2-25-2020

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Preprint · September 2018

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Abstract

In the present paper, we endow a family of axiomatic extensions of semi De Morgan logic with proper multi-type display calculi which are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal builds on an algebraic analysis of semi De Morgan algebras and its subvarieties and applies the guidelines of the multi-type methodology in the design of display calculi.

Keywords : semi De Morgan algebras, proper display calculus, multi-type methodology.
Math. **Subject Class 2010** : 03B45, 03G25, 03F05, 06D30, 08A68.

1 Introduction

Semi De Morgan logic, introduced in an algebraic setting by H.P. Sankappanavar [22], is a very well known paraconsistent logic [21], and is designed to capture the salient features of intuitionistic negation in a paraconsistent setting. Semi De Morgan algebras form a variety of normal distributive lattice expansions (cf. [15, Definition 9]), and are a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. Besides being studied from a universal-algebraic perspective [22, 2, 3], semi De Morgan logic has been studied from a duality-theoretic perspective [18] and from the perspective of canonical extensions [19].

From a proof-theoretic perspective, the main challenge posed by semi De Morgan logic is that, unlike De Morgan logic, its axiomatization is not analytic inductive in the sense of [15, Definition 55]. In [14], an analytic calculus for semi De Morgan logic is introduced which is sound, complete, conservative, and enjoys cut elimination and subformula property. The design of this calculus builds on an algebraic analysis of semi De Morgan algebras, and applies the guidelines of the multi-type methodology, introduced in [7, 5] and further developed in [1, 6, 8, 17, 16]. This methodology guarantees in particular that all the properties mentioned above follow from the general background theory of proper multi-type display calculi (cf. [17, Definition A.1.]).

Due to space constraints, in [14], the proofs of the algebraic analysis on which the design of this calculus is grounded had to be omitted. The present chapter provides the missing proofs, and also extends the results of [14] by explicitly and modularly accounting for the logics associated with the five subvarieties of semi De Morgan algebras introduced in [22]. This modular account is partly made possible by the fact that all but two of these subvarieties correspond to axiomatic extensions of semi De Morgan logic with so-called *analytic inductive* axioms (cf. [15, Definition

*This research is supported by the NWO Vidi grant 016.138.314, the NWO Aspasia grant 015.008.054, and a Delft Technology Fellowship awarded to the second author in 2013.

55]), and the two remaining ones can be given analytic equivalent presentations in the multi-type setting for the basic calculus. The general theory of proper (multi-type) display calculi provides an algorithm which computes the analytic structural rules corresponding to these axioms, and guarantees that each calculus obtained by adding any subset of these rules to the basic calculus still enjoys cut elimination and subformula property.

Therefore, this chapter introduces a proof-theoretic environment which is suitable to complement, from a proof-theoretic perspective, the investigations on the lattice of axiomatic extensions of semi De Morgan logic, as well as on the connections between the lattices of axiomatic extensions of semi De Morgan logic and of De Morgan logic.

Structure of the chapter. In Section 2, we report on the axioms and rules of semi De Morgan logic and its axiomatic extensions arising from the subvarieties of semi De Morgan algebras introduced in [22], and discuss why the basic axiomatization is not amenable to the standard treatment of display calculi. In Section 3, we define the algebraic environment which motivates our multi-type approach and prove that this environment is an equivalent presentation of the standard algebraic semantics of semi De Morgan logic and its extensions. Then we introduce the multi-type semantic environment and define translations between the single-type and the multi-type languages of semi De Morgan logic and its extensions. In Section 4, we discuss how equivalent analytic (multi-type) reformulations can be given of non-analytic (single-type) axioms in the language of semi De Morgan logic. In Section 5, we introduce the display calculi for semi De Morgan logic and its extensions, and in Section 6, we discuss their soundness, completeness, conservativity, cut elimination and subformula property.

2 Preliminaries

2.1 Semi De Morgan logic and its axiomatic extensions

Fix a denumerable set Atprop of propositional variables, let p denote an element in Atprop. The language $\mathcal{L}$ of semi De Morgan logic over Atprop is defined recursively as follows:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A$$

Definition 1. Semi De Morgan logic, denoted SM, consists of the following axioms:

$$\bot \vdash A, \quad A \vdash \top, \quad \neg\top \vdash \bot, \quad \top \vdash \neg\bot, \quad A \vdash A, \quad A \wedge B \vdash A, \quad A \wedge B \vdash B,$$

$$A \vdash A \vee B, \quad B \vdash A \vee B, \quad \neg A \vdash \neg\neg\neg A, \quad \neg\neg\neg A \vdash \neg A, \quad \neg A \wedge \neg B \vdash \neg(A \vee B),$$

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B), \quad A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

and the following rules:

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C} \qquad \frac{A \vdash B \quad A \vdash C}{A \vdash B \wedge C} \qquad \frac{A \vdash B \quad C \vdash B}{A \vee C \vdash B} \qquad \frac{A \vdash B}{\neg B \vdash \neg A}$$

The following table reports the name of each axiomatic extension of SM arising from the subvarieties introduced in [22], its acronym, and its characterizing axiom:

lower quasi De Morgan logic	LQM	$A \vdash \neg\neg A$
upper quasi De Morgan logic	UQM	$\neg\neg A \vdash A$
demi pseudo-complemented lattice logic	DP	$\neg A \wedge \neg\neg A \vdash \bot$
almost pseudo-complemented lattice logic	AP	$A \wedge \neg A \vdash \bot$
weak Stone logic	WS	$\top \vdash \neg A \vee \neg\neg A$

In [15], a characterization is given of the properly displayable (single-type) logics (i.e. those logics that can be captured by a proper display calculus, cf. [23, Chapter 4]). Properly displayable logics are exactly those logics which admit a presentation consisting of analytic inductive axioms (cf. [15, Definition 55]). It is not difficult to verify that $\neg A \vdash \neg\neg\neg A$, $\neg\neg\neg A \vdash \neg A$ and $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$ in SM, $\neg\neg A \vdash A$ in UQM, and $\neg A \wedge \neg\neg A \vdash \bot$ in DP are not analytic inductive. To our knowledge, no equivalent axiomatizations have been introduced for semi De Morgan logic and its extensions using only analytic inductive axioms. This provides the motivation for circumventing this difficulty by introducing proper multi-type display calculi for semi De Morgan logic and its extensions.

2.2 The variety of semi De Morgan algebras and its subvarieties

We recall the definition of the variety of semi De Morgan algebras and those of its subvarieties introduced in [22, Definition 2.2, Definition 2.6].

Definition 2. An algebra $\mathbb{A} = (L, \wedge, \vee, ', \top, \bot)$ is a *semi De Morgan algebra* (SMA) if for all $a, b \in L$,

(S1) $(L, \wedge, \vee, 1, 0)$ is a bounded distributive lattice;

(S2) $\bot' = \top, \top' = \bot$;

(S3) $(a \vee b)' = a' \wedge b'$;

(S4) $(a \wedge b)'' = a'' \wedge b''$;

(S5) $a' = a'''$.

A *lower quasi De Morgan algebra* (LQMA) is an SMA satisfying the following inequality:

(S6a) $a \leq a''$.

Dually, a *upper quasi De Morgan algebra* (UQMA) is an SMA satisfying the following inequality:

(S6b) $a'' \leq a$.

A *demi pseudocomplemented lattice* (DPL) is an SMA satisfying the following equation:

(S7) $a' \wedge a'' = \bot$.

A *almost pseudocomplemented lattice* (APL) is an SMA satisfying the following equation:

(S8) $a \wedge a' = \bot$.

A *weak Stone algebra* (WSA) is an SMA satisfying the following equation:

(S9) $a' \vee a'' = \top$.

The following proposition is a straightforward consequence of (S8), (S2), (S3) and (S5):

Proposition 3 ([22] see discussion above Corollary 2.7)**.** (S7) *holds in any APL and WSA.*

Definition 4. An algebra $\mathbb{D} = (D, \cap, \cup, ^*, 1, 0)$ is a *De Morgan algebra* (DMA) if for all $a, b \in D$,

(D1) $(D, \cap, \cup, 1, 0)$ is a bounded distributive lattice;

(D2) $0^* = 1, 1^* = 0$;

(D3) $(a \cup b)^* = a^* \cap b^*$;

(D4) $(a \cap b)^* = a^* \cup b^*$;

(D5) $a = a^{**}$.

As is well known, a Boolean algebra (BA) $\mathbb{D}$ is a DMA satisfying one of the following equations:

(B1) $a \vee a^* = 1$;

(B2) $a \wedge a^* = 0$.

The following theorem can be shown using a routine Lindenbaum-Tarski construction.

Theorem 5 (Completeness). *SM (resp. LQM, UQM, DP, AP, WS) is complete with respect to the class of SMAs (resp. LQMAs, UQMAs, DPLs, APLs, WSAs).*

Definition 6. A distributive lattice $\mathbb{A}$ is *perfect* (cf. [12, Definition 2.14]) if $\mathbb{A}$ is complete, completely distributive and completely join-generated by the set $J^{\infty}(\mathbb{A})$ of its completely join-irreducible elements (as well as completely meet-generated by the set $M^{\infty}(\mathbb{A})$ of its completely meet-irreducible elements).

A De Morgan algebra (resp. Boolean algebra) $\mathbb{A}$ is *perfect* if its lattice reduct is a perfect distributive lattice, and the following distributive laws are valid:

$$(\bigvee X)^* = \bigwedge X^* \qquad (\bigwedge X)^* = \bigvee X^*.$$

A lattice homomorphism $h : \mathbb{L} \to \mathbb{L}'$ is *complete* if for each $X \subseteq \mathbb{L}$,

$$h(\bigvee X) = \bigvee h(X) \qquad h(\bigwedge X) = \bigwedge h(X).$$

3 Towards a multi-type presentation

In the present section, we introduce the algebraic environment which justifies semantically the multi-type approach to semi De Morgan logic and its extensions of Section 2.1. In the next subsection, we define the kernel of an SMA (cf. Definition 8) and show that it can be endowed with a structure of DMA (cf. Definition 4). Similarly, we define the kernel of a DPL (cf. Definition 2) and show that it can be endowed with a structure of Boolean algebra. Then we define two maps between the kernel of any SMA (resp. DPL) $\mathbb{A}$ and the lattice reduct of $\mathbb{A}$. These are the main components of the heterogeneous semi De Morgan algebras and the heterogeneous demi p-lattices which we introduce in Subsection 3.2, where we also show that SMAs (resp. DPLs) can be equivalently presented in terms of heterogeneous semi De Morgan algebras (heterogeneous demi p-lattices). Based on these, we can also define the heterogeneous algebras for other subvariety of SMAs we introduced in Section 2.2. In Subsection 3.3, we apply results pertaining to the theory of canonical extensions to the heterogeneous semi De Morgan algebras and the heterogeneous demi p-lattices.

3.1 The kernel of a semi De Morgan algebra

For any semi De Morgan algebra $\mathbb{A} = (L, \wedge, \vee, ', \top, \bot)$, we let $K := \{a'' \mid a \in L\}$, define $h : L \twoheadrightarrow K$ by the assignment $a \mapsto a''$ for any $a \in L$, and let $e : K \hookrightarrow L$ denote the natural embedding. Hence, $eh(a) = a''$ and $h(a) = h(a'')$ for every $a \in L$.

Lemma 7. *For any semi De Morgan algebra $\mathbb{A}$, and K, h, e defined as above, the following equation holds for any $\alpha \in K$:*

$$he(\alpha) = \alpha \tag{1}$$

Proof. Let $\alpha \in K$, and let $a \in L$ such that $h(a) = \alpha$. Hence,

$$\begin{array}{rcll} he(\alpha) & = & heh(a) & \alpha = h(a) \\ & = & h(a'') & eh(a) = a'' \\ & = & h(a) & h(a) = h(a'') \\ & = & \alpha & \text{definition of } h \end{array}$$

□

Definition 8. For any SMA $\mathbb{A} = (L, \wedge, \vee, \top, \bot, ')$, let the *kernel* $\mathbb{K}_{\mathbb{A}} = (K, \cap, \cup, ^*, 1, 0)$ of $\mathbb{A}$ be defined as follows:

K1 $K := \{a'' \mid a \in L\}$;

K2 $\alpha \cup \beta := h((e(\alpha) \vee e(\beta))'')$ for all $\alpha, \beta \in K$;

K3 $\alpha \cap \beta := h(e(\alpha) \wedge e(\beta))$ for all $\alpha, \beta \in K$;

K4 $1 := h(\top)$;

K5 $0 := h(\bot)$;

K6 $\alpha^* := h(e(\alpha)')$.

In what follows, to simplify the notation, we omit as many parentheses as we can without generating ambiguous readings. For example, we write $e(h(a)^*)$ in place of $e((h(a))^*)$, and $eh(a)'$ in place of $(eh(a))'$.

Proposition 9. *If $\mathbb{A} = (L, \wedge, \vee, \top, \bot, ')$ is an SMA, then $\mathbb{K}_{\mathbb{A}}$ is a De Morgan algebra.*

Proof. Let us show that $\mathbb{K}_{\mathbb{A}}$ is a distributive lattice. Associativity and commutativity are straightforwardly verified and their corresponding verification is omitted. To show that the absorption law and the distributive law hold, let $\alpha, \beta, \gamma \in K$, and let $a, b, c \in L$ such that (i) $h(a) = \alpha$, (ii) $h(b) = \beta$ and (iii) $h(a) = \gamma$.

