SEMI-INFINITE MULTIOBJECTIVE PROGRAMMING WITH GENERALIZED INVEXITY

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Motivated by important applications, the theory of mathematical programming has been extended to the case of infinitely many restrictions. At the same time, this theory knew remarcable developments since invexity and its further generalizations have been introduced as substitute of convexity. Here, we consider the multiobjective programming with a set of restrictions indexed in a compact. We obtain optimality criteria of Kuhn-Tucker type under new weaker invexity conditions. Also some dual probles are introduced and it is proved that the weak and strong duality properties hold within the same environement.

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INTRODUCTION

The theory of semi-infinite programming was developed in the last two decades due to its important applications. Numerous researches and extensive studies pushed on the limits of the classical mathematical programming theory considering infinitely many restrictions. We refer the readers to [6], [3] and [12] for both the theoretical progress in semi-infinite programming and for mathematical and technical problems where this theory can be applied.

As well as in the classical case, the basic theoretical results in semi-infinite programming are established under some additional convexity assumptions. On the other hand, the theory of the classical single and multiobjective programming has been considerably extended when the convexity was replaced by weaker invexity like properties.

As this is well-known, the concept of invexity has been introduced in literature in 1981 by Craven [2], after Hanson [4] showed that both weak duality and Kuhn-Tucker sufficiency for optimum in the mathematical programming hold when convexity is replaced by a weaker condition. The theory of mathematical programming has grown remarkably when invexity and its further extensions were been used in the settings of optimality conditions and duality theory. After the works of Hanson and Craven, other types of differentiable

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functions have been introduced with the intent of generalizing invex functions from different points of view. Hanson and Mond [5] introduced the concept of F-convexity and Jeyakumar [7] generalized Vial's ρ-convexity ([10]) introducing the concept of ρ-invexity. The concept of generalized (F, ρ)-convexity, introduced by Preda [8] is in turn an extension of the above properties and was used by several authors to obtain relevant results. The (F, ρ)-convexity was recently generalized to (Φ, ρ)-invexity by Caristi et al. [1] to extend fundamental theoretical results of mathematical programming.

Now we consider the multiobjective programming to obtain necessary and sufficient condition for optimum and duality results, in the semi-infinite case, under new invexity type conditions. In this way we extend the classical theory of the multiobjective programming both by allowing infinitely many restrictions and by weakening convexity (or invexity) assumptions.

The problem to be considered here is the multiobjective programming (vector programming) problem

$$(VP): \quad \min\{f(x) \mid x \in X_0,\ g(x,y) \leq 0\ \forall y \in Y\},$$

where X_0 is an open subset of $\mathbb{R}^n$, Y is a nonempty compact set in a Banach space, the objective vector function $f = (f_1, \ldots, f_p) : X_0 \mapsto \mathbb{R}^p$ is differentiable on X_0 and the restriction function $g : X_0 \times Y \mapsto \mathbb{R}$ is such that $g(\cdot\,, y)$ is differentiable on X_0, for each $y \in Y$ and $g(x, \cdot)$ is continuous on Y, for each $x \in X_0$.

All these conditions will be always assumed but they will be not explicitly mentioned elsewhere. However, for necessity criteria or for some duality results, stronger or additional properties upon g are required and we shall specify them each time when they will be needed.

Since Y is a nonempty compact, for each $x \in X_0$ the set $Y(x) = \big\{\widehat{y} \in Y \mid g(x, \widehat{y}) = \sup_{y \in Y} g(x, Y)\big\}$ is also a nonempty compact.

Let us denote by $X = \{x \in X_0 \mid g(x, y) \leq 0,\ \forall y \in Y\}$ the set of all feasible solutions of the problem (VP).

For each $x \in X_0$ let be $J(x) = \{\widehat{y} \in Y \mid g(x, \widehat{y}) = 0\}$. If x is a feasible solution, then $J(x)$ is the index set of active restrictions and if $x \in X$ and $J(x) \neq \emptyset$, then $J(x) = Y(x)$.

The gradient of f_i will be denoted by ∇f_i while for the vector $\frac{\partial g}{\partial x}$ we prefer the symbol $\nabla_x g$.

