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Semi-cotangent bundle and problems of lifts

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Abstract: Using the fiber bundle M over a manifold B, we define a semi-cotangent (pull-back) bundle t * B, which has a degenerate symplectic structure. We consider lifting problem of projectable geometric objects on M to the semi-cotangent bundle. Relations between lifted objects and a degenerate symplectic structure are also presented.

Key words: Vector field, complete lift, basic 1-form, semi-cotangent bundle

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $\pi_1 : M_n \to B_m$ the differentiable bundle determined by a submersion π_1. Suppose that $(x^i) = (x^a, x^\alpha), a, b, ... = 1, ..., n-m; \alpha, \beta, ... = n-m+1, ..., n; i, j, ... = 1, 2, ..., n$ is a system of local coordinates adapted to the bundle $\pi_1 : M_n \to B_m$, where x^α are coordinates in B_m, and x^a are fiber coordinates of the bundle $\pi_1 : M_n \to B_m$. If $(x^{a'}, x^{\alpha'})$ is another system of local adapted coordinates in the bundle, then we have

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}(x^\beta). \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has components

$$(A_j^{i'}) = \left(\frac{\partial x^{i'}}{\partial x^j} \right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix}.$$

Let $T_x^*(B_m)(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$ be the cotangent space at a point x of B_m. If p_α are components of $p \in T_x^*(B_m)$ with respect to the natural coframe $\{dx^\alpha\}$, i.e. $p = p_i \ dx^i$, then by definition the set of all points $(x^I) = (x^a, x^\alpha, x^{\overline{\alpha}})$, $x^{\overline{\alpha}} = p_\alpha$, $\overline{\alpha} = \alpha + m$, $I = 1, ..., n+m$ is a semi-cotangent bundle $t^*(B_m)$ over the manifold M_n.

The semi-cotangent bundle $t^*(B_m)$ has the natural bundle structure over B_m, its bundle projection $\pi : t^*(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$. If we introduce a mapping $\pi_2 : t^*(B_m) \to M_n$ by $\pi_2 : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^a, x^\alpha)$, then $t^*(B_m)$ has a bundle structure over M_n. It is easily verified that $\pi = \pi_1 \circ \pi_2$.

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On the other hand, let now $\pi : E \to B$ be a fiber bundle and let $f : B' \to B$ be a differentiable map. It is well known that the pull-back (induced) bundle or Whitney product is defined by the total space (see, for example [2,3,6])

$$f^*E = \{(b', e) \in B' \times E \,|\, f(b') = \pi(e)\} \subset B' \times E$$

and the projection map $\pi' : f^*E \to B'$ is given by the projection onto the first factor, i.e.

$$\pi'(b', e) = b'.$$

The generalization of pull-back bundles to higher order cases is known as Pontryagin bundles [4].

From the above definition it follows that the semi-cotangent bundle $(t^*(B_m), \pi_2)$ is a pull-back bundle of the cotangent bundle over B_m by π_1.

To a transformation (1.1) of local coordinates of M_n, there corresponds on $t^*(B_m)$ the coordinate transformation

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}(x^\beta), \\ x^{\overline{\alpha}'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} x^{\overline{\beta}}. \end{cases} \tag{1.2}$$

The Jacobian of (1.2) is given by

$$\overline{A} = (A_J^{I'}) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & p_\sigma A_\beta^{\beta'} A_{\beta'\alpha'}^{\alpha} & A_{\alpha'}^{\beta} \end{pmatrix}, \tag{1.3}$$

where

$$A_{\beta'\alpha'}^{\alpha} = \frac{\partial^2 x^\alpha}{\partial x^{\beta'} \partial x^{\alpha'}}.$$

It is easily verified that the condition $Det\ \overline{A} \neq 0$ is equivalent to the non-vanishing of the diagonal matrices:

$$Det(A_b^{a'}) \neq 0, \quad Det(A_\beta^{\alpha'}) \neq 0, \quad Det(A_{\alpha'}^{\beta}) \neq 0.$$

Also, $\dim t^*(B_m) = n + m$. In the special case $n = m$, $t^*(B_m)$ is a cotangent bundle $T^*(M_n)$ [8, p. 224].

We note that semi-tangent bundles and their properties were studied in [1,5,7]. The main purpose of this paper is to study semi-cotangent bundles and some of their lift problems.

We denote by $\Im_q^p(B_m)$ the module over $F(B_m)$ of all tensor fields of type (p, q) on B_m, where $F(B_m)$ denotes the ring of real-valued C^∞-functions on B_m.

2. Basic 1-form in the semi-cotangent bundle

Let us consider a 1-form p in $\pi^{-1}(U) \in t^*(B_m)$, $U \subset B_m$, whose components are $(0, p_\alpha, 0)$. Taking account of (1.3), we easily see that $p = \overline{A} p'$, where

$$p = (0, p_\alpha, 0), \; p' = (0, p_{\alpha'}, 0).$$

We call the 1-form p a basic 1-form on $t^*(B_m)$.

The exterior differential dp of the basic 1-form p is the 2-form given by

$$dp = dp_\alpha \wedge dx^\alpha.$$

Hence, if we write $dp = \omega = \frac{1}{2}\omega_{AB}dx^A \wedge dx^B$, then we have

$$\omega = (\omega_{AB}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta^\alpha_\beta \\ 0 & \delta^\beta_\alpha & 0 \end{pmatrix},$$

where $A = (a, \alpha, \overline{\alpha})$, $B = (b, \beta, \overline{\beta})$. Since $d\omega = d^2p = 0$, we have:

Theorem 1 *The semi-cotangent bundle $t^*(B_m)$ has a degenerate symplectic structure ω.*

3. Vertical lift of 1-form

If f is a function on B_m, we write ${}^{vv}f$ for the function on $t^*(B_m)$ obtained by forming the composition of $\pi : t^*(B_m) \to B_m$ and ${}^v f = f \circ \pi_1$, so that

$$^{vv}f =^v f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \tag{3.1}$$

Then we have

$$^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha).$$

Thus, the value ${}^{vv}f$ is constant along each fiber of $\pi : t^*(B_m) \to B_m$. We call ${}^{vv}f$ the vertical lift of the function f.