- absorption law:

$$
\begin{array}{lll}
 & \alpha \cup (\alpha \cap \beta) & \\
= & h((e(\alpha) \vee e(\alpha \cap \beta))'') & \text{K2} \\
= & h((e(\alpha) \vee eh(e(\alpha) \wedge e(\beta)))'') & \text{K3} \\
= & h((e(\alpha) \vee (e(\alpha) \wedge e(\beta))'')'') & eh(a) = a'' \\
= & h((e(\alpha)' \wedge (e(\alpha) \wedge e(\beta))''')') & \text{S3} \\
= & h((e(\alpha)''' \wedge (e(\alpha)'' \wedge e(\beta)'')')') & \text{S5, S4} \\
= & h((e(\alpha)'' \vee (e(\alpha)'' \wedge e(\beta)''))'') & \text{S3} \\
= & h((e(\alpha)'' \vee (e(\alpha)'''' \wedge e(\beta)''''))) & \text{S4} \\
= & h((e(\alpha)'' \vee (e(\alpha)'' \wedge e(\beta)''))) & \text{S5} \\
= & h(e(\alpha)'') & \text{S1} \\
= & he(\alpha) & h(a) = h(a'') \\
= & \alpha & \text{Lemma 7}
\end{array}
$$

- distributivity law:

$$
\begin{array}{lll}
 & \alpha \cap (\beta \cup \gamma) & \\
= & h(e(\alpha) \wedge e(\beta \cup \gamma)) & \text{K3} \\
= & h(e(\alpha) \wedge eh((e(\beta) \vee e(\gamma))'')) & \text{K2} \\
= & h(e(\alpha) \wedge (e(\beta) \vee e(\gamma))'''') & eh(a) = a'' \\
= & h(eh(a) \wedge (e(\beta) \vee e(\gamma))'''') & \text{(i)} \\
= & h(a'' \wedge (e(\beta) \vee e(\gamma))'''') & eh(a) = a'' \\
= & h(a'''' \wedge (e(\beta) \vee e(\gamma))'') & \text{S5} \\
= & h((a'' \wedge (e(\beta) \vee e(\gamma)))'') & \text{S4} \\
= & h(((a'' \wedge e(\beta)) \vee (a'' \wedge e(\gamma)))'') & \text{S1} \\
= & h((a'' \wedge eh(b)) \vee (a'' \wedge eh(c))'') & \text{(ii) and (iii)} \\
= & h(((a'' \wedge b'') \vee (a'' \wedge c''))'') & eh(a) = a'' \\
= & h(((a'''' \wedge b'''') \vee (a'''' \wedge c''''))'') & \text{S5} \\
= & h(((a'' \wedge b'')'' \vee (a'' \wedge c'')'')'') & \text{S4} \\
= & h((eh(eh(a) \wedge eh(b)) \vee eh(eh(a) \wedge eh(c)))'') & eh(a) = a'' \\
= & h((eh(e(\alpha) \wedge e(\beta)) \vee eh(e(\alpha) \wedge e(\gamma)))'') & \text{(i), (ii) and (iii)} \\
= & h(((e(\alpha \cap \beta)) \vee e(\alpha \cap \gamma))'') & \text{K3} \\
= & (\alpha \cap \beta) \cup (\alpha \cap \gamma) & \text{K2}
\end{array}
$$

Let us show that $\mathbb{K}_{\mathbb{A}}$ satisfies (D1)-(D5).
As to (D1), we need to show that $\mathbb{K}_{\mathbb{A}}$ is bounded:

$$
\begin{array}{lll}
 & 0 \cap \alpha & \\
= & h(e(0) \wedge e(\alpha)) & \text{K3} \\
= & h(eh(\bot) \wedge e(\alpha)) & \text{K5} \\
= & h(\bot'' \wedge e(\alpha)) & eh(a) = a'' \\
= & h(\bot \wedge e(\alpha)) & \text{S2} \\
= & h(\bot) & \text{S1} \\
= & 0 & \text{K5}
\end{array}
\qquad
\begin{array}{lll}
 & 1 \cup \alpha & \\
= & h((e(1) \vee e(\alpha))'') & \text{K2} \\
= & h((eh(\top)) \vee e(\alpha))'') & \text{K4} \\
= & h((\top'' \vee e(\alpha))'') & eh(a) = a'' \\
= & h((\top \vee e(\alpha))'') & \text{S2} \\
= & h(\top'') & \text{S5} \\
= & 1 & \text{S2, K4}
\end{array}
$$

As to (D2):

$$
\begin{array}{llll}
0^* & = & h(e(0)') & \text{K6} \\
 & = & h((eh(\bot))') & \text{K5} \\
 & = & h(\bot''') & eh(a) = a'' \\
 & = & h(\bot') & \text{S5} \\
 & = & h(\top) & \text{S2} \\
 & = & 1 & \text{K4}
\end{array}
\qquad
\begin{array}{llll}
1^* & = & h(e(1)') & \text{K6} \\
 & = & h((eh(\top))') & \text{K4} \\
 & = & h(\top''') & eh(a) = a'' \\
 & = & h(\top') & \text{S5} \\
 & = & h(\bot) & \text{S2} \\
 & = & 0 & \text{K4}
\end{array}
$$

As to (D3):

$$
\begin{array}{llll}
(\alpha \cup \beta)^* & = & h(e(\alpha \cup \beta)') & \text{K6} \\
 & = & h((eh((e(\alpha) \vee e(\beta))'')') & \text{K2} \\
 & = & h((e(\alpha) \vee e(\beta))''''') & eh(a) = a'' \\
 & = & h((e(\alpha)' \wedge e(\beta)')'''') & \text{S3} \\
 & = & h((e(\alpha)''' \wedge e(\beta)''')'') & \text{S5} \\
 & = & h((eh(e(\alpha)') \wedge eh(e(\beta)'))'') & eh(a) = a'' \\
 & = & h((e(\alpha^*) \wedge e(\beta^*))'') & \text{K6} \\
 & = & heh(e(\alpha^*) \wedge e(\beta^*)) & eh(a) = a'' \\
 & = & h(e(\alpha^*) \wedge e(\beta^*)) & \text{Lemma 7} \\
 & = & \alpha^* \cap \beta^* & \text{K3}
\end{array}
$$

As to (D4):

$$
\begin{array}{rcll}
(\alpha\cap\beta)^* & = & h(e(\alpha\cap\beta)') & \text{K6} \\
 & = & h((eh(e(\alpha)\wedge e(\beta)))') & \text{K3} \\
 & = & h((e(\alpha)\wedge e(\beta))''') & eh(a)=a'' \\
 & = & h((e(\alpha)''\wedge e(\beta)'')') & \text{S4} \\
 & = & h((e(\alpha)'\vee e(\beta)')'') & \text{S3} \\
 & = & h((eh(a)'\vee eh(b)')'') & \text{(i) and (ii)} \\
 & = & h((a'''\vee b''')'') & eh(a)=a'' \\
 & = & h((a'''''\vee b''''')'') & \text{S5} \\
 & = & h((eh(eh(a)')\vee eh(eh(b)'))'') & eh(a)=a'' \\
 & = & h((eh(e(\alpha)')\vee eh(e(\beta)'))'') & \text{(i) and (ii)} \\
 & = & h((e(\alpha^*)\vee e(\beta^*))'') & \text{K6} \\
 & = & \alpha^*\cup\beta^* & \text{K2}
\end{array}
$$

As to (D5):

$$
\begin{array}{rcll}
\alpha^{**} & = & h((eh(e(\alpha)'))') & \text{K6} \\
 & = & h(e(\alpha)'''') & eh(a)=a'' \\
 & = & h((eh(a))'''') & \text{(i)} \\
 & = & h(a'''''') & eh(a)=a'' \\
 & = & h(a'') & \text{S5} \\
 & = & heh(a) & eh(a)=a'' \\
 & = & h(a) & \text{Lemma 7} \\
 & = & \alpha & \text{(i)}
\end{array}
$$

□

Corollary 10. *If $\mathbb{A}=(L,\wedge,\vee,\top,\bot,')$ is a DPL, then $\mathbb{K}_{\mathbb{A}}$ is a Boolean algebra.*

Proof. By Proposition 9, $\mathbb{K}_{\mathbb{A}}$ is a De Morgan algebra. Hence, it suffices to show that $\mathbb{K}$ satisfies (B1). For any $\alpha\in\mathbb{K}_{\mathbb{A}}$,

$$
\begin{array}{rcll}
\alpha\cap\alpha^* & = & h(\alpha\cap h(e(\alpha)')) & \text{K3} \\
 & = & h(e(\alpha)\wedge eh(e(\alpha)')) & \text{K6} \\
 & = & h(e(\alpha)\wedge e(\alpha)''') & eh(a)=a'' \\
 & = & h(e(\alpha)\wedge e(\alpha)') & \text{S5} \\
 & = & heh(e(\alpha)\wedge e(\alpha)') & \text{Lemma 7} \\
 & = & h((e(\alpha)\wedge e(\alpha)')'') & eh(a)=a'' \\
 & = & h(e(\alpha)''\wedge e(\alpha)''') & \text{S4} \\
 & = & h(\bot) & \text{S7} \\
 & = & 0 & \text{K5}
\end{array}
$$

□

Proposition 11. *Let $\mathbb{A}$ be an SMA (resp. a DPL), and e,h be defined as above. Then h is a lattice homomorphism from $\mathbb{A}$ onto $\mathbb{K}_{\mathbb{A}}$, and for all $\alpha,\beta\in K$,*

$$e(\alpha)\wedge e(\beta)=e(\alpha\cap\beta) \qquad e(1)=\top \qquad e(0)=\bot.$$

Proof. It is an immediate consequence of K1 that h is surjective. We need to show that h is a lattice homomorphism. For any $a,b\in L$,

$$
\begin{array}{llll}
 & h(a\wedge b) & & \\
= & heh(a\wedge b) & \text{Lemma 7} \\
= & h((a\wedge b)'') & eh(a)=a'' \\
= & h(a''\wedge b'') & \text{S4} \\
= & h(eh(a)\wedge eh(b)) & eh(a)=a'' \\
= & h(a)\cap h(b) & \text{K3}
\end{array}
\qquad
\begin{array}{llll}
 & h(a\vee b) & & \\
= & heh(a\vee b) & \text{Lemma 7} \\
= & h((a\vee b)'') & eh(a)=a'' \\
= & h((a'\wedge b')') & \text{S3} \\
= & h(a'''\wedge b''')' & \text{S5} \\
= & h(a''\vee b'')'' & \text{S3} \\
= & h((eh(a)\vee eh(b))'') & eh(a)=a'' \\
= & h(a)\cup h(b) & \text{K2}
\end{array}
$$

Moreover, $h(\bot)=\bot''=\bot$ and $h(\top)=\top''=\top$. This completes the proof that h is a homomorphism from $\mathbb{A}$ to $\mathbb{K}_{\mathbb{A}}$. Next, we show that $e(\alpha)\wedge e(\beta)=e(\alpha\cap\beta)$. For any $\alpha,\beta\in K$,

$$
\begin{array}{llll}
e(\alpha\cap\beta) & = & eh(e(\alpha)\wedge e(\beta)) & \text{K3} \\
 & = & (e(\alpha)\wedge e(\beta))'' & eh(a)=a'' \\
 & = & e(\alpha)''\wedge e(\beta)'' & \text{S4} \\
 & = & ehe(\alpha)\wedge ehe(\beta) & eh(a)=a'' \\
 & = & e(\alpha)\wedge e(\beta) & \text{Lemma 7}
\end{array}
$$

Finally, $e(0)=eh(\bot)=\bot''=\bot$ and $e(1)=eh(\top)=\top''=\top$ are straightforward consequences of (K4), (K5) and (S2). □

In what follows, we will drop the subscript of the kernel whenever it does not cause confusion.

3.2 Heterogeneous SMAs as equivalent presentations of SMAs

Definition 12. A *heterogeneous semi De Morgan algebra* (HSMA) is a tuple $(\mathbb{L},\mathbb{D},e,h)$ satisfying the following conditions:

(H1) $\mathbb{L}$ is a bounded distributive lattice;

(H2a) $\mathbb{D}$ is a De Morgan algebra;

(H3) $e:\mathbb{D}\hookrightarrow\mathbb{L}$ is an order embedding, and for all $\alpha_1,\alpha_2\in\mathbb{D}$,

- $e(\alpha_1)\wedge e(\alpha_2)=e(\alpha_1\cap\alpha_2)$;
- $e(1)=\top$, $e(0)=\bot$.

(H4) $h:\mathbb{L}\twoheadrightarrow\mathbb{D}$ is a surjective lattice homomorphism;

(H5) $he(\alpha)=\alpha$ for every $\alpha\in\mathbb{D}$.[1]

A *heterogeneous lower quasi De Morgan algebra* (HLQMA) is an HSMA satisfying the following condition:

(H6a) $a\leq eh(a)$ for any $a\in L$.

A *heterogeneous upper quasi De Morgan algebra* (HUQMA) is an HSMA satisfying the following condition:

(H6b) $eh(a)\leq a$ for any $a\in L$.

