mathfrak{v} + 2\dim \mathfrak{z} = 2d + 2p$$

is called the homogeneous dimension of G:

$$(2.4) \qquad \int_G f(\delta_t \, y) \, dy = t^{-Q} \int_G f(y) \, dy \, .$$

We may identify $\mathfrak{g}$ with the space of left-invariant vector field via

$$Xf = \frac{d}{dt} f(\mathrm{Exp}(tX)) \bigg|_{t=0} \, .$$

A differential operator T on G (and more generally any operator T defined on $C_c^\infty(G)$ and valued in the distributions of $G \sim \mathbb{R}^{2d+p}$) is said to be homogeneous of degree ν (or ν-homogeneous) when

$$T(f \circ \delta_t) = t^\nu(Tf) \circ \delta_t.$$

For instance, a left invariant vector field in $\mathfrak{v}$ is 1-homogeneous while a left-invariant vector field in $\mathfrak{z}$ is 2-homogeneous.

We can define the Schwartz space $\mathcal{S}(G)$ as the set of smooth functions on G such that for all α, β in $\mathbb{N}^{2d+p}$, the function $x \mapsto x^\beta \mathcal{X}^\alpha f(x)$ belongs to $L^\infty(G)$, where $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left invariant vector fields forming a basis of $\mathfrak{g}$ and x^β a product of $|\beta|$ coordinate functions on $G \sim \mathfrak{v} \times \mathfrak{z}$. The Schwartz space $\mathcal{S}(G)$ has properties very similar to those of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d+p})$, particularly density in Lebesgue spaces.

2.1. **The Fourier transform.** The group G being non commutative, its Fourier transform is defined by means of irreducible unitary representations.

2.1.1. *Irreducible unitary representations.* We assume $\lambda \in \mathfrak{z}^* \backslash \{0\}$ and use the skew-symmetric bilinear form defined on (2.1). One can find an orthonormal basis $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$ where $B(\lambda)$ is represented by the matrix $|\lambda| J$, that is,

$$(2.5) \qquad B(\lambda)(U, V) = |\lambda| U^t J V,$$

for two vector $U, V \in \mathfrak{v}$ written in the $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$-basis; recall that the matrix J was defined in (2.2). For example, on the Heisenberg group $\mathbb{H}_d$, in view of (2.2), the canonical coordinates (p, q) yield an orthonormal basis where (2.5) holds for $\lambda > 0$; however, for $\lambda < 0$ this needs modifying.

We decompose $\mathfrak{v}$ in a λ-depending way as $\mathfrak{v} = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda$ with

$$\mathfrak{p} := \mathfrak{p}_\lambda := \mathrm{Span}\left(P_1, \ldots, P_d\right), \quad \mathfrak{q} := \mathfrak{q}_\lambda := \mathrm{Span}\left(Q_1, \ldots, Q_d\right).$$

We shall denote by $p = (p_1, \cdots, p_d)$ the coordinates of P on the vector basis $(P_1, \cdots, P_d)$, by $q = (q_1, \cdots, q_d)$ those of Q on $(Q_1, \cdots, Q_d)$ and by $z = (z_1, \cdots, z_p)$ those of Z on a basis $(Z_1, \cdots, Z_p)$. We will often use the writing of an element $x \in G$ or $X \in \mathfrak{g}$ as

$$(2.6) \qquad x = \mathrm{Exp}(X), \qquad X = p_1 P_1 + \ldots + p_d P_d \ + \ q_1 Q_1 + \ldots + q_d Q_d \ + \ z_1 Z_1 + \ldots + z_p Z_p.$$

We introduce irreducible unitary representations π_x^λ of G on $L^2(\mathfrak{p}_\lambda)$ by

$$(2.7) \qquad \pi_x^\lambda \Phi(\xi) = e^{i\lambda(z) + \frac{i}{2}|\lambda| \, pq + i\sqrt{|\lambda|} \, \xi q} \Phi\left(\xi + \sqrt{|\lambda|} p\right),$$

where x has been written as in (2.6). Note that, setting

$$(2.8) \qquad T_r \Phi(\xi) = r^d \Phi\left(r\xi_1, \cdots, r\xi_d\right), \ \ \forall \Phi \in L^2(\mathfrak{p}_\lambda),$$

we obtain for each $r > 0$ a unitary operator T_r on $L^2(\mathfrak{p}_\lambda)$ and it satisfies $T_r^* = T_{1/r} = T_r^{-1}$. Furthermore, intertwining π^λ with T_r when $r = \sqrt{|\lambda|}$ yields

$$(2.9) \qquad T_{\sqrt{|\lambda|}}\pi_x^\lambda T_{\sqrt{|\lambda|}}^* \Phi(\xi) := \mathrm{e}^{i\lambda(Z + [\xi + \frac{1}{2}P, Q])}\Phi(\xi + P), \quad \forall \Phi \in L^2(\mathfrak{p}_\lambda).$$

Remark 2.1 (Link with the Wigner transform). *Identifying $\mathfrak{p}_\lambda \sim \mathbb{R}^d$, For $f, g \in L^2(\mathbb{R}^d)$, we have*

$$\left(\pi_x^\lambda f, g\right) = \mathrm{e}^{i\lambda(z)} W[f,g]\left(\sqrt{|\lambda|}p, \sqrt{|\lambda|}q\right),$$

where for $p, q \in \mathbb{R}^d$ and $\varepsilon > 0$

$$W[f,g](p,q) = \int_{\mathbb{R}^d} \mathrm{e}^{iqv} f\left(v + \frac{1}{2}p\right) \, \overline{g}\left(v - \frac{1}{2}p\right) dv.$$

The representations π^λ, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, are infinite dimensional. There also exist other unitary irreducible representations of G. They are given by the characters of the first stratum in the following way: for every $\omega \in \mathfrak{v}^*$, we set

$$\pi_x^{0,\omega} = \mathrm{e}^{i\omega(V)}, \quad x = \mathrm{Exp}(V + Z) \in G, \quad \text{with } V \in \mathfrak{v} \text{ and } Z \in \mathfrak{z},$$

and this defines a 1-dimensional representation $\pi^{0,\omega}$ of G. Up to unitary equivalence, there are no other unitary irreducible representations of G than π^λ, $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and $\pi^{0,\omega}$, $\omega \in \mathfrak{v}^*$. This can be proved in two different ways.

The first way is to consider an irreducible unitary (non-trvial) representation π of G, to quotient the group and the representation by $\ker \pi$ and to compare the latters with the well-known representations of $\mathbb{H}_d$ and with the characters of $\mathbb{R}^{2d}$. Indeed, the restriction of π to the center $Z = \mathrm{Exp}(\mathfrak{z})$ of G must be a character of Z (as $\pi(z)$, $z \in Z$, intertwines π), so it is of the form $\pi(\mathrm{Exp}(z)) = e^{i\lambda(z)}$ for some $\lambda \in \mathfrak{z}^*$. If $\lambda = 0$, then one can see that $G/\ker \pi$ is isomorphic to an abelian subgroup of $\mathfrak{v} \sim \mathbb{R}^{2d}$ and that the quotient of π is therefore a character $e^{i\omega(\cdot)}$ of this group. If $\lambda \neq 0$, then one uses that $G/\ker \pi$ is isomorphic to the Heisenberg group $\mathbb{H}_d$ and that the quotient of π by $\ker \pi$ becomes via this isomorphism a representation of $\mathbb{H}_d$ whose restriction on the centre is $e^{i\lambda(\cdot)}$; this characterises the λ-Schrödinger representation of $\mathbb{H}_d$ by the Stone - Von Neumann Theorem.

The second way is to use Kirillov's theory. Note that this theory is shown recursively using quotients by kernels and eventually the Stone - Von Neumann Theorem; so, although Kirillov's theory is a general and powerful tool, the two proofs are in fact readily equivalent for groups of Heisenberg type. By Kirillov's theory, the set of unitary irreducible representations modulo unitary equivalence, that we shall denote by $\widehat{G}$, is in bijection with the orbit of $\mathfrak{g}^*$ under the co-adjoint action of G. For instance, one can compute that $\lambda \in \mathfrak{z}^* \setminus \{0\}$ corresponds to the class of π^λ while $\omega \in \mathfrak{v}^*$ corresponds to the class of $\pi^{0,\omega}$. Simple calculations show that any two co-adjoint orbits of each $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and also of each $\omega \in \mathfrak{v}^*$ are disjoint and that the (disjoint) union of all these orbits is $\mathfrak{g}^*$. Consequently, $\widehat{G}$, can be parametrized by $(\mathfrak{z}^* \setminus \{0\}) \sqcup \mathfrak{v}^*$:

$$(2.10) \qquad \widehat{G} = \{\text{class of } \pi^\lambda \; : \; \lambda \in \mathfrak{z}^* \setminus \{0\}\} \sqcup \{\text{class of } \pi^{0,\omega} \; : \; \omega \in \mathfrak{v}^*\}.$$

One can think of $\mathfrak{v}^*$ as a sheet above $\lambda = 0$. The case of $\lambda = 0$ and $\omega = 0$ corresponds to the trivial representation of G.

2.1.2. *The Fourier transform.* In contrast with the Euclidean case, the Fourier transform is defined on $\widehat{G}$ and is valued in the space of bounded operators on $L^2(\mathbb{R}^d)$. More precisely, the Fourier transform of a function f in $L^1(G)$ is defined as follows: for any $\lambda \in \mathfrak{z}^*$, $\lambda \neq 0$,

$$\widehat{f}(\lambda) := \mathcal{F}(f)(\lambda) := \int_G f(x) \left(\pi_x^\lambda\right)^* \, dx\,.$$

Note that for any $\lambda \in \mathfrak{z}^*$, $\lambda \neq 0$, we have $\left(\pi_x^\lambda\right)^* = \pi_{x^{-1}}^\lambda$ and the map π_x^λ is a group homomorphism from G into the group $U(L^2(\mathfrak{p}_\lambda))$ of unitary operators of $L^2(\mathfrak{p}_\lambda)$ (often identified with $L^2(\mathbb{R}^d)$). Therefore, the function f in $L^1(G)$ has a Fourier transform $(\mathcal{F}(f)(\lambda))_\lambda$ which is a bounded family of bounded operators on $L^2(\mathfrak{p}_\lambda)$ with uniform bound:

(2.11) $$\|\mathcal{F}f(\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} \leq \int_G |f(x)| \|(\pi_x^\lambda)^*\|_{\mathcal{L}(L^2(G))} dx = \|f\|_{L^1(G)}.$$

since the unitarity of π^λ implies $\|(\pi_x^\lambda)^*\|_{\mathcal{L}(L^2(G))} = 1$.

The Fourier transform can be extended to an isometry from $L^2(G)$ onto the Hilbert space of families $A = \{A(\lambda)\}_{(\lambda) \in \mathfrak{z}^* \setminus \{0\}}$ of operators on $L^2(\mathfrak{p}_\lambda)$ which are Hilbert-Schmidt for almost every $\lambda \in \mathfrak{z}^* \setminus \{0\}$, with $\|A(\lambda)\|_{HS(L^2(\mathfrak{p}_\lambda))}$ measurable and with norm

$$\|A\| := \left(\int_{\mathfrak{z}^* \setminus \{0\}} \|A(\lambda)\|^2_{HS(L^2(\mathfrak{p}_\lambda))} |\lambda|^d \, d\lambda\right)^{\frac{1}{2}} < \infty\,.$$

We have the following Fourier-Plancherel formula:

(2.12) $$\int_G |f(x)|^2 \, dx = c_0 \int_{\mathfrak{z}^* \setminus \{0\}} \|\mathcal{F}(f)(\lambda)\|^2_{HS(L^2(\mathfrak{p}_\lambda))} |\lambda|^d \, d\lambda\,,$$

where $c_0 > 0$ is a computable constant. This yields an inversion formula for any Schwartz function $f \in \mathcal{S}(G)$ and $x \in G$:

(2.13) $$f(x) = c_0 \int_{\mathfrak{z}^* \setminus \{0\}} \text{Tr}\left(\pi_x^\lambda \mathcal{F}f(\lambda)\right) |\lambda|^d \, d\lambda\,,$$

where Tr denotes the trace of operators in $\mathcal{L}(L^2(\mathfrak{p}_\lambda))$. The inversion formula makes sense since for $f \in \mathcal{S}(G)$, the operators $\mathcal{F}f(\lambda)$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, are trace-class and $\int_{\mathfrak{z}^* \setminus \{0\}} \text{tr} \left|\mathcal{F}f(\lambda)\right| |\lambda|^d \, d\lambda$ is finite.

Usually, the Fourier transform of a locally compact group G would be defined on $\widehat{G}$, the set of unitary irreducible representations of G modulo equivalence, via

$$\widehat{f}(\pi) = \mathcal{F}(f)(\pi) = \int_G f(x)\pi(x)^* dx,$$

for a representation π of G, and then considering the unitary equivalence we obtain a measurable field of operators $\mathcal{F}(f)(\pi)$, $\pi \in \widehat{G}$. Here, the Plancherel measure is supported in the subset $\{\text{class of } \pi^\lambda : \lambda \in \mathfrak{z}^* \setminus \{0\}\}$ of $\widehat{G}$ (see (2.10)) since it is $c_0|\lambda|^d d\lambda$. This allows us to identify $\widehat{G}$ and $\mathfrak{z}^* \setminus \{0\}$ when considering measurable objects up to null sets for the Plancherel measure. However, our semiclassical analysis will lead us to consider objects which are also supported in the other part of $\widehat{G}$. For this reason, we also set for $\omega \in \mathfrak{v}^*$ and $f \in L^1(G)$:

$$\widehat{f}(0, \omega) = \mathcal{F}(f)(0, \omega) := \int_G f(x)(\pi_x^{(0,\omega)})^* dx$$

$$= \int_{\mathfrak{v} \times \mathfrak{z}} f(\text{Exp}(V + Z))e^{-i\omega(V)} dV \, dZ.$$

The Fourier transform sends the convolution, whose definition is recalled in (2.3), to composition in the following way:

$$(2.14) \qquad \mathcal{F}(f \star g)(\lambda) = \mathcal{F}(g)(\lambda) \, \mathcal{F}(f)(\lambda).$$

Other conventions for the convolution or in having (or not) the adjoint $\left(\pi_x^\lambda\right)^*$ in the formula for the Fourier transform would lead to obtaining $\mathcal{F}(f)(\lambda) \, \mathcal{F}(g)(\lambda)$. However, we made the consistent choices of privileging left objects (e.g. in our choice of convolution and identification of the Lie algebra $\mathfrak{g}$ with the Lie algebra of left invariant vector fields). The choice of considering the adjoint in the Fourier transform is natural from an analytical viewpoint. A first reason is to extend the case of the Euclidean Fourier transform on the abelian group $\mathbb{R}^n$ where the formula usually contains $e^{-ix\xi}$. A deeper reason is that given our choice of convolution, it is usual to consider right convolution operators since they are invariant under (i.e. commute with) left translations. For such an operator T with, say, integrable convolution kernel $\kappa \in L^1(G)$, this means that $Tf = f * \kappa$ and we have $\mathcal{F}(Tf) = \mathcal{F}(\kappa) \, \mathcal{F}f$ by (2.14); in other words, T is a Fourier multipliers with Fourier symbol $\mathcal{F}(\kappa)$ acting on the left (and not the right) of $\mathcal{F}f$, and this setting seems quite natural. Note that the formula $\mathcal{F}(Tf) = \mathcal{F}(\kappa) \, \mathcal{F}f$ implies

$$(2.15) \qquad \|T\|_{\mathcal{L}(L^2(G))} = \sup_{\lambda \in \widehat{G}} \|\mathcal{F}\kappa(\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))},$$

where the supremum here is in fact the essential supremum with respect to the Plancherel measure, justifying the identification of the element in $\widehat{G}$ with $\lambda \in \mathfrak{z}^*$.

With our conventions, we also have

$$(2.16) \qquad \mathcal{F}(Xf)(\pi) = \pi(X)\mathcal{F}(f)(\pi)$$

where $\pi(X)$ is the infinitesimal representation of π at $X \in \mathfrak{g}$,

$$(2.17) \qquad \text{i.e. } \pi(X) = \frac{d}{dt}\pi(\mathrm{Exp}(tX))|_{t=0};$$

(the class of) π is equal to (the class of) π^λ or $\pi^{(0,\omega)}$ identified with λ or ω respectively. For instance, we have for $Z \in \mathfrak{z}$ identified with a left-invariant vector field

$$(2.18) \qquad \mathcal{F}(Zf)(\lambda) = i\lambda(Z)\mathcal{F}(f)(\lambda), \quad \text{or in other words} \quad \widehat{Z}(\lambda) = i\lambda(Z).$$

The infinitesimal representation of π extends to the universal enveloping Lie algebra of $\mathfrak{g}$ that we identify with the left invariant differential operators on G. Then for such a differential operator T we have $\mathcal{F}(Tf)(\pi) = \pi(T)\mathcal{F}(f)(\pi)$ and we may write $\pi(T) = \mathcal{F}(T)$. For instance, if as before $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left invariant vector fields forming a basis of $\mathfrak{g}$, then

$$\mathcal{F}(\mathcal{X}^\alpha f)(\pi) = \pi(\mathcal{X})^\alpha \mathcal{F}(f)(\pi) \text{ and } \mathcal{F}(\mathcal{X}^\alpha) = \mathcal{F}(\mathcal{X})^\alpha.$$

The reader can refer to [6, 7, 39] for another insight on the Fourier transform in the Heisenberg group and, more generally on 2-step Lie groups (which includes the H-group considered here).