Let $\widetilde{X} \in \Im^1_0(t^*(B_m))$ be a vector field such that $\widetilde{X}({}^{vv}f) = 0$ for all functions $f \in \Im^0_0(B_m)$. Then we say that $\widetilde{X}$ is a vertical vector field on $t^*(B_m)$. If $\begin{pmatrix} \widetilde{X}^a \\ \widetilde{X}^\alpha \\ \widetilde{X}^{\overline{\alpha}} \end{pmatrix}$ are components of $\widetilde{X}$ with respect to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$, then for the vertical vector field we have

$$\begin{aligned} \widetilde{X}^a \partial_a {}^{vv}f + \widetilde{X}^\alpha \partial_\alpha {}^{vv}f + \widetilde{X}^{\overline{\alpha}} \partial_{\overline{\alpha}} {}^{vv}f &= 0, \\ \widetilde{X}^\alpha \partial_\alpha {}^{vv}f &= 0, \\ \widetilde{X}^\alpha &= 0. \end{aligned}$$

Thus, the vertical vector field $\widetilde{X}$ on $t^*(B_m)$ has components

$$\widetilde{X} = (\widetilde{X}^A) = \begin{pmatrix} \widetilde{X}^a \\ 0 \\ \widetilde{X}^{\overline{\alpha}} \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

Let ω be a 1-form with local components ω_α on B_m, so that ω is a 1-form with local expression $\omega = \omega_\alpha dx^\alpha$. On putting

$$^{vv}\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \tag{3.2}$$

we have a vector field $^{vv}\omega$ on $t^*(B_m)$. In fact, from (1.3) we easily see that $(^{vv}\omega)' = \overline{A}(^{vv}\omega)$. The vector field thus introduced is called the vertical lift of the 1-form ω to $t^*(B_m)$. Clearly, we have

$$^{vv}\omega(^{vv}f) = 0$$

for any $f \in \Im_0^0(B_m)$, so that $^{vv}\omega$ is a vertical vector field. In particular, if $\omega = p$, then ^{vv}p is a Liouville covector field on $t^*(B_m)$.

From (3.2) we have:

Theorem 2 *For any 1-forms ω, θ and function f on B_m,*

(i) $^{vv}(\omega + \theta) =^{vv} \omega +^{vv} \theta$,

(ii) $^{vv}(f\omega) = {}^{vv}f\ {}^{vv}\omega$.

For the natural coframe dx^α in each U, from (3.2) we have in $\pi^{-1}(U)$

$$^{vv}(dx^\alpha) = \frac{\partial}{\partial p_\alpha}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

4. $\gamma-$ Operator

Let X be a vector field on B_m. We define a function γX on $t^*(B_m)$ by

$$\gamma X = p_\beta X^\beta. \tag{4.1}$$

For any $F \in \Im_1^1(B_m)$, if we take account of (1.3), we can prove that $(\gamma F)' = \overline{A}(\gamma F)$ where γF is a vector field defined by

$$\gamma F = (\gamma F^A) = \begin{pmatrix} 0 \\ 0 \\ p_\beta F_\alpha^\beta \end{pmatrix}, \tag{4.2}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. Then we have

$$(\gamma F)^{vv}(f) = 0$$

for any $f \in \Im_0^0(B_m)$, i.e. γF is a vertical vector field on $t^*(B_m)$.

Let $T \in \Im_2^1(B_m)$. On putting

$$\gamma T = (\gamma T_B^A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_\varepsilon T_{\beta\alpha}^\varepsilon & 0 \end{pmatrix}, \tag{4.3}$$

we easily see that $\gamma T_{B'}^{A'} = A_A^{A'} A_{B'}^{B} \gamma T_B^A$, where $\overline{(A)}^{-1} = (A_{B'}^{B})$ is the inverse matrix of $\overline{A}$.

If $\omega \in \Im_1^0(B_m)$ and $T \in \Im_2^1(B_m)$, then

$$(\gamma T)({}^{vv}\omega) = 0.$$

5. Complete lift of vector fields

We now denote by $\Im_q^p(M_n)$ the module over $F(M_n)$ of all tensor fields of type (p,q) on M_n, where $F(M_n)$ denotes the ring of real-valued C^∞-functions on M_n.

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [7] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. On putting

$$ {}^{cc}\widetilde{X} = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -p_\varepsilon(\partial_\alpha X^\varepsilon) \end{pmatrix}, \tag{5.1}$$

we easily see that ${}^{cc}\widetilde{X}' = \overline{A}({}^{cc}\widetilde{X})$. The vector field ${}^{cc}\widetilde{X}$ is called the complete lift of $\widetilde{X}$ to the semi-cotangent bundle $t^*(B_m)$.

A vector field X on a semi-cotangent bundle $t^*(B_m)$ with the degenerate symplectic structure $\omega = dp$ is called a Hamiltonian vector field if $\iota_X \omega = dH$ for same C^∞-function H on $t^*(B_m)$, i.e. if the interior product $\iota_X\omega$ is exact. X is called a symplectic vector field if $L_X\omega = 0$, i.e. if $\iota_X\omega$ is closed. It is well known that, locally, symplectic vector fields are Hamiltonian. Using $L_X = d \circ \iota_X + \iota_X \circ d$ (Cartan's magic formula), we have

$$L_{{}^{cc}X} dp = (d \circ \iota_{{}^{cc}X})\, dp + (\iota_{{}^{cc}X} \circ d)\, dp = d_{{}^{cc}X}\,(\iota\,(dp)) + \iota_{{}^{cc}X}(d^2p) = d\,(\iota_{{}^{cc}X}\,(dp))$$

for complete lift ${}^{cc}X$. From here we see that ${}^{cc}X$ is a Hamiltonian vector field (only locally) if $L_{{}^{cc}X} dp = 0$, i.e.

$${}^{cc}X^A \partial_A \omega_{KL} + (\partial_K({}^{cc}X^A))\omega_{AL} + (\partial_L({}^{cc}X^A))\omega_{KA} = 0.$$

Using (5.1) and coordinates of $\omega = dp$, from the last equation, we have the identity $0 = 0$. Thus, we have:

Theorem 3 *The complete lift ${}^{cc}\widetilde{X}$ of projectable vector field $\widetilde{X}$ to a semi-cotangent bundle is Hamiltonian with the degenerate symplectic structure $\omega = dp$.*

We have from (5.1)

$${}^{cc}\widetilde{X}\ {}^{vv}f ={}^{vv}(Xf)$$

for any $f \in \Im_0^0(B_m)$ and projectable vector field $\widetilde{X} \in \Im_0^1(M_n)$.

We also have from (3.2) and (5.1)

$$\begin{aligned} {}^{cc}(\widetilde{X} + \widetilde{Y}) &= {}^{cc}\widetilde{X} + {}^{cc}\widetilde{Y}, \\ {}^{cc}(f\widetilde{X}) &= {}^{vv}f({}^{cc}\widetilde{X}) - (\gamma X)^{vv}(df), \end{aligned}$$

for any $f \in \Im_0^0(B_m)$ and $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$.

Theorem 4 *Let $\widetilde{X}$ and $\widetilde{Z}$ be projectable vector fields on M_n with projections X and Z on B_m, respectively. If $f \in \Im_0^0(B_m)$, $\omega \in \Im_1^0(B_m)$, and $F \in \Im_1^1(B_m)$, then*

(i) ${}^{vv}\omega{}^{vv}f = 0,$

(ii) ${}^{vv}\omega(\gamma Z) = {}^{vv}(\omega(Z)),$

(iii) $(\gamma F)({}^{vv}f) = 0,$

(iv) $(\gamma F)\gamma Z = \gamma(FZ),$

(v) ${}^{cc}\widetilde{X}(\gamma Z) = \gamma[X, Z],$

(vi) ${}^{cc}\widetilde{X}\ {}^{vv}f = {}^{vv}(Xf).$

Proof (i) If $\omega \in \Im_1^0(B_m)$, then, by (3.1) and (3.2), we find

$$\begin{aligned} {}^{vv}\omega{}^{vv}f &= {}^{vv}\omega^I\partial_I({}^{vv}f) \\ &= {}^{vv}\omega^a\partial_a({}^{vv}f) + {}^{vv}\omega^\alpha\partial_\alpha({}^{vv}f) + {}^{vv}\omega^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{vv}f) \\ &= 0. \end{aligned}$$

Thus, we have (i) of Theorem 4.