A *heterogeneous demi pseudocomplemented lattice* (HDPL) is defined analogously, except replacing (H2a) with the following condition (H2b):

(H2b) $\mathbb{D}$ is a Boolean algebra.

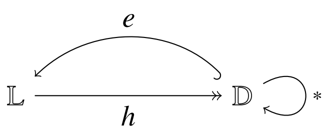
[1]Condition (H5) implies that h is surjective and e is injective.

A *heterogeneous almost pseudocomplemented lattice* (HAPL) is an HDPL satisfying the following condition:

(H7) $e(h(a)^*) \wedge a = \bot$ for all $a \in \mathbb{L}$.

A *heterogeneous weak Stone algebra* (HWSA) is an HDPL satisfying the following condition:

(H8) $e(\alpha^*) \vee e(\alpha) = \top$ for all $\alpha \in \mathbb{A}$.



An HSMA (resp. HLQMA,HUQMA, HDPL, HAPL and HWSA) is *perfect* if:

(PH1) both $\mathbb{L}$ and $\mathbb{D}$ are perfect as a distributive lattice and De Morgan algebra (or Boolean algebra), respectively (see Definition 6);

(PH2) e is an order-embedding and is completely meet-preserving;

(PH3) h is a complete homomorphism.

Definition 13. For any SMA (resp. LQMA, UQMA, DPL, APL and WSA) $\mathbb{A}$, let

$$\mathbb{A}^+ := (\mathbb{L}, \mathbb{K}, e, h),$$

where $\mathbb{L}$ is the lattice reduct of $\mathbb{A}$, $\mathbb{K}$ is the kernel of $\mathbb{A}$ (cf. Definition 8), and $e : \mathbb{K} \to \mathbb{L}$ and $h : \mathbb{L} \to \mathbb{K}$ are defined as in the beginning of Section 3.1.

Proposition 14. *If $\mathbb{A}$ is an SMA (resp. DPL), then $\mathbb{A}^+$ is an HSMA (resp. HDPL).*

Proof. It immediately follows from Proposition 9 and Proposition 11. □

Corollary 15. *If $\mathbb{A}$ is an LQMA (resp. UQMA, APL and WSA), then $\mathbb{A}^+$ is an HLQMA (resp. HUQMA, HAPL and HWSA).*

Proof. If $\mathbb{A}$ is an LQMA, by Proposition 14, it suffices to show that $\mathbb{A}_+$ satisfies (H6a). By (S6a) and H5, it is easy to see $a \leq e(h(a))$. The argument is dual when $\mathbb{A}$ is a UQMA. If $\mathbb{A}$ is an APL, it suffices to show $\mathbb{A}^+$ satisfies (H7).

$$\begin{array}{lll} & e(h(a)^*) \wedge a & \\ = & eh((eh(a))') \wedge a & \text{K6} \\ = & a'''' \wedge a & eh(a) = a'' \\ = & a' \wedge a & \text{S5} \\ = & \bot & \text{S8} \end{array}$$

If $\mathbb{A}$ is a WSA, it suffices to show $\mathbb{A}^+$ satisfies (H8).

$$\begin{array}{lll} & e(\alpha^*) \vee e(\alpha) & \\ = & eh(e(\alpha)') \vee e(\alpha) & \text{K6} \\ = & eh(e(\alpha)') \vee ehe(\alpha) & \text{Lemma 7} \\ = & e(\alpha)''' \vee ehe(\alpha) & eh(a) = a'' \\ = & e(\alpha)''' \vee e(\alpha)'' & \text{Lemma 7} \\ = & e(\alpha)' \vee e(\alpha)'' & \text{S5} \\ = & \top & \text{S9} \end{array}$$

□

Definition 16. For any HSMA (resp. HLQMA, HUQMA, HDPL, HAPL and HWSA) $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$, let

$$\mathbb{H}_+ := (\mathbb{L}, '),$$

where $' : \mathbb{L} \to \mathbb{L}$ is defined by the assignment $a' \mapsto e(h(a)^*)$.

Proposition 17. *If $\mathbb{H}$ is an HSMA (resp. HDPL), then $\mathbb{H}_+$ is an SMA (resp. DPL). Moreover, $\mathbb{K}_{\mathbb{H}^+} \cong \mathbb{K}$.*

Proof. Since $\mathbb{H}$ is an HSMA by assumption, $\mathbb{L}$ is a bounded distributive lattice, hence it suffices to show that the operation $'$ satisfies (S2)-(S5) (cf. Definition 2).

- As to (S2):

$$\begin{array}{rcll} \bot' & = & e(h(\bot)^*) & \text{definition of } ' \\ & = & e(0^*) & \text{H3} \\ & = & e(1) & \text{H2a} \\ & = & \top & \text{H3} \end{array} \qquad \begin{array}{rcll} \top' & = & e(h(\top)^*) & \text{definition of } ' \\ & = & e(1^*) & \text{H3} \\ & = & e(0) & \text{H2a} \\ & = & \bot & \text{H3} \end{array}$$

- As to (S3):

$$\begin{array}{rcll} (a \vee b)' & = & e(h(a \vee b)^*) & \text{definition of } ' \\ & = & e((h(a) \cup h(b))^*) & \text{H4} \\ & = & e(h(a)^* \cap h(b)^*) & \text{H2a} \\ & = & e(h(a)^*) \wedge e(h(b)^*) & \text{H3} \\ & = & a' \wedge b' & \text{definition of } ' \end{array}$$

- As to (S4):

$$\begin{array}{rcll} (a \wedge b)'' & = & e((he(h(a \wedge b)^*))^*) & \text{definition of } ' \\ & = & e(h(a \wedge b)^{**}) & \text{H5} \\ & = & eh(a \wedge b) & \text{H2a} \\ & = & e(h(a) \cap h(b)) & \text{H4} \\ & = & eh(a) \wedge eh(b) & \text{H3} \\ & = & e(h(a)^{**}) \wedge e(h(b)^{**}) & \text{H2a} \\ & = & e((he(h(a)^*))^*) \wedge e((he(h(b)^*))^*) & \text{H5} \\ & = & a'' \wedge b'' & \text{definition of } ' \end{array}$$

- As to (S5):

$$\begin{array}{rcll} a''' & = & e((he((he(h(a)^*))^*))^*) & \text{definition of } ' \\ & = & e(h(a)^{***}) & \text{H5} \\ & = & e(h(a)^*) & \text{H2a} \\ & = & a' & \text{definition of } ' \end{array}$$

Hence, $(\mathbb{L},')$ is a semi De Morgan algebra. If $(\mathbb{L}, \mathbb{D}, e, h)$ is an HDPL, we also need to show that $'$ satisfies (S7):

$$\begin{array}{rcll} a' \wedge a'' & = & e(h(a)^*) \wedge e((he(h(a)^*))^*) & \text{definition of } ' \\ & = & e(h(a)^*) \wedge e(h(a)^{**}) & \text{H5} \\ & = & e(h(a)^*) \wedge eh(a) & \text{H2a} \\ & = & e(h(a)^* \cap h(a)) & \text{H3} \\ & = & e(0) & \text{H2a} \\ & = & \bot & \text{H3} \end{array}$$

which completes the proof that (L, $'$) is a DPL. As to the second part of the statement, let us show preliminarily that the following identities hold:

K2$_{\mathbb{D}}$. $\alpha \cup \beta = h((e(\alpha) \vee e(\beta))'')$ for all $\alpha, \beta \in \mathbb{D}$;

K3$_{\mathbb{D}}$. $\alpha \cap \beta = h(e(\alpha) \wedge e(\beta))$ for all $\alpha, \beta \in \mathbb{D}$;

K4$_{\mathbb{D}}$. $1 = h(\top)$;

K5$_{\mathbb{D}}$. $0 = h(\bot)$;

K6$_{\mathbb{D}}$. $\alpha^* = h(e(\alpha)')$.

As to K2$_{\mathbb{D}}$,

$$\begin{array}{rcll} h((e(\alpha) \vee e(\beta))'') & = & he((he(h(e(\alpha) \vee e(\beta))^*))^*) & \text{definition of } ' \\ & = & (h(e(\alpha) \vee e(\beta)))^{**} & \text{H5} \\ & = & h(e(\alpha) \vee e(\beta)) & \text{H2a} \\ & = & he(\alpha) \cup he(\beta) & \text{H4} \\ & = & \alpha \cup \beta & \text{H5} \end{array}$$

Conditions K3$_{\mathbb{D}}$,K4$_{\mathbb{D}}$ and K5$_{\mathbb{D}}$ easily follow from H4, H5 and H3, and their proofs are omitted.

As to K6$_{\mathbb{D}}$,

$$\begin{array}{rcll} h(e(\alpha)') & = & he((he(\alpha))^*) & \text{definition of } ' \\ & = & \alpha^* & \text{H5} \end{array}$$

To show that $\mathbb{D}$ and $\mathbb{K}$ are isomorphic to each other, notice that the domain of $\mathbb{K}$ is defined as $K := \mathsf{Range}('') = \mathsf{Range}(e \circ^* \circ h \circ e \circ^* \circ h) = \mathsf{Range}(e \circ h)$. Since by assumption h is surjective, $K = \mathsf{Range}(e)$, and since e is an order embedding, K, regarded as a sub-poset of $\mathbb{L}$, is order-isomorphic to the domain of $\mathbb{D}$ with its lattice order. Let $f : \mathbb{D} \to \mathbb{K}$ denote the order-isomorphism between $\mathbb{D}$ and $\mathbb{K}$. Define $e_k : \mathbb{K} \hookrightarrow \mathbb{L}$ and $h_k : \mathbb{L} \twoheadrightarrow \mathbb{K}$ as as in the beginning of Section 3.1. Thus, $e = e_k \circ f$ and $h_k = f \circ h$. We need to show that: for all $\alpha, \beta \in \mathbb{D}$, let $\cap_k, \cup_k, {}^{*_k}$ denote the operations on K,

1. $f(\alpha) \cap_k f(\beta) = f(\alpha \cap \beta)$,

$$\begin{array}{rcll} f(\alpha) \cap_k f(\beta) & = & h_k(e_k f(\alpha) \wedge e_k f(\beta)) & \text{definition of } \cap_k \\ & = & fh(e_k f(\alpha) \wedge e_k f(\beta)) & h_k = f \circ h \\ & = & fh(e(\alpha) \wedge e(\beta)) & e = e_k \circ f \\ & = & f(\alpha \cap \beta) & \text{K3}_{\mathbb{D}} \end{array}$$

2. $f(\alpha) \cup_k f(\beta) = f(\alpha \cap \beta)$,

$$\begin{array}{rcll} f(\alpha) \cup_k f(\beta) & = & h_k((e_k f(\alpha) \vee e_k f(\beta))'') & \text{definition of } \cup_k \\ & = & fh((e_k f(\alpha) \vee e_k f(\beta))'') & h_k = f \circ h \\ & = & fh((e(\alpha) \vee e(\beta))'') & e = e_k \circ f \\ & = & f(\alpha \cup \beta) & \text{K2}_{\mathbb{D}} \end{array}$$

3. $f(\alpha)^{*_k} = f(\alpha^*)$,

$$\begin{array}{rcll} (f(\alpha))^{*_k} & = & h_k((e_k f(\alpha))') & \text{definition of } {}^{*_k} \\ & = & fh((e_k f(\alpha))') & h_k = f \circ h \\ & = & fh(e(\alpha)') & e = e_k \circ f \\ & = & f(\alpha^*) & \text{K6}_{\mathbb{D}} \end{array}$$

Hence, $f : \mathbb{D} \to \mathbb{K}$ is an isomorphism of De Morgan algebras (resp. Boolean algebras). This completes the proof. □

Corollary 18. *If $\mathbb{H}$ is an HLQMA (resp. HUQMA, HAPL and HWSA), then $\mathbb{A}^+$ is a LQMA (resp. UQMA, APL and WSA).*

Proof. By Proposition 17, if $\mathbb{H}$ is an HLQMA, it suffices to show that $\mathbb{H}_+$ satisfies (S6a).

$$\begin{array}{lll} & a \leq eh(a) & \text{H6a} \\ \text{iff} & a \leq e(h(a)^{**}) & \text{H2a} \\ \text{iff} & a \leq e((he(h(a)^*))^*) & \text{H5} \\ \text{iff} & a \leq a'' & \text{definition of }' \end{array}$$

If $\mathbb{H}$ is an HUQMA, the argument is dual. If $\mathbb{H}$ is an HAPL, it is clear that $\mathbb{H}_+$ satisfies (S8) by (H7). If $\mathbb{H}$ is an HWSA, it suffices to show that $\mathbb{H}_+$ satisfies (S6).

$$\begin{array}{lll} & a' \vee a'' & \\ = & e(h(a)^*) \vee e((he(h(a)^*))^*) & \text{def. of }' \\ = & e(h(a)^*) \vee e(h(a)^{**}) & \text{Lemma 7} \\ = & e(h(a)^*) \vee eh(a) & \text{H2a} \\ = & \top & \text{H8} \end{array}$$

□

Proposition 19. *For any SMA (resp. LQMA, UQMA, DPL, APL, and WSA) $\mathbb{A}$ and any HSMA (resp. HLQMA, HUQMA, HDPL, HAPL, and HWSA) $\mathbb{H}$:*

$$\mathbb{A} \cong (\mathbb{A}^+)_+ \quad \textit{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

These correspondences restrict appropriately to the relevant classes of perfect algebras and perfect heterogeneous algebras.