2.2. **The sublaplacian.** The sublaplacian on G is defined by

$$\Delta_G := \sum_{j=1}^{2d} V_j^2,$$

where the V_j's form an orthonormal basis of $\mathfrak{v}$. One checks easily that Δ_G is a differential operator which is left invariant and homogeneous of degree 2.

The definition of Δ_G is independent of the chosen orthonormal basis for $\mathfrak{v}$ - although it depends on the scalar product that we have fixed at the very beginning on $\mathfrak{v}$. In particular, choosing the basis fixed in Section 2.1 for any $\lambda \in \mathfrak{z}^* \setminus \{0\}$ we have $\Delta_G = \sum_{j=1}^d (P_j^2 + Q_j^2)$. This allows us to compute its infinitesimal representation at π^λ:

$$(2.19) \qquad \mathcal{F}(-\Delta_G)(\lambda) = H(\lambda),$$

where $H(\lambda)$ is the diagonal operator defined on $L^2(\mathbb{R}^d)$ by

(2.20)
$$H(\lambda) = |\lambda| \sum_{1 \le j \le d} \left(-\partial_{\xi_j}^2 + \xi_j^2 \right).$$

The eigenfunctions of $H(\lambda)$ express in terms of the basis of Hermite functions $(h_n)_{n \in \mathbf{N}}$, normalized in $L^2(\mathbb{R})$ and satisfying for all real numbers ξ:

$$-h_n''(\xi) + \xi^2 h_n(\xi) = (2n+1)h_n(\xi).$$

Indeed, for each multi-index $\alpha \in \mathbb{N}^d$, the function h_α defined by

$$h_\alpha(\xi) := \prod_{j=1}^d h_{\alpha_j}(\xi_j), \ \ \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

is an eigenfunction of $H(\lambda)$, that is, $H(\lambda)h_\alpha = \zeta(\alpha, \lambda)h_\alpha$, for the eigenvalue

$$\zeta(\alpha, \lambda) := |\lambda| \sum_{j=1}^d (2\alpha_j + 1) = |\lambda|(2|\alpha| + d), \quad \alpha \in \mathbb{N}^d.$$

These eigenvalues describe the entire spectrum of $H(\lambda)$.

We compute easily the infinitesimal representation (or Fourier transform) of Δ_G on the rest of $\widehat{G}$: at $\pi^{(0,\omega)}$, it is the number

$$\mathcal{F}(-\Delta_G)(0, \omega) = |\omega|^2.$$

The eigenvalues for the Fourier transform of the sub-laplacian allow us to link the infinite dimensional representations π^λ and the dimension 1 ones $\pi^{0,\omega}$:

Lemma 2.2. *Let $(\lambda_n)_{n \in \mathbb{N}}$ and $(\alpha_n)_{n \in \mathbb{N}}$ be sequences in $\mathfrak{z}^* \setminus \{0\}$ and $\mathbb{N}$ respectively such that*

$$\lambda_n \underset{n \to +\infty}{\longrightarrow} 0 \quad and \quad \alpha_n \underset{n \to +\infty}{\longrightarrow} +\infty \quad while \quad |\lambda_n|(2|\alpha_n| + d) \underset{n \to +\infty}{\longrightarrow} \mu^2$$

for some given $\mu \in \mathbb{R}_+^*$. Then for any function $f \in L^1(G)$ which is radial (i.e. $f(\mathrm{Exp}(V + Z) = \tilde{f}(|V|, Z)$ for some mesurable function $\tilde{f}$), we have

$$\left(\widehat{f}(\lambda_n) h_{\alpha_n}, h_{\alpha_n} \right) \underset{n \to +\infty}{\longrightarrow} \int_{\varsigma \in \mathbf{S}^{2d-1}} \widehat{f}(0, \mu\varsigma) d\varsigma = \int_{\mathbb{R}^{2d} \times \mathbb{R}^p \times \mathbf{S}^{2d-1}} f(\mathrm{Exp}(V + Z)) e^{-i\mu\varsigma V} dV dZ d\varsigma.$$

Proof. The following facts are well known, see [38, 21, 24, 22]. The function defined by

$$\Psi_{\lambda, |\alpha|} : x = \mathrm{Exp}(V + Z) \longmapsto \int_{SO(\mathfrak{v})} ((\pi_{\mathrm{Exp}(kV+Z)}^\lambda)^* h_\alpha, h_\alpha) dk$$

is smooth and bounded by 1 (its value at $\mathrm{Exp}(0)$) on G. It is equal to

$$\Psi_{\lambda, |\alpha|}(\mathrm{Exp}(V + Z)) = e^{-i\lambda(Z)} \mathcal{L}_{|\alpha|}^{d-1} \left(\frac{|\lambda|}{2} |V|^2 \right),$$

where $\mathcal{L}_k^\delta$ is the normalised Laguerre function of type $\delta = d - 1$ and degree $k \in \mathbb{N}$; this means that $\mathcal{L}_k^\delta(t) = \frac{L_k(t)}{L_k(0)} e^{-\frac{t}{2}}$, $t \in \mathbb{R}$, where $L_k(t) = t^{-\delta} e^t \frac{1}{k!} (\frac{d}{dt})^k (e^{-t} t^{k+\delta})$ is the Laguerre polynomial of type $\delta > -1$ and degree $k \in \mathbb{N}$. The function defined on G by

$$\Psi_{|\omega|} : x = \mathrm{Exp}(V + Z) \longmapsto \int_{SO(\mathfrak{v})} (\pi_{\mathrm{Exp}(kV+Z)}^{(0,\omega)})^* dk.$$

is smooth and bounded by 1 (its value at $\mathrm{Exp}(0)$). We have

$$\Psi_{|\omega|}(\mathrm{Exp}(V + Z)) = \int_{\varsigma \in \mathbf{S}^{2d-1}} e^{-i|\omega|\varsigma V} d\varsigma,$$
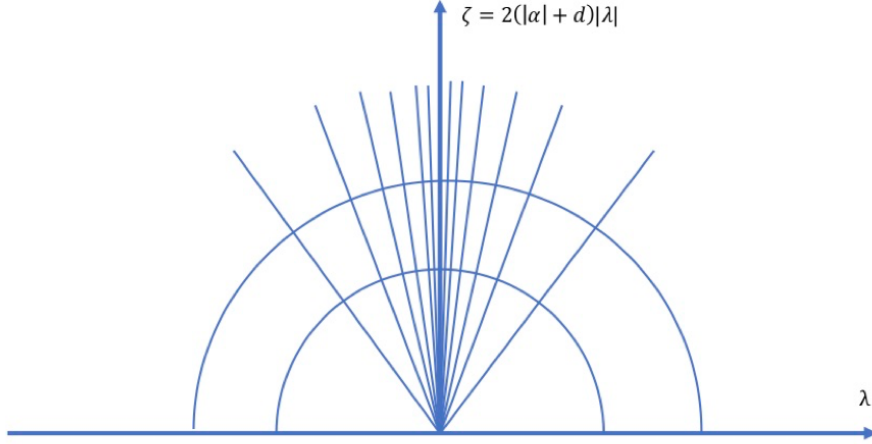
FIGURE 1. Heisenberg's fan

and this expression is a known function of $|\omega||V|$ (this known function is sometimes called the reduced Bessel function). Properties of the Laguerre and Bessel functions implies [24] the convergence $\Psi_{\lambda_n,\alpha_n} \underset{n \to +\infty}{\longrightarrow} \Psi_\mu$ on any compact of G when the sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\alpha_n)_{n \in \mathbb{N}}$ are as in the statement. The result follows easily from this and the calculations

$$\left(\widehat{f}(\lambda_n)h_{\alpha_n}, h_{\alpha_n}\right) = \int_G f(x)\Psi_{\lambda_n,|\alpha_n|}(x)dx, \qquad \int_{\varsigma \in \mathbf{S}^{2d-1}} \widehat{f}(0, \mu\varsigma)d\varsigma = \int_G f(x)\Psi_{|\omega|}(x)dx.$$

$\square$

Remark 2.3. *Note that the proof above shows that we also have for any $f \in \mathcal{S}(G)$,*

$$\int_{SO(\mathfrak{v})} \left(\widehat{f_k}(\lambda_n)h_{\alpha_n}, h_{\alpha_n}\right) dk \underset{n \to +\infty}{\longrightarrow} \int_{\varsigma \in \mathbf{S}^{2d-1}} \widehat{f}(\mu\varsigma)d\varsigma,$$

where $f_k(\mathrm{Exp}(V + Z)) = f(\mathrm{Exp}(kV + Z))$ for all $k \in SO(\mathfrak{v})$. Here dk the normalised (probability) Haar measure of the compact group $SO(\mathfrak{v})$.

The function $\Psi_{\lambda,|\alpha|}$ and $\Psi_{|\omega|}$ are called the bounded spherical functions of G [38, 21, 24, 22]. They can be characterised in the following way: the associated linear maps $f \mapsto \int_G f(x)\Psi(x)dx$ for $\Psi = \Psi_{\lambda,|\alpha|}, \lambda \in \mathfrak{z}^* \setminus \{0\}, \alpha \in \mathbb{N}^d$, and $\Psi_{|\omega|} \omega \in \mathfrak{v}^*$, are in fact the characters of the commutative convolution algebra of radial integrable functions on G. Another viewpoint is that these functions are general eigenfunctions of Δ_G and $Z_1, \ldots, Z_p$:

$$\Delta_G\Psi_{\lambda,|\alpha|} = -\zeta(\alpha,\lambda)\Psi_{\lambda,|\alpha|}, \quad \Delta_G\Psi_{|\omega|} = -|\omega|^2\Psi_{|\omega|}, \quad Z_j\Psi_{\lambda,|\alpha|} = i\lambda_j\Psi_{\lambda,|\alpha|} \quad \text{and} \quad Z_j\Psi_{|\omega|} = 0.$$

Moreover, they yield the joint spectral decomposition of Δ_G and $Z_1, \ldots, Z_p$. The joint spectrum in the case of $p = 1$, that is, in the case of $G = \mathbb{H}_d$ may be drawn in what is often called the Heisenberg fan.

Putting λ on the horizontal axis and ζ on the vertical axis, the set of half lines $\zeta = |\lambda|(2|\alpha| + d)$ when α describes $\mathbf{N}^d$ concentrates on the vertical line as pictured in Figure 1. This figure is known as the Heisenberg fan. It can be viewed as the spectrum of the convolution algebra of radial and integrable functions on G as well as the joint spectrum of $-\Delta_G, iZ_1, \ldots, Z_p$, see for instance [1, 27].

3. Semi-classical pseudodifferential operators

As mentioned above, the Fourier transform being operator-valued, so will be the symbols of semi-classical pseudodifferential operators that we shall now define.

3.1. **The algebra of symbols $\mathcal{A}_0$.** We denote by $\mathcal{A}_0$ the space of symbols $\sigma = \{\sigma(x,\pi) : (x,\pi) \in G \times \widehat{G}\}$ of the form

$$\sigma(x,\lambda) = \mathcal{F}\kappa_x(\lambda) = \int_G \kappa_x(y)(\pi_x^\lambda)^* dx,$$

where $x \mapsto \kappa_x(y)$ is a smooth and compactly supported function from G to $\mathcal{S}(G)$. Being compactly supported means that $\kappa_x(y) = 0$ for x outside a compact of G and any $y \in G$.

Remark 3.1. *In the case of representations of finite dimension, we distinguish between all the finite dimensional representations by replacing $\lambda = 0$ by the parameters $(0,\omega)$, $\omega \in \mathfrak{v}^*$. Besides, the operator $\mathcal{F}\kappa_x(0,\omega) = \sigma(x,(0,\omega))$ reduces to a complex number because $\mathcal{H}_0 = \mathbb{C}$.*

As the Fourier transform is injective, it yields a one-to-one correspondence between the symbol σ and the function κ: we have $\sigma(x,\pi) = \mathcal{F}\kappa_x(\lambda)$ and conversely the Fourier inversion formula (2.13) yields

$$\forall x, z \in G, \quad \kappa_x(z) = c_0 \int_{\widehat{G}} \mathrm{Tr}\left(\pi_z^\lambda \sigma(x,\lambda)\right) |\lambda|^d d\lambda.$$

The set $\mathcal{A}_0$ is an algebra for the composition of symbols since if $\sigma_1(x,\lambda) = \mathcal{F}\kappa_{1,x}(\lambda)$ and $\sigma_2(x,\lambda) = \mathcal{F}\kappa_{2,x}(\lambda)$ are in $\mathcal{A}_0$, then so is $\sigma_1(x,\lambda)\sigma_2(x,\lambda) = \mathcal{F}(\kappa_{2,x} * \kappa_{1,x})(\lambda)$ by (2.14).

We can define two norms on $\mathcal{A}_0$: one expressed directly on the symbol σ

$$(3.1) \qquad \|\sigma\|_{\mathcal{A}} := \sup_{(x,\lambda) \in G \times \widehat{G}} \|\sigma(x,\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))},$$

and the other expressed on the associated function $\kappa_x = \mathcal{F}^{-1}\sigma(x,\cdot)$

$$(3.2) \qquad \|\sigma\|_{\mathcal{A}_0} := \int_G \sup_{x \in G} |\kappa_x(z)| dz.$$

From an algebraic viewpoint, they are both submultiplicative on $\mathcal{A}_0$:

$$\|\sigma_1\sigma_2\|_{\mathcal{A}_0} = \int_G \sup_{x \in G} |\kappa_{2,x} * \kappa_{1,x}(z)| dz \leq \int_G (\sup_{x_2 \in G} |\kappa_{2,x_2}|) * (\sup_{x_1 \in G} |\kappa_{1,x}|)(z) dz \leq \|\sigma_2\|_{\mathcal{A}_0} \|\sigma_1\|_{\mathcal{A}_0},$$

and

$$\|\sigma_1\sigma_2\|_{\mathcal{A}} \leq \sup_{(x_1,\lambda_1) \in G \times \widehat{G}} \|\sigma_1(x_1,\lambda_1)\|_{\mathcal{L}(L^2(\mathfrak{p}_{\lambda_1}))} \sup_{(x_2,\lambda_2) \in G \times \widehat{G}} \|\sigma_2(x_2,\lambda_2)\sigma_1(x,\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_{\lambda_2}))} = \|\sigma_1\|_{\mathcal{A}} \|\sigma_2\|_{\mathcal{A}}.$$

By (2.11), we have:

$$\|\sigma\|_{\mathcal{A}} \leq \sup_{x \in G} \int_G |\kappa_x(z)| dz \leq \|\sigma\|_{\mathcal{A}_0}.$$

but one can show that the two norms are different. In fact, their counterparts on the abelian group $\mathbb{R}^n$ are also different.

Although we will not use the following, we observe that the algebra of symbol $\mathcal{A}_0$ may be defined as the space of smoothing symbols $S^{-\infty}(G)$ which are compactly supported in x; here the smoothing symbols are to be taken in the sense of [28] (Section 5.2 for any graded nilpotent Lie group and Section 6.5 for the Heisenberg group). In the context of an H-type group, a symbol $\sigma(x,\lambda)$ is smoothing when for all $\alpha, \beta \in \mathbb{N}^{2d+p}$ and for any $k_1, k_2 \in \mathbb{N}$, there exists a constant $C_{\alpha,\beta,k_1,k_2} > 0$ such that

$$(3.3) \qquad \forall (x,\lambda) \in G \times (\mathfrak{z}^* \setminus \{0\}) \qquad \left\| H(\lambda)^{k_1} \mathcal{X}_x^\beta \Delta^\alpha \sigma(x,\lambda) H(\lambda)^{k_2} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_{\alpha,\beta,k_1,k_2}.$$

Here we assume that a basis $X_1, \ldots, X_{2d+p}$ of $\mathfrak{g}$ has been fixed. As before $\mathcal{X}_x^\beta$ is the (ordered) product of $|\beta|$ of these left invariant vector fields. The difference operators Δ^α are related with the coordinate functions $\phi_1, \ldots, \phi_{2d+p}$ associated with the basis $X_1, \ldots, X_{2d+p}$, i.e. $x = \mathrm{Exp}(\phi_1(x)X_1 + \ldots + \phi_{2d+p}(x)X_{2d+p})$; denote by $\Delta_{\phi_1}, \ldots, \Delta_{\phi_{2d+p}}$ the corresponding difference operators,

$$(\Delta_{\phi_j}\sigma)(x,\lambda) = \mathcal{F}(\phi_j\kappa_x) \quad \text{if } \sigma(x,\lambda) = \mathcal{F}\kappa_x(\lambda).$$

Then Δ^α is the product of $|\alpha|$ of such difference operators. Note that two difference operators Δ_{ϕ_1} and Δ_{ϕ_2} of that form will commute since $\Delta_{\phi_1}\Delta_{\phi_2} = \Delta_{\phi_1\phi_2}$. Then, the estimate (3.3) above yields semi-norms on $S^{-\infty}(G)$ and induces there a topology by inductive limit.