(ii) If $\omega \in \Im_1^0(B_m)$ and $\widetilde{Z}$ is a projectable vector field on M_n with projection $Z \in \Im_0^1(B_m)$, then we have by (3.2) and (4.1):

$$\begin{aligned} {}^{vv}\omega(\gamma Z) &= {}^{vv}\omega^I\partial_I(\gamma Z) \\ &= {}^{vv}\omega^a\partial_a(p_\beta Z^\beta) + {}^{vv}\omega^\alpha\partial_\alpha(p_\beta Z^\beta) + {}^{vv}\omega^{\overline{\alpha}}\partial_{\overline{\alpha}}(p_\beta Z^\beta) \\ &= \omega_\alpha Z^\alpha = {}^{vv}(\omega(Z)). \end{aligned}$$

Thus, we have ${}^{vv}\omega(\gamma Z) = {}^{vv}(\omega(Z))$.

(iii) If $F \in \Im_1^1(B_m)$, then we have by (3.1) and (4.2):

$$\begin{aligned} (\gamma F)({}^{vv}f) &= (\gamma F)^I\partial_I({}^{vv}f) \\ &= (\gamma F)^a\partial_a({}^{vv}f) + (\gamma F)^\alpha\partial_\alpha({}^{vv}f) + (\gamma F)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{vv}f) \\ &= 0. \end{aligned}$$

Thus, we have (iii) of Theorem 4.

(iv) If $F \in \Im_1^1(B_m)$, and $\widetilde{Z}$ is a projectable vector field on M_n, then we have by (4.1) and (4.2):

$$\begin{aligned} (\gamma F)\gamma Z &= (\gamma F)^I\partial_I(\gamma Z) \\ &= (\gamma F)^a\partial_a(p_\beta Z^\beta) + (\gamma F)^\alpha\partial_\alpha(p_\beta Z^\beta) + (\gamma F)^{\overline{\alpha}}\partial_{\overline{\alpha}}(p_\beta Z^\beta) \\ &= p_\beta F_\alpha^\beta\partial_{\overline{\alpha}}(p_\beta Z^\beta) \\ &= p_\beta F_\alpha^\beta Z^\alpha = p_\beta(FZ)^\beta = \gamma(FZ), \end{aligned}$$

and hence equation (iv) of Theorem 4.

(v) If $\widetilde{X}$ and $\widetilde{Z}$ are projectable vector fields on M_n, then taking account of (4.1) and (5.1), we have:

$$\begin{aligned} {}^{cc}\widetilde{X}(\gamma Z) &= {}^{cc}X^I\partial_I(\gamma Z) \\ &= {}^{cc}X^a\partial_a(p_\beta Z^\beta) +^{cc} X^\alpha\partial_\alpha(p_\beta Z^\beta) +^{cc} X^{\overline{\alpha}}\partial_{\overline{\alpha}}(p_\beta Z^\beta) \\ &= X^\alpha\partial_\alpha(p_\beta Z^\beta) - p_\beta(\partial_\alpha X^\beta)Z^\alpha \\ &= p_\beta(X^\alpha\partial_\alpha Z^\beta - Z^\alpha\partial_\alpha X^\beta) \\ &= p_\beta[X,Z]^\beta = \gamma[X,Z], \end{aligned}$$

which proves (v) of Theorem 4.

(vi) We shall prove the last equation. If $\widetilde{X}$ is a projectable vector field on M_n, then we have by (3.1) and (5.1):

$$\begin{aligned} {}^{cc}\widetilde{X}\ {}^{vv}f &= {}^{cc}X^I\partial_I({}^{vv}f) \\ &= {}^{cc}X^a\partial_a({}^{vv}f) +^{cc} X^\alpha\partial_\alpha({}^{vv}f) +^{cc} X^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{vv}f) \\ &= X^\alpha\partial_\alpha f =^{vv} (Xf), \end{aligned}$$

which gives equation (vi) of Theorem 4. □

Theorem 5 *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X \in \Im_0^1(B_m)$ and $Y \in \Im_0^1(B_m)$. For the Lie product, we have*

(i) $[{}^{vv}\omega,{}^{vv}\theta] = 0,$

(ii) $[{}^{vv}\omega,\gamma F] =^{vv} (\omega\circ F),$

(iii) $[\gamma F,\gamma G] = \gamma[F,G],$

(iv) $[{}^{cc}\widetilde{X},{}^{vv}\omega] =^{vv} (L_X\omega),$

(v) $[{}^{cc}\widetilde{X},\gamma F] = \gamma(L_X F),$

(vi) $[{}^{cc}\widetilde{X},{}^{cc}\widetilde{Y}] =^{cc} \widetilde{[X,Y]}$

for any $\omega,\theta \in \Im_1^0(B_m)$ and $F,G \in \Im_1^1(B_m)$, where $\omega\circ F$ is a 1-form defined by $(\omega\circ F)(Z) = \omega(FZ)$ for any $Z \in \Im_0^1(B_m)$ and L_X is the operator of Lie derivation with respect to X.

Proof (i) If $\omega,\theta \in \Im_1^0(B_m)$ and $\begin{pmatrix} [{}^{vv}\omega,{}^{vv}\theta]^b \\ [{}^{vv}\omega,{}^{vv}\theta]^\beta \\ [{}^{vv}\omega,{}^{vv}\theta]^{\overline{\beta}} \end{pmatrix}$ are components of $[{}^{vv}\omega,{}^{vv}\theta]^J$ with respect to the coordinates

$(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$, then we have

$$\begin{aligned}[]
[{}^{vv}\omega, {}^{vv}\theta]^J &= {}^{vv}\omega^I \partial_I({}^{vv}\theta^J) - {}^{vv}\theta^I \partial_I({}^{vv}\omega^J) \\
&= {}^{vv}\omega^a \partial_a({}^{vv}\theta^J) + {}^{vv}\omega^\alpha \partial_\alpha({}^{vv}\theta^J) + {}^{vv}\omega^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{vv}\theta^J) \\
&\quad - {}^{vv}\theta^a \partial_a({}^{vv}\omega^J) - {}^{vv}\theta^\alpha \partial_\alpha({}^{vv}\omega^J) - {}^{vv}\theta^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{vv}\omega^J) \\
&= \omega_\alpha \partial_{\overline{\alpha}}({}^{vv}\theta^J) - \theta_\alpha \partial_{\overline{\alpha}}({}^{vv}\omega^J).
\end{aligned}$$