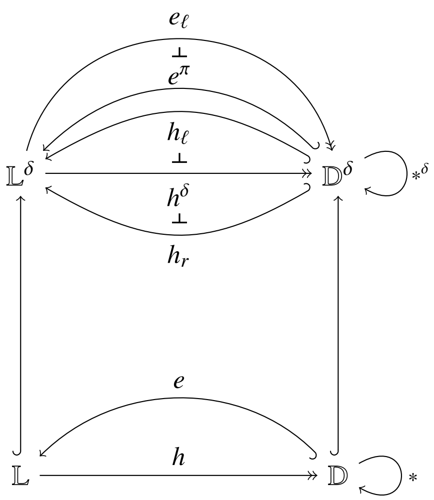
Proof. It immediately follows from Proposition 14, Corollary 15, Proposition 17 and Corollary 18. □

3.3 Canonical extensions of heterogeneous algebras

Canonicity in the multi-type environment is used both to provide complete semantics for a large class of axiomatic extensions of the basic logic (semi De Morgan logic in the present case), and to prove the conservativity of its associated display calculus (cf. Section 6.3). In the present section, we define the canonical extension $\mathbb{H}^\delta$ of any heterogeneous algebra $\mathbb{H}$ introduced in Section 3.2 by instantiating the general definition discussed in [17]. This makes it possible to define the canonical extension of any SMA $\mathbb{A}$ as a perfect SMA $(\mathbb{A}^+)^\delta_+$. We then show that this definition coincides with the definition given in [19, Section 4]. In what follows, we let $\mathbb{L}^\delta$ and $\mathbb{A}^\delta$ denote the canonical extensions of the distributive lattice $\mathbb{L}$ and of the De Morgan algebra (resp. Boolean algebra) $\mathbb{D}$ respectively, and e^π and h^δ denote the π-extensions of e and h[2], respectively. We refer to [9] for the relevant definitions.

Proposition 20. *If $(\mathbb{L}, \mathbb{D}, e, h)$ is an HSMA (resp. HDPL, HLQMA, HUQMA, HAPL, and HWSA), then $(\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$ is a perfect HSMA (resp. perfect HDPL, perfect HLQMA, perfect HUQMA, perfect HAPL and perfect HWSA).*

[2]The order-theoretic properties of h guarantee that the σ-extension and the π-extension of h coincide. This is why we use h^δ to denote the resulting extension.



Proof. Firstly, $\mathbb{L}^\delta$ and $\mathbb{D}^\delta$ are a perfect distributive lattice and a perfect De Morgan algebra (resp. perfect Boolean algebra) respectively. Secondly, since h is a surjective homomorphism, h is both finitely meet-preserving and finitely join-preserving. Hence, as is well known, h^δ is surjective, and completely meet- (join-) preserving [11, Theorem 3.7]. Since h is also smooth, this shows that $h^\delta = h^\pi = h^\sigma$ is a complete homomorphism. Thirdly, since e is finitely meet-preserving, e^π is completely meet-preserving, and it immediately follows from the definition of π-extension that e^π is an order-embedding [11, Corollary 2.25]. The identity $e^\pi(0) = 0$ clearly holds, since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$. Moreover, $Id_{\mathbb{D}} = h \circ e$ is canonical by [10, Proposition 14]. This is enough to show that if $(\mathbb{L}, \mathbb{D}, e, h)$ is a SMA (resp. DPL), then $(\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$ is a perfect HSMA (resp. perfect HDPL).

Since (H6a), (H6b), (H7) and (H8) are analytic inductive (cf. Definition 29), they are canonical. So their corresponding heterogeneous algebras are perfect. □

In the environment of perfect heterogeneous algebras, completely join (resp. meet) preserving maps have right (resp. left) adjoints. These adjoints guarantee the soundness of all display rules in the display calculi introduced in the next section.

In [19], C. Palma studied the canonical extensions of semi De Morgan algebras using insights from the Sahlqvist theory of Distributive Modal Logic. She recognized that not all inequalities in the axiomatization of SMA are Sahlqvist, and circumvented this problem by introducing the following term-equivalent presentation of SMAs.

Definition 21 ([19], Definition 4.1.2)**.** For any SMA $\mathbb{A} = (L, \wedge, \vee, ', \top, \bot)$, let $\mathbb{S}_{\mathbb{A}} = (L, \wedge, \vee, \rhd, \Box, \top, \bot)$ be such that $\Box$ and $\rhd$ are unary operations respectively defined by the assignments $a \mapsto a''$ and $a \mapsto a'$.

Palma showed that the algebras corresponding to SMAs via the construction above are exactly those $\{\Box, \rhd\}$-reducts of Distributive Modal Algebras satisfying the following additional axioms:

1. $\rhd \top \leq \bot$;
2. $\Box a \leq \rhd\rhd a$;
3. $\rhd\rhd a \leq \Box a$;
4. $\Box \rhd a \leq \rhd a$;
5. $\rhd a \leq \Box \rhd a$.

The axioms above can be straightforwardly verified to be Sahlqvist and hence canonical. This enables Palma to define the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$ as the $\{\rhd\}$-reduct of $\mathbb{S}^\sigma_{\mathbb{A}} = (\mathbb{L}^\sigma, \rhd^\pi, \Box^\pi)$. The following lemma immediately implies that $\mathbb{A}^\delta$ coincides with $(\mathbb{A}^{+\delta})_+$.

Lemma 22. *For any SMA $\mathbb{A}$, letting $\mathbb{S}_{\mathbb{A}}$ be defined as above,*

1. $\Box^\pi = e^\pi \circ h^\delta$;
2. $\rhd^\pi = e^\pi \circ *^\delta \circ h^\delta$.

Proof. By the definitions of $\Box, \rhd$, e and h (cf. beginning of Section 3.1),

$$
\begin{array}{rcll}
\Box^\pi & = & ('')^\pi & \text{definition of } \Box \\
 & = & (e \circ h)^\pi & \text{definitions of } e \text{ and } h \\
 & = & e^\pi \circ h^\pi & \text{[11, Lemma 3.3, Corollary 2.25]} \\
 & = & e^\pi \circ h^\delta & h \text{ is smooth}
\end{array}
$$

$$
\begin{array}{rcll}
\rhd^\pi & = & (')^\pi & \text{definition of } \rhd \\
 & = & (e \circ * \circ h)^\pi & \text{definition of } ' \\
 & = & e^\pi \circ *^\pi \circ h^\pi & \text{[11, Lemma 3.3, Corollary 2.25]} \\
 & = & e^\pi \circ *^\delta \circ h^\delta & * \text{ and } h \text{ are smooth}
\end{array}
$$

□

4 Multi-type presentation of semi De Morgan logic and its extensions

In Section 3.2 we showed that heterogeneous semi De Morgan algebras are equivalent presentations of semi De Morgan algebras. This provides a semantic motivation for introducing the multi-type language $\mathcal{L}_{\mathrm{MT}}$, which is naturally interpreted on heterogeneous semi De Morgan algebras. The language $\mathcal{L}_{\mathrm{MT}}$ consists of terms of types DL and K, defined as follows:

$$
\begin{array}{rcl}
\mathsf{DL} \ni A & ::= & p \mid \Box\alpha \mid \top \mid \bot \mid A \wedge A \mid A \vee A \\
\mathsf{K} \ni \alpha & ::= & \circ A \mid 1 \mid 0 \mid \sim \alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha
\end{array}
$$

The interpretation of $\mathcal{L}_{\mathrm{MT}}$-terms into heterogeneous algebras is defined as the easy generalization of the interpretation of propositional languages in universal algebra; namely, the heterogeneous operation e interprets the connective $\Box$, the heterogeneous operation h interprets the connective $\circ$, and DL-terms (resp. K-terms) are interpreted in the first (resp. second) component of heterogeneous algebras.

The toggle between single-type algebras and their heterogeneous counterparts (cf. Sections 3.2) is reflected syntactically by the translations $(\cdot)^\tau : \mathcal{L} \to \mathcal{L}_{\mathrm{MT}}$ defined as follows:

$$
\begin{array}{rcl}
p^\tau & ::= & p \\
\top^\tau & ::= & \top \\
\bot^\tau & ::= & \bot \\
(A \wedge B)^\tau & ::= & A^\tau \wedge B^\tau \\
(A \vee B)^\tau & ::= & A^\tau \vee B^\tau \\
(\neg A)^\tau & ::= & \Box \sim \circ A^\tau
\end{array}
$$

Recall that $\mathbb{A}^+$ denotes the heterogeneous algebra associated with the single-type algebra $\mathbb{A}$ (cf. Definition 13). The following proposition is proved by a routine induction on $\mathcal{L}$-formulas.

Proposition 23. *For all $\mathcal{L}$-formulas A and B and any $\mathbb{A} \in \{$SMA, LQMA, UQMA, DPL, APL, WSA$\}$,*

$$\mathbb{A} \models A \vdash B \quad \textit{iff} \quad \mathbb{A}^+ \models A^\tau \vdash B^\tau.$$

We are now in a position to translate the characteristic axioms of every logic mentioned in Section 2.1 into $\mathcal{L}_{\mathrm{MT}}$. Together with Proposition 19, the proposition above guarantees that the translation of each of the axioms below is valid on the corresponding class of heterogeneous algebras.

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ\Box{\sim}\circ B^\tau \qquad (i)$$

$$\neg A \vdash \neg\neg\neg A \rightsquigarrow \quad \Box{\sim}\circ A^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ\Box{\sim}\circ A^\tau \qquad (ii)$$

$$\neg\neg\neg A \vdash \neg A \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash \Box{\sim}\circ A^\tau \qquad (iii)$$

$$\neg A \wedge \neg B \vdash \neg(A \vee B) \rightsquigarrow \quad \Box{\sim}\circ(A \vee B)^\tau \vdash \Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ B^\tau \qquad (iv)$$

$$\top \vdash \neg\bot \rightsquigarrow \quad \top \vdash \Box{\sim}\circ\bot \qquad (v)$$

$$\neg\top \vdash \bot \rightsquigarrow \quad \Box{\sim}\circ\top \vdash \bot \qquad (vi)$$

$$A \vdash \neg\neg A \rightsquigarrow \quad A^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ A^\tau \qquad (vii)$$

$$\neg\neg A \vdash A \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash A^\tau \qquad (viii)$$

$$\neg A \wedge \neg\neg A \vdash \bot \rightsquigarrow \quad \Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash \bot \qquad (ix)$$

$$A \wedge \neg A \vdash \bot \rightsquigarrow \quad A^\tau \wedge \Box{\sim}\circ A^\tau \vdash \bot \qquad (x)$$

$$\top \vdash \neg A \vee \neg\neg A \rightsquigarrow \quad \top \vdash \Box{\sim}\circ A^\tau \vee \Box{\sim}\circ\Box{\sim}\circ A^\tau \qquad (xi)$$

Notice that the defining identities of heterogeneous algebras (cf. Definition 12) can be expressed as *analytic inductive* $\mathcal{L}_{\mathrm{MT}}$-inequalities (cf. Definition 29). Hence, these inequalities can be used to generate the analytic rules of the calculus introduced in Section 5, with a methodology analogous to the one introduced in [15]. As we will discuss in Section 6.2, the inequalities (*i*)-(*xi*) are derivable in the calculus obtained in this way.

5 Proper Display Calculi for semi De Morgan logic and its extensions

In the present section, we introduce proper multi-type display calculi for semi De Morgan logic and its extensions. The language manipulated by these calculi has types DL and K, and is built up from structural and operational (aka logical) connectives. In the tables of Section 5.1, each structural connective corresponding to a logical connective which belongs to the family $\mathcal{F}$ (resp. $\mathcal{G}$, $\mathcal{H}$) defined in Section A is denoted by decorating that logical connective with $\hat{}$ (resp. $\check{}$, $\tilde{}$).[3]

5.1 Language

Structural and operational terms.