More generally, one can always define a difference operator associated with a function ϕ of polynomial growth (i.e. $|\phi(x)| \leq (1+|x|)^N$ for some $N \in \mathbb{N}$) as the operator defined on $\mathcal{A}_0$ by

(3.4)
$$(\Delta_\phi\sigma)(x,\lambda) = \mathcal{F}(\phi\kappa_x) \quad \text{if } \sigma(x,\lambda) = \mathcal{F}\kappa_x(\lambda).$$

3.2. **Semi-classical pseudodifferential operators.** Let $\varepsilon > 0$ be a small parameter, the *semi-classical parameter* that we shall use to weight the oscillations of the functions that we shall consider. Following [13] and [28], we quantify the symbols that we have introduced previously by setting

(3.5)
$$\mathrm{Op}_\varepsilon(\sigma)f(x) = c_0 \int_{\widehat{G}} \mathrm{Tr}\left(\pi_x^\lambda \sigma(x, \varepsilon^2\lambda)\mathcal{F}f(\lambda)\right) |\lambda|^d \, d\lambda, \quad f \in \mathcal{S}(G).$$

The kernel of the operator $\mathrm{Op}_\varepsilon(a)$ is the function

$$G \times G \ni (x,y) \mapsto \kappa_x^\varepsilon(y^{-1}x)$$

where $\kappa_x^\varepsilon(z) = \varepsilon^{-Q}\kappa_x\left(\delta_{\varepsilon^{-1}}z\right)$ and κ_x is such that $\mathcal{F}(\kappa_x)(\lambda) = \sigma(x,\lambda)$. For this reason, we call κ_x the *convolution kernel* of σ.

The norm $\| \cdot \|_{\mathcal{A}_0}$ defined in (3.2) allows us to bound the action of the symbols in $\mathcal{A}_0$ (and more generally in $S^{-\infty}(G)$) on $L^2(G)$:

Proposition 3.2. *Let $\sigma \in \mathcal{A}_0$, then $\mathrm{Op}_\varepsilon(\sigma)$ is bounded in $L^2(G)$. Moreover, there exists a constant $C > 0$ such that for all*

$$\forall \sigma \in \mathcal{A}_0, \ \ \forall \varepsilon > 0, \ \ \|\mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C \|\sigma\|_{\mathcal{A}_0}.$$

Proof. We observe that if $f \in \mathcal{S}(G)$ then

$$|\mathrm{Op}_\varepsilon(\sigma)f(x)| = |\int_G f(y)\kappa_x^\varepsilon(y^{-1}x)dy| \leq \int_G |f(y)| \sup_{x_1 \in G} |\kappa_{x_1}^\varepsilon(y^{-1}x)| \, dy = |f| * \sup_{x_1 \in G} |\kappa_{x_1}^\varepsilon(\cdot)|(x),$$

so the Young convolution inequality implies

$$\|\mathrm{Op}_\varepsilon(\sigma)f\|_{L^2(G)} \leq \|f\|_{L^2(G)} \| \sup_{x_1 \in G} |\kappa_{x_1}^\varepsilon(\cdot)|\|_{L^1(G)}.$$

We recognise this L^1-norm as $\|\sigma\|_{\mathcal{A}_0}$:

$$\| \sup_{x_1 \in G} |\kappa_{x_1}^\varepsilon(\cdot)|\|_{L^1(G)} = \| \sup_{x_1 \in G} |\kappa_{x_1}(\cdot)|\|_{L^1(G)} = \|\sigma\|_{\mathcal{A}_0}.$$

$\square$

Notice that the boundedness of semi-classical pseudodifferential operators of symbols of the form $\tau(x,\lambda) = a(x)b(\lambda)$ is much easier and, since $\mathrm{Op}_\varepsilon(\tau)$ simply is the composition of the operator of multiplication by $a(x)$ and of the Fourier multiplier $b(\varepsilon^2\lambda)$, one has

(3.6)
$$\|\mathrm{Op}_\varepsilon(\tau)\|_{\mathcal{L}(L^2(G))} \leq \|\tau\|_{\mathcal{A}}.$$

However, similarly to what happens in the Euclidean case, when the link between x and λ is more intricate inside the symbol $\sigma(x,\lambda)$, the estimate of $\mathrm{Op}_\varepsilon(\sigma)$ requires a more elaborate norm, implying bound on the derivatives. The use of the norm $\| \cdot \|_{\mathcal{A}_0}$ is particularly convenient in this semi-classical

frame. However, instead of $\|\cdot\|_{\mathcal{A}_0}$, we could have chosen a suitable semi-norm on $S^{-\infty}$ which guarantees the $L^2(G)$-boundedness of the corresponding operator, for instance a semi-norm on the class of symbol $S^0(G)$ from [28, Theorem 5.4.17]. Then, with this seminorm instead of $\|\cdot\|_{\mathcal{A}_0}$, we would obtain an estimate similar to the one in Proposition 3.2 for all $\varepsilon \in (0,1)$.

Note secondly that our quantization in (3.5) is the analogue of the Kohn-Nirenberg quantization or left quantization on the abelian group $\mathbb{R}^n$ where it is more common to use the Weyl quantization, which is linked with the Wigner transform, see Section 4.4.

Let us conclude with a remark about the generalisation of our result.

Remark 3.3. *The elements above can be similarly developed for any graded Lie group G. Denoting by π the elements of the set $\widehat{G}$ of its representations, it is possible to extend the dilations to $\widehat{G}$ and to define $\varepsilon \cdot \pi$ (see Section 2.3 of [26]) and for $\sigma \in S^{-\infty}(G)$, we set*

$$\mathrm{Op}_\varepsilon(\sigma) = c_0 \int_{\pi \in \widehat{G}} \mathrm{Tr}_{L^2(\mathcal{H}_\pi)} \left(\pi(x)\sigma(x, \varepsilon \cdot \pi)\mathcal{F}f(\pi) \right) d\mu(\pi),$$

where $d\mu(\pi)$ is the Plancherel measure on $\widehat{G}$.

In the next two paragraphs, we emphasize properties of the semi-classical pseudodifferential operators and revisit the difference operators which can be precisely computed in the context of H-type groups and appear in the symbolic calculus.

3.3. Role of the diagonal in the kernel of a semi-classical pseudodifferential operator.
The diagonal plays an important role for the integral kernel of $\mathrm{Op}_\varepsilon(\sigma)$ because that is where the singularities will lie. In fact, one can always assume that the function $(x,z) \mapsto \kappa_x(z)$ is compactly supported close to $z = 0$ (note that $\kappa_x(z)$ is compactly supported in x by definition):

Proposition 3.4. *Let $\chi \in \mathcal{C}_0^\infty(G)$ be identically equal to 1 close to 0. For every $\varepsilon > 0$ and $\sigma = \mathcal{F}(\kappa_x) \in \mathcal{A}_0$, the symbol defined via $\sigma_\varepsilon(x,\lambda) = \mathcal{F}(\kappa_x \chi(\delta_\varepsilon \cdot))(\lambda)$ is in $\mathcal{A}_0$ and its kernel*

$$(x,y) \mapsto \varepsilon^{-Q}\kappa_x(\delta_{\varepsilon^{-1}}(xy^{-1}))\chi(xy^{-1})$$

is compactly supported close to the diagonal $x = y$. For all $N \in \mathbf{N}$, there exists a constant $C = C_{N,\sigma} > 0$ such that

$$\forall \varepsilon > 0 \qquad \|\sigma_\varepsilon - \sigma\|_{\mathcal{A}_0} \leq C\varepsilon^{QN}.$$

Proof. As the function χ is identically 0 close to $z = 0$, for all $N \in \mathbf{N}$, there exists a bounded smooth function θ_N such that

$$\forall y \in G, \ \ \chi(y) - 1 = \theta(y)\|y\|^N,$$

where $\|y\| = (|V|^4 + |Z|^2)^{1/4}$ for $y = \mathrm{Exp}(V + Z) \in G$. Note that $\|\varepsilon y\| = \varepsilon\|y\|$. We also set

$$b_x^{(N)}(z) := \|z\|^N \kappa_x(z)$$

We compute easily

$$\|\sigma_\varepsilon - \sigma\|_{\mathcal{A}_0} = \varepsilon^{NQ} \int_G \sup_{x \in G} |b_x^{(N)}(z)\theta(\delta_\epsilon z)|dz \leq \varepsilon^{NQ}\|\theta\|_{L^\infty} \int_G \sup_{x \in G} |b_x^{(N)}(z)|dz.$$

As this last integral is finite, this concludes the proof. $\qquad \square$

3.4. **Difference operators of H-type groups.** We now study the difference operators rapidly introduced in (3.4). We fix orthonormal bases $V_1, \ldots V_{2d}$ of $\mathfrak{v}$ and $Z_1, \ldots, Z_p$ of $\mathfrak{z}$. We denote by $v_1, \ldots, v_{2d}$ the coordinates of a vector $V = v_1 V_1 + \ldots + v_{2d} V_{2d}$ in $\mathfrak{v}$ and by $z_1, \ldots, z_p$ the coordinates of a vector $Z = z_1 Z_1 + \ldots + z_p Z_p$ in $\mathfrak{z}$. We associate with the functions $v_1, \cdots, v_{2d}$ the difference operators $\Delta_{v_1}, \cdots, \Delta_{v_{2d}}$ as defined in (3.4). In the case of H-type groups, as for the Heisenberg group (see [28, Section 6.3]), these operators simply express in terms of the parameter $\lambda \in \mathfrak{z}^* \setminus \{0\}$.

Recall that for each $\lambda \in \mathfrak{z}^* \setminus \{0\}$, we have already fixed in Section 2.1 a λ-depending orthonormal basis $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$ with respect to which the representation π^λ can be conveniently described. The corresponding difference operators considered not at every $\lambda' \in \mathfrak{z}^* \setminus \{0\}$ but just at $\lambda' = \lambda$ are also easy to compute:

$$(3.7) \qquad \Delta_{p_j} \widehat{f}(\lambda) = |\lambda|^{-1/2}[\xi_j, \widehat{f}(\lambda)] = -i|\lambda|^{-1}[\pi^\lambda(Q_j), \widehat{f}(\lambda)] \quad \text{denoted by } \Delta_{p_j}^\lambda \widehat{f}$$

$$(3.8) \qquad \Delta_{q_j} \widehat{f}(\lambda) = |\lambda|^{-1/2}[i\partial_{\xi_j}, \widehat{f}(\lambda)] = i|\lambda|^{-1}[\pi^\lambda(P_j), \widehat{f}(\lambda)] \quad \text{denoted by } \Delta_{q_j}^\lambda \widehat{f},$$

and for $1 \leq m \leq p$,

$$\Delta_{z_m} \widehat{f}(\lambda) = i\partial_{\lambda_m} \widehat{f}(\lambda) + \frac{1}{2}\frac{\lambda_m}{|\lambda|}\Delta_p \cdot \Delta_q \widehat{f}(\lambda) + \frac{i}{2}\frac{\lambda_m}{|\lambda|}\pi^\lambda(Q) \cdot \nabla_q \widehat{f}(\lambda) + \frac{i}{2}\frac{\lambda_m}{|\lambda|}\Delta_p \cdot \Pi^\lambda(P)\widehat{f}(\lambda)$$

that we denote by $\Delta_{z_m}^\lambda \widehat{f}$. Indeed, the proof of [28, Lemma 6.3.1] for the case of the Heisenberg group extends naturally to H-type groups because of the link between the representation π^λ quotiented by its kernel and the Schrödinger representation of the Heisenberg group already mentioned in Section 2.1.1. For the same reasons (see [28, Examples 6.3.4 and 6.3.5]), we also obtain

$$(3.9) \qquad \Delta_{p_j}^\lambda H(\lambda) = 2\pi^\lambda(P_j) = 2|\lambda|\partial_{\xi_j} \qquad \text{and} \qquad \Delta_{q_j}^\lambda H(\lambda) = 2\pi^\lambda(Q_j) = 2i|\lambda|\xi_j,$$

where $H(\lambda) = -\mathcal{F}\Delta_G(\lambda)$ has been defined in (2.19) and (2.20). The notation $\Delta_{p_j}^\lambda$, $\Delta_{q_j}^\lambda$ and $\Delta_{z_m}^\lambda$ emphasises the fact that these are not difference operators. However, they are helpful when expressing the difference operators $\Delta_{v_1}, \cdots, \Delta_{v_n}$:

Proposition 3.5. *Consider an orthogonal matrix M^λ realising the change of basis from $(V_1, \ldots, V_{2d})$ to $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$. Then, in vectoriel notation,*

$$\Delta_v \widehat{f}(\lambda) = \begin{pmatrix} \Delta_{v_1}\widehat{f}(\lambda) \\ \vdots \\ \Delta_{v_{2d}}\widehat{f}(\lambda) \end{pmatrix} = (M^\lambda)^{-1}\begin{pmatrix} \Delta_p^\lambda \widehat{f} \\ \Delta_q^\lambda \widehat{f} \end{pmatrix} = |\lambda|^{-1/2}(M^\lambda)^{-1}\begin{pmatrix} [\xi, \widehat{f}(\lambda)] \\ [i\partial_\xi, \widehat{f}(\lambda)] \end{pmatrix}.$$

Proof. The definition of M^λ means that if

$$V = v_1 V_1 + \ldots + v_{2d} V_{2d} = p_1 P_1 + \ldots + p_d P_d + q_1 Q_1 + \ldots + q_d Q_d,$$

then

$$\begin{pmatrix} p \\ q \end{pmatrix} = M^\lambda v, \quad \text{where} \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix}, q = \begin{pmatrix} q_1 \\ \vdots \\ q_d \end{pmatrix}, v = \begin{pmatrix} v_1 \\ \vdots \\ v_{2d} \end{pmatrix}$$

and the formula follows. $\qquad\qquad\square$

Note that due to the orthogonality of M^λ, we have

$$(3.10) \qquad V_1 \Delta_{v_1} + \ldots + V_{2d}\Delta_{v_{2d}} = V \cdot \Delta_v = (M^\lambda)^{-1}\begin{pmatrix} P \\ Q \end{pmatrix} \cdot (M^\lambda)^{-1}\begin{pmatrix} \Delta_p^\lambda \\ \Delta_q^\lambda \end{pmatrix} = P \cdot \Delta_p^\lambda + Q \cdot \Delta_q^\lambda$$

where the vector fields V_j, P_j and Q_j acts on the variable $x \in G$ whereas the difference operators Δ_{v_j} act on the variable $\lambda \in \widehat{G}$ of a symbol. In the same way, we have

$$(3.11) \qquad \begin{aligned} V \cdot \pi^\lambda(V) &= V_1 \pi^\lambda(V_1) + \ldots + V_{2d} \pi^\lambda(V_{2d}) \\ &= P_1 \pi^\lambda(P_1) + \ldots + P_d \pi^\lambda(P_d) + Q_1 \pi^\lambda(Q_1) + \ldots + Q_d \pi^\lambda(Q_d) \\ &= P \cdot \pi^\lambda(P) + Q \cdot \pi^\lambda(Q), \end{aligned}$$

acts on x and λ via the vector fields and the multiplication by $\pi^\lambda(V_j)$.

3.5. **Symbolic calculus.** These symbols enjoy a symbolic calculus.