Firstly, if $J = b$, we have

$$[{}^{vv}\omega, {}^{vv}\theta]^b = \omega_\alpha \partial_{\overline{\alpha}} {}^{vv}\theta^b - \theta_\alpha \partial_{\overline{\alpha}} {}^{vv}\omega^b = 0$$

because of (3.2). Secondly, if $J = \beta$, we have

$$[{}^{vv}\omega, {}^{vv}\theta]^\beta = \omega_\alpha \partial_{\overline{\alpha}} {}^{vv}\theta^\beta - \theta_\alpha \partial_{\overline{\alpha}} {}^{vv}\omega^\beta = 0$$

because of (3.2). Thirdly, let $J = \overline{\beta}$. Then we have

$$\begin{aligned}[]
[{}^{vv}\omega, {}^{vv}\theta]^{\overline{\beta}} &= \omega_\alpha \partial_{\overline{\alpha}} {}^{vv}\theta^{\overline{\beta}} - \theta_\alpha \partial_{\overline{\alpha}} {}^{vv}\omega^{\overline{\beta}} \\
&= \omega_\alpha \partial_{\overline{\alpha}} \theta_\beta - \theta_\alpha \partial_{\overline{\alpha}} \omega_\beta = 0
\end{aligned}$$

by (3.2). Thus, we have (i) of Theorem 5.

(ii) If $\omega \in \Im_1^0(B_m)$, $F \in \Im_1^1(B_m)$ and $\begin{pmatrix} [{}^{vv}\omega, \gamma F]^b \\ [{}^{vv}\omega, \gamma F]^\beta \\ [{}^{vv}\omega, \gamma F]^{\overline{\beta}} \end{pmatrix}$ are components of $[{}^{vv}\omega, \gamma F]^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$, then we have by (3.2) and (4.2)

$$\begin{aligned}[]
[{}^{vv}\omega, \gamma F]^J &= {}^{vv}\omega^I \partial_I(\gamma F)^J - (\gamma F)^I \partial_I({}^{vv}\omega)^J \\
&= {}^{vv}\omega^a \partial_a(\gamma F)^J + {}^{vv}\omega^\alpha \partial_\alpha(\gamma F)^J + {}^{vv}\omega^{\overline{\alpha}} \partial_{\overline{\alpha}}(\gamma F)^J \\
&\quad - (\gamma F)^a \partial_a({}^{vv}\omega)^J - (\gamma F)^\alpha \partial_\alpha({}^{vv}\omega)^J - (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{vv}\omega)^J \\
&= {}^{vv}\omega^{\overline{\alpha}} \partial_{\overline{\alpha}}(\gamma F)^J - (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{vv}\omega)^J \\
&= \omega_\alpha \partial_{\overline{\alpha}}(\gamma F)^J - p_\varepsilon F_\beta^\varepsilon \partial_{\overline{\alpha}}({}^{vv}\omega)^J.
\end{aligned}$$

Firstly, if $J = b$, we have

$$[{}^{vv}\omega, \gamma F]^b = \omega_\alpha \partial_{\overline{\alpha}}(\gamma F)^b - p_\varepsilon F_\beta^\varepsilon \partial_{\overline{\alpha}} {}^{vv}\omega^b = 0$$

because of (3.2) and (4.2). Secondly, if $J = \beta$, we have

$$[{}^{vv}\omega, \gamma F]^\beta = \omega_\alpha \partial_{\overline{\alpha}}(\gamma F)^\beta - p_\varepsilon F_\beta^\varepsilon \partial_{\overline{\alpha}} {}^{vv}\omega^\beta = 0$$

because of (3.2) and (4.2). Thirdly, let $J = \overline{\beta}$. Then we have

$$\begin{aligned}[]
[{}^{vv}\omega, \gamma F]^{\overline{\beta}} &= \omega_\alpha \partial_{\overline{\alpha}}(\gamma F)^{\overline{\beta}} - p_\varepsilon F_\beta^\varepsilon \partial_{\overline{\alpha}}({}^{vv}\omega)^{\overline{\beta}} \\
&= \omega_\alpha \partial_{\overline{\alpha}} p_\varepsilon F_\beta^\varepsilon - p_\varepsilon F_\beta^\varepsilon \partial_{\overline{\alpha}} \omega_\beta \\
&= \omega_\alpha F_\beta^\alpha = (\omega \circ F)_\beta
\end{aligned}$$

by (3.2) and (4.2). On the other hand, the vertical lift ${}^{vv}(\omega \circ F)$ of $(\omega \circ F)$ has components of the form

$$ {}^{vv}(\omega \circ F) = \begin{pmatrix} 0 \\ 0 \\ (\omega \circ F)_\beta \end{pmatrix} $$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$. Thus, we have (ii) of Theorem 5.

(iii) If $F, G \in \Im^1_1(B_m)$ and $\begin{pmatrix} [\gamma F, \gamma G]^b \\ [\gamma F, \gamma G]^\beta \\ [\gamma F, \gamma G]^{\overline{\beta}} \end{pmatrix}$ are components of $[\gamma F, \gamma G]^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$, then we have by (4.2)

$$ \begin{aligned} [\gamma F, \gamma G]^J &= (\gamma F)^I \partial_I (\gamma G)^J - (\gamma G)^I \partial_I (\gamma F)^J \\ &= (\gamma F)^a \partial_a (\gamma G)^J + (\gamma F)^\alpha \partial_\alpha (\gamma G)^J + (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma G)^J \\ &\quad - (\gamma G)^a \partial_a (\gamma F)^J - (\gamma G)^\alpha \partial_\alpha (\gamma F)^J - (\gamma G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^J \\ &= (\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma G)^J - (\gamma G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^J \\ &= p_\varepsilon F^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma G)^J - p_\varepsilon G^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma F)^J. \end{aligned} $$

Firstly, if $J = b$, we have

$$ [\gamma F, \gamma G]^b = p_\varepsilon F^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma G)^b - p_\varepsilon G^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma F)^b = 0 $$

because of (4.2). Secondly, if $J = \beta$, we have

$$ [\gamma F, \gamma G]^\beta = p_\varepsilon F^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma G)^\beta - p_\varepsilon G^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma F)^\beta = 0 $$

by (4.2). Thirdly, let $J = \overline{\beta}$. Then we have

$$ \begin{aligned} [\gamma F, \gamma G]^{\overline{\beta}} &= p_\varepsilon F^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma G)^{\overline{\beta}} - p_\varepsilon G^\varepsilon_\alpha \partial_{\overline{\alpha}} (\gamma F)^{\overline{\beta}} \\ &= p_\varepsilon F^\varepsilon_\alpha \partial_{\overline{\alpha}} p_\varepsilon G^\varepsilon_\beta - p_\varepsilon G^\varepsilon_\alpha \partial_{\overline{\alpha}} p_\varepsilon F^\varepsilon_\beta \\ &= p_\varepsilon F^\varepsilon_\alpha G^\alpha_\beta - p_\varepsilon G^\varepsilon_\alpha F^\alpha_\beta \\ &= p_\varepsilon (F^\varepsilon_\alpha G^\alpha_\beta - G^\varepsilon_\alpha F^\alpha_\beta) \\ &= p_\varepsilon [F, G]^\varepsilon_\beta \end{aligned} $$

because of (4.2). It is well known that $\gamma[F, G]$ have components

$$ \gamma[F, G] = \begin{pmatrix} 0 \\ 0 \\ p_\varepsilon [F, G]^\varepsilon_\beta \end{pmatrix} $$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$. Thus, we have (iii) of Theorem 5.