[3]For any sequent $x \vdash y$, we define the signed generation trees $+x$ and $-y$ by labelling the root of the generation tree of x (resp. y) with the sign + (resp. −), and then propagating the sign to all nodes according to the polarity of the coordinate of the connective assigned to each node. Positive (resp. negative) coordinates propagate the same (resp. opposite) sign to the corresponding child node. Then, a substructure z in $x \vdash y$ is in *precedent* (resp. *succedent*) *position* if the sign of its root node as a subtree of $+x$ or $-y$ is + (resp. −).

$$\mathsf{DL}\begin{cases} A ::= p \mid \top \mid \bot \mid \Box\alpha \mid A \wedge A \mid A \vee A \\[2ex] X ::= A \mid \hat{\top} \mid \check{\bot} \mid \check{\Box}\Gamma \mid \hat{\bullet}_\ell\Gamma \mid \check{\bullet}_r\Gamma \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid X \mathbin{\hat{\succ\!\!-}} X \mid X \mathbin{\check{\rightarrow}} X \end{cases}$$

$$\mathsf{K}\begin{cases} \alpha ::= 1 \mid 0 \mid \circ A \mid {\sim}\,\alpha \mid \alpha \cap \alpha \mid \alpha \cup \alpha \\[2ex] \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\circ}X \mid \hat{\blacklozenge}X \mid \tilde{*}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid \Gamma \mathbin{\hat{\supset\!\!-}} \Gamma \mid \Gamma \mathbin{\check{-\!\!\supset}} \Gamma \end{cases}$$

Interpretation of pure-type structural connectives as their logical counterparts[4]:

DL						K							
$\hat{\top}$	$\hat{\wedge}$	$\hat{\succ\!\!-}$	$\check{\bot}$	$\check{\vee}$	$\check{\rightarrow}$	$\hat{1}$	$\hat{\cap}$	$\hat{\supset\!\!-}$	$\check{0}$	$\check{\cup}$	$\check{-\!\!\supset}$	$\tilde{*}$	
$\top$	$\wedge$	$(\succ\!\!-)$	$\bot$	$\vee$	$(\rightarrow)$	1	$\cap$	$(\supset\!\!-)$	0	$\cup$	$(-\!\!\supset)$	$\sim$	$\sim$

Interpretation of heterogeneous structural connectives as their logical counterparts:

DL $\to$ K		K $\to$ DL	K $\to$ DL	K $\to$ DL	DL $\to$ K
$\tilde{\circ}$		$\hat{\bullet}_\ell$	$\check{\bullet}_r$	$\check{\Box}$	$\hat{\blacklozenge}$
$\circ$	$\circ$	$(\bullet_\ell)$	$(\bullet_r)$	$\Box$	$(\blacklozenge)$

Algebraic interpretation of heterogeneous structural connectives as operations in perfect HSM-algebras (see Lemma 20).

DL $\to$ K	K $\to$ DL		K $\to$ DL	DL $\to$ K
$\tilde{\circ}$	$\hat{\bullet}_\ell$	$\check{\bullet}_r$	$\check{\Box}$	$\hat{\blacklozenge}$
h	h_ℓ	h_r	e	e_ℓ

5.2 Multi-type display calculi for semi De Morgan logic and its extensions

In what follows, structures of type DL are denoted by the variables X, Y, Z, and W; structures of type A are denoted by the variables Γ, Δ, Θ and Π.

- The proper display calculus for semi De Morgan logic D.SM consists of the following rules:

  - Identity and cut rules

$$\text{Id}\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{Cut}_{\mathsf{L}} \qquad \frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\ \text{Cut}_{\mathsf{D}}$$

  - Pure DL-type display rules

$$\text{res}_{\mathsf{L}}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{\overline{Y \vdash X \mathbin{\check{\rightarrow}} Z}} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{\overline{Y \mathbin{\hat{\succ\!\!-}} X \vdash Z}}\ \text{res}_{\mathsf{L}}$$

  - Pure K-type display rules

$$\text{res}_{\mathsf{D}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Theta}{\overline{\Delta \vdash \Gamma \mathbin{\check{-\!\!\supset}} \Theta}} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Theta}{\overline{\Delta \mathbin{\hat{\supset\!\!-}} \Gamma \vdash \Theta}}\ \text{res}_{\mathsf{D}}$$

$$\text{adj}_*\ \frac{\tilde{*}\Gamma \vdash \Delta}{\overline{\tilde{*}\Delta \vdash \Gamma}} \qquad \frac{\Gamma \vdash \tilde{*}\Delta}{\overline{\Delta \vdash \tilde{*}\Gamma}}\ \text{adj}_*$$

  - Multi-type display rules

$$\text{adj}_{\mathsf{LD}}\ \frac{X \vdash \check{\Box}\Gamma}{\overline{\hat{\blacklozenge}X \vdash \Gamma}} \qquad \text{adj}_{\mathsf{DL}}\ \frac{\tilde{\circ}X \vdash \Gamma}{\overline{X \vdash \check{\bullet}_r\Gamma}} \qquad \frac{\Gamma \vdash \tilde{\circ}X}{\overline{\hat{\bullet}_\ell\Gamma \vdash X}}\ \text{adj}_{\mathsf{DL}}$$

[4]In the synoptic table, the operational symbols which occur only at the structural level will appear between round brackets.

- Pure DL-type structural rules

$$\hat{\top}\ \frac{X \vdash Y}{X \hat{\wedge} \hat{\top} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \check{\vee} \check{\bot}}\ \check{\bot}$$

$$\mathrm{E_L}\ \frac{X \hat{\wedge} Y \vdash Z}{Y \hat{\wedge} X \vdash Z} \qquad \frac{X \vdash Y \check{\vee} Z}{X \vdash Z \check{\vee} Y}\ \mathrm{E_L}$$

$$\mathrm{A_L}\ \frac{(X \hat{\wedge} Y) \hat{\wedge} Z \vdash W}{X \hat{\wedge} (Y \hat{\wedge} Z) \vdash Z} \qquad \frac{X \vdash (Y \check{\vee} Z) \check{\vee} W}{X \vdash Y \check{\vee} (Z \check{\vee} W)}\ \mathrm{A_L}$$

$$\mathrm{W_L}\ \frac{X \vdash Y}{X \hat{\wedge} Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \check{\vee} Z}\ \mathrm{W_L}$$

$$\mathrm{C_L}\ \frac{X \hat{\wedge} X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash Y \check{\vee} Y}{X \vdash Y}\ \mathrm{C_L}$$

- Pure K-type structural rules

$$\hat{\mathrm{I}}\ \frac{\Gamma \vdash \Delta}{\Gamma \hat{\cap} \hat{\mathrm{I}} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \check{\cup} \check{\mathrm{O}}}\ \check{\mathrm{O}}$$

$$\mathrm{E_D}\ \frac{\Gamma \hat{\cap} \Delta \vdash \Theta}{\Delta \hat{\cap} \Gamma \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta \check{\cup} \Theta}{\Gamma \vdash \Theta \check{\cup} \Delta}\ \mathrm{E_D}$$

$$\mathrm{A_D}\ \frac{(\Gamma \hat{\cap} \Delta) \hat{\cap} \Theta \vdash \Pi}{\Gamma \hat{\cap} (\Delta \hat{\cap} \Theta) \vdash \Theta} \qquad \frac{\Gamma \vdash (\Delta \check{\cup} \Theta) \check{\cup} \Pi}{\Gamma \vdash \Delta \check{\cup} (\Theta \check{\cup} \Pi)}\ \mathrm{A_D}$$

$$\mathrm{W_D}\ \frac{\Gamma \vdash \Delta}{\Gamma \hat{\cap} \Theta \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \check{\cup} \Theta}\ \mathrm{W_D}$$

$$\mathrm{C_D}\ \frac{\Gamma \hat{\cap} \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta \check{\cup} \Delta}{\Gamma \vdash \Delta}\ \mathrm{C_D}$$

$$\frac{\Gamma \vdash \Delta}{\tilde{*} \Delta \vdash \tilde{*} \Gamma}\ \mathrm{cont}$$

- Multi-type structural rules

$$\tilde{\circ}\ \frac{X \vdash Y}{\tilde{\circ} X \vdash \tilde{\circ} Y} \qquad \frac{\hat{\bullet}_{\ell} \Gamma \vdash \check{\bullet}_{r} \Delta}{\Gamma \vdash \Delta}\ \tilde{\bullet}$$

$$\hat{\blacklozenge}\hat{\mathrm{I}}\ \frac{\hat{\mathrm{I}} \vdash \Gamma}{\hat{\blacklozenge} \hat{\top} \vdash \Gamma} \qquad \frac{X \vdash \check{\Box} \check{\mathrm{O}}}{X \vdash \check{\bot}}\ \check{\Box}\check{\mathrm{O}}$$

$$\frac{\Gamma \vdash \tilde{\circ} \check{\Box} \Delta}{\Gamma \vdash \Delta}\ \tilde{\circ}\check{\Box}$$

- Pure DL-type operational rules

$$\top\ \frac{\hat{\top} \vdash X}{\top \vdash X} \qquad \frac{}{\hat{\top} \vdash \top}\ \top$$

$$\bot\ \frac{}{\bot \vdash \check{\bot}} \qquad \frac{X \vdash \check{\bot}}{X \vdash \bot}\ \bot$$

$$\wedge\ \frac{A \hat{\wedge} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \hat{\wedge} Y \vdash A \wedge B}\ \wedge$$

$$\vee\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \check{\vee} Y} \qquad \frac{X \vdash A \check{\vee} B}{X \vdash A \vee B}\ \vee$$

- Pure K-type operational rules

$$
{\scriptstyle 1}\,\frac{\hat{1} \vdash \Gamma}{1 \vdash \Gamma} \qquad \frac{}{\hat{1} \vdash 1}\,{\scriptstyle 1}
$$

$$
{\scriptstyle 0}\,\frac{}{0 \vdash \check{0}} \qquad \frac{\Gamma \vdash \check{0}}{\Gamma \vdash 0}\,{\scriptstyle 0}
$$

$$
{\scriptstyle \cap}\,\frac{\alpha \,\hat{\cap}\, \beta \vdash \Gamma}{\alpha \cap \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \,\hat{\cap}\, \Delta \vdash \alpha \cap \beta}\,{\scriptstyle \cap}
$$

$$
{\scriptstyle \cup}\,\frac{\alpha \vdash \Gamma \qquad \beta \vdash \Delta}{\alpha \cup \beta \vdash \Gamma \,\check{\cup}\, \Delta} \qquad \frac{\Gamma \vdash \alpha \,\check{\cup}\, \beta}{\Gamma \vdash \alpha \cup \beta}\,{\scriptstyle \cup}
$$

$$
{\scriptstyle \sim}\,\frac{\tilde{*} \alpha \vdash \Gamma}{\sim \alpha \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{*} \alpha}{\Gamma \vdash \sim \alpha}\,{\scriptstyle \sim}
$$

- Multi-type operational rules

$$
{\scriptstyle \circ}\,\frac{\tilde{\circ} A \vdash \Gamma}{\circ A \vdash \Gamma} \qquad \frac{X \vdash \tilde{\circ} A}{X \vdash \circ A}\,{\scriptstyle \circ}
$$

$$
{\scriptstyle \Box}\,\frac{\alpha \vdash \Gamma}{\Box \alpha \vdash \check{\Box} \Gamma} \qquad \frac{X \vdash \check{\Box} \alpha}{X \vdash \Box \alpha}\,{\scriptstyle \Box}
$$

- The proper display calculus D.LQM for lower quasi De Morgan logic consists of all axiom and rules in D.SM plus the following rule:

$$
\frac{X \vdash Y}{X \vdash \check{\Box} \tilde{\circ} Y}\,{\scriptstyle \mathrm{LQM}}
$$

- The proper display calculus D.UQM for upper quasi De Morgan logic consists of all axiom and rules in D.SM plus the following rule:

$$
\frac{\hat{\bullet}_\ell \, \hat{\blacklozenge} X \vdash Y}{X \vdash Y}\,{\scriptstyle \mathrm{UQM}}
$$

- The proper display calculus D.DP for demi pseudocomplemented lattice logic consists of all axiom and rules in D.SM plus the following rule:

$$
{\scriptstyle \mathrm{res_B}}\,\frac{\Gamma \,\hat{\cap}\, \Delta \vdash \Sigma}{\Delta \vdash \tilde{*} \Gamma \,\check{\cup}\, \Sigma}
$$

- The proper display calculus D.AP for almost pseudocomplemented lattice logic consists of all axiom and rules in D.DP plus the following rule:

$$
\frac{X \vdash \check{\Box} \tilde{*} \tilde{\circ} Y}{X \,\hat{\wedge}\, Y \vdash \check{\bot}}\,{\scriptstyle \mathrm{AP}}
$$

- The proper display calculus D.WS for weak stone logic consists of all axiom and rules in D.DP plus the following rule:

$$
\frac{\hat{\blacklozenge} X \vdash \Delta}{\hat{\blacklozenge} (\check{\Box} \tilde{*} \tilde{\circ} X \,\hat{\succ}\, \hat{\top}) \vdash \Delta}\,{\scriptstyle \mathrm{WS}}
$$

6 Properties

6.1 Soundness

In the present subsection, we outline the verification of the soundness of the rules of D.SM (resp. D.LQM, D.UQM D.DP, D.AP and D.WS) w.r.t. the semantics of *perfect* HSMAs (resp. HQ-MAs, HDPLs, HAPLs and HWSAs, see Definition 12). The first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position, as indicated at the beginning of Section 5. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{X \vdash Y}{\tilde{\circ} X \vdash \tilde{\circ} Y} \quad \rightsquigarrow \quad \forall a \forall b[a \leq b \Rightarrow h(a) \leq h(b)]$$

$$\frac{\hat{\blacklozenge} X \vdash \Delta}{\hat{\blacklozenge}(\check{\Box} \tilde{\ast} \tilde{\circ} X \hat{\succ\!\!-} \hat{\top}) \vdash \Delta} \quad \rightsquigarrow \quad \forall a[e_\ell[e(h(a)^*) \succ\!\!- \top] \leq e_\ell(a)]$$