Proposition 3.6. *Let $\sigma \in \mathcal{A}_0$. Then, in $\mathcal{L}(L^2(G))$,*

$$(3.12) \qquad \mathrm{Op}_\varepsilon(\sigma)^* = \mathrm{Op}_\varepsilon(\sigma^*) - \varepsilon \, \mathrm{Op}_\varepsilon(P \cdot \Delta_p^\lambda \sigma^* + Q \cdot \Delta_q^\lambda \sigma^*) + O(\varepsilon^2).$$

Let $\sigma_1, \sigma_2 \in \mathcal{A}_0$. Then in $\mathcal{L}(L^2(G))$,

$$(3.13) \qquad \mathrm{Op}_\varepsilon(\sigma_1) \circ \mathrm{Op}_\varepsilon(\sigma_2) = \mathrm{Op}_\varepsilon(\sigma_1 \, \sigma_2) - \varepsilon \, \mathrm{Op}_\varepsilon\left(\Delta_p^\lambda \sigma_1 \cdot P \, \sigma_2 + \Delta_q^\lambda \sigma_1 \cdot Q \, \sigma_2\right) + O(\varepsilon^2),$$

Note that in (3.12), we can replace $P \cdot \Delta_p^\lambda \sigma^* + Q \cdot \Delta_q^\lambda \sigma^*$ with $V \cdot \Delta_v \sigma^*$, see (3.10). Similarly, we can replace $\Delta_p^\lambda \sigma_1 \cdot P \, \sigma_2 + \Delta_q^\lambda \sigma_1 \cdot Q \, \sigma_2$ with $\Delta_v \sigma_1 \cdot V \sigma_2$ in (3.13).

A proof of Proposition 3.6 is given in Appendix Section 6. It follows the lines of the proofs of [28, Section 5.5] with major simplification due to the semi-classical setting (as in the Euclidean case). The proof relies on Taylor formula, and, depending on the order to which this Taylor formula is pushed, one obtains more or less precise asymptotic expansions of the symbols. As in the Euclidean case, these expansions can be realized at any order.

The difference operators Δ_{v_j} play the role of $-i\partial_{\xi_j}$ in the Euclidean setting. The reader will be able to check that the symbolic calculus formula are exactly the same as in the case of Kohn-Niremberg quantization in the Euclidean setting (see formula (3.17) and (3.18) in [31]), and as in the Euclidean setting, they can be pushed at any order.

It is also interesting to notice that formula (3.13) also hold when one of the operators $\mathrm{Op}_\varepsilon(\sigma_1)$ or $\mathrm{Op}_\varepsilon(\sigma_2)$ is a semi-classical differential operator $\sum_{|\alpha| \leq N} a_\alpha(x) \varepsilon^{[\alpha]} \mathcal{X}_x^\alpha$. In the latter notation, the coefficients a_α are smooth functions of the variable x, $\mathcal{X}^\alpha$ is the (ordered) product of $|\alpha|$ left invariant vector fields of the basis $X_1, \cdots, X_{2d+p}$ of $\mathfrak{g}$, and $[\alpha]$ is the homogeneous length of α taking into account the homogeneity of X^α. Such an operator admits the symbol $\sum_{|\alpha| \leq N} a_\alpha(x) \pi_\lambda(\mathcal{X}_x^\alpha)$ and one can extend the formula (3.12) and (3.13) to these operators. In particular, if the differential operator has constant coefficients (as the sublaplacian $-\varepsilon^2 \Delta_G$) then the asymptotic formula stops at some precise order linked with the term of higher order of the differential operator. [1]. Note also that one can also include in the calculus functions of the sublaplacian $f(-\varepsilon^2 \Delta_G)$ or Fourier multipliers of the form $f(\varepsilon^{[\alpha]} \mathcal{X}_x^\alpha)$ for continuous functions f with at most polynomial growth.

Finally, one can use the property (3.13) and Helffer-Sjöstrand formula [35] to prove that an operator constructed by the functional calculus as a function of a semi-classical pseudodifferential operator coincides at leading order with a semi-classical pseudodifferential (see for example the proof of Lemma 8 in [25] or the one of Theorem 4.7 in Chapter 2 of [2]).[2]

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[1] A-t-on vraiment besoin que les coefficients soient constants pour obtenir une formule exacte?
[2] Is-it correct?

4. Semi-classical measures

Semi-classical measures are the adaptation of microlocal defect measures in a context where a scale of oscillations is specified. This scale which is the semi-classical scale ε is prescribed by the sequence of family of functions to study or by the parameters of a given problem. Naturally, the section below highly relies on the work [26] which is devoted to microlocal defect measures on graded Lie groups. In particular, we use here a similar C^*-algebra approach.

4.1. **The C^*-algebra $\mathcal{A}$ of semi-classical symbols.** We introduce the algebra $\mathcal{A}$ which is the closure of $\mathcal{A}_0$ for the norm $\|\cdot\|_{\mathcal{A}}$ given in (3.1). The algebra $\mathcal{A}$ enjoys the properties of a C^*-algebra and one can identify its spectrum in the following way:

Proposition 4.1. *The set $\mathcal{A}$ is a separable C^*-algebra of type 1. It is not unital but admits an approximation of identity. Besides, if $\lambda_0 \in \widehat{G}$ and $x_0 \in G$, then the mapping*

$$\begin{cases} \mathcal{A}_0 & \longrightarrow & \mathcal{L}(\mathcal{H}_{\lambda_0}) \\ \sigma & \longmapsto & \sigma(x_0, \lambda_0) \end{cases}$$

extends to a continuous mapping $\rho_{x_0, \pi_{\lambda_0}} : \mathcal{A} \to \mathcal{L}(\mathcal{H}_{\lambda_0})$ which is an irreducible non-zero representation of $\mathcal{A}$. In fact, this is true for $\pi_0 = \pi^{\lambda_0}$ and for $\pi_0 = \pi^{0,\omega_0}$. Furthermore, the mapping

$$R : \begin{cases} G \times \widehat{G} & \longrightarrow & \widehat{\mathcal{A}} \\ (x_0, \pi_0) & \longmapsto & \rho_{x_0, \pi_0} \end{cases}$$

is a homeomorphism which allows to identify $\widehat{\mathcal{A}}$ with $G \times \widehat{G}$.

The proof follows the lines of [26, Section 5]. It utilises the fact that, by definition, the C^* algebra $C^*(G)$ of the group G is the closure of $\mathcal{FS}(G)$ for $\sup_{\lambda \in \widehat{G}} \|\cdot\|_{\mathcal{L}(\mathcal{H}_\lambda)}$ and that the spectrum of $C^*(G)$ is $\widehat{G}$. This implies readily that $\mathcal{A}$ may be identified with the C^*-algebra of continuous functions which vanish at infinity on G and are valued on $C^*(G)$. It also implies that its spectrum is as described in Proposition 4.1.

We can also describe the states of the C^*-algebra $\mathcal{A}$. Still following [26, Section 5], we will need the following vocabulary:

Definition 4.2. *Let Z be a complete separable metric space, and let $\xi \mapsto \mathcal{H}_\xi$ a measurable field of complex Hilbert spaces of Z.*

- *The set $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ is the set of pairs (γ, Γ) where γ is a positive Radon measure on Z and $\Gamma = \{\Gamma(\xi) \in \mathcal{L}(\mathcal{H}_\xi) : \xi \in Z\}$ is a measurable field of trace-class operators such that for all compact set $K \subset Z$,*

$$\int_K \mathrm{Tr}\, |\Gamma(\xi)|\, d\gamma(\xi) < +\infty.$$

- *Two pairs (γ, Γ) and (γ', Γ') in $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ are equivalent when there exists a measurable function $f : Z \to \mathbb{C} \setminus \{0\}$ such that*

$$d\gamma'(\xi) = f(\xi) d\gamma(\xi) \ \text{ and } \ \Gamma'(\xi) = \frac{1}{f(\xi)} \Gamma(\xi)$$

for γ-almost every $\xi \in Z$. The equivalence class of (γ, Γ) is denoted by $\Gamma d\gamma$.

- *A pair (γ, Γ) in $\mathcal{M}_1(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ is positive when $\Gamma(\xi) \geq 0$ for γ-almost all $\xi \in Z$. In this case, we may write $\Gamma d\gamma \geq 0$ or $(\gamma, \Gamma) \in \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi \in Z})$.*

We will use the short-hands

$$\mathcal{M}_1^+(G \times \widehat{G}) = \mathcal{M}_1^+(Z, (\mathcal{H}_\xi)_{\xi \in Z}) \quad \text{when} \quad Z = \{(x, \lambda) \in G \times \widehat{G}\}, \quad \text{and} \quad \mathcal{H}_{x,\lambda} = \mathcal{H}_\lambda.$$

We recall that the Hilbert space $\mathcal{H}_\lambda$ is associated with the representation of $\lambda \in \widehat{G}$, that is, using the description in (2.10), $\mathcal{H}_\lambda = L^2(\mathfrak{p}_\lambda)$ if the representation corresponds to $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and $\mathcal{H}_{(0,\omega)} = \mathbb{C}$ if $\lambda = 0$ and the representation corresponds to $(0,\omega)$ with $\omega \in \mathfrak{v}^*$.

With this concept in mind, the states of the C^*-algebra $\mathcal{A}$ can be described as follows:

Proposition 4.3. *If ℓ is a state of $\mathcal{A}$, then there exists $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$, unique up to its equivalence class, satisfying*

$$(4.1) \qquad \int_{G \times \widehat{G}} \mathrm{Tr}\,(\Gamma(x,\lambda))\,d\gamma(x,\lambda) = 1,$$

and

$$(4.2) \qquad \forall \sigma \in \mathcal{A} \qquad \ell(\sigma) = \int_{G \times \widehat{G}} \mathrm{Tr}\,(\sigma(x,\lambda)\Gamma(x,\lambda))\,d\gamma(x,\lambda).$$

Conversely, if a pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ satisfies (4.1), then the linear form ℓ defined via (4.2) is a state of $\mathcal{A}$.

The proof of this proposition follows the lines of [26, Section 5] and is given in Appendix Section 7, for the convenience of the reader.

4.2. **Semi-classical measures.** We associate with a bounded family $(u^\varepsilon)_{\varepsilon>0}$ of $L^2(G)$ the quantities

$$(4.3) \qquad \ell_\varepsilon(\sigma) = (\mathrm{Op}_\varepsilon(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)}, \quad \sigma \in \mathcal{A}_0,$$

the limits of which are characterized by an element of $\mathcal{M}_1^+(G \times \widehat{G})$.

Theorem 4.1. *Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family of $L^2(G)$. There exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,+\infty)$ with $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ and a pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ such that we have*

$$\forall \sigma \in \mathcal{A}_0, \quad \left(\mathrm{Op}_{\varepsilon_k}(\sigma)u^{\varepsilon_k}, u^{\varepsilon_k}\right)_{L^2(G)} \underset{k\to+\infty}{\longrightarrow} \int_{G \times \widehat{G}} \mathrm{Tr}\,(\sigma(x,\lambda)\Gamma(x,\lambda))\,d\gamma(x,\lambda).$$

Given the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, the pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ is unique up to equivalence in $\mathcal{M}_1^+(G \times \widehat{G})$ and satisfies

$$\int_{G \times \widehat{G}} \mathrm{Tr}\,(\Gamma(x,\lambda))\,d\gamma(x,\lambda) \leq \limsup_{\varepsilon>0} \|u^\varepsilon\|_{L^2(G)}^2.$$

Any equivalence class $\Gamma d\gamma$ satisfying to Theorem 4.1 for some subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ is called a *semi-classical measure* of the family $(u^\varepsilon)_{\varepsilon>0}$.

Remark 4.4.   (1) *Note that this result can be generalized on any graded Lie group: there exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,+\infty)$ with $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ and a pair $(\gamma, \Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ such that we have*

$$\forall \sigma \in \mathcal{A}_0(G), \quad \left(\mathrm{Op}_{\varepsilon_k}(\sigma)u^{\varepsilon_k}, u^{\varepsilon_k}\right)_{L^2(G)} \underset{k\to+\infty}{\longrightarrow} \int_{G \times \widehat{G}} \mathrm{Tr}\,(\sigma(x,\pi)\Gamma(x,\pi))\,d\gamma(x,\pi).$$

The pair $\Gamma d\gamma$ is called a semi-classical measure of the family (u^ε).
   (2) *In the case of this article where G is H-type, the special structure of $\widehat{G}$ implies that $\Gamma d\gamma$ consists of two pieces, one localized above $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and another one which is scalar above $\mathfrak{v}^*$, see (2.10).*

We now sketch the proof of Theorem 4.1.

Proof. By dividing u^ε by $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(G)}$ if necessary, we can assume that

$$\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(G)} = 1.$$

Indeed, if $\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2 G} = 0$, then the measure $\gamma = 0$ answers our problem. We then consider the quantities $\ell_\varepsilon(\sigma)$ defined in (4.3) and we observe the three following facts:

(1) For any $\sigma \in \mathcal{A}_0$, the family $\ell_\varepsilon(\sigma)$ is bounded and there exists a subsequence $(\varepsilon_k(\sigma))_{k\in\mathbb{N}}$ such that $\ell_{\varepsilon_k(\sigma)}(\sigma)$ has a limit $\ell(\sigma)$

(2) Using the separability of $\mathcal{A}_0$ and a diagonal extraction, one can find a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that for all $\sigma \in \mathcal{A}_0$, the sequence $(\ell_{\varepsilon_k}(\sigma))_{k\in\mathbb{N}}$ has a limit $\ell(\sigma)$ and the sequence $(\|u^{\varepsilon_k}\|_{L^2(G)})_{k\in\mathbb{N}}$ converges to 1.

(3) The map $\sigma \mapsto \ell(\sigma)$ constructed at point (2) is linear and satisfies $\ell(\sigma\sigma^*) \geq 0$ for all $\sigma \in \mathcal{A}_0$.

Observing that the set of symbols of the form $\tau(x, \lambda) = a(x)b(\lambda)$ is dense in $\mathcal{A}$, and that for these symbols, $|\ell(\tau)| \leq \|\tau\|_{\mathcal{A}}$ (see (3.6)), we can extend the linear form ℓ into a state of $\mathcal{A}$, thus the existence of $\Gamma d\gamma$ by Proposition 4.1. □

4.3. **Link with energy density and ε-oscillation.** We want to link here the weak limits of the measure $|u^\varepsilon(x)|^2 dx$ and the semi-classical measures of the family u^ε. For this, we introduce the definition of an ε-oscillating family of $L^2(G)$.

Definition 4.5. *Let (u^ε) be a bounded family in $L^2(G)$. We shall say that (u^ε) is $\underline{\varepsilon\text{-oscillating}}$ if*

$$\limsup_{\varepsilon\to 0} \|\mathbf{1}_{-\varepsilon^2\Delta_G > R}u^\varepsilon\|_{L^2(G)} \xrightarrow[R\to+\infty]{} 0.$$

Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi = 0$ on $]-\infty, 1]$ and $\chi = 1$ on $[2, +\infty[$. Equivalently, u^ε is ε-oscillating if and only if

$$\limsup_{\varepsilon\to 0} \left\|\chi\left(-\frac{\varepsilon^2}{R}\Delta_G\right)u^\varepsilon\right\|_{L^2(G)} \xrightarrow[R\to+\infty]{} 0.$$

Proposition 4.6. *If there exists $s > 0$ and $C > 0$ such that*

$$\forall\varepsilon > 0, \quad \|(-\varepsilon\Delta_G)^{\frac{s}{2}}\psi^\varepsilon\|_{L^2(G)} \leq C,$$

then u^ε is ε-oscillating.

Proof. We use Plancherel formula and the facts that for $s > 0$,

$$\chi\left(-\frac{\varepsilon^2}{R}\Delta_G\right) \leq \frac{(-\varepsilon^2\Delta_G)^{\frac{s}{2}}}{R^s}\chi\left(-\frac{\varepsilon^2}{R}\Delta_G\right) \leq \frac{(-\varepsilon^2\Delta_G)^{\frac{s}{2}}}{R^s}.$$

□

The interest of the notion of ε-oscillation relies in the fact that it gives an indication of the size of the oscillations that have to be taken into account. It legitimates the use of semi-classical pseudodifferential operators and semi-classical measures. In particular, we have the following straightforward proposition.