(iv) If $\omega \in \Im^0_1(B_m)$, $\widetilde{X}$ is a projectable vector field on M_n with projection $X \in \Im^1_0(B_m)$, and $\begin{pmatrix} [{}^{cc}\widetilde{X}, {}^{vv}\omega]^b \\ [{}^{cc}\widetilde{X}, {}^{vv}\omega]^\beta \\ [{}^{cc}\widetilde{X}, {}^{vv}\omega]^{\overline{\beta}} \end{pmatrix}$ are components of $[{}^{cc}\widetilde{X}, {}^{vv}\omega]^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$, then we have

$$[{}^{cc}\widetilde{X}, {}^{vv}\omega]^J = ({}^{cc}\widetilde{X})^I \partial_I ({}^{vv}\omega)^J - ({}^{vv}\omega)^I \partial_I ({}^{cc}\widetilde{X})^J.$$

Firstly, if $J = b$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}, {}^{vv}\omega]^b &= ({}^{cc}\widetilde{X})^I \partial_I ({}^{vv}\omega)^b - ({}^{vv}\omega)^I \partial_I ({}^{cc}\widetilde{X})^b \\
&= -({}^{vv}\omega)^a \partial_a ({}^{cc}\widetilde{X})^b - ({}^{vv}\omega)^\alpha \partial_\alpha ({}^{cc}\widetilde{X})^b - ({}^{vv}\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{X})^b \\
&= -({}^{vv}\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} \widetilde{X}^b \\
&= 0
\end{aligned}$$

because of (3.2) and (5.1). Secondly, if $J = \beta$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}, {}^{vv}\omega]^\beta &= ({}^{cc}\widetilde{X})^I \partial_I ({}^{vv}\omega)^\beta - ({}^{vv}\omega)^I \partial_I ({}^{cc}\widetilde{X})^\beta \\
&= -({}^{vv}\omega)^a \partial_a ({}^{cc}\widetilde{X})^\beta - ({}^{vv}\omega)^\alpha \partial_\alpha ({}^{cc}\widetilde{X})^\beta - ({}^{vv}\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{X})^\beta \\
&= -({}^{vv}\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} \widetilde{X}^\beta \\
&= 0
\end{aligned}$$

by (3.2) and (5.1). Thirdly, let $J = \overline{\beta}$. Then we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}, {}^{vv}\omega]^{\overline{\beta}} &= ({}^{cc}\widetilde{X})^I \partial_I ({}^{vv}\omega)^{\overline{\beta}} - ({}^{vv}\omega)^I \partial_I ({}^{cc}\widetilde{X})^{\overline{\beta}} \\
&= ({}^{cc}\widetilde{X})^a \partial_a ({}^{vv}\omega)^{\overline{\beta}} + ({}^{cc}\widetilde{X})^\alpha \partial_\alpha ({}^{vv}\omega)^{\overline{\beta}} + ({}^{cc}\widetilde{X})^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{vv}\omega)^{\overline{\beta}} \\
&\quad -({}^{vv}\omega)^a \partial_a ({}^{cc}\widetilde{X})^{\overline{\beta}} - ({}^{vv}\omega)^\alpha \partial_\alpha ({}^{cc}\widetilde{X})^{\overline{\beta}} - ({}^{vv}\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{X})^{\overline{\beta}} \\
&= ({}^{cc}\widetilde{X})^\alpha \partial_\alpha ({}^{vv}\omega)^{\overline{\beta}} - ({}^{vv}\omega)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{X})^{\overline{\beta}} \\
&= X^\alpha \partial_\alpha \omega_\beta + \omega_\alpha \partial_{\overline{\alpha}} p_\varepsilon (\partial_\beta X^\varepsilon) \\
&= X^\alpha \partial_\alpha \omega_\beta + (\partial_\beta X^\alpha)\omega_\alpha \\
&= (L_X \omega)_\beta
\end{aligned}$$

because of (3.2) and (5.1). On the other hand, the vertical lift ${}^{vv}(L_X\omega)$ of $(L_X\omega)$ has components of the form

$${}^{vv}(L_X\omega) = \begin{pmatrix} 0 \\ 0 \\ (L_X\omega)_\beta \end{pmatrix}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$. Thus, we have (iv) of Theorem 5.

(v) If $F \in \Im_1^1(B_m)$, $\widetilde{X}$ is a projectable vector field on M_n with projection $X \in \Im_0^1(B_m)$, and $\begin{pmatrix} [{}^{cc}\widetilde{X}, \gamma F]^b \\ [{}^{cc}\widetilde{X}, \gamma F]^\beta \\ [{}^{cc}\widetilde{X}, \gamma F]^{\overline{\beta}} \end{pmatrix}$ are components of $[{}^{cc}\widetilde{X}, \gamma F]^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$, then we have

$$[{}^{cc}\widetilde{X}, \gamma F]^J = ({}^{cc}\widetilde{X})^I \partial_I (\gamma F)^J - (\gamma F)^I \partial_I ({}^{cc}\widetilde{X})^J.$$

For $J=b$, we have

$$\begin{aligned}[^{cc}\widetilde{X},\gamma F]^{b} &= (^{cc}\widetilde{X})^{I}\partial_{I}(\gamma F)^{b}-(\gamma F)^{I}\partial_{I}(^{cc}\widetilde{X})^{b}\\ &= -(\gamma F)^{a}\partial_{a}(^{cc}\widetilde{X})^{b}-(\gamma F)^{\alpha}\partial_{\alpha}(^{cc}\widetilde{X})^{b}-(\gamma F)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}\widetilde{X})^{b}=0\end{aligned}$$

because of (4.2) and (5.1). For $J=\beta$, we have

$$\begin{aligned}[^{cc}\widetilde{X},\gamma F]^{\beta} &= (^{cc}\widetilde{X})^{I}\partial_{I}(\gamma F)^{\beta}-(\gamma F)^{I}\partial_{I}(^{cc}\widetilde{X})^{\beta}\\ &= -(\gamma F)^{a}\partial_{a}X^{\beta}-(\gamma F)^{\alpha}\partial_{\alpha}X^{\beta}-(\gamma F)^{\overline{\alpha}}\partial_{\overline{\alpha}}X^{\beta}=0\end{aligned}$$