Finally, note that as usually the operator "min" in (VP) means that we are required to solve the problem according to one of the following characterizations of the optimum:

Definition 1. The feasible solution $a \in X$ is said to be weakly efficient if there is no $x \in X$ such that

$$f_i(x) < f_i(a), \text{ for all } i = 1, \ldots, p(f(x) < f(a)).$$

Definition 2. The feasible solution $a \in X$ is said to be efficient if there is no $x \in X$ such that

$$f_i(x) \leq f_i(a), \text{ for all } i = 1, \ldots, p \text{ and } f(x) \neq f(a)(f(x) \leq f(y), f(x) \neq f(y)).$$

Definition 3. The feasible solution $a \in X$ is said to be properly efficient if it is efficient and, there exists a constant K such that, for every $x \in X$ and $i \in \{1, \ldots, p\}$ for which $f_i(x) < f_i(a)$, then there is some $j \in \{1, \ldots, p\}$ such that $f_j(a) < f_j(x)$ and

$$f_i(a) - f_i(x) \leq K(f_j(x) - f_j(a)).$$

(Φ, ρ) AND $(\Phi, \rho)^w$-INVEXITY

First all, let us recall the definition of one usual generalization of the concept of convexity, beyond topological structure. In particular, for the present approach we consider real valued functions defined on $\mathbb{R}^{n+1}$, or on a convex subset C of this space.

Definition 4. $\Phi : C \mapsto \mathbb{R}$ is said to be quasi-convex if, for each $\alpha \in \mathbb{R}$, the set $\{c \in C \mid \Phi(c) \leq \alpha\}$ is convex.

A stronger property is also considered in

Definition 5. $\Phi : C \mapsto \mathbb{R}$ is said to be strictly quasi-convex if it is quasi-convex and $\Phi(\lambda c^1 + (1 - \lambda)c^2) < 0$, whenever $\Phi(c^1) < 0$, $\Phi(c^2) \leq 0$ and $\lambda \in (0, 1)$.

Remark 1. If Φ is strictly quasi-convex and $c^1, \ldots, c^k \in C$ such that $\Phi(c^i) < 0$ for $i \in I$ and $\Phi(c^i) \leq 0$ for $i \in \overline{I}$, where $\emptyset \neq I \subset \{1, \ldots, n\}$ and $\overline{I} = \{1, \ldots, n\} \backslash I$, then $\Phi(\sum_{i=1}^n \lambda_i c^i) < 0$ for every $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$ for at least one $i \in I$.

In the next definitions Φ is a real valued function defined on $X_0 \times X_0 \times \mathbb{R}^{n+1}$ and $\rho : X_0 \times X_0 \mapsto \mathbb{R}$. If an element of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is represented as the ordered pair (y, r), with $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$, it is always assumed that

$$\Phi(x, a, (0, r)) \geq 0 \text{ for every } (x, a) \in X_0 \times X_0 \text{ and } r \geq 0. \tag{1}$$

Let h be a differentiable real valued function defined on X_0 and $a \in X_0$.

Definition 6. 1. h is said to be (Φ, ρ)-invex at a with respect to X if, for each $x \in X$,

$$\Phi(x, a, (\nabla h(a), \rho(x, a))) \leq h(x) - h(a) \tag{2}$$

and

$$\Phi(x, a, \cdot) \text{ is convex on } \mathbb{R}^{n+1}. \tag{3}$$

2. h is said to be weakly (Φ, ρ)-invex ($(\Phi, \rho)^w$-invex) at a with respect to X if, for each $x \in X$, it satisfies (2) and

$$\Phi(x, a, \cdot) \text{ is strictly quasi-convex on } \mathbb{R}^{n+1}. \tag{3'}$$

It is obvious that the $(\Phi, \rho)^w$-invexity generalizes the (Φ, ρ)-invexity introduced in [1]. Also, for

$$\Phi(x, a, (y, r)) = F(x, a, y) + \rho d(x, y),$$

where F is sublinear in the third argument and ρ is constant, (Φ, ρ)-invexity becomes the (F, ρ)-convexity of [8].

Generalizing (Φ, ρ)-invexity ($(\Phi, \rho)^w$-invexity), several other weaker conditions could be also defined under the same restrictions for the function Φ.