Proposition 4.7. *Let (u^ε) be an ε-oscillating family admitting a semi-classical measure $\Gamma d\gamma$ for the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, then for all $\phi \in \mathcal{C}_0^\infty(G)$*

$$\lim_{k\to+\infty} \int_G \phi(x)|u^{\varepsilon_k}(x)|^2 dx = \int_{G\times\widehat{G}} \phi(x)\mathrm{Tr}\left(\Gamma(x, \lambda)\right) d\gamma(x, \lambda).$$

Proof. Let χ be as above. We write for any $R > 0$

$$\int_G \phi(x)|u^{\varepsilon_k}(x)|^2 dx = I_0^{k,R} + I_1^{k,R},$$

where

$$I_0^{k,R} := \int_G \phi(x) \; \chi \left(R^{-1} \varepsilon_k^2 \Delta_G \right) u^{\varepsilon_k}(x) \; \overline{u^{\varepsilon_k}(x)} dx,$$

$$I_1^{k,R} := \int_G \phi(x) \; (1 - \chi) \left(R^{-1} \varepsilon_k^2 \Delta_G \right) u^{\varepsilon_k}(x) \; \overline{u^{\varepsilon_k}(x)} dx.$$

As (u^ε) is ε-oscillating, $\lim_{R \to +\infty} \lim_{k \to +\infty} I_0^{k,R} = 0$. For the other integral, it is known that $\phi(x) \; (1 - \chi) \left(R^{-1} H(\lambda) \right) \in \mathcal{A}_0$, see for instance [26, Corollary 3.8], so Theorem 4.1 implies

$$\lim_{R \to +\infty} \lim_{k \to +\infty} I_1^{k,R} = \lim_{R \to +\infty} \int_{G \times \widehat{G}} \phi(x) \mathrm{Tr} \left((1 - \chi) \left(R^{-1} H(\lambda) \right) \Gamma(x, \lambda) \right) d\gamma(x, \lambda)$$

$$= \int_{G \times \widehat{G}} \phi(x) \mathrm{Tr} \left(\Gamma(x, \lambda) \right) d\gamma(x, \lambda).$$

Combining the limits shows the statement. $\square$

4.4. **Other quantizations and Wigner transform.** It is of course possible to quantize symbols $\sigma(x, \lambda)$ in another manner as it is in the Euclidean case and as discussed in the article [42] in a non-semi-classical setting. One can associate with any $\tau \in (0, 1)$ the τ-quantization of the symbol $a \in \mathcal{A}_0$ by setting

$$\mathrm{Op}_\varepsilon^\tau(a) f(x) = c_0 \int_{\widehat{G}} \mathrm{Tr} \left(\pi_x^\lambda \sigma(\delta_\tau x \, \delta_{1-\tau} y, \varepsilon^2 \lambda) \mathcal{F}(f)(\lambda) \right) |\lambda|^d d\lambda, \;\; \forall f \in \mathcal{S}(G), \;\; \forall x \in G.$$

As in the Euclidean case, these quantizations are equivalent at leading order in the sense that we have the following proposition.

Proposition 4.8. *Let $\sigma \in \mathcal{A}_0$ and $\tau \in (0, 1)$. There exists a constant $C > 0$ such that*

$$\|\mathrm{Op}_\varepsilon^\tau(\sigma) - \mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C\varepsilon.$$

The proof crucially uses the special of the kernel of the operator $\mathrm{Op}_\varepsilon^\tau(\sigma)$ which is the function

$$G \times G \ni (x, y) \mapsto \varepsilon^{-Q} \kappa_{\delta_\tau x \delta_{1-\tau} y}(\delta_{\varepsilon^{-1}}(y^{-1} x))$$

with $\mathcal{F}(\lambda)(\kappa_x) = \sigma(x, \lambda)$. Set

$$\kappa_x^{\varepsilon, \tau}(z) = \kappa_{x \delta_{\varepsilon(1-\tau)} z^{-1}}(z), \;\; z \in G,$$

then, the convolution kernel of $\mathrm{Op}_\varepsilon^\tau(\sigma)$ is the function

$$G \times G \ni (x, z) \mapsto \varepsilon^{-Q} \kappa_x^{\varepsilon, \tau}(\varepsilon^{-1} z).$$

The proof of Proposition 4.8 follows form the analysis of the difference between $\kappa_x^{\varepsilon, \tau}(z)$ and $\kappa_x^\tau(z)$ thanks to a Taylor formula as in Proposition 3.6.

The quantization we have discussed until now corresponds to the choice of $\tau = 1$. The analogue of what is the "left" quantization in the Euclidean case corresponds to $\tau = 0$ and, as in the Euclidean case, one has for all $f \in \mathcal{S}(G)$,

$$\left(\mathrm{Op}_\varepsilon^1(\sigma(x, \lambda)) f, f \right) = \left(f, \mathrm{Op}_\varepsilon^0(\sigma(x, \lambda)^*) f \right).$$

Similarly, the Weyl quantization should correspond to $\tau = 1/2$. However, due to the non-commutativity of the law group, the choice of $\tau = 1/2$ does not induce a self-adjoint quantization, in the sense that we would have $\mathrm{Op}_\varepsilon^\tau(\sigma)^* = \mathrm{Op}_\varepsilon^\tau(\sigma^*)$. This constitutes a major difference with the Euclidean case.

This implies that the definition of a "natural" Wigner distribution is not straightforward. Indeed, in the Euclidean case, the Wigner distribution of a function $f \in L^2(\mathbb{R}^d)$ is the self-adjoint distribution such that

$$(\mathrm{Op}^{1/2}(a) f, f)_{L^2(\mathbb{R}^d)} = \langle a, W_f^\varepsilon \rangle.$$

Introduced in the 30s (see [49]), it plays the role of a generalized energy density defined in the phase space as its marginal gives the density in position or in impulsion:

$$|f(x)|^2 = \int_{\xi \in \mathbb{R}^d} W_f^\varepsilon(x,\xi)d\xi, \quad \varepsilon^{-d/2}|\widehat{f}(\xi/\varepsilon)|^2 = \int_{x \in \mathbb{R}^d} W_f^\varepsilon(x,\xi)dx.$$

Even though it is not a positive distribution, it is self-adjoint, and it allows a unified treatment of space and Fourier variables which proves to be efficient in the analysis of a large range of problems.

Let us now discuss what could be the generalization of Wigner distribution in the case of H-type groups. We endow the space of fields of operators defined on $G \times \widehat{G}$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle A, B \rangle = c_0 \int_{G \times \widehat{G}} \mathrm{Tr}\left(A(x,\lambda)B(x,\lambda)^*\right) |\lambda|^d d\lambda \, dx.$$

We then associate with a function $f \in \mathcal{S}(G)$ its *Wigner distribution* $W_f^\varepsilon(x,\lambda)$ defined by

$$W_f^\varepsilon(x,\lambda) = \frac{c_0}{2} \int_G \pi_w^\lambda \left(f(x \; \delta_\varepsilon w^{-1})\overline{f}(x) + f(x)\overline{f}(x \; \delta_\varepsilon w) \right) dw,$$

(1) The Wigner distribution of a function $f \in L^2(G)$ is a ε-dependent self-adjoint field of operators $W_f^\varepsilon(x,\lambda)$.

(2) The Wigner distribution $W_f^\varepsilon(x,\lambda)$ is L^1 in the variable x and for γ-almost all (x,λ), $W_f^\varepsilon(x,\lambda)$ is a trace-class operator on $\mathcal{H}_\lambda$, besides

$$\int_{G \times G} \mathrm{Tr}\left(W_f^\varepsilon(x,\lambda)\right) |\lambda|^d d\lambda dx = \|f\|_{L^2(G)}^2.$$

(3) For any $\sigma(x,\lambda)$, we have

$$\langle W_f^\varepsilon, \sigma \rangle = \frac{1}{2}\left((\mathrm{Op}_\varepsilon(\sigma)f, f) + (f, \mathrm{Op}_\varepsilon(\sigma^*)f)\right) = \frac{1}{2}\left((\mathrm{Op}_\varepsilon^0(\sigma)f, f) + \left(\mathrm{Op}_\varepsilon^1(\sigma)f, f\right)\right).$$

(4) The marginals of the Wigner distribution give the energy density in position and what we shall call the density in impulsion of f:

$$\int_{\widehat{G}} \mathrm{Tr}\, W_f^\varepsilon(x,\lambda)|\lambda|^d d\lambda = |f(x)|^2,$$

$$\int_G W_f^\varepsilon(x,\lambda)dx = \varepsilon^{-Q}\widehat{f}(\varepsilon^{-2}\lambda)\widehat{f}(\varepsilon^{-2}\lambda)^*.$$

(5) If u^ε is a bounded family in $L^2(G)$, the semi-classical measures of (u^ε) are the weak limits of $W_{u^\varepsilon}^\varepsilon$.

5. EXAMPLES

In the Euclidean case, the basic examples are those families which present concentration on a point $x_0 \in \mathbb{R}^n$ as the family u^ε described in (5.1) or which oscillate along a vector $\xi_0 \in \mathbb{R}^n$ as the sequence v^ε:

(5.1) $$u_\varepsilon(x) = \varepsilon^{-d/2}\Phi\left((x - x_0)/\varepsilon\right) \quad \text{et} \quad v_\varepsilon(x) = \Phi(x)e^{ix \cdot \xi_0/\varepsilon}$$

where $x_0, \xi_0 \in \mathbb{R}^n$ and $\Phi \in C_0^\infty(\mathbb{R}^n)$. These two families tend weakly to 0 but not strongly because the point x_0 is an obstruction to the strong convergence for u^ε and similarly for v^ε for which the point ξ_0 appears as a concentration point of the ε-Fourier transform of v^ε:

$$(2\pi\varepsilon)^{-d/2}\widehat{v}^\varepsilon(\xi/\varepsilon) = (2\pi\varepsilon)^{-d/2}\widehat{\Phi}\left(\frac{\xi - \xi_0}{\varepsilon}\right).$$

The use of semi-classical measures allows one to unify the treatment of these two different behaviours, the semi-classical measures of $(u^\varepsilon)_{\varepsilon>0}$ presenting a Dirac mass at x_0 and the one of $(v^\varepsilon)_{\varepsilon>0}$ a Dirac mass at ξ_0:

$$\mu_{(u^\varepsilon)}(x,\xi) = \delta(x-x_0) \otimes (2\pi)^{-d}|\widehat{\Phi}(\xi)|^2 d\xi \;\; \text{and} \;\; \mu_{(v^\varepsilon)}(x,\xi) = |\Phi(x)|^2 dx \otimes \delta(\xi - \xi_0).$$

We present in the next sections similar examples in the context of Lie groups with a particular attention to sequences which concentrate on the finite dimensional representations of G.

5.1. **Concentration.** In this section, we describe examples of bounded families of $L^2(\mathbb{R}^d)$ with semi-classical measures that concentrate on any point of G. Let $a \in \mathcal{S}(G)$ and $x_0 \in G$, we set

$$u^\varepsilon(x) = \varepsilon^{-\frac{Q}{2}} a(\delta_{\varepsilon^{-1}}(xx_0^{-1})), \;\; x \in G.$$

Then u^ε is a strictly ε-oscillating family since it satisfies the Sobolev criterium of Proposition 4.6.

Proposition 5.1. *Any semi-classical measure of the family u^ε is equivalent to the pair (Γ, γ) with*

$$\Gamma(\lambda) = \widehat{a}(\lambda)\widehat{a}(\lambda)^*, \;\; \gamma(x,\lambda) = c_0 \, \delta_{x_0}(x) \otimes \left(|\lambda|^d d\lambda\right).$$

Proof. We write the proof for $x_0 = 0$. We have

$$(\text{Op}_\varepsilon(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)} = c_0\varepsilon^{-Q} \int_{G\times G\times\widehat{G}} \text{Tr}\left(\pi_x^\lambda \sigma(x, \varepsilon^2\lambda)(\pi_y^\lambda)^*\right) a(\delta_{\varepsilon^{-1}}y)\overline{a(\delta_{\varepsilon^{-1}}x)}|\lambda|^d dx dy d\lambda.$$

The change of variable $\delta_{\varepsilon^{-1}}x \to x$, $\delta_{\varepsilon^{-1}}y \to y$ and $\varepsilon^2\lambda \to \lambda$ and the fact that $\pi_{\delta_\varepsilon x}^{\varepsilon^{-2}\lambda} = \pi_x^\lambda$ gives

$$(\text{Op}_\varepsilon(\sigma)u^\varepsilon, u^\varepsilon)_{L^2(G)} = c_0 \int_{G\times G\times\widehat{G}} \text{Tr}\left(\pi_x^\lambda \sigma(\delta_\varepsilon x, \lambda)(\pi_y^\lambda)^*\right) a(y)\overline{a(x)}|\lambda|^d dx dy d\lambda,$$

whence the result in view of

$$\int_{G\times G\times\widehat{G}} \text{Tr}\left(\pi_x^\lambda \sigma(0, \lambda)(\pi_y^\lambda)^*\right) a(y)\overline{a(x)}|\lambda|^d dx dy d\lambda = \int_{\widehat{G}} \text{Tr}\left(\sigma(0, \lambda)\mathcal{F}(a)(\lambda)\mathcal{F}(a)(\lambda)^*\right)|\lambda|^d d\lambda.$$

$$\square$$

5.2. **Oscillations.** In this section, we build examples of bounded families of $L^2(\mathbb{R}^d)$ with semi-classical measures that concentrate on some points of $\widehat{G}$, i.e. on the representations π_x^λ for $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and also for the finite dimensional representations $\pi_{0,\omega}$, $\omega \in \mathfrak{v}^*$.

For $x \in G$, we write $x = \text{Exp}(V+Z) = x_\mathfrak{z}x_\mathfrak{v} = x_\mathfrak{v}x_\mathfrak{z}$ with $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$, $x_\mathfrak{z} = e^Z \in G_\mathfrak{z}$ and $x_\mathfrak{v} = e^V \in G_\mathfrak{v} = G/G_\mathfrak{z}$. For $\lambda_\varepsilon \in \mathfrak{z}^*$, we define the family

$$v_\varepsilon(x) = |\lambda_\varepsilon|^{d/2} e_\varepsilon(x) a(x_\mathfrak{z}), \;\; e_\varepsilon(x) = \left(\pi_x^{\lambda_\varepsilon}\Phi_\varepsilon, \Phi_\varepsilon\right).$$

where $a \in \mathcal{S}(G_\mathfrak{z})$ is such that $\|a\|_{L^2(G_\mathfrak{z})} = 1$ and $\Phi_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$ with $\|\Phi_\varepsilon\|_{L^2(\mathbb{R}^d)} = 1$ (where we identify $\mathfrak{p}_\lambda$ with $\mathbb{R}^p$).

Proposition 5.2.   (1) *Assume $\Phi_\varepsilon = \Phi$ is independent of ε and*

$$\varepsilon^2\lambda_\varepsilon \xrightarrow[\varepsilon\to 0]{} \lambda_0 \neq 0.$$

Then any semi-classical measure of $(v^\varepsilon)_{\varepsilon>0}$ is equivalent to the pair (Γ, γ) with

$$\Gamma = \mathcal{F}\Phi \otimes (\mathcal{F}\Phi)^* \;\; \text{and} \;\; d\gamma(x,\lambda) = \left(|a(x_\mathfrak{z})|^2 dx_\mathfrak{z} \otimes \delta_{x_\mathfrak{v}=0}\right) \otimes \delta_{\lambda=\lambda_0}.$$

(2) *Assume $\Phi_\varepsilon = h_{\alpha_\varepsilon}$ a Hermite function with $\alpha_\varepsilon \sim \varepsilon^{-\alpha}$ with $0 < \alpha < \frac{1}{d+1}$. Assume*

$$\varepsilon^2 |\lambda_\varepsilon|(2|\alpha_\varepsilon| + d) \underset{\varepsilon \to 0}{\longrightarrow} \mu^2 \in \mathbb{R}_+^* .$$

Then any semi-classical measure of $(v^\varepsilon)_{\varepsilon>0}$ is equivalent to the pair (Γ, γ) supported in $G_{\mathfrak{z}} \times \{\lambda = 0\}$, so we may choose $\Gamma = 1$. Furthermore, the radialisation $\tilde{\gamma}(x, \omega) = \int_{SO(\mathfrak{v})} \gamma(x, k\omega)dk$ of γ satisfies

$$\tilde{\gamma}(x, \omega) = \left(|a(x_{\mathfrak{z}})|^2 dx_{\mathfrak{z}} \otimes \delta_{x_{\mathfrak{v}}=0}\right) \otimes \left(\delta_{\omega=\mu\varsigma} \mathbf{1}_{\varsigma \in \mathbf{S}^{2d-1}} d\varsigma\right)$$

where $d\varsigma$ denotes the probability measure on the sphere $\mathbf{S}^{2d-1}$.

Recall that $\sigma(x, (0, \omega))$ is the value of the map $\sigma(x, \cdot)$ defined on $\widehat{G}$ above the points $\pi^{0,\omega}$ of G and that this value is a complex number.

The first case above corresponds to a concentration on some $\lambda_0 \in \mathfrak{z}^* \setminus \{0\}$ and the second one, to some concentration on the set of dimension 1 representations $\pi^{0,\omega}$, $\omega \in \mathbf{S}^{2d}$. The analysis relies on Lemma 2.2 and illustrates the phenomena of convergence of infinite dimensioned representations towards finite dimensioned one, as illustrated by the Heisenberg fan of Figure 1.