by (4.2) and (5.1). For $J=\overline{\beta}$ we have

$$\begin{aligned}[^{cc}\widetilde{X},\gamma F]^{\overline{\beta}} &= (^{cc}\widetilde{X})^{I}\partial_{I}(\gamma F)^{\overline{\beta}}-(\gamma F)^{I}\partial_{I}(^{cc}\widetilde{X})^{\overline{\beta}}\\ &= (^{cc}\widetilde{X})^{a}\partial_{a}(\gamma F)^{\overline{\beta}}+(^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}(\gamma F)^{\overline{\beta}}+(^{cc}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}(\gamma F)^{\overline{\beta}}\\ &\quad -(\gamma F)^{a}\partial_{a}(^{cc}\widetilde{X})^{\overline{\beta}}-(\gamma F)^{\alpha}\partial_{\alpha}(^{cc}\widetilde{X})^{\overline{\beta}}-(\gamma F)^{\overline{\alpha}}\partial_{\overline{\alpha}}(^{cc}\widetilde{X})^{\overline{\beta}}\\ &= \widetilde{X}^{a}\partial_{a}p_{\varepsilon}F_{\beta}^{\varepsilon}+X^{\alpha}\partial_{\alpha}p_{\varepsilon}F_{\beta}^{\varepsilon}-p_{\varepsilon}(\partial_{\alpha}X^{\varepsilon})\partial_{\overline{\alpha}}p_{\varepsilon}F_{\beta}^{\varepsilon}+p_{\varepsilon}F_{\alpha}^{\varepsilon}\partial_{\overline{\alpha}}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon})\\ &= X^{\alpha}\partial_{\alpha}p_{\varepsilon}F_{\beta}^{\varepsilon}-p_{\varepsilon}(\partial_{\alpha}X^{\varepsilon})F_{\beta}^{\alpha}+p_{\varepsilon}F_{\alpha}^{\varepsilon}(\partial_{\beta}X^{\alpha})\\ &= p_{\varepsilon}(X^{\alpha}\partial_{\alpha}F_{\beta}^{\varepsilon}-\partial_{\alpha}X^{\varepsilon}F_{\beta}^{\alpha}+\partial_{\beta}X^{\alpha}F_{\alpha}^{\varepsilon})\\ &= p_{\varepsilon}(L_{X}F)_{\beta}^{\varepsilon}\end{aligned}$$

because of (4.2) and (5.1). It is well known that $\gamma(L_XF)$ have components

$$\gamma(L_XF)=\begin{pmatrix}0\\0\\p_{\varepsilon}(L_XF)_{\beta}^{\varepsilon}\end{pmatrix}$$

with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t^*(B_m)$. Thus, we have (v) of Theorem 5.

(vi) If $\widetilde{X}$ and $\widetilde{Y}$ are projectable vector fields on M_n with projection $X,Y\in\Im_0^1(B_m)$ and $\begin{pmatrix}[^{cc}\widetilde{X},^{cc}\widetilde{Y}]^{b}\\ [^{cc}\widetilde{X},^{cc}\widetilde{Y}]^{\beta}\\ [^{cc}\widetilde{X},^{cc}\widetilde{Y}]^{\overline{\beta}}\end{pmatrix}$ are components of $[^{cc}\widetilde{X},^{cc}\widetilde{Y}]^{J}$ with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t^*(B_m)$, then we have

$$[^{cc}\widetilde{X},^{cc}\widetilde{Y}]^{J}=(^{cc}\widetilde{X})^{I}\partial_{I}(^{cc}\widetilde{Y})^{J}-(^{cc}\widetilde{Y})^{I}\partial_{I}(^{cc}\widetilde{X})^{J}.$$

Firstly, if $J = b$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}]^{b} &= ({}^{cc}\widetilde{X})^{I}\partial_{I}({}^{cc}\widetilde{Y})^{b} - ({}^{cc}\widetilde{Y})^{I}\partial_{I}({}^{cc}\widetilde{X})^{b} \\
&= ({}^{cc}\widetilde{X})^{a}\partial_{a}({}^{cc}\widetilde{Y})^{b} + ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{b} + ({}^{cc}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{b} \\
&\quad -({}^{cc}\widetilde{Y})^{a}\partial_{a}({}^{cc}\widetilde{X})^{b} - ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{b} - ({}^{cc}\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{X})^{b} \\
&= ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{b} - ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{b} \\
&= X^{\alpha}\partial_{\alpha}\widetilde{Y}^{b} - Y^{\alpha}\partial_{\alpha}\widetilde{X}^{b} \\
&= \widetilde{[X, Y]}^{b}
\end{aligned}$$

because of (5.1). Secondly, if $J = \beta$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}]^{\beta} &= ({}^{cc}\widetilde{X})^{I}\partial_{I}({}^{cc}\widetilde{Y})^{\beta} - ({}^{cc}\widetilde{Y})^{I}\partial_{I}({}^{cc}\widetilde{X})^{\beta} \\
&= ({}^{cc}\widetilde{X})^{a}\partial_{a}({}^{cc}\widetilde{Y})^{\beta} + ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{\beta} + ({}^{cc}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{\beta} \\
&\quad -({}^{cc}\widetilde{Y})^{a}\partial_{a}({}^{cc}\widetilde{X})^{\beta} - ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{\beta} - ({}^{cc}\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{X})^{\beta} \\
&= ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{\beta} - ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{\beta} \\
&= X^{\alpha}\partial_{\alpha}Y^{\beta} - Y^{\alpha}\partial_{\alpha}X^{\beta} \\
&= [X, Y]^{\beta}
\end{aligned}$$

by (5.1). Thirdly, let $J = \overline{\beta}$. Then we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}]^{\overline{\beta}} &= ({}^{cc}\widetilde{X})^{I}\partial_{I}({}^{cc}\widetilde{Y})^{\overline{\beta}} - ({}^{cc}\widetilde{Y})^{I}\partial_{I}({}^{cc}\widetilde{X})^{\overline{\beta}} \\
&= ({}^{cc}\widetilde{X})^{a}\partial_{a}({}^{cc}\widetilde{Y})^{\overline{\beta}} + ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{Y})^{\overline{\beta}} + ({}^{cc}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{\overline{\beta}} \\
&\quad -({}^{cc}\widetilde{Y})^{a}\partial_{a}({}^{cc}\widetilde{X})^{\overline{\beta}} - ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}({}^{cc}\widetilde{X})^{\overline{\beta}} - ({}^{cc}\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{X})^{\overline{\beta}} \\
&= -({}^{cc}\widetilde{X})^{a}\partial_{a}p_{\varepsilon}(\partial_{\beta}Y^{\varepsilon}) - ({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}p_{\varepsilon}(\partial_{\beta}Y^{\varepsilon}) - ({}^{cc}\widetilde{X})^{\overline{\alpha}}\partial_{\overline{\alpha}}p_{\varepsilon}(\partial_{\beta}Y^{\varepsilon}) \\
&\quad +({}^{cc}\widetilde{Y})^{a}\partial_{a}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) + ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) + ({}^{cc}\widetilde{Y})^{\overline{\alpha}}\partial_{\overline{\alpha}}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) \\
&= -({}^{cc}\widetilde{X})^{\alpha}\partial_{\alpha}p_{\varepsilon}(\partial_{\beta}Y^{\varepsilon}) - ({}^{cc}\widetilde{X})^{\overline{\alpha}}(\partial_{\beta}Y^{\alpha}) + ({}^{cc}\widetilde{Y})^{\alpha}\partial_{\alpha}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) + ({}^{cc}\widetilde{Y})^{\overline{\alpha}}(\partial_{\beta}X^{\alpha}) \\
&= -X^{\alpha}\partial_{\alpha}p_{\varepsilon}(\partial_{\beta}Y^{\varepsilon}) + p_{\varepsilon}\partial_{\alpha}X^{\varepsilon}(\partial_{\beta}Y^{\alpha}) + Y^{\alpha}\partial_{\alpha}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) - p_{\varepsilon}\partial_{\alpha}Y^{\varepsilon}(\partial_{\beta}X^{\alpha}) \\
&= p_{\varepsilon}(-X^{\alpha}\partial_{\alpha}\partial_{\beta}Y^{\varepsilon} + \partial_{\beta}Y^{\alpha}\partial_{\alpha}X^{\varepsilon} + Y^{\alpha}\partial_{\alpha}\partial_{\beta}X^{\varepsilon} - \partial_{\beta}X^{\alpha}\partial_{\alpha}Y^{\varepsilon}) \\
&= -p_{\varepsilon}(\partial_{\beta}(X^{\alpha}\partial_{\alpha}Y^{\varepsilon} - Y^{\alpha}\partial_{\alpha}X^{\varepsilon})) \\
&= -p_{\varepsilon}(\partial_{\beta}[X, Y]^{\varepsilon})
\end{aligned}$$