The proof of the soundness of the rules in these display calculi then consists in verifying the validity of their corresponding quasi-inequalities in the corresponding class of perfect heterogeneous algebras. The verification of the soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The validity of the quasi-inequalities corresponding to multi-type structural rules follows straightforwardly from the observation that the quasi-inequality corresponding to each rule is obtained by running the algorithm ALBA (cf. Section 3.4 [15]) on some of the defining inequalities of its corresponding heterogeneous algebras.[5] For instance, the soundness of the characteristic rule of D.WS on HWSAs follows from the validity of the inequality (*xi*) in every HWSA (discussed in Section 4) and from the soundness of the following ALBA reduction in every HWSA:

$$\begin{array}{lll}
 & \forall a[\top \leq e(h(a)^*) \vee e((he(h(a)^*))^*)] & \\
\text{iff} & \forall a \forall b \forall c[b \leq a \,\&\, c \leq e(h(a)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall a \forall b \forall c[b \leq a \,\&\, a \leq h_r(e_\ell(c)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow e(h(c)^*) \succ\!\!- \top \leq e(h(b)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow b \leq h_r(e_\ell[e(h(c)^*) \succ\!\!- \top]^*)] & \\
\text{iff} & \forall c[h_r(e_\ell(c)^*) \leq h_r(e_\ell[e(h(c)^*) \succ\!\!- \top]^*)] & \\
\text{iff} & \forall c[e_\ell(c)^* \leq e_\ell[e(h(c)^*) \succ\!\!- \top]^*] & h_r \text{ is injective} \\
\text{iff} & \forall c[e_\ell[e(h(c)^*) \succ\!\!- \top] \leq e_\ell(c)] & {}^* \text{ is injective}
\end{array}$$

6.2 Completeness

In the present subsection, we show that the translations of the axioms and rules of SM, LQM, UQM, DP, AP and WS are derivable in D.SM, D.LQM, D.UQM, D.DP, D.AP and D.WS, respectively. Then, the completeness of these display calculi w.r.t. the classes of SMAs, LQMAs, UQMAs, DPLs, APLs and WSAs immediately follows from the completeness of SM, LQM, UQM, DP, AP and WS (cf. Theorem 5).

Proposition 24. *For every $A \in \mathcal{L}$, the sequent $A^\tau \vdash A^\tau$ is derivable in all display calculi introduced in Section 5.2.*

Proof. By inducution on $A \in \mathcal{L}$. The proof of base cases: $A := \top$, $A := \bot$ and $A := p$, are straightforward and are omitted.

Inductive cases:

- as to $A := \neg B$,

[5] Indeed, as discussed in [15], the soundness of the rewriting rules of ALBA only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B^\tau \vdash B^\tau}\,\text{ind.hyp.}}{\tilde{\circ} B^\tau \vdash \tilde{\circ} B^\tau}\,\tilde{\circ}}{\circ B^\tau \vdash \tilde{\circ} B^\tau}}{\circ B^\tau \vdash \circ B^\tau}}{\tilde{*}\circ B^\tau \vdash \tilde{*}\circ B^\tau}\,\text{cont}}{\tilde{*}\circ B^\tau \vdash \sim\circ B^\tau}}{\sim\circ B^\tau \vdash \sim\circ B^\tau}}{\Box\sim\circ B^\tau \vdash \check{\Box}\sim\circ B^\tau}}{\Box\sim\circ B^\tau \vdash \Box\sim\circ B^\tau}$$

- as to $A := B \vee C$,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B^\tau \vdash B^\tau}\,\text{ind.hyp.}}{B^\tau \vdash B^\tau \check{\vee} C^\tau}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{}{C^\tau \vdash C^\tau}\,\text{ind.hyp.}}{C^\tau \vdash C^\tau \check{\vee} B^\tau}\,\text{W}}{C^\tau \vdash B^\tau \check{\vee} C^\tau}\,\text{E}}{B^\tau \vee C^\tau \vdash (B^\tau \check{\vee} C^\tau) \check{\vee} (B^\tau \check{\vee} C^\tau)}}{B^\tau \vee C^\tau \vdash B^\tau \vee C^\tau}\,\text{C}$$

- as to $A := B \wedge C$,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B^\tau \vdash B^\tau}\,\text{ind.hyp.}}{B^\tau \hat{\wedge} C^\tau \vdash B^\tau}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{}{C^\tau \vdash C^\tau}\,\text{ind.hyp.}}{C^\tau \hat{\wedge} B^\tau \vdash C^\tau}\,\text{W}}{B^\tau \hat{\wedge} C^\tau \vdash C^\tau}\,\text{E}}{(B^\tau \hat{\wedge} C^\tau) \hat{\wedge} (B^\tau \hat{\wedge} C^\tau) \vdash B^\tau \wedge C^\tau}}{B^\tau \wedge C^\tau \vdash B^\tau \wedge C^\tau}\,\text{C}$$

$\Box$

Proposition 25. *For every $A, B \in \mathcal{L}$, if $A \vdash B$ is derivable in any logic introduced in 2.1, then $A^\tau \vdash B^\tau$ is derivable in its respective display calculus.*

Proof. It is enough to show the statement of the proposition on the axioms. For the sake of readability, in what follows, we suppress the translation symbol $(\cdot)^\tau$. As to the axioms in SM:

- $\neg\top \vdash \bot \quad \rightsquigarrow \quad \Box\sim\circ\top \vdash \bot$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\hat{\top} \vdash \top}{\hat{\bullet}_\ell \tilde{*}\check{0} \hat{\wedge} \hat{\top} \vdash \top}\,\text{W}_\text{L}}{\hat{\bullet}_\ell \tilde{*}\check{0} \vdash \top}\,\hat{\top}}{\tilde{*}\check{0} \vdash \tilde{\circ}\top}}{\tilde{*}\check{0} \vdash \circ\top}}{\tilde{*}\circ\top \vdash \check{0}}}{\sim\circ\top \vdash \check{0}}}{\Box\sim\circ\top \vdash \check{\Box}\check{0}}}{\Box\sim\circ\top \vdash \check{\bot}}\,\check{\Box}\check{0}}{\Box\sim\circ\top \vdash \bot}$$

- $\top \vdash \neg\bot \rightsquigarrow \quad \top \vdash \Box\sim\circ\top$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash \check{\bot}}{\bot \vdash \check{\bot} \check{\vee} \check{\bullet}_r \tilde{*}\hat{1}}\,\text{W}_\text{L}}{\bot \vdash \check{\bullet}_r \tilde{*}\hat{1}}\,\check{\bot}}{\tilde{\circ}\bot \vdash \tilde{*}\hat{1}}}{\circ\bot \vdash \tilde{*}\hat{1}}}{\hat{1} \vdash \tilde{*}\circ\bot}}{\hat{1} \vdash \sim\circ\bot}}{\hat{\blacklozenge}\hat{\top} \vdash \sim\circ\bot}\,\hat{\blacklozenge}\hat{\top}}{\hat{\top} \vdash \check{\Box}\sim\circ\bot}}{\hat{\top} \vdash \Box\sim\circ\bot}}{\top \vdash \Box\sim\circ\bot}$$

- $\neg A \vdash \neg\neg\neg A \quad \rightsquigarrow \quad \Box \sim \circ A \vdash \Box \sim \circ \Box \sim \circ \Box \sim \circ A$
  and $\neg\neg\neg A \vdash \neg A \quad \rightsquigarrow \quad \Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \Box \sim \circ A$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\circ A \vdash \tilde{\circ} A & \\ \hline
\circ A \vdash \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{*} \circ A & \text{cont} \\ \hline
\tilde{*} \circ A \vdash \sim \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{\circ} \check{\Box} \sim \circ A & \tilde{\circ}\,\check{\Box} \\ \hline
\hat{\bullet}_\ell \tilde{*} \circ A \vdash \check{\Box} \sim \circ A & \\ \hline
\hat{\bullet}_\ell \tilde{*} \circ A \vdash \Box \sim \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{\circ} \Box \sim \circ A & \\ \hline
\tilde{*} \circ A \vdash \circ \Box \sim \circ A & \\ \hline
\tilde{*} \circ \Box \sim \circ A \vdash \circ A & \\ \hline
\sim \circ \Box \sim \circ A \vdash \circ A & \\ \hline
\Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \circ A & \\ \hline
\tilde{\circ} \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \circ A & \tilde{\circ} \\ \hline
\circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \circ A & \\ \hline
\circ \Box \sim \circ \Box \sim \circ A \vdash \circ A & \tilde{\circ}\,\check{\Box} \\ \hline
\tilde{*} \circ A \vdash \tilde{*} \circ \Box \sim \circ \Box \sim \circ A & \text{cont} \\ \hline
\tilde{*} \circ A \vdash \sim \circ \Box \sim \circ \Box \sim \circ A & \\ \hline
\sim \circ A \vdash \sim \circ \Box \sim \circ \Box \sim \circ A & \\ \hline
\Box \sim \circ A \vdash \check{\Box} \sim \circ \Box \sim \circ \Box \sim \circ A & \\ \hline
\Box \sim \circ A \vdash \Box \sim \circ \Box \sim \circ \Box \sim \circ A &
\end{array}
\qquad
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\circ A \vdash \tilde{\circ} A & \\ \hline
\circ A \vdash \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{*} \circ A & \text{cont} \\ \hline
\sim \circ A \vdash \tilde{*} \circ A & \\ \hline
\Box \sim \circ A \vdash \check{\Box} \tilde{*} \circ A & \\ \hline
\tilde{\circ} \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \tilde{*} \circ A & \tilde{\circ} \\ \hline
\circ \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \tilde{*} \circ A & \\ \hline
\circ \Box \sim \circ A \vdash \tilde{*} \circ A & \tilde{\circ}\,\check{\Box} \\ \hline
\circ A \vdash \tilde{*} \circ \Box \sim \circ A & \\ \hline
\circ A \vdash \sim \circ \Box \sim \circ A & \\ \hline
\circ A \vdash \tilde{\circ} \check{\Box} \sim \circ \Box \sim \circ A & \tilde{\circ}\,\check{\Box} \\ \hline
\hat{\bullet}_\ell \circ A \vdash \check{\Box} \sim \circ \Box \sim \circ A & \\ \hline
\hat{\bullet}_\ell \circ A \vdash \Box \sim \circ \Box \sim \circ A & \\ \hline
\circ A \vdash \tilde{\circ} \Box \sim \circ \Box \sim \circ A & \\ \hline
\circ A \vdash \circ \Box \sim \circ \Box \sim \circ A & \\ \hline
\tilde{*} \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{*} \circ A & \text{cont} \\ \hline
\sim \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{*} \circ A & \\ \hline
\sim \circ \Box \sim \circ \Box \sim \circ A \vdash \sim \circ A & \\ \hline
\Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \sim \circ A & \\ \hline
\Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \Box \sim \circ A &
\end{array}
$$

- $\neg A \wedge \neg B \vdash \neg(A \vee B) \quad \rightsquigarrow \quad \Box \sim \circ A \wedge \Box \sim \circ B \vdash \Box \sim \circ (A \vee B)$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\tilde{\circ} A \vdash \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{*} \tilde{\circ} A & \text{cont} \\ \hline
\sim \circ A \vdash \tilde{*} \tilde{\circ} A & \\ \hline
\Box \sim \circ A \vdash \check{\Box} \tilde{*} \tilde{\circ} A & \\ \hline
\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} A & \mathrm{W_L} \\ \hline
\Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} A & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{*} \tilde{\circ} A & \\ \hline
\tilde{\circ} A \vdash \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
A \vdash \check{\bullet}_r \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) &
\end{array}
\qquad
\begin{array}{cl}
B \vdash B & \\ \hline
\tilde{\circ} B \vdash \tilde{\circ} B & \tilde{\circ} \\ \hline
\tilde{\circ} B \vdash \circ B & \\ \hline
\tilde{*} \circ B \vdash \tilde{*} \tilde{\circ} B & \text{cont} \\ \hline
\sim \circ B \vdash \tilde{*} \tilde{\circ} B & \\ \hline
\Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} B & \\ \hline
\Box \sim \circ B \,\hat{\wedge}\, \Box \sim \circ A \vdash \check{\Box} \tilde{*} \tilde{\circ} B & \mathrm{W_L} \\ \hline
\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} B & \mathrm{E_L} \\ \hline
\Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} B & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{*} \tilde{\circ} B & \\ \hline
\tilde{\circ} B \vdash \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
B \vdash \check{\bullet}_r \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) &
\end{array}
$$

$$
\begin{array}{cl}
\hline
A \vee B \vdash \check{\bullet}_r \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) \,\check{\vee}\, \tilde{\bullet} \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
A \vee B \vdash \tilde{\bullet} \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) & \mathrm{C_L} \\ \hline
\tilde{\circ}(A \vee B) \vdash \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
\circ(A \vee B) \vdash \tilde{*} \hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{*} \circ (A \vee B) & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \sim \circ (A \vee B) & \\ \hline
\Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \sim \circ (A \vee B) & \\ \hline
\Box \sim \circ A \wedge \Box \sim \circ B \vdash \Box \sim \circ (A \vee B) &
\end{array}
$$