Pushing the argument further and using properties of the bounded spherical functions given in Section 2, one can show that the radialisation of the family v^ε used in Part 2 above admits the semi-classical measures

$$\gamma(x, \omega) = \left(|a(x_{\mathfrak{z}})|^2 dx_{\mathfrak{z}} \otimes \delta_{x_{\mathfrak{v}}=0}\right) \otimes \left(\delta_{\omega=\mu\varsigma} \mathbf{1}_{\varsigma \in \mathbf{S}^{2d-1}} d\varsigma\right)$$

supported above $G_{\mathfrak{z}} \times \{\lambda = 0\}$.

One can also replace the function a by a concentrating family

$$a_\varepsilon(x_{\mathfrak{z}}) = \varepsilon^{-p\beta} a(\delta_{\varepsilon^{-\beta}}((x_{\mathfrak{z}}^0)^{-1} x_{\mathfrak{z}}))$$

with $a \in \mathcal{S}(\mathbb{R}^p)$ and $\beta \in (0, 1/2)$, we shall then obtain a Dirac mass in $x_{\mathfrak{z}}^0$ instead of the absolutely continuous measure $|a(x_{\mathfrak{z}})|^2 dx_{\mathfrak{z}}$. Indeed, in the proof below, we just use

(5.2) $$\forall Z \in \mathfrak{z}, \quad \forall \varepsilon > 0, \quad \varepsilon \|Z a_\varepsilon\|_{L^2(G_{\mathfrak{z}})} \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

which is guaranteed by the condition $\beta \in (0, 1/2)$. It is likely that the proof of section 6.5 in [26], which is based on these convergences, adapt to the semi-classical setting for proving that any element of $\mathcal{M}_1^+(G \times \widehat{G})$ is a semi-classical measure of some bounded family of $L^2(G)$.

Before going to the proof, let us first detail basic facts about the family v^ε:

• *The norm of v^ε.* The sequence v^ε is a bounded family of $L^2(G)$ since we have

$$\begin{aligned}
\|v^\varepsilon\|_{L^2(G)}^2 &= |\lambda_\varepsilon|^d \int_G |a(x_{\mathfrak{z}})|^2 |\left(\pi_{x_{\mathfrak{v}}}^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)|^2 dx \\
&= \|a\|_{L^2(\mathbb{R}^p)}^2 |\lambda_\varepsilon|^d \int_{G_{\mathfrak{v}}} \left|\left(\pi_{\sqrt{|\lambda_\varepsilon|}x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)\right|^2 dx_{\mathfrak{v}} \\
&= \|a\|_{L^2(\mathbb{R}^p)}^2 \int_{G_{\mathfrak{v}}} \left|\left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)\right|^2 dx_{\mathfrak{v}}
\end{aligned}$$

where $\hat{\lambda}_\varepsilon = \frac{\lambda_\varepsilon}{|\lambda_\varepsilon|}$ is of modulus 1. We then observe

$$\int_{G_{\mathfrak{v}}} \left|\left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)\right|^2 dx_{\mathfrak{v}} = \int_{p,q,\xi \in \mathbb{R}^d} e^{iq \cdot (\xi - \xi')} \Phi_\varepsilon(\xi + p) \overline{\Phi_\varepsilon(\xi)} \Phi_\varepsilon(\xi') \overline{\Phi_\varepsilon(\xi' + p)} d\xi \, dp \, dq d\xi' = c_1 \|\Phi_\varepsilon\|_{L^2(\mathbb{R}^d)}^4$$

where $c_1 > 0$ is a universal constant. We thus obtain

$$\|v^\varepsilon\|_{L^2(G)} = c_1 \|\Phi_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 \|a\|_{L^2(G_{\mathfrak{z}})}.$$

- *The ε-oscillation of v^ε.* We have

$$\forall V \in \mathfrak{v}, \quad \|Vv^\varepsilon\|_{L^2(G)} = O(\sqrt{|\lambda_\varepsilon|}),$$

$$\forall Z \in \mathfrak{z}, \quad \|Zv^\varepsilon\|_{L^2(G)} = O\left(|\lambda^\varepsilon| + \|Za\|_{L^2(\mathbb{R}^p)}\right).$$

Therefore, the family (v^ε) is ε-oscillating by Sobolev criteria of Proposition 4.6. Example (1) is strictly oscillating while example (2) is not.

- *The Fourier transform of v^ε.*

$$\mathcal{F}(v^\varepsilon)(\lambda) = |\lambda_\varepsilon|^{d/2} B_\varepsilon(\lambda)\widehat{a}\left(\lambda - \lambda_\varepsilon\right) \quad \text{with} \quad B_\varepsilon(\lambda) = \int_{G_\mathfrak{v}} (\pi^\lambda_{x_\mathfrak{v}})^* \left(\pi^{\lambda_\varepsilon}_{x_\mathfrak{v}} \Phi_\varepsilon, \Phi_\varepsilon\right) dx_\mathfrak{v}.$$

One can observe that

$$|\lambda_\varepsilon|^d B_\varepsilon(\lambda_\varepsilon) = \Phi_\varepsilon \otimes \Phi_\varepsilon^*.$$

We describe the semi-classical measures of $(v^\varepsilon)_{\varepsilon>0}$ in the next subsection.

5.3. **Proof of Proposition 5.2.** The beginning of the proofs for Parts (1) and (2) of Proposition 5.2 is the same: it consists of the following lemma. After its proof, we will analyse each case separately.

Lemma 5.3. *Let $\sigma = \mathcal{F}(\kappa_x) \in \mathcal{A}_0$ and let $\chi \in \mathcal{C}_0^\infty(G)$ be identically equal to 1 close to 0. For every $\varepsilon > 0$, we set $\sigma_\varepsilon(x, \lambda) = \mathcal{F}(\kappa_x \chi(\delta_\varepsilon \cdot))(\lambda)$. Under the hypotheses of Section 5.2, we have*

$$\left| (\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon, v^\varepsilon) - \int_{G_\mathfrak{z}} |a(x_\mathfrak{z})|^2 \left(\sigma_\varepsilon(x_\mathfrak{z}, \varepsilon^2 \lambda_\varepsilon)\Phi_\varepsilon, \Phi_\varepsilon\right) dx_\mathfrak{z} \right| \leq C_{1,a,\varepsilon}|\lambda_\varepsilon|^{-1/2} + C_{2,a}\varepsilon,$$

where $C_{2,a}$ and $C_{1,a,\varepsilon}$ are constant of the form

$$C_{2,a} \leq C_2 \sup_{G_\mathfrak{z}} |a| \sup_{G_\mathfrak{z}} |\nabla a|$$

with $C_2 > 0$ independent of ε and a, and

$$C_{1,a,\varepsilon} \leq C_{1,N}\|a\|^2_{L^2} \max_{|\beta_1|+\ldots+|\beta_8|\leq N} \|\xi^{\beta_1}\partial_\xi^{\beta_2}\Phi_\varepsilon\|_{L^2}\|\xi^{\beta_3}\partial_\xi^{\beta_4}\Phi_\varepsilon\|_{L^2}\|\xi^{\beta_5}\partial_\xi^{\beta_6}\Phi_\varepsilon\|_{L^2}\|\xi^{\beta_7}\partial_\xi^{\beta_8}\Phi_\varepsilon\|_{L^2},$$

with $C_{1,N}$ independent of ε and a, for any integer $N \geq 2d + 1$.

Proof of Lemma 5.3. By Propositions 3.4 and 3.2,

$$(\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon, v^\varepsilon) = (\mathrm{Op}_\varepsilon(\sigma_\varepsilon)v^\varepsilon, v^\varepsilon) + O(\varepsilon^N),$$

for any $N > 0$. We compute easily with the changes of variables $\varepsilon^2\lambda \to \lambda$, $\delta_{\varepsilon^{-1}}(xy^{-1}) \to w$

$$(\mathrm{Op}_\varepsilon(\sigma_\varepsilon)v^\varepsilon, v^\varepsilon) = c_0 \int_{G \times G \times \widehat{G}} \mathrm{Tr}\left(\pi^\lambda_{y^{-1}.x}\sigma_\varepsilon(x, \varepsilon^2\lambda)\right) v^\varepsilon(y)\overline{v^\varepsilon(x)}|\lambda|^d dx\, dy\, d\lambda$$

$$= c_0 \int_{G \times G \times \widehat{G}} \mathrm{Tr}\left(\pi^\lambda_w \sigma_\varepsilon(x, \lambda)\right) v^\varepsilon((\delta_\varepsilon w)^{-1}x)\overline{v^\varepsilon(x)}|\lambda|^d dx\, dw\, d\lambda$$

$$= \int_{G \times G} \chi(\delta_\varepsilon w)\kappa_x(w)v^\varepsilon((\delta_\varepsilon w)^{-1}x)\overline{v^\varepsilon(x)}dx\, dw,$$

having used the Fourier inversion formula. The Taylor formula [29, (1.41)] yields

$$a((\delta_\varepsilon w)^{-1}x)_\mathfrak{z}) = a(x_\mathfrak{z}) + A_\varepsilon(x, w),$$

with

$$|A_\varepsilon(x, w)| \lesssim \sum_{j=1}^{2d} \varepsilon|w_j| \sup_{\|z\|\lesssim\|\delta_\varepsilon w\|} |V_j a(xz)|.$$

here $\|\cdot\|$ denotes a pseudo-norm used, for instance the one used in the proof of Proposition 3.4. We can now write

$$(\mathrm{Op}_\varepsilon(\sigma_\varepsilon)v^\varepsilon, v^\varepsilon) = E_\varepsilon + R_\varepsilon,$$

where the first term is

$$E_\varepsilon := \int_{G \times G} \chi(\delta_\varepsilon w) \kappa_x(w) |\lambda_\varepsilon|^d |a(x_{\mathfrak{z}})|^2 \overline{\left(\pi_x^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right)} \left(\pi_{(\delta_\varepsilon w)^{-1} x}^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right) dx\, dw,$$

and the remainder is

$$R_\varepsilon := \int_{G \times G} \chi(\delta_\varepsilon w) \kappa_x(w) \overline{a(x_{\mathfrak{z}})} A_\varepsilon(x, w) |\lambda_\varepsilon|^d \overline{\left(\pi_x^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right)} \left(\pi_{(\delta_\varepsilon w)^{-1} x}^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right) dx dw.$$

Note that x and $(\delta_\varepsilon w)^{-1} x$ are compactly supported on the support of the integral defining R_ε. Therefore, we can find two compactly supported smooth functions χ_1 and χ_2 such that

$$R_\varepsilon = \int_{G \times G} \chi(\delta_\varepsilon w) \kappa_x(w) \overline{a(x_{\mathfrak{z}})} A_\varepsilon(x, w) \overline{f_1^\varepsilon(x)} f_2^\varepsilon((\delta_\varepsilon w)^{-1} x) dx dw,$$

where

$$f_j^\varepsilon(x) := |\lambda_\varepsilon|^{d/2} \chi_j(x_{\mathfrak{z}}) \left(\pi_x^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right), \qquad j = 1, 2.$$

We observed that the two families $x \mapsto f_j^\varepsilon(x)$ are bounded in $L^2(G)$ uniformly with respect to $\varepsilon > 0$. We therefore estimate the remainder in the following way, using first the estimate for A_ε:

$$|R_\varepsilon| \lesssim \varepsilon \sum_{j=1}^{2d} \sup_{G_{\mathfrak{z}}} |a| \sup_K |V_j a| \int_{G \times G} |w_j| \sup_x |\kappa_x|(w) |f_1^\varepsilon(w)| \; |f_2^\varepsilon((\delta_\varepsilon w)^{-1} x)| dx dw,$$

where K is a compact containing $\{x_1 x_2 : x_1 \in \operatorname{supp} \chi, x_2 \in x - \operatorname{supp} \sigma(x, \cdot)\}$; note that $\sup_K |V_j a| \lesssim \sup_{G_{\mathfrak{z}}} |\nabla a|$. Using the change of variable $w' = \delta_\varepsilon w$ and the Cauchy-Schwartz inequality, this last integral is estimated by

$$\int_{G \times G} |w_j| \sup_x |\kappa_x|(w) |f_1^\varepsilon(x)| \; |f_2^\varepsilon((\delta_\varepsilon w)^{-1} x)| dx dw$$

$$= \varepsilon^{-Q} \int_{G \times G} \varepsilon^{-1} |w_j'| \sup_x |\kappa_x|(\delta_{\varepsilon^{-1}} w') \; |f_1^\varepsilon(x)| \; |f_2^\varepsilon({w'}^{-1} x)| \; dx dw$$

$$\leq \||f_1^\varepsilon\|_{L^2} \varepsilon^{-Q} \|(\varepsilon^{-1} |w_j'| \sup_x |\kappa_x|(\delta_{\varepsilon^{-1}} \cdot)) * f_2^\varepsilon\|_{L^2}$$

$$\leq \||f_1^\varepsilon\|_{L^2} \varepsilon^{-Q} \|\varepsilon^{-1} |w_j'| \sup_x |\kappa_x|(\delta_{\varepsilon^{-1}} w')\|_{L^1(dw')} \|f_2^\varepsilon\|_{L^2}$$

$$= \||f_1^\varepsilon\|_{L^2} \|f_2^\varepsilon\|_{L^2} \||w_j| \sup_x |\kappa_x(w)|\|_{L^1(dw)}.$$

undoing the change of variable $w' = \delta_\varepsilon w$ after having used Young's convolution inequality. Hence we have obtained

$$|R_\varepsilon| \lesssim \varepsilon \sup_{G_{\mathfrak{z}}} |a| \sup_{G_{\mathfrak{z}}} |\nabla a| \|f_1^\varepsilon\|_{L^2} \|f_2^\varepsilon\|_{L^2} \max_{j=1,\dots,2d} \||w_j| \sup_x |\kappa_x(w)|\|_{L^1(dw)}.$$

We now concentrate on the main term E_ε. Writing

$$\pi_{(\delta_\varepsilon w)^{-1} x}^{\lambda_\varepsilon} = \pi_{(\delta_\varepsilon w)^{-1}}^{\lambda_\varepsilon} \pi_x^{\lambda_\varepsilon} = \pi_{w^{-1}}^{\varepsilon^2 \lambda_\varepsilon} \pi_x^{\lambda_\varepsilon} \quad \text{and} \quad \pi_x^{\lambda_\varepsilon} = e^{-i\lambda_\varepsilon \cdot x_{\mathfrak{z}}} \pi_{\lambda_\varepsilon} x_{\mathfrak{v}}$$

the integration in w gives

$$E_\varepsilon = \int_G |\lambda_\varepsilon|^d |a(x_{\mathfrak{z}})|^2 \overline{\left(\pi_x^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right)} \left(\sigma_\varepsilon(x, \varepsilon^2 \lambda_\varepsilon) \pi_x^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right) dx$$

$$= \int_G |\lambda_\varepsilon|^d |a_\varepsilon(x_{\mathfrak{z}})|^2 \overline{\left(\pi_{x_{\mathfrak{v}}}^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right)} \left(\sigma_\varepsilon(x, \varepsilon^2 \lambda_\varepsilon) \pi_{x_{\mathfrak{v}}}^{\lambda_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right) dx_{\mathfrak{v}} dx_{\mathfrak{z}},$$

$$= \int_G |a_\varepsilon(x_{\mathfrak{z}})|^2 \overline{\left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right)} \left(\sigma_\varepsilon((|\lambda_\varepsilon|^{-1/2} x_{\mathfrak{v}}) x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon \right) dx_{\mathfrak{v}} dx_{\mathfrak{z}},$$

having made the change of variable $x_{\mathfrak{v}} \mapsto |\lambda_\varepsilon|^{-1/2} x_{\mathfrak{v}}$. We write

$$E_\varepsilon = F_\varepsilon + S_\varepsilon,$$

where the first term is

$$F_\varepsilon := \int_G |a_\varepsilon(x_{\mathfrak{z}})|^2 \overline{\left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)} \left(\sigma_\varepsilon(x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right) dx_{\mathfrak{v}} dx_{\mathfrak{z}},$$

and the remainder is

$$S_\varepsilon := \int_0^1 \int_G |a(x_{\mathfrak{z}})|^2 \overline{\left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)} \frac{d}{ds} \left(\sigma_\varepsilon\left((s|\lambda_\varepsilon|^{-1/2} x_{\mathfrak{v}})\ x_{\mathfrak{z}}\right), \varepsilon^2 \lambda_\varepsilon) \pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right) dx\, ds.$$