because of (5.1). It is well known that ${}^{cc}\widetilde{[X, Y]}$ have components

$$ {}^{cc}\widetilde{[X, Y]} = \begin{pmatrix} \widetilde{[X, Y]}^{b} \\ [X, Y]^{\beta} \\ -p_{\varepsilon}(\partial_{\beta}[X, Y]^{\varepsilon}) \end{pmatrix} $$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t^*(B_m)$. Thus, we have (vi) of Theorem 5. □

Theorem 6 *Let $\widetilde{X}$ be a projectable vector field on M_n. If $\omega \in \Im^0_1(B_m)$, $F \in \Im^1_1(B_m)$, and $S, T \in \Im^1_2(B_m)$, then*

(i) $(\gamma S)^{cc}\widetilde{X} = \gamma(S_X)$,

(ii) $(\gamma S)({}^{vv}\omega) = 0$,

(iii) $(\gamma S)(\gamma F) = 0$,

(iv) $(\gamma S)(\gamma T) = 0$,

where S_X is tensor field of type (1,1) on B_m defined by $S_X(Z) = S(X, Z)$ for any $Z \in \Im^1_0(B_m)$.

Proof (i) Using (4.3) and (5.1), we have

$$(\gamma S)^{cc}\widetilde{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_\sigma S^\sigma_{\beta\alpha} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -p_\varepsilon(\partial_\alpha X^\varepsilon) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ p_\sigma S^\sigma_{\beta\alpha} X^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_\sigma (S_X)^\sigma_\beta \end{pmatrix} = \gamma(S_X).$$

Similarly, we have

$$(\gamma S)({}^{vv}\omega) = 0, \quad (\gamma S)(\gamma F) = 0, \quad (\gamma S)(\gamma T) = 0.$$

□

6. Complete lift of affinor fields

Let $\widetilde{F} \in \Im^1_1(M_n)$ be a projectable affinor field [7] with projection $F = F^\alpha_\beta(x^\alpha)\partial_\alpha \otimes dx^\beta$, i.e. $\widetilde{F}$ has components

$$\widetilde{F} = (\widetilde{F}^i_j) = \begin{pmatrix} \widetilde{F}^a_b(x^a, x^\alpha) & \widetilde{F}^a_\beta(x^a, x^\alpha) \\ 0 & F^\alpha_\beta(x^\alpha) \end{pmatrix}$$

with respect to the coordinates (x^a, x^α). On putting

$${}^{cc}\widetilde{F} = ({}^{cc}\widetilde{F}^I_J) = \begin{pmatrix} \widetilde{F}^a_b & \widetilde{F}^a_\beta & 0 \\ 0 & F^\alpha_\beta & 0 \\ 0 & p_\sigma(\partial_\beta F^\sigma_\alpha - \partial_\alpha F^\sigma_\beta) & F^\beta_\alpha \end{pmatrix}, \tag{6.1}$$

we easily see that ${}^{cc}\widetilde{F}^{I'}_{J'} = A^{I'}_I A^J_{J'} \, {}^{cc}\widetilde{F}^I_J$.

We call ${}^{cc}\widetilde{F}$ the complete lift of the tensor field $\widetilde{F}$ of type (1,1) to $t^*(B_m)$.

Proof For simplicity we take only ${}^{cc}F^{\overline{\alpha}'}_{\beta'}$. In fact,