Definition 7. φ is said to be

(Φ, ρ)-quasi invex ($(\Phi, \rho)^w$-quasi invex) at a with respect to X, if Φ satisfies (3) (respectively, (3′)) and

$$\Phi(x, a, (\nabla \varphi(a), \rho)) \leq 0 \text{ whenever } \varphi(x) - \varphi(a) \leq 0 \text{ for some } x \in X;$$

(Φ, ρ)-semistrict quasi invex ($(\Phi, \rho)^w$-semistrict quasi invex) at a with respect to X, if Φ satisfies (3) (respectively, (3′)) and

$$\Phi(x, a, (\nabla \varphi(a)\rho)) < 0, \text{ whenever } \varphi(x) - \varphi(a) < 0 \text{ for some } x \in X;$$

(Φ, ρ)-strict quasi invex ($(\Phi, \rho)^w$-strict quasi invex) at a with respect to X, if it is (Φ, ρ)-quasi invex and (Φ, ρ)-semistrict quasi invex (respectively, $(\Phi, \rho)^w$-semistrict quasi invex and $(\Phi, \rho)^w$-semistrict quasi invex) at a with respect to X.

Obviously, (Φ, ρ)-invexity implies (Φ, ρ)-strict quasi invexity.

As usual, we will say that the above properties hold on X_0 if they are verified for each $a \in X_0$.

Everywhere in the rest of this paper, the invexity properties will be considered as defined with respect to the set X of feasible solutions. But, for the sake of the simplicity, we will always omit this specification.

SUFFICIENT CONDITIONS FOR EFFICIENCY

In this section we are dealing with sufficient criteria for efficiency in the problem (VP). As well as in the classical case, these criteria consist of an analogous of the Kuhn-Tucker conditions and a substitute of convexity. What is remarkable is the fact that in the Kuhn-Tucker conditions are involved only finitely many restrictions. As substitute of convexity to ensure the sufficiency of the Kuhn-Tucker conditions we use $(\Phi,\rho)^w$-invexity or some generalized $(\Phi,\rho)^w$-invexity. Thus, we extend the classical theory in two ways; allowing infinitely many restrictions and weakening convexity (or invexity) conditions.

The first result establishes sufficient conditions for weakly efficient solutions in (VP).

THEOREM 1. *Let $a \in X$ be a feasible solution of (VP). Suppose that there exist a non-negative integer k, non-negative scalars $\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_k$, and elements $y^j \in J(a)$, $j=1,\dots,k$, such that*

$$\sum_{i=1}^{p}\lambda_i\nabla f_i(a)+\sum_{j=1}^{k}\mu_j\nabla_x g(a,y^j)=0 \tag{4}$$

and

$$\sum_{i=1}^{p}\lambda_i>0. \tag{5}$$

If for each i, f_i is $(\Phi,\rho_i)^w$-semistrict quasi invex at a, for each j,$g(\cdot\,,y^j)$ is $(\Phi,\rho_{y^i})^w$-quasi invex at a and $\sum_{i=1}^{k}\lambda_i\rho_i(x,a)+\sum_{j=1}^{k}\mu_j\rho_{y^j}(x,a)\geq 0$, for all $x\in X$, then a is a weakly efficient solution of (VP).

Proof. Setting $w_0=\sum_{i=1}^{p}\lambda_i+\sum_{j=1}^{k}\mu_j, v_i=\lambda_i/w_0$, $i=1,\dots,p$, $w_j=\mu_j/w_0$, $j=1,\dots,k$, equality (4) becomes

$$\sum_{i=1}^{p}v_i\nabla f(a)+\sum_{j=1}^{k}w_j\nabla_x g(a,y^j)=0.$$

Hence, for every $x\in X$,

(6)
$$\Phi\bigg(x,a,\Big(\sum_{i=1}^{p}v_i\nabla f(a)+\sum_{j=1}^{k}w_j\nabla_x g(a,y^j),\sum_{i=1}^{p}v_i\rho_i(x,a)+\sum_{j=1}^{k}w_j\rho_{y^j}(x,a)\Big)\bigg)\geq 0.$$

Suppose that a is not weakly efficient. Then $f_i(x) < f_i(a)$ for some $x\in X$ and for all $i=1,\dots,p$. This means that $\Phi(x,a,(\nabla f(a),\rho_i(x,a)))<0$, for $i=1,\dots,p$. Also, for each j, $g(x,y^j)-g(a,y^j)\leq 0$ and this implies that

$\Phi(x,a,(\nabla_x g(a,y^j),\rho_{y^j}(x,a)))\le 0$. Since Φ is strictly quasi-convex in the third argument and at least one v_i is positive, we have

(7)

$$\Phi\Big(x,a,\Big(\sum_{i=1}^{p} v_i\nabla f(a)+\sum_{j=1}^{k} w_j\nabla_x g(a,y^j),\sum_{i=1}^{p} v_i\rho_i(x,a)+\sum_{j=1}^{k} w_j\rho_{y^j}(x,a)\Big)\Big)<0$$

contradicting (6). □

Strengthening the assumptions concerning λ we obtain sufficient conditions for efficiency.