For the first term, we use the facts that we have for any $\Phi_1, \widetilde{\Phi}_1, \Phi_2, \widetilde{\Phi}_2 \in \mathcal{S}(\mathbb{R}^d)$

$$\text{when } |\lambda| = 1, \quad \int_{G_{\mathfrak{v}}} \overline{(\pi_{x_{\mathfrak{v}}}^\lambda \Phi_1, \Phi_2)} \left(\pi_{x_{\mathfrak{v}}}^\lambda \widetilde{\Phi}_1, \widetilde{\Phi}_2\right) dx_{\mathfrak{v}} = (\Phi_1, \widetilde{\Phi}_1)(\widetilde{\Phi}_2, \Phi_2),$$

and also that $\|\Phi_\varepsilon\|_{L^2} = 1$ to obtain

$$F_\varepsilon = \int_{G_{\mathfrak{z}}} |a(x_{\mathfrak{z}})|^2 \left(\sigma_\varepsilon(x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \Phi_\varepsilon, \Phi_\varepsilon\right) dx_{\mathfrak{z}}.$$

For the remainder, we first write

$$\begin{aligned}
S_\varepsilon &= \int_G |\lambda_\varepsilon|^{-1/2} \int_0^1 |a(x_{\mathfrak{z}})|^2 \overline{\left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right)} \left(x_{\mathfrak{v}} \cdot \partial_{x_{\mathfrak{v}}} \sigma_\varepsilon(s(|\lambda_\varepsilon|^{-1/2} x_{\mathfrak{v}}) x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi_\varepsilon, \Phi_\varepsilon\right) ds\, dx \\
&= \int_{|x_{\mathfrak{v}}|>1} + \int_{|x_{\mathfrak{v}}|\le 1} := S_{1,\varepsilon} + S_{0,\varepsilon}.
\end{aligned}$$

For $S_{0,\varepsilon}$, we easily obtain

$$|S_{0,\varepsilon}| \le C\, |\lambda_\varepsilon|^{-1/2} \|a\|_{L^2} \|\Phi_\varepsilon\|_{L^2}^4.$$

For $S_{1,\varepsilon}$ we need the following observation. For each $\hat{\lambda}_\varepsilon$, we set $x_{\mathfrak{v}} = \text{Exp}[P+Q]$, $P = \sum_{1\le j\le d} p_j P_j$ and $Q = \sum_{1\le j\le d} q_j Q_j$ and we observe for $1 \le j \le d$ and for any $\Phi \in \mathcal{S}(\mathbb{R}^d)$

$$q_j \left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi, \Phi\right) = +i \left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \partial_{\xi_j} \Phi, \Phi\right) + i \left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi, \partial_{\xi_j} \Phi\right),$$

$$p_j \left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi, \Phi\right) = \left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} (\xi_j \Phi), \Phi\right) - \left(\pi_{x_{\mathfrak{v}}}^{\hat{\lambda}_\varepsilon} \Phi, (\xi_j \Phi)\right).$$

By using this observation, we deduce that $S_{1,\varepsilon}$ may be written as a linear combination of terms of the form

$$\begin{aligned}
S_{1,\varepsilon}^N := |\lambda_\varepsilon|^{-1/2} \int_0^1 \int_G \mathbf{1}_{|x_{\mathfrak{v}}|>1} |x_{\mathfrak{v}}|^{-N} |a(x_{\mathfrak{z}})|^2 \\
\times \overline{\left(\pi_{x_{\mathfrak{v}}}^{\widehat{\lambda}_\varepsilon} \Phi_{\varepsilon,1}, \Phi_{\varepsilon,2}\right)} \left(x_{\mathfrak{v}} \cdot \partial_{x_{\mathfrak{v}}} \sigma_\varepsilon(s(|\lambda_\varepsilon|^{-1/2} x_{\mathfrak{v}}) x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \pi_{x_{\mathfrak{v}}}^{\widehat{\lambda}_\varepsilon} \Phi_{\varepsilon,3}, \Phi_{\varepsilon,4}\right) dx\, ds
\end{aligned}$$

for each $N \in \mathbf{N}$ large enough ($N \ge 2d+1$) so that the integral is absolutely convergent and where the functions $\Phi_{\varepsilon,j}$ are linear combination of terms obtained by successive derivations of Φ_ε or by multiplication by powers of the coordinates of ξ (at most N-times in total). Therefore, these functions are in L^2 and there exists a constant $C > 0$ such that

$$|S_{1,\varepsilon}^N| \le C\, |\lambda_\varepsilon|^{-1/2} \|a\|_{L^2}^2 \|\Phi_{\varepsilon,1}\|_{L^2} \|\Phi_{\varepsilon,2}\|_{L^2} \|\Phi_{\varepsilon,3}\|_{L^2} \|\Phi_{\varepsilon,4}\|_{L^2}.$$

Combining all the estimates shows Lemma 5.3. $\qquad\square$

We can now particularise our study to the three cases of Proposition 5.2.

• Case (1): Let us choose $\Phi_\varepsilon = \Phi$ (with $\|\Phi\|^2 = 1$) independent of ε and $\varepsilon^2 \lambda_\varepsilon \xrightarrow[\varepsilon \to 0]{} \lambda_0$. Lemma 5.3 yields:

$$(\mathrm{Op}_\varepsilon(\sigma) v^\varepsilon, v^\varepsilon) = \int_{G_{\mathfrak{z}}} |a(x_{\mathfrak{z}})|^2 \left(\sigma_\varepsilon(x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \Phi, \Phi \right) dx_{\mathfrak{z}} + (\varepsilon)$$

$$= \int_{G_{\mathfrak{z}}} |a(x_{\mathfrak{z}})|^2 \left(\sigma(x_{\mathfrak{z}}, \lambda_0) \Phi, \Phi \right) dx_{\mathfrak{z}} + o(1),$$

by Lebesgue dominated convergence. Part (1) follows.

• Case (2): Let us choose now $\Phi_\varepsilon = h_{\alpha_\varepsilon}$ a Hermite function with α_ε and λ_ε as in Part (2). Well known properties of the Hermite function yield for any $N_0 \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$

$$\max_{|\beta_1| + |\beta_2| \leq N_0} \|\xi^{\beta_1} \partial_\xi^{\beta_2} h_\alpha\|_{L^2} \lesssim_{N_0} |\alpha|^{N_0/2}.$$

With the notation of Lemma 5.3, we compute

$$C_{1,\varepsilon,N} \leq C'_{1,N} |\alpha_\varepsilon|^{N/2}, \quad \text{so that } C_{1,\varepsilon,N} |\lambda_\varepsilon|^{-1/2} \lesssim |\alpha_\varepsilon|^{(N+1)/2} \varepsilon \xrightarrow{\varepsilon \to 0} 0,$$

for $N = 2d + 1$ and with the decay required in Part (2). Therefore we have

$$(\mathrm{Op}_\varepsilon(\sigma) v^\varepsilon, v^\varepsilon) = \int_{G_{\mathfrak{z}}} |a(x_{\mathfrak{z}})|^2 \left(\sigma(x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) \Phi_\varepsilon, \Phi_\varepsilon \right) dx_{\mathfrak{z}} + o(1).$$

The first consequence of this relation and of $\varepsilon^2 \lambda_\varepsilon \to 0$ is that any semi-classical measure $\Gamma d\gamma$ of the family (v^ε) is supported in $\{(0, \omega), \ \omega \in \mathfrak{v}^*\}$. Indeed, if $\sigma(x, \lambda)$ is compactly supported in $\mathfrak{z}^* \setminus \{0\}$, then the principal term of the right-hand side above is 0 as ε is small enough. We deduce that any semi-classical measure of (v^ε) is equivalent to

$$\Gamma(x, \lambda) = 1 \ \text{ and } \ \gamma(x, \lambda) = \mathbf{1}_{\lambda = 0} \gamma(x, (0, \omega)).$$

Besides, by Lemma 2.2 and Remark 2.3, we have

$$\int_{SO(\mathfrak{v})} \left(\sigma_\varepsilon^{(k)}(x_{\mathfrak{z}}, \varepsilon^2 \lambda_\varepsilon) h_{\alpha_\varepsilon}, h_{\alpha_\varepsilon} \right) dk \xrightarrow[\varepsilon \to 0]{} \int_{|\varsigma| = 1} \sigma(x_{\mathfrak{z}}, (0, \mu\varsigma)) d\varsigma,$$

where the symbol $\sigma^{(k)}$ is associated with the convolution kernel

$$\kappa_x^{(k)}(z) = \kappa_x(\mathrm{Exp}(k z_{\mathfrak{v}} + z_{\mathfrak{z}})), \quad z = \mathrm{Exp}(z_{\mathfrak{v}} + z_{\mathfrak{z}}), \quad k \in SO(\mathfrak{v}).$$

We deduce by Lebesgue dominated convergence,

$$\int_{SO(\mathfrak{v})} \left(\mathrm{Op}_\varepsilon(\sigma^{(k)}) v^\varepsilon, v^\varepsilon \right) dk \xrightarrow[\varepsilon \to 0]{} \int_{G_{\mathfrak{z}}} \int_{|\varsigma| = 1} |a(x_{\mathfrak{z}})|^2 \sigma(x_{\mathfrak{z}}, (0, \mu\varsigma)) d\varsigma dx_{\mathfrak{z}}.$$

In view of

$$\sigma^{(k)}(x, (0, \omega)) = \sigma(x, (0, k\omega)), \quad \forall \omega \in \mathfrak{v},$$

the latter relation implies that the measure $\gamma(x, (0, \omega))$ satisffies

$$\int_{SO(\mathfrak{v})} \int_{G \times \mathfrak{v}^*} \sigma(x_{\mathfrak{z}}, (0, k\omega)) d\gamma(x, (0, \omega)) dk = \int_{G_{\mathfrak{z}}} \int_{|\varsigma| = 1} |a(x_{\mathfrak{z}})|^2 \sigma(x_{\mathfrak{z}}, (0, \mu\varsigma)) d\varsigma dx_{\mathfrak{z}}.$$

This implies Part (2).

6. APPENDIX A - SYMBOLIC CALCULUS

We prove here Proposition 3.6.

Proof. Let us consider the composition of the operators of symbols $\sigma_1(x, \lambda)$ and $\sigma_2(x, \lambda)$ and denote by $\kappa_x^\varepsilon(z)$ its convolution kernel. This function $\kappa_x^\varepsilon(z)$ can be expressed in terms of the convolution kernels $\kappa_{1,x}(z)$ and $\kappa_{2,x}(z)$ associated with $\sigma_1(x, \lambda)$ and $\sigma_2(x, \lambda)$ respectively. Indeed, we have

$$\varepsilon^{-Q}\kappa_x^\varepsilon(\delta_{\varepsilon^{-1}}(y^{-1}x)) = \varepsilon^{-2Q}\int_G \kappa_{1,x}(\delta_{\varepsilon^{-1}}(v^{-1}x))\kappa_{2,v}(\delta_{\varepsilon^{-1}}(y^{-1}v))dv.$$

Therefore, performing the change of variable $z = \delta_{\varepsilon^{-1}}(y^{-1}x)$ and $u = \delta_{\varepsilon^{-1}}(v^{-1}x)$, we obtain

$$\kappa_x^\varepsilon(z) = \int_G \kappa_{1,x}(u)\kappa_{2,x\delta_\varepsilon u^{-1}}(zu^{-1})du.$$

This function $\kappa_x^\varepsilon(z)$ is smooth and compactly supported in x and Schwartz class in z, besides there exists a constant $C > 0$ such that

$$\int_G \sup_{x \in G} |\kappa_x^\varepsilon(z)|dz \leq C,$$

which implies that the family

$$\sigma^\varepsilon(x, \lambda) = \mathcal{F}(\kappa_x^\varepsilon(\cdot))(\lambda)$$

is a family of symbols of $\mathcal{A}_0$ which generates a bounded family of operators on $\mathcal{L}(L^2(G))$. We are now going to prove that the symbol $\sigma^\varepsilon(x, \lambda)$ has an asymptotic expansion. For this, we perform a Taylor expansion of $u \mapsto \kappa_{2,x(\delta_\varepsilon u^{-1})}(z)$:

$$\kappa_{2,x(\delta_\varepsilon u^{-1})}(z) = \kappa_{2,x}(z) - \varepsilon \sum_{1 \leq j \leq 2d} v_j(u)V_j\kappa_{2,x}(z) + \varepsilon^2 R_\varepsilon(x, z, u)$$

where, according to the Taylor expansion result of Theorem 3.1.51 in [28], there exists constants $\eta, C > 0$ such that

$$|R_\varepsilon(x, v, u)| \leq C \sum_{|\alpha| \leq 2, [\alpha] > 1} |u|^{[\alpha]} \sup_{z \in G} |X^\alpha \kappa_{2,z}(v)|,$$

where $[\alpha]$ is the homogeneous length of α:

$$[\alpha] = \sum_{j=1}^{2d} \alpha_j + 2\sum_{j=2d+1}^{2d+p} \alpha_j \ \text{if} \ \alpha = (\alpha_1, \cdots, \alpha_{2d}, \alpha_{2d+1}, \cdots, \alpha_{2d+p}).$$

Therefore,

$$\kappa_x^\varepsilon(z) = \int_G \kappa_{1,x}(u)\kappa_{2,x}(zu^{-1})du - \varepsilon \sum_{1 \leq j \leq 2d}\int_G v_j(u)\kappa_{1,x}(u)V_j\kappa_{2,x}(zu^{-1})du + \varepsilon^2 r_x^\varepsilon(z)$$

and there exists a constant $C > 0$ such that for all $\varepsilon > 0$

$$\int_G \sup_{x \in G} |r_x^\varepsilon(v)|dv \leq \sum_{|\alpha| \leq 2, [\alpha] > 1} \sup_{x \in G}\left(\int_G |u|^{[\alpha]}|\kappa_{1,x}(u)|du\right)\left(\int_G \sup_{x \in G}|X^\alpha \kappa_{2,x}(v)|dv\right) < \infty.$$

As a consequence, the operator with convolution kernel $\varepsilon^{-Q}r_x^\varepsilon(\delta_{\varepsilon^{-1}}z)$ is uniformly bounded in $\mathcal{L}(L^2(\mathbb{R}^d))$. Besides,

$$\int_G \kappa_{1,x}(u)\kappa_{2,x}(zu^{-1})du = (\kappa_{2,x} * \kappa_{1,x})(z)$$

and

$$\mathcal{F}(\kappa_{2,x} * \kappa_{1,x})(\lambda) = \mathcal{F}(\kappa_{1,x})(\lambda) \circ \mathcal{F}(\kappa_{2,x})(\lambda) = \sigma_1(x, \lambda) \circ \sigma_2(x, \lambda).$$

Similarly, observing that

$$\int_G v_j(u)\kappa_{1,x}(u)V_j\kappa_{2,x}(zu^{-1})du = V_j\kappa_{2,x}*(v_j\kappa_{1,x})$$

and

$$\mathcal{F}(v_j\kappa_{1,x})(\lambda) = \Delta_{v_j}\mathcal{F}(\kappa_{1,x})(\lambda) = \Delta_{v_j}\sigma_1(x,\lambda),$$

we obtain

$$\mathcal{F}(V_j\kappa_{2,x}*(v_j\kappa_{1,x}))(\lambda) = \mathcal{F}(v_j\kappa_{1,x})(\lambda) \circ \mathcal{F}(V_j\kappa_{2,x})(\lambda) = \Delta_{v_j}\sigma_1(x,\lambda) \circ V_j\sigma_2(x,\lambda).$$

The relations

$$\begin{pmatrix} p \\ q \end{pmatrix} = M^\lambda v, \quad \begin{pmatrix} \Delta_p \\ \Delta_q \end{pmatrix} = M^\lambda \Delta_v, \quad (M^\lambda)^{-1} = {}^t M^\lambda,$$

stated in section 3.4 allows to conclude and obtain the relation (3.13).

The proof concerning the adjoint of the operator of symbol $\sigma(x,\lambda)$ is similar since its convolution kernel $\kappa_x^{\varepsilon,*}(z)$ is given by

$$\kappa_x^{\varepsilon,*}(z) = \overline{\kappa}_{x(\delta_\varepsilon z)^{-1}}(z).$$

One can then use Taylor expansion and identify the first terms of the expansion in the same manner than in the preceding proof. $\qquad\square$

7. APPENDIX B : THE STATES OF THE C^*-ALGEBRA $\mathcal{A}$

We give here a proof of Proposition 4.3, which is adapted form the proofs of Propositions 5.15 and 5.17 in [26].