$$\begin{aligned}
{}^{cc}F^{\overline{\alpha}'}_{\beta'} &= A^{\overline{\alpha}'}_{\alpha}A^{\beta}_{\beta'}\,({}^{cc}F^{\alpha}_{\beta}) + A^{\overline{\alpha}'}_{\overline{\alpha}}A^{\beta}_{\beta'}\,({}^{cc}F^{\overline{\alpha}}_{\beta}) + A^{\overline{\alpha}'}_{\overline{\alpha}}A^{\overline{\beta}}_{\beta'}({}^{cc}F^{\overline{\alpha}}_{\overline{\beta}})\\
&= p_{\varepsilon}A^{\gamma}_{\alpha}A^{\varepsilon}_{\gamma\alpha'}A^{\beta}_{\beta'}F^{\alpha}_{\beta} + A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) + A^{\alpha}_{\alpha'}(p_{\varepsilon'}A^{\theta}_{\beta'}A^{\varepsilon'}_{\theta}{}_{\beta})F^{\beta}_{\alpha}\\
&= -p_{\varepsilon}(\partial_{\gamma}A^{\gamma}_{\alpha})A^{\varepsilon}_{\alpha'}A^{\beta}_{\beta'}F^{\alpha}_{\beta} + p_{\sigma}A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}(\partial_{\beta}F^{\sigma}_{\alpha}) - p_{\sigma}A^{\alpha}_{\alpha'}A^{\beta}_{\beta'}\partial_{\alpha}F^{\sigma}_{\beta} + p_{\varepsilon'}A^{\theta}_{\beta'}A^{\varepsilon'}_{\theta}{}_{\beta}F^{\beta}_{\alpha'}\\
&= -p_{\varepsilon}(\partial_{\gamma}A^{\gamma}_{\alpha})A^{\varepsilon}_{\alpha'}F^{\alpha}_{\beta'} + p_{\sigma}A^{\alpha}_{\alpha'}\partial_{\beta'}F^{\sigma}_{\alpha} - p_{\sigma}A^{\beta}_{\beta'}A^{\alpha}_{\alpha'}\partial_{\alpha}F^{\sigma}_{\beta} - p_{\varepsilon'}(\partial_{\beta}A^{\theta}_{\beta'})A^{\varepsilon'}_{\theta}F^{\beta}_{\alpha'}\\
&= -p_{\alpha'}(\partial_{\gamma}A^{\gamma}_{\alpha})F^{\alpha}_{\beta'} + p_{\sigma}\partial_{\beta'}F^{\sigma}_{\alpha'} - p_{\sigma}A^{\beta}_{\beta'}\partial_{\alpha'}F^{\sigma}_{\beta} - p_{\theta}(\partial_{\beta}A^{\theta}_{\beta'})F^{\beta}_{\alpha'}\\
&= -p_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'} + p_{\sigma}\partial_{\beta'}F^{\sigma}_{\alpha'} - p_{\sigma}\partial_{\alpha'}F^{\sigma}_{\beta'} - p_{\alpha'}A^{\alpha'}_{\theta}A^{\alpha}_{\beta}A^{\alpha'}_{\beta'}A^{\beta}_{\alpha}A^{\beta'}_{\alpha'}(\partial_{\beta}A^{\theta}_{\beta'})F^{\beta}_{\alpha'}\\
&= -p_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'} + p_{\sigma}(\partial_{\beta'}F^{\sigma}_{\alpha'} - \partial_{\alpha'}F^{\sigma}_{\beta'}) - p_{\alpha'}A^{\alpha'}_{\theta}A^{\beta}_{\alpha}A^{\beta'}_{\alpha'}(\partial_{\beta}A^{\theta}_{\beta'})F^{\alpha}_{\beta'}\\
&= -p_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'} + p_{\sigma'}(\partial_{\beta'}F^{\sigma'}_{\alpha'} - \partial_{\alpha'}F^{\sigma'}_{\beta'}) + p_{\alpha'}A^{\alpha'}_{\theta}A^{\theta}_{\beta'}A^{\beta'}_{\alpha'}(\partial_{\beta}A^{\beta}_{\alpha})F^{\alpha}_{\beta'}\\
&= -p_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'} + p_{\sigma'}(\partial_{\beta'}F^{\sigma'}_{\alpha'} - \partial_{\alpha'}F^{\sigma'}_{\beta'}) + p_{\alpha'}A^{\alpha'}_{\theta}A^{\theta}_{\beta'}A^{\beta'}_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'}\\
&= -p_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'} + p_{\sigma'}(\partial_{\beta'}F^{\sigma'}_{\alpha'} - \partial_{\alpha'}F^{\sigma'}_{\beta'}) + p_{\alpha'}\partial_{\alpha}F^{\alpha}_{\beta'}\\
&= p_{\sigma'}(\partial_{\beta'}F^{\sigma'}_{\alpha'} - \partial_{\alpha'}F^{\sigma'}_{\beta'}).
\end{aligned}$$

Thus, we have ${}^{cc}F^{\overline{\alpha}'}_{\beta'} = p_{\sigma'}(\partial_{\beta'}F^{\sigma'}_{\alpha'} - \partial_{\alpha'}F^{\sigma'}_{\beta'})$. Similarly, we can easily find other components of ${}^{cc}\widetilde{F}^{I'}_{J'}$.

□

Theorem 7 *Let $\widetilde{F}, \widetilde{G}$, and $\widetilde{X}$ be projectable affinor and vector fields on M_n with projections F, G, and X on B_m, respectively. If $\omega \in \Im^0_1(B_m)$, then*

(i) ${}^{cc}\widetilde{F}(\gamma G) = \gamma(G \circ F)$,

(ii) ${}^{cc}\widetilde{F}\ {}^{vv}\omega = {}^{vv}(\omega \circ F)$,

(iii) ${}^{cc}\widetilde{F}\,{}^{cc}\widetilde{X} = {}^{cc}\widetilde{(FX)} + \gamma(L_X F)$.

Proof (i) If $\widetilde{F}$ and $\widetilde{G}$ are projectable affinor fields on M_n, then we have by (4.2) and (6.1)

$$
{}^{cc}\widetilde{F}(\gamma G) = \begin{pmatrix} \widetilde{F}^a_b & \widetilde{F}^a_{\beta} & 0\\ 0 & F^{\alpha}_{\beta} & 0\\ 0 & p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) & F^{\beta}_{\alpha}\end{pmatrix}\begin{pmatrix}0\\0\\p_{\varepsilon}G^{\varepsilon}_{\beta}\end{pmatrix}
= \begin{pmatrix}0\\0\\p_{\varepsilon}G^{\varepsilon}_{\beta}F^{\beta}_{\alpha}\end{pmatrix} = \begin{pmatrix}0\\0\\p_{\varepsilon}(G\circ F)^{\varepsilon}_{\alpha}\end{pmatrix} = \gamma(G\circ F).
$$

Thus, we have ${}^{cc}\widetilde{F}(\gamma G) = \gamma(G \circ F)$.

(ii) If $\omega \in \Im^0_1(B_m)$, and $\widetilde{F}$ is a projectable affinor field on M_n, then we get by (3.2) and (6.1):

$$
{}^{cc}\widetilde{F}\ {}^{vv}\omega = \begin{pmatrix} \widetilde{F}^a_b & \widetilde{F}^a_{\beta} & 0\\ 0 & F^{\alpha}_{\beta} & 0\\ 0 & p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) & F^{\beta}_{\alpha}\end{pmatrix}\begin{pmatrix}0\\0\\\omega_{\beta}\end{pmatrix}
$$

$$= \begin{pmatrix} 0 \\ 0 \\ \omega_\beta F_\alpha^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\omega \circ F)_\alpha \end{pmatrix} =^{vv} (\omega \circ F),$$

which gives ${}^{cc}\widetilde{F}\ {}^{vv}\omega =^{vv} (\omega \circ F)$.

(iii) If $\widetilde{F}$ and $\widetilde{X}$ are projectable affinor and vector fields on M_n, respectively.

Then we have by (5.1) and (6.1):

$$\begin{aligned}
{}^{cc}\widetilde{F}\ {}^{cc}\widetilde{X} &= \begin{pmatrix} \widetilde{F}_b^a & \widetilde{F}_\beta^a & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix} \begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ -p_\varepsilon(\partial_\beta X^\varepsilon) \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{F}_b^a \widetilde{X}^b + \widetilde{F}_\beta^a X^\beta \\ F_\beta^\alpha X^\beta \\ p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) X^\beta - p_\varepsilon(\partial_\beta X^\varepsilon) F_\alpha^\beta \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{(FX)}^a \\ (FX)^\alpha \\ -p_\sigma \partial_\alpha (FX)^\sigma \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_\sigma(X^\beta \partial_\beta F_\alpha^\sigma - (\partial_\alpha X^\beta) F_\beta^\sigma - (\partial_\beta X^\sigma) F_\alpha^\beta) \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{(FX)}^a \\ (FX)^\alpha \\ -p_\sigma \partial_\alpha (FX)^\sigma \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_\sigma (L_X F)_\alpha^\sigma \end{pmatrix} =^{cc} \widetilde{(FX)} + \gamma(L_X F),
\end{aligned}$$

which gives ${}^{cc}\widetilde{F}\ {}^{cc}\widetilde{X} =^{cc} \widetilde{(FX)} + \gamma(L_X F)$. □

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