Theorem 2. *Let $a\in X$ be a feasible solution of (VP). Suppose that there exist non-negative integer k, non-negative scalars $\lambda_1,\dots,\lambda_p$, $\mu_1,\dots,\mu_k$, and elements $y^j\in J(a)$, $j=1,\dots,k$, such that* (4) *holds and*

$$\lambda_i>0,\quad i=1,\dots,p. \tag{8}$$

If for each i, f_i is $(\Phi,\rho_i)^w$-semistrict quasi invex at a, for each j,$g(\cdot,y^j)$ is $(\Phi,\rho_{y^j})^w$-quasi invex at a, and $\sum_{i=1}^k\lambda_i\rho_i(x,a)+\sum_{j=1}^k\mu_j\rho_{y^j}(x,a)\ge 0$ for all $x\in X$, then a is an efficient solution of (VP).

Proof. As in the above, if a is not efficient, we find $x\in X$ for which $\Phi(x,a,(\nabla f(a),\rho_i(x,a)))\le 0$ for all i, $\Phi(x,a,(\nabla f(a),\rho_i(x,a)))<0$ for at least one i. But all λ_i are positive so that we will arrive to the impossible inequality (7). □

The next result concerns the proper efficiency.

Theorem 3. *Let $a\in X$ be a feasible solution of (VP). Suppose that there exist the non-negative integer k, non-negative scalars $\lambda_1,\dots,\lambda_p$, $\mu_1,\dots,$ μ_k, and elements $y^j\in J(a)$, $j=1,\dots,k$, such that* (4) *and* (8) *hold. If $\sum_{i=1}^p\lambda_i f_i$ is $(\Phi,\rho_0)^w$-semistrict quasi invex at a, for each $j,g(\cdot,y^j)$ is $(\Phi,\rho_{y^j})^w$-quasi invex at a, and $\rho_0(x,a)+\sum_{j=1}^k\mu_j\rho_{y^j}(x,a)\ge 0$ for all $x\in X$, then a is a properly efficient solution of (VP).*

Proof. Let us prove first that

$$\Phi\Big(x,a,\Big(\sum_{i=1}^{p}\lambda_i\nabla f_i(a),\rho_0(x,a)\Big)\Big)\ge 0,\quad \forall x\in X. \tag{9}$$

If $\mu=(\mu_1,\dots,\mu_k)=0$ then $\sum_{i=1}^p\lambda_i\nabla f_i(a)=0$ and $\rho_0(x,a)\ge 0$, $\forall x\in X$, so that (9) is trivially satisfied.

Otherwise, if $\mu\ne 0$, set $t=\frac{1}{1+\sum_{j=1}^k\mu_j}$, $v_j=\frac{\mu_j}{\sum_{j=1}^k\mu_j}=\frac{t}{1-t}\mu_j$, $j=1,\dots,k$. Then $t\sum_{i=1}^p\lambda_i\nabla f_i(a)+(1-t)\sum_{j=1}^k v_j\nabla_x g(a,y^j)=0$, and $t\rho_0(x,a)+(1-$

$t)\sum_{j=1}^{k} v_j\rho_{y^j}(x.a) \geq 0$, for all $x \in X$. Hence

$$\Phi\Bigg(x, a, \Big(t\sum_{i=1}^{p} \lambda_i \nabla f_i(a) + (1-t)\sum_{j=1}^{k} v_j \nabla_x g(a, y^j), t\rho_0(x,a) + (1-t)\sum_{j=1}^{k} v_j\rho_j(x.a)\Big)\Bigg) \geq 0$$

for all $x \in X$.

Since for each $j \in J(a)$, $g(\cdot\,, y^j)$ is $(\Phi, \rho_{y^j})^w$-quasi invex at a, $\Phi(x, a, (\nabla_x g(a, y^j), \rho_{y^j}(x,a)) \leq 0$, hence

$$\Phi(x, a, \Bigg(\sum_{j=1}^{k} v_j \nabla g(a, y^j), \sum_{j\in J(a)} v_j\rho_j(x,a)\Bigg) \leq 0, \quad \forall x \in X.$$

Then, the strict quasi-convexity of Φ implies again (9).

Now, let us show that a is efficient. Suppose that this is not true. Then, $\sum_{i=1}^{p} \lambda_i(f_i(x) - f_i(a)) < 0$ for some $x \in X$, and since $\sum_{i=1}^{p} \lambda_i f_i$ is $(\Phi, \rho_0)^w$-semistrict quasi invex at a $\Phi(x, a, (\sum_{i=1}^{p} \lambda_i \nabla f_i(a), \rho_0(x,a)) < 0$, contradicting (9).

Finally, we will prove that a is properly efficient. If not for each positive K there exists $x \in X$ and $i \in \{1, 2, \ldots, p\}$, such that $f_i(a) - f_i(x) > 0$ and

$$f_i(a) - f_i(x) > K(f_j(x) - f_j(a)) \tag{10}$$

for every $j \in \{1, 2, \ldots, p\}$ satisfying $f_j(x) > f_j(a)$. In particular, take $K = (p-1)\max_{1\leq k,\ell\leq p, k\neq \ell} \frac{\lambda_k}{\lambda_\ell}$. Since (10) holds for every $j \neq i$, we have

$$f_i(a) - f_i(x) > (p-1)\frac{\lambda_j}{\lambda_i}((f_j(x) - f_j(a)), \quad \forall j \neq i.$$

Summing these inequalities it results

$$\sum_{i=1}^{p} \lambda_i(f_i(x) - f_i(a)) < 0.$$

But $\sum_{i=1}^{p} \lambda_i f_i$ is $(\Phi, \rho_0)^w$-semistrict quasi invex at a, that implies the inequality

$$\Phi(x, a, \Bigg(\sum_{i=1}^{p} \lambda_i \nabla f_i(a), \rho_0(x,a)\Bigg) < 0$$

in contradiction with (9). □

Corollary 1. *Let $a \in X$ be a feasible solution of (VP). Suppose that there exist non-negative integer k, non-negative scalars $\lambda_1, \ldots, \lambda_p$, $\mu_1, \ldots, \mu_k$, and elements $y^j \in J(a)$, $j = 1, \ldots, k$, such that* (4) *and* (8) *hold. If f_i is*

(Φ, ρ_i)-invex at a, for each i, $g(\cdot\,, y^j)$ is $(\Phi, \rho_{y^j})^w$-quasi invex at a for each j, and $\sum_{i=1}^p \lambda_i\rho_i(x,a) + \sum_{j=1}^k \mu_j\rho_{y^j}(x,y) \geq 0$ for all $x \in X$, then a is properly efficient.

Proof. From the inequalities

$$\Phi\biggl(x, a, \Bigl(\sum_{i=1}^p \mu_i\nabla f_i(a), \sum_{i=1}^p \mu_i\rho_i\Bigr)\biggr) \leq \sum_{i=1}^p \mu_i\Phi(x, a, (\nabla f_i(a), \rho_i))$$

$$\leq \sum_{i=1}^p \mu_i(f_i(x) - f_i(a))$$

it results that $\sum\limits_{i=1}^p \lambda_i f_i$ is $(\Phi, \rho_0)^w$-invex at a, where $\rho_0(x,a) = \sum\limits_{i=1}^p \mu_i\rho_i(x,a)$. $\square$

NECESSARY CONDITIONS

Besides the general conditions required for X_0, Y and f in the definition of the problem (VP) we will assume, everywhere in this section, the following properties for g:

(11) $$g(\cdot\,, y) \in C^1(X_0) \text{ for each } y \in Y,$$

(12) $$g(\cdot\,,\cdot) \text{ and } \nabla_x g(\cdot\,,\cdot) \text{ are continuous on } X_0 \times Y.$$

The main result of this section establishes that (4) and (5) hold if a is weakly efficient and g satisfies some "constraints qualification" involving generalized $(\Phi, \rho)^w$-invexity. Moreover, the number of restrictions involved in (4) can be limited to at most n. That means that in the corresponding sufficiency result (Theorem 1) the integer k may be also chosen as less than n. This fact has important computational implications, because it reduces the number of possible combinations of y^j that we need to consider for finding efficient solutions.

We begin by finding Fritz John type conditions as necessary for weak efficiency. In some sense our result is the counterpart of those established in [9] for the minmax programming with finitely many restrictions.

The following lemma is helpful for the proof of the main result.

LEMMA 1. *Let a be a feasible solution of (VP). If $J(a) \neq \emptyset$ and there exists $\eta \in \mathbb{R}^n$ such that*

$$\langle \eta, \nabla_x g(a, y)\rangle < 0 \quad \textit{for all } y \in J(a),$$

then there exists $\delta > 0$ such that

$$a + t\eta \in X_0 \textit{ and } g(a + t\eta, y) < 0 \textit{ for every } t \in (0, \delta) \textit{ and all } y \in Y.$$

Proof. Since the function $\varphi(\cdot) = \langle \eta, \nabla_x g(a,\cdot)\rangle$ is continuous on the compact $J(a)$, there exists some $\alpha < 0$ such that $\sup_{y\in J(a)} \varphi(y) < \alpha$. Choose $\beta \in (0, -\frac{\alpha}{2})$. Since φ is uniformly continuous on the compact Y, there exists $\varepsilon > 0$ such that $|\varphi(y) - \varphi(y')| < \beta$ whenever $y, y' \in Y$, $\|y - y'\| < \varepsilon$. Then $\varphi(y) < \frac{\alpha}{2}$ for every y in the open set $Y_\varepsilon = \{y \in Y \mid \|y - y'\| < \varepsilon$, for some $y' \in J(a)\}$. Hence $\sup_{y\in Y_\varepsilon} \varphi(y) < 0$.

Now, since $\langle \eta, \nabla_x g(\cdot\,,\cdot)\rangle$ is continuous on $X_0 \times Y$, there exists an open neighborhood $V_a^1 \subseteq X_0$ of a such that $\sup_{y\in Y_\varepsilon} \langle \eta, \nabla_x g(x,y)\rangle < 0$ for every $x \in V_a^1$. Obviously, there exists $\delta_1 > 0$ such that $a+t\eta \in V_a^1$ for every $t \in (0,\delta_1)$, so that

$$\langle \eta, \nabla_x g(a + t\eta, y)\rangle < 0 \text{ for every } t \in (0, \delta_1) \text{ and all } y \in Y_\varepsilon.$$

Pick an $t \in (0, \delta_1)$ and let be $y \in Y_\varepsilon$. Since

$$g(a + t\eta, y) - g(a, y) = t\,\langle \eta, \nabla_x g(a + \xi t\eta, y)\rangle \text{ for some } \xi \in (0, 1),$$

it results that

$$g(a + t\eta, y) < g(a, y) \le 0 \text{ for every } t \in (0, \delta_1) \text{ and all } y \in Y_\varepsilon. \tag{13}$$

On the other hand, $g(a, y) < 0$ for any $y \in Y \setminus Y_\varepsilon$. Since $Y \setminus Y_\varepsilon$ is compact, this implies that $\sup_{y\in Y\setminus Y\varepsilon} g(a, y) < 0$. Invoke again the continuity of $\sup_{y\in Y\setminus Y\varepsilon} g(\cdot\,, y)$ and find an open neighborhood $V_a^2 \subseteq X_0$ of a such that $\sup_{y\in Y\setminus Y_\varepsilon} g(x, y) < 0$ for every $x \in V_a^2$. Also, find $\delta_2 > 0$ such that $a + t\eta \in V_a^2$ for $t \in (0, \delta_2)$ and then

$$g(a + t\eta, y) < 0, \text{for every } t \in (0, \delta_2) \text{ and all } y \in Y \setminus Y_\varepsilon. \tag{14}$$

Finally, take $\delta = \min\{\delta_1, \delta_2\}$ and the conclusion of the lemma follows from (13) and (14). □

THEOREM 4. *Assume that a is a weakly efficient solution of (VP). Then there exist a non-negative integer $k \le n + 1$, vectors $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}_+^p$, $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{R}_+^k$, and elements $y^j \in J(a)$, $j = 1, \dots, k$, such that the equality* (4) *holds and*

$$\sum_{i=1}^{k} \lambda_i + \sum_{j=1}^{\ell} \mu_j = 1. \tag{15}$$

Proof. The system

$$\langle \eta, \nabla f_i(a)\rangle < 0, \quad i = 1, \dots, p, \tag{16}$$

$$(17) \qquad \langle \eta, \nabla_x g(a,y) \rangle < 0, \quad y \in J(a)$$

has no solution η in $\mathbb{R}^n$. To show this, suppose the contrary and let η satisfy (16) and (17).

If $J(a) \neq \emptyset$ apply Lemma 1 and find $\delta' > 0$ such that $a + t\eta \in X_0$ and $\sup_{y \in Y} g(a + t\eta, y) < 0$ for all $t \in (0, \delta')$.

If $J(a) = \emptyset$ then $\sup_{y \in Y} g(a, y) < 0$ and since $\sup_{y \in Y} g(\cdot\,, y)$ is continuous we can find again δ' as above.

Now, since the objective functions are differentiable