- $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \quad \rightsquigarrow \quad \Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \Box \sim \circ \Box \sim \circ (A \wedge B)$,

$$
\begin{array}{ll}
 & A \vdash A \\
\tilde{\circ} & \tilde{\circ} A \vdash \tilde{\circ} A \\
 & \circ A \vdash \tilde{\circ} A \\
\text{cont} & \tilde{*} \tilde{\circ} A \vdash \tilde{*} \circ A \\
 & \tilde{*} \tilde{\circ} A \vdash \sim \circ A \\
\tilde{\circ}\,\check{\Box} & \tilde{*} \tilde{\circ} A \vdash \tilde{\circ} \check{\Box} \sim \circ A \\
 & \hat{\bullet}_\ell \tilde{*} \tilde{\circ} A \vdash \check{\Box} \sim \circ A \\
 & \hat{\bullet}_\ell \tilde{*} \tilde{\circ} A \vdash \Box \sim \circ A \\
 & \tilde{*} \tilde{\circ} A \vdash \tilde{\circ} \Box \sim \circ A \\
 & \tilde{*} \tilde{\circ} A \vdash \circ \Box \sim \circ A \\
 & \tilde{*} \circ \Box \sim \circ A \vdash \tilde{\circ} A \\
 & \sim \circ \Box \sim \circ A \vdash \tilde{\circ} A \\
 & \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} A \\
\mathrm{W_L} & \Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} A \\
 & \Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} A \\
 & \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} A \\
 & \hat{\bullet}_\ell\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A
\end{array}
$$

$$
\begin{array}{ll}
 & B \vdash B \\
\tilde{\circ} & \tilde{\circ} B \vdash \tilde{\circ} B \\
 & \circ B \vdash \tilde{\circ} B \\
\text{cont} & \tilde{*} \tilde{\circ} B \vdash \tilde{*} \circ B \\
 & \tilde{*} \tilde{\circ} B \vdash \sim \circ B \\
\tilde{\circ}\,\check{\Box} & \tilde{*} \tilde{\circ} B \vdash \tilde{\circ} \check{\Box} \sim \circ B \\
 & \hat{\bullet}_\ell \tilde{*} \tilde{\circ} B \vdash \check{\Box} \sim \circ B \\
 & \hat{\bullet}_\ell \tilde{*} \tilde{\circ} B \vdash \Box \sim \circ B \\
 & \tilde{*} \tilde{\circ} B \vdash \tilde{\circ} \Box \sim \circ B \\
 & \tilde{*} \tilde{\circ} B \vdash \circ \Box \sim \circ B \\
 & \tilde{*} \circ \Box \sim \circ B \vdash \tilde{\circ} B \\
 & \sim \circ \Box \sim \circ B \vdash \tilde{\circ} B \\
 & \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} B \\
\mathrm{W_L} & \Box \sim \circ \Box \sim \circ B \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} B \\
\mathrm{E_L} & \Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} B \\
 & \Box \sim \circ \Box \sim \circ B \wedge \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} B \\
 & \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} B \\
 & \hat{\bullet}_\ell\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash B
\end{array}
$$

$$
\begin{array}{ll}
 & \hat{\bullet}_\ell\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \,\hat{\wedge}\, \hat{\bullet}_\ell\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A \wedge B \\
\mathrm{C_L} & \hat{\bullet}_\ell\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A \wedge B \\
 & \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ}(A \wedge B) \\
 & \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \circ(A \wedge B) \\
\text{cont} & \tilde{*} \circ (A \wedge B) \vdash \tilde{*}\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \\
 & \sim \circ (A \wedge B) \vdash \tilde{*}\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \\
 & \Box \sim \circ (A \wedge B) \vdash \check{\Box} \tilde{*}\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \\
\tilde{\circ} & \tilde{\circ} \Box \sim \circ (A \wedge B) \vdash \tilde{\circ} \check{\Box} \tilde{*}\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \\
\tilde{\circ}\,\check{\Box} & \tilde{\circ} \Box \sim \circ (A \wedge B) \vdash \tilde{*}\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \\
 & \circ \Box \sim \circ (A \wedge B) \vdash \tilde{*}\, \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \\
 & \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{*} \circ \Box \sim \circ (A \wedge B) \\
 & \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \sim \circ \Box \sim \circ (A \wedge B) \\
 & \Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \sim \circ \Box \sim \circ (A \wedge B) \\
 & \Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \Box \sim \circ \Box \sim \circ (A \wedge B)
\end{array}
$$

As to the characterizing axioms of LQM and UQM:

- $A \vdash \neg\neg A \quad \rightsquigarrow \quad A \vdash \square \sim \circ \square \sim \circ A$ and $\neg\neg A \vdash A \quad \rightsquigarrow \quad \square \sim \circ \square \sim \circ A \vdash A$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
A \vdash \check{\square}\tilde{\circ} A & \text{LQM} \\ \hline
\hat{\blacklozenge} A \vdash \tilde{\circ} A & \\ \hline
\hat{\blacklozenge} A \vdash \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{*} \hat{\blacklozenge} A & \text{cont} \\ \hline
\sim \circ A \vdash \tilde{*} \hat{\blacklozenge} A & \\ \hline
\square \sim \circ A \vdash \check{\square} \tilde{*} \hat{\blacklozenge} A & \\ \hline
\tilde{\circ} \square \sim \circ A \vdash \tilde{\circ} \check{\square} \tilde{*} \hat{\blacklozenge} A & \tilde{\circ} \\ \hline
\tilde{\circ} \square \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} A & \tilde{\circ}\check{\square} \\ \hline
\circ \square \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} A & \\ \hline
\hat{\blacklozenge} A \vdash \tilde{*} \circ \square \sim \circ A & \\ \hline
\hat{\blacklozenge} A \vdash \sim \circ \square \sim \circ A & \\ \hline
A \vdash \check{\square} \sim \circ \square \sim \circ A & \\ \hline
A \vdash \square \sim \circ \square \sim \circ A &
\end{array}
\qquad
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\circ A \vdash \tilde{\circ} A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \tilde{*} \circ A & \text{cont} \\ \hline
\tilde{*} \tilde{\circ} A \vdash \sim \circ A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \tilde{\circ} \check{\square} \sim \circ A & \tilde{\circ}\check{\square} \\ \hline
\hat{\bullet}_{\ell} \tilde{*} \tilde{\circ} A \vdash \check{\square} \sim \circ A & \\ \hline
\hat{\bullet}_{\ell} \tilde{*} \tilde{\circ} A \vdash \square \sim \circ A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \tilde{\circ} \square \sim \circ A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \circ \square \sim \circ A & \\ \hline
\tilde{*} \circ \square \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\sim \circ \square \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\square \sim \circ \square \sim \circ A \vdash \check{\square} \tilde{\circ} A & \\ \hline
\hat{\blacklozenge} \square \sim \circ \square \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\hat{\bullet}_{\ell} \hat{\blacklozenge} \square \sim \circ \square \sim \circ A \vdash A & \\ \hline
\square \sim \circ \square \sim \circ A \vdash A & \text{UQM}
\end{array}
$$

As to the characterizing axiom of AP:

- $\neg A \wedge A \vdash \bot \quad \rightsquigarrow \quad \square \sim \circ A \wedge A \vdash \bot$,

$$
\begin{array}{lcl}
 & A \vdash A & \\ \hline
 & \tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
 & \tilde{\circ} A \vdash \circ A & \\ \hline
 & \sim \circ A \vdash \tilde{*} \tilde{\circ} A & \\ \hline
 & \square \sim \circ A \vdash \check{\square} \tilde{*} \tilde{\circ} A & \\ \hline
 & A \mathbin{\hat{\wedge}} \square \sim \circ A \vdash \check{\bot} & \text{AP} \\ \hline
 & \square \sim \circ A \vdash A \mathbin{\check{\to}} \check{\bot} & \\ \hline
 & A \mathbin{\hat{\wedge}} \square \sim \circ A \vdash \check{\bot} & \\ \hline
\mathrm{E_L} & \square \sim \circ A \mathbin{\hat{\wedge}} A \vdash \check{\bot} & \\ \hline
 & \square \sim \circ A \mathbin{\hat{\wedge}} A \vdash \bot & \\ \hline
 & \square \sim \circ A \wedge A \vdash \bot &
\end{array}
$$

As to the characterizing axiom of DP:

- $\neg A \wedge \neg\neg A \vdash \bot \quad \rightsquigarrow \quad \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot$,

$$A \vdash A \quad \tilde{\circ}$$
$$\tilde{\circ} A \vdash \tilde{\circ} A$$
$$\circ A \vdash \tilde{\circ} A$$
$$\circ A \vdash \circ A \quad \mathrm{W_L}$$
$$\circ A \,\hat{\cap}\, \check{0} \vdash \circ A$$
$$\check{0} \vdash \tilde{*} \circ A \,\check{\cup}\, \circ A \quad \mathrm{E_L}$$
$$\check{0} \vdash \circ A \,\check{\cup}\, \tilde{*} \circ A$$
$$\tilde{*} \circ A \,\hat{\cap}\, \check{0} \vdash \tilde{*} \circ A \quad \check{0}$$
$$\tilde{*} \circ A \vdash \tilde{*} \circ A$$
$$\tilde{*} \circ A \vdash \sim \circ A$$
$$\sim \circ A \vdash \sim \circ A \quad \tilde{\circ}\check{\Box}$$
$$\sim \circ A \vdash \tilde{\circ} \check{\Box} \sim \circ A$$
$$\hat{\bullet}_\ell \sim \circ A \vdash \check{\Box} \sim \circ A$$
$$\hat{\bullet}_\ell \sim \circ A \vdash \Box \sim \circ A$$
$$\sim \circ A \vdash \tilde{\circ} \Box \sim \circ A$$
$$\sim \circ A \vdash \circ \Box \sim \circ A$$
$$\Box \sim \circ A \vdash \check{\Box} \circ \Box \sim \circ A$$
$$\hat{\blacklozenge} \Box \sim \circ A \vdash \circ \Box \sim \circ A \quad \mathrm{W_L}$$
$$\hat{\blacklozenge} \Box \sim \circ A \vdash \circ \Box \sim \circ A \,\check{\cup}\, \check{0}$$
$$\mathrm{E_L} \quad \tilde{*} \circ \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ A \vdash \check{0}$$
$$\hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \tilde{*} \circ \Box \sim \circ A \vdash \check{0}$$
$$\tilde{*} \circ \Box \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0}$$
$$\sim \circ \Box \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0}$$
$$\Box \sim \circ \Box \sim \circ A \vdash \check{\Box}(\tilde{*} \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0})$$
$$\hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0}$$
$$\hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \check{0}$$
$$\hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0}$$
$$\Box \sim \circ \Box \sim \circ A \vdash \check{\Box}(\hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0})$$
$$\mathrm{W_L} \quad \Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ A \vdash \check{\Box}(\hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0})$$
$$\mathrm{E_L} \quad \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box}(\hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0})$$
$$\hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0}$$
$$\hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0}$$
$$\hat{\blacklozenge} \Box \sim \circ A \vdash \hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0}$$
$$\Box \sim \circ A \vdash \check{\Box}(\hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0})$$
$$\mathrm{W_L} \quad \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box}(\hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0})$$
$$\hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0}$$
$$\hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\hat{\cap}\, \hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0}$$
$$\mathrm{C_D} \quad \hat{\blacklozenge}(\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0}$$
$$\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \check{0} \quad \check{\Box}\check{0}$$
$$\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\bot}$$
$$\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \bot$$
$$\Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot$$

As to the characterizing axiom of WS:

- $\top \vdash \neg\neg A \vee \neg A \quad \rightsquigarrow \quad \top \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A$,

$$
\begin{array}{c}
A \vdash A \\
\hline
\tilde{\circ} A \vdash \tilde{\circ} A \quad (\tilde{\circ}) \\
\hline
\tilde{\circ} A \vdash \circ A \\
\hline
\circ A \vdash \circ A \\
\hline
\sim \circ A \vdash \tilde{*} \circ A \\
\hline
\sim \circ A \vdash \sim \circ A \\
\hline
\Box \sim \circ A \vdash \check{\Box} \sim \circ A \\
\hline
\hat{\blacklozenge} \Box \sim \circ A \vdash \sim \circ A \\
\hline
\hat{\blacklozenge} (\check{\Box} \tilde{*} \tilde{\circ} \Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top}) \vdash \sim \circ A \quad (\mathrm{WS}) \\
\hline
\check{\Box} \tilde{*} \tilde{\circ} \Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top} \vdash \check{\Box} \sim \circ A \\
\hline
\check{\Box} \tilde{*} \tilde{\circ} \Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top} \vdash \Box \sim \circ A \\
\hline
\hat{\top} \vdash \check{\Box} \tilde{*} \tilde{\circ} \Box \sim \circ A \,\check{\vee}\, \Box \sim \circ A \\
\hline
\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top} \vdash \check{\Box} \tilde{*} \tilde{\circ} \Box \sim \circ A \\
\hline
\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top}) \vdash \tilde{*} \tilde{\circ} \Box \sim \circ A \\
\hline
\tilde{\circ} \Box \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top}) \\
\hline
\circ \Box \sim \circ A \vdash \tilde{*} \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top}) \\
\hline
\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top}) \vdash \tilde{*} \circ \Box \sim \circ A \\
\hline
\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top}) \vdash \sim \circ \Box \sim \circ A \\
\hline
\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top} \vdash \check{\Box} \sim \circ \Box \sim \circ A \\
\hline
\Box \sim \circ A \,\hat{\succ\!\!-}\, \hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \\
\hline
\hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \,\check{\vee}\, \Box \sim \circ A \\
\hline
\hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A \\
\hline
\top \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A
\end{array}
$$

□

6.3 Conservativity

To argue that the calculi introduced in Section 5 conservatively capture their respective logics (see Section 2.1), we follow the standard proof strategy discussed in [15, 13]. Let L be one of the logics of Definition 1, let $\vdash_{\mathsf{L}}$ denote its syntactic consequence relation, and let $\models_{\mathsf{L}}$ (resp. $\models_{\mathsf{HL}}$) denote the semantic consequence relation arising from the class of the perfect (heterogeneous) algebras associated with L. We need to show that, for all $\mathcal{L}$-formulas A and B, if $A^{\tau} \vdash B^{\tau}$ is derivable in the display calculus D.L, then $A \vdash_{\mathsf{L}} B$. This claim can be proved using the following facts: (a) the rules of D.L are sound w.r.t. perfect heterogeneous L-algebras (cf. Section 6.1); (b) L is complete w.r.t. its associated class of algebras (cf. Theorem 5); and (c) L-algebras are equivalently presented as heterogeneous L-algebras (cf. Section 3.2), so that the semantic consequence relations arising from each type of algebras preserve and reflect the translation (cf. Proposition 23). If $A^{\tau} \vdash B^{\tau}$ is derivable in D.L, then by (a), $\models_{\mathsf{HL}} A^{\tau} \vdash B^{\tau}$. By (c), this implies that $\models_{\mathsf{L}} A \vdash B$. By (b), this implies that $A \vdash_{\mathsf{L}} B$, as required.