Given $(\gamma,\Gamma) \in \mathcal{M}_1^+(G \times \widehat{G})$ satisfying (4.1), one checks easily that the linear form ℓ defined via (4.2) is a state of $\mathcal{A}$. Therefore, the proof consists in proving that any state can be represented by a unique $\gamma d\Gamma$. Let ℓ be a state of the C^*-algebras $\mathcal{A}$. The GNS construction [23, Proposition 2.4.4] yields a representation ρ of $\mathcal{A}$ on the Hilbert space $\widetilde{\mathcal{H}}_\ell := \mathcal{A}/\{\sigma : \ell(\sigma\sigma^*) = 0\}$ such that

$$\ell(\sigma) = \lim_{n\to+\infty} (\rho(\sigma)\xi_n, \xi_n)_{\widetilde{\mathcal{H}}_\ell}, \quad \sigma \in \mathcal{A},$$

where the sequence $(\xi_n)_{n\in\mathbb{N}}$ is the image of any approximate identity of $\mathcal{A}$ via the canonical projection $\mathcal{A} \mapsto \widetilde{\mathcal{H}}_\ell$. We then decompose [23, Theorem 8.6.6] the representation ρ (taking into account the possible multiplicities) as

$$(\rho, \widetilde{\mathcal{H}}_\ell) \sim (\rho_1, \widetilde{\mathcal{H}}_1) \oplus 2(\rho_2, \widetilde{\mathcal{H}}_2) \oplus \ldots \oplus \aleph_0(\rho_\infty, \widetilde{\mathcal{H}}_\infty),$$

and each ρ_r, $r \in \mathbf{N} \cup \{\infty\}$, may be disintegrated as

$$\rho_r \sim \int_{\widehat{\mathcal{A}}} \zeta d\gamma_r(\zeta);$$

furthermore, the positive measures $\gamma_1, \gamma_2, \ldots, \gamma_\infty$ are mutually singular in $\widehat{\mathcal{A}}$. Consequently we can write $\xi \in \widetilde{\mathcal{H}}_\ell$ as

$$\xi \sim (\xi_1, \xi_2, \ldots, \xi_\infty), \quad \text{with } \xi_r = (\xi_{r,s})_{1\le s\le r} \text{ for each } r \in \mathbf{N} \cup \{\infty\}, \text{ and } \xi_{r,s} \in \widetilde{\mathcal{H}}_r.$$

Note that

$$1 = |\xi|^2_{\widetilde{\mathcal{H}}_\ell} = \sum_{r\in\mathbf{N}\cup\{\infty\}} \sum_{s=1}^{r} |\xi_{r,s}|^2_{\widetilde{\mathcal{H}}_r} \quad \text{with} \quad |\xi_{r,s}|^2_{\widetilde{\mathcal{H}}_r} = \int_{\widehat{\mathcal{A}}} |\xi_{r,s}(\zeta)|^2_{\widetilde{\mathcal{H}}_\zeta} d\gamma_r(\zeta).$$

Since we have identifed $\widehat{\mathcal{A}}$ with $G \times \widehat{G}$:

$$\rho_r \sim \int_{G\times\widehat{G}} (x,\lambda)d\gamma_r(x,\lambda), \quad \mathcal{H}_r \sim \int_{G\times\widehat{G}} \mathcal{H}_\lambda, \quad \sum_{r=1}^{\infty}\sum_{s=1}^{r} \int_{G\times\widehat{G}} |\xi_{r,s}(x,\lambda)|^2_{\mathcal{H}_\lambda} d\gamma_r(x,\lambda) = 1.$$

Hence $\Gamma_r := \sum_{s=1}^{r} \xi_{r,s} \otimes (\xi_{r,s})^*$ is a γ_r-measurable field on $G \times \widehat{G}$ of positive trace-class operators of rank r. We have obtained:

$$\ell(\sigma) = (\rho(\sigma)\xi, \xi) = \sum_{r \in \mathbf{N} \cup \{\infty\}} \sum_{s=1}^{r} \int_{G \times \widehat{G}} (\sigma(x, \lambda)\xi_{r,s}(x, \lambda), \xi_{r,s}(x, \lambda))_{L^2(\mathbb{R}^d)} d\gamma_r(x, \lambda)$$

$$= \sum_{r \in \mathbf{N} \cup \{\infty\}} \int_{G \times \widehat{G}} \mathrm{Tr}\,(\sigma(x, \lambda)\Gamma_r(x, \lambda))\, d\gamma_r(x, \lambda).$$

We now define the positive measure $\gamma := \sum_r \gamma_r$. As the measures γ_r are mutually singular, the field $\Gamma := \sum_r \Gamma_r$ is measurable and satisfies

$$\Gamma(x, \lambda) \geq 0, \quad \mathrm{Tr}\,(\Gamma(x, \lambda)) < \infty, \qquad \int_{G \times \widehat{G}} \mathrm{Tr}\,(\Gamma(x, \lambda))\, d\gamma(x, \lambda) = 1\,.$$

This shows the existence of the pair (Γ, γ).

Let us now prove that this pair is unique up to the equivalence class and consider $(\gamma', \Gamma') \in \mathcal{M}_1^+(G \times \widehat{G})$ which also satisfies (4.1) and (4.2) for the same state ℓ. It suffices to consider the case of γ and Γ obtained as in the preceding argumentation and we may assume that γ' and Γ' have the same support in $G \times \widehat{G}$. For each $r \in \mathbf{N} \cup \{\infty\}$, let B_r be the measurable subset of $G \times \widehat{G}$ where $\Gamma'(x, \lambda)$ is of rank r a.e. We may assume these subsets disjoint. We define the measure $\gamma'_r = 1_{B_r} \gamma'$ and the field $\Gamma'_r := 1_{B_r} \Gamma'$ as the restrictions of γ' and Γ' to B_r. As Γ'_r is a measurable field of positive operators of rank r, there exists a measurable field of orthogonal vectors $(\xi_{r,s})_{s=1}^{r}$ such that

$$\Gamma'_r = \sum_{s=1}^{r} \xi'_{r,s} \otimes (\xi'_{r,s})^*$$

and we have

$$\mathrm{Tr}\,\Gamma'_r = \sum_{s=1}^{r} |\xi'_{r,s}|^2.$$

We define the representation ρ' of $\mathcal{A}$ and the vector ξ' of ρ' via

$$\rho' := \oplus_{r \in \mathbf{N} \cup \{\infty\}} r \int_{G \times \widehat{G}} (x, \lambda)\, d\gamma'_r(x, \lambda), \qquad \text{and} \qquad \xi' := \oplus_{r \in \mathbf{N} \cup \{\infty\}} \oplus_{s=1}^{r} \int_{G \times \widehat{G}} \xi'_{r,s}(x, \lambda)\, d\gamma'_r(x, \lambda).$$

We observe that ξ' is a unit vector:

$$|\xi'|^2 = \sum_{r \in \mathbf{N} \cup \{\infty\}} \sum_{s=1}^{r} |\tilde{\xi}'_{r,s}|^2 = \sum_{r \in \mathbf{N} \cup \{\infty\}} \int_{G \times \widehat{G}} \mathrm{Tr}\,\Gamma'_r\, d\gamma'_r = \int_{G \times \widehat{G}} \mathrm{Tr}\,\Gamma'\, d\gamma' = 1.$$

Moreover for any $\sigma \in \mathcal{A}$:

$$(\rho'(\sigma)\xi', \xi') = \sum_{r \in \mathbf{N} \cup \{\infty\}} \sum_{s=1}^{r} \int_{\Sigma_1} (\sigma\xi'_{r,s}, \xi'_{r,s})\, d\gamma'_r = \sum_{r \in \mathbf{N} \cup \{\infty\}} \int_{G \times \widehat{G}} \mathrm{Tr}\,(\sigma\Gamma'_r)\, d\gamma'_r$$

$$= \int_{G \times \widehat{G}} \mathrm{Tr}\,(\sigma\Gamma')\, d\gamma' = \ell(\sigma).$$

In other words, the state associated with ρ' and ξ' coincides with ℓ. This implies that ρ' and ρ are equivalent [23, Proposition 2.4.1], therefore the measures γ'_r and γ_r are equivalent for every $r \in \mathbf{N} \cup \{\infty\}$ [23, Theorem 8.6.6]. In other words, there exists a measurable positive function f_r supported in B_r such that

$$d\gamma'_r(x, \lambda) = f_r(x, \lambda)d\gamma_r(x, \lambda).$$

As ξ' corresponds to ξ via the (ρ', ρ)-equivalence, we must have

$$\Gamma_r(x, \lambda) = f_r(x, \lambda)\Gamma'_r(x, \lambda),$$

which concludes the proof.

References

[1] F. Astengo, B. Di Blasio and F. Ricci. Gelfand pairs on the Heisenberg group and Schwartz functions, *J. Funct. Anal.*, 256, 2009, No 5, pp 1565–1587.

[2] N. Anantharaman, F. Faure and C. Fermanian Kammerer. Le chaos quantique, Actes des journes X-UPS 2014, Éditions de l'École Polytechnique.

[3] N. Anantharaman, C. Fermanian Kammerer and F. Macià. Long-time dynamics of completely integrable Schrodinger flows on the torus, *Amer. J. Math.* 137 (2015), no. 3, p. 577-638.

[4] N. Anantharaman and F. Macià. The dynamics of the Schrödinger flow from the point of view of semiclassical measures. In *Spectral geometry*, volume 84 of *Proc. Sympos. Pure Math.*, pages 93–116. Amer. Math. Soc., Providence, RI, 2012.

[5] N. Anantharaman and F. Macià. Semiclassical measures for the Schrödinger equation on the torus. *J. Eur. Math. Soc. (JEMS)*, 16(6):1253–1288, 2014.

[6] H. Bahouri, J.-Y. Chemin and R. Danchin. A Frequency Space for the Heisenberg Group (arXiv:1609.03850).

[7] H. Bahouri, J.-Y. Chemin, R. Danchin. Tempered distributions and Fourier transform on the Heisenberg group (arXiv:1705.02195).

[8] H. Bahouri, J.-Y. Chemin, R. Danchin. Monograph to appear.

[9] H. Bahouri, J.-Y. Chemin, I. Gallagher. Precised Hardy inequalities on Rd and on the Heisenberg group H^d. Séminaire Équations aux Dérivées Partielles. 20042005, Exp. No. XIX, 17 pp., Smin. qu. Driv. Partielles, cole Polytech., Palaiseau, 2005.

[10] H. Bahouri, J.-Y. Chemin, C.-J. Xu. Trace theorem on the Heisenberg group. *Ann. Inst. Fourier* (Grenoble) 59 (2009), no. 2, 491514.

[11] H. Bahouri, J.-Y. Chemin, C.-J. Xu. Trace theorem on the Heisenberg group on homogeneous hypersurfaces. Phase space analysis of partial differential equations, 115, Progr. Nonlinear Differential Equations Appl., 69, Birkhuser Boston, Boston, MA, 2006.

[12] H. Bahouri, J.-Y. Chemin, C.-J. Xu. Trace theorem in Sobolev spaces associated with Hörmander's vector fields. Partial differential equations and their applications (Wuhan, 1999), 114, World Sci. Publ., River Edge, NJ, 1999.

[13] H. Bahouri, C. Fermanian Kammerer and I. Gallagher. Phase space analysis on the Heisenberg group. *Astérisque*, 345 (2012).

[14] H. Bahouri, C. Fermanian Kammerer and I. Gallagher. Dispersive estimates for the Schrödinger operator on step 2 stratified Lie Groups, *Analysis and PDEs*, 9 (2016), no. 3, p. 545-574.

[15] H. Bahouri, P. Gérard and C.-J. Xu. Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg, *Journal d'Analyse Mathématique* 82 (2000), pages 93–118.

[16] R. Beals and P. Greiner, *Calculus on Heisenberg manifolds*, Annals of Mathematics Studies, 119, Princeton University Press, Princeton, NJ, 1988.

[17] V. Chabu, C. Fermanian-Kammerer and F. Macià. Semiclassical analysis of dispersion phenomena (arXiv:1803.08771).

[18] M. Christ, D. Geller, P. Glowacki and L. Polin, Pseudodifferential operators on groups with dilations, Duke Math. J., 68, 1992, No 1, pp 31–65.

[19] P. Ciatti, F. Ricci and M. Sundari. Uncertainty inequalities on stratified nilpotent groups, *Bulletin of Kerala Mathematics Association*, Special Issue, (2007), pages 53-72.

[20] Y. Colin de Verdière, L. Hillairet, E. Trélat: Spectral asymptotics for sub-Riemannian Laplacians. I: quantum ergodicity and quantum limits in the 3D contact case, *Duke Math. J.* 167 (2018), no. 1, p. 109-174

[21] E. Damek and F. Ricci, Harmonic analysis on solvable extensions of H-type groups, *J. Geom. Anal.*, 2, 1992, No 3, pp 213–248.

[22] M. Del Hiero. Dispersive and Strichartz estimates on H-type groups, *Studia Math*, 169 (2005), pages 1-20.

[23] J. Dixmier, C^*-algebras, Translated from the French by Francis Jellett, *North-Holland Mathematical Library*, 15, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[24] J. Faraut and K. Harzallah, *Deux cours d'analyse harmonique*, Progress in Mathematics, 69, Papers from the Tunis summer school held in Tunis, August 27–September 15, 1984, Birkhäuser Boston, Inc., Boston, MA, 1987.

[25] C. Fermanian Kammerer and P. Gérard. A Landau-Zener formula for non-degenerated involutive codimension three crossings, *Ann. Henri Poincaré* 4, 513–552, 2003.

[26] C. Fermanian Kammerer and V. Fischer. Defect measures on graded lie groups (arXiv:1707.04002).

[27] V. Fischer, F. Ricci and O. Yakimova, Nilpotent Gelfand pairs and spherical transforms of Schwartz functions I: rank-one actions on the centre, *Math. Z.*, 271, 2012, No 1-2, pp 221–255.

[28] V. Fischer and M. Ruzansky. Quantization on nilpotent Lie groups, *Progress in Mathematics*, 314, Birkhäuser Basel, 2016.

[29] G. B. Folland and E. Stein. Hardy Spaces on Homogeneous Groups, *Annals of Mathematics Studies, Princeton University Press*, (1982).

[30] D. Geller. Analytic pseudodifferential operators for the Heisenberg group and local solvability, volume 37 of Mathematical Notes. Princeton University Press, Princeton, NJ, 1990.

[31] P. Gérard. Mesures semi-classiques et ondes de Bloch. In *Séminaire sur les Équations aux Dérivées Partielles, 1990–1991*, pages Exp. No. XVI, 19. École Polytech., Palaiseau, 1991.

[32] P. Gérard and E. Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.*, 71(2):559–607, 1993.

[33] P. Gérard, P. A. Markowich, N. J. Mauser, and F. Poupaud. Homogenization limits and Wigner transforms. *Comm. Pure Appl. Math.*, 50(4):323–379, 1997.

[34] B. Helffer. *30 ans d'analyse semi-classique, bibliographie commentée*, http://www.math.u-psud.fr/~helffer/histoire2003.ps.

[35] B. Helffer, J. Sjöstrand. Equation de Schrödinger avec champ magnétique et équation de Harper, volume 345 of *Lecture Notes in Physics*. Springer Verlag, 1989.

[36] L. Hörmander. The Weyl calculus of pseudodifferential operators, *Comm. Pure Appl. Math.* 32 (1979), no. 3, 360–444.

[37] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, *Trans. Amer. Math. Soc.*, 258, 1980, No 1, pp 147–153.

[38] A. Korányi, Some applications of Gelfand pairs in classical analysis, in *Harmonic analysis and group representations*, 333–348, Liguori, Naples, 1982.

[39] G. Lévy. The Fourier transform on 2-step Lie groups (arXiv:1712.09880).

[40] P.-L. Lions et T. Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.

[41] F. Macià. High-frequency propagation for the Schrödinger equation on the torus. J. Funct. Anal., 258 (3), 933–955, 2010. . Macià.

[42] M. Mantoiu, M. Ruzhansky. Pseudo-Differential Operators, Wigner Transform and Weyl Systems on Type I Locally Compact Groups. *Documenta Mathematica*, 22 (2017), 1539–1592

[43] D. Müller and F. Ricci. Solvability for a class of doubly characteristic differential operators on two-step nilpotent groups, *Annals of Mathematics*, 143 (1996), p. 1-49.

[44] M. Ruzhansky, V. Turunen. On the Fourier analysis of operators on the torus, in *Modern trends in pseudo-differential operators*, Oper. Theory Adv. Appl., vol. 172, Birkhäuser, Basel, 2007, 87–105.

[45] M.l A. Shubin. Pseudodifferential operators and spectral theory, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1987, in russian: 1978.

[46] E. Stein and G. Weiss. Introduction to Fourier analysis on euclidean spaces, *Princeton Mathematical Series, Princeton University Press, Princeton, N.J.*, 32 (1971).

[47] R. Strichartz, Sub-Riemannian geometry, *J. Differential Geom.*, 24, 1986, No 2, pp 221–263.

[48] M. E. Taylor. Noncommutative microlocal analysis. I. Mem. Amer. Math. Soc., 52(313):iv+182, (Revised version accessible at http://math.unc.edu/Faculty/met/ncmlms.pdf) 1984.

[49] E. P. Wigner : *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*. New York, Academic Press, 1959.

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