6.4 Cut elimination and subformula property

In the present subsection, we briefly sketch the proof of cut elimination and subformula property for all display calculi introduced in Section 5.2. As discussed earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi. The meta-theorem to which we will appeal was proved in [6, Theorem 4.1].

Theorem 26. *Cut elimination and subformula property hold for all display calculi introduced in Section 5.2.*

Proof. All conditions in [6] except C_8' are readily satisfied by inspecting the rules. Condition C_8' requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we show C_8' for the unary connectives by induction on the complexity of cut formula.

Pure type atomic propositions:

$$\dfrac{p \vdash p \qquad p \vdash p}{p \vdash p} \quad \rightsquigarrow \quad p \vdash p$$

Pure type constants:

$$\dfrac{\hat{\top} \vdash \top \qquad \dfrac{\overset{\vdots\, \pi_1}{\hat{\top} \vdash X}}{\top \vdash X}}{\hat{\top} \vdash X} \quad \rightsquigarrow \quad \overset{\vdots\, \pi_1}{\hat{\top} \vdash X}$$

The cases for $\bot$, 1, 0 are standard and similar to the one above.

Pure-type unary connectives:

$$\dfrac{\dfrac{\overset{\vdots\, \pi_1}{\Gamma \vdash \tilde{*}\alpha}}{\Gamma \vdash \sim \alpha} \qquad \dfrac{\overset{\vdots\, \pi_2}{\tilde{*}\alpha \vdash \Delta}}{\sim \alpha \vdash \Delta}}{\Gamma \vdash \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, \pi_2}{\tilde{*}\alpha \vdash \Delta}}{\tilde{*}\Delta \vdash \alpha} \qquad \dfrac{\overset{\vdots\, \pi_1}{\Gamma \vdash \tilde{*}\alpha}}{\alpha \vdash \tilde{*}\Gamma}}{\tilde{*}\Delta \vdash \tilde{*}\Gamma}}{\Gamma \vdash \Delta}\,\text{cont}$$

Pure-type binary connectives:

$$\dfrac{\dfrac{\overset{\vdots\, \pi_1}{X \vdash A} \qquad \overset{\vdots\, \pi_2}{Y \vdash B}}{X \hat{\wedge} Y \vdash A \wedge B} \qquad \dfrac{\overset{\vdots\, \pi_3}{A \hat{\wedge} B \vdash Z}}{A \wedge B \vdash Z}}{X \hat{\wedge} Y \vdash Z} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, \pi_1}{X \vdash A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\, \pi_2}{Y \vdash B} \qquad \dfrac{\overset{\vdots\, \pi_3}{A \hat{\wedge} B \vdash Z}}{B \vdash A \check{\rightarrow} Z}}{Y \vdash A \check{\rightarrow} Z}}{A \hat{\wedge} Y \vdash Z}}{Y \hat{\wedge} A \vdash Z}}{A \vdash Y \check{\rightarrow} Z}}{X \vdash Y \check{\rightarrow} Z}}{Y \hat{\wedge} X \vdash Z}}{X \hat{\wedge} Y \vdash Z}$$

The cases for $A \vee B$, $\alpha \cap \beta$, $\alpha \cup \beta$ are standard and similar to the one above.

Multi-type unary connectives:

$$\dfrac{\dfrac{\overset{\vdots\, \pi_1}{X \vdash \check{\Box}\alpha}}{X \vdash \Box\alpha} \qquad \dfrac{\overset{\vdots\, \pi_2}{\alpha \vdash \Delta}}{\Box\alpha \vdash \check{\Box}\Delta}}{X \vdash \check{\Box}\Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, \pi_1}{X \vdash \check{\Box}\alpha}}{\hat{\blacklozenge} X \vdash \alpha} \qquad \overset{\vdots\, \pi_2}{\alpha \vdash \Delta}}{\hat{\blacklozenge} X \vdash \Delta}}{X \vdash \check{\Box}\Delta}$$

$$\dfrac{\dfrac{\overset{\vdots\, \pi_1}{\Gamma \vdash \tilde{\circ} A}}{\Gamma \vdash \circ A} \qquad \dfrac{\overset{\vdots\, \pi_2}{\tilde{\circ} A \vdash \Delta}}{\circ A \vdash \Delta}}{\Gamma \vdash \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\vdots\, \pi_1}{\Gamma \vdash \tilde{\circ} A}}{\hat{\bullet}_\ell \Gamma \vdash A} \qquad \dfrac{\overset{\vdots\, \pi_2}{\tilde{\circ} A \vdash \Delta}}{A \vdash \check{\bullet}_r \Delta}}{\hat{\bullet}_\ell \Gamma \vdash \check{\bullet}_r \Delta}}{\Gamma \vdash \Delta}\ \blacksquare$$

□

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A Analytic inductive inequalities

In the present section, we specialize the definition of *analytic inductive inequalities* (cf. [15]) to the multi-type language $\mathcal{L}_{\mathrm{MT}}$, in the types DL and K, defined in Section 4 and reported below for the reader's convenience.

$$\mathsf{DL} \ni A ::= p \mid \Box\alpha \mid \top \mid \bot \mid A \wedge A \mid A \vee A$$
$$\mathsf{K} \ni \alpha ::= \circ A \mid 1 \mid 0 \mid \sim\alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha$$

We will make use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order type ϵ, we denote its *opposite* order type by ϵ^∂, that is, $\epsilon^\partial(i) = 1$ iff $\epsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_{\mathsf{DL}} \cup \mathcal{F}_{\mathsf{K}} \cup \mathcal{F}_{\mathrm{MT}}$, $\mathcal{G} := \mathcal{G}_{\mathsf{DL}} \cup \mathcal{G}_{\mathsf{K}} \cup \mathcal{G}_{\mathrm{MT}}$, and $\mathcal{H} := \mathcal{H}_{\mathsf{DL}} \cup \mathcal{H}_{\mathsf{K}} \cup \mathcal{H}_{\mathrm{MT}}$ defined as follows:

$$\begin{array}{lll} \mathcal{F}_{\mathsf{DL}} := \{\wedge, \top\} & \mathcal{G}_{\mathsf{DL}} := \{\vee, \bot\} & \mathcal{H}_{\mathsf{DL}} := \varnothing \\ \mathcal{F}_{\mathsf{K}} := \{\cap, 1\} & \mathcal{G}_{\mathsf{K}} := \{\cup, 0\} & \mathcal{H}_{\mathsf{K}} := \{\sim\} \\ \mathcal{F}_{\mathrm{MT}} := \varnothing & \mathcal{G}_{\mathrm{MT}} := \{\Box\} & \mathcal{H}_{\mathrm{MT}} := \{\circ\} \end{array}$$

For any $\ell \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$, we let $n_\ell \in \mathbb{N}$ denote the arity of ℓ, and the order-type ϵ_ℓ on n_ℓ indicates whether the ith coordinate of ℓ is positive ($\epsilon_\ell(i) = 1$) or negative ($\epsilon_\ell(i) = \partial$). The order-theoretic motivation for this partition is that the algebraic interpretations of $\mathcal{F}$-connectives (resp. $\mathcal{G}$-connectives), preserve finite joins (resp. meets) in each positive coordinate and reverse finite meets (resp. joins) in each negative coordinate, while the algebraic interpretations of $\mathcal{H}$-connectives, preserve both finite joins and meets in each positive coordinate and reverse both finite meets and joins in each negative coordinate.

For any term $s(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an *ϵ-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\epsilon(i) = 1$ or $-p_i$ with $\epsilon(i) = \partial$. An *ϵ-critical branch* in the tree is a branch ending in an ϵ-critical node. For any term $s(p_1, \ldots p_n)$ and any order

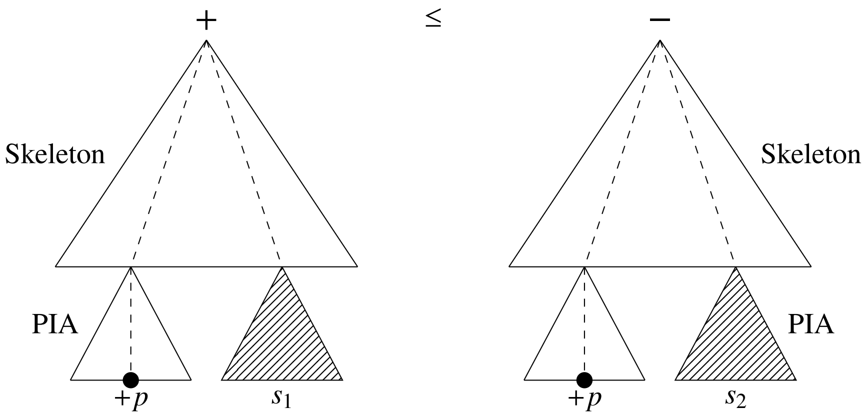
Skeleton	PIA
Δ-adjoints	SRA
$+$ $\vee$ $\cup$	$+$ $\wedge$ $\cap$ $\circ$ $\sim$ $\Box$
$-$ $\wedge$ $\cap$	$-$ $\vee$ $\cup$ $\circ$ $\sim$
SLR	SRR
$+$ $\wedge$ $\cap$ $\circ$ $\sim$	$+$ $\vee$ $\cup$
$-$ $\vee$ $\cup$ $\circ$ $\sim$ $\Box$	$-$ $\wedge$ $\cap$

Table 1: Skeleton and PIA nodes.

type ϵ over n, we say that $+s$ (resp. $-s$) *agrees with* ϵ, and write $\epsilon(+s)$ (resp. $\epsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ϵ-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\epsilon(s') \prec *s$ (resp. $\epsilon^\partial(s') \prec *s$) to indicate that the signed subtree s', with the sign inherited from $*s$, agrees with ϵ (resp. with ϵ^∂).

Definition 27 (**Signed Generation Tree**)**.** The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{MT}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: For any node labelled with $\ell \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ of arity n_ℓ, and for any $1 \leq i \leq n_\ell$, assign the same (resp. the opposite) sign to its ith child node if $\epsilon_\ell(i) = 1$ (resp. if $\epsilon_\ell(i) = \partial$). Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).

Definition 28 (**Good branch**)**.** Nodes in signed generation trees will be called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 1. A branch in a signed generation tree $*s$, with $* \in \{+,-\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes[6], and P_2 consists (apart from variable nodes) only of Skeleton-nodes.



Definition 29 (**Analytic inductive inequalities**)**.** For any order type ϵ and any irreflexive and transitive relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{-,+\}$) of an $\mathcal{L}_{MT}$ term $s(p_1, \ldots p_n)$ is *analytic* (Ω, ϵ)*-inductive* if

1. every branch of $*s$ is good (cf. Definition 28);
2. for all $1 \leq i \leq n$, every SRR-node occurring in any ϵ-critical branch with leaf p_i is of the form $\circledast(s, \beta)$ or $\circledast(\beta, s)$, where the critical branch goes through β and

[6]For an expanded discussion on this definition, see [20, Remark 3.24] and [4, Remark 3.3].

(a) $\epsilon^\partial(s) \prec *s$ (cf. discussion before Definition 28), and

(b) $p_k <_\Omega p_i$ for every p_k occurring in s and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is *analytic* (Ω, ϵ)*-inductive* if the signed generation trees $+s$ and $-t$ are analytic (Ω, ϵ)-inductive. An inequality $s \leq t$ is *analytic inductive* if is analytic (Ω, ϵ)-inductive for some Ω and ϵ.

In each setting in which they are defined, analytic inductive inequalities are a subclass of inductive inequalities (cf. [15, Definition 16]). In their turn, inductive inequalities are *canonical* (that is, preserved under canonical extensions, as defined in each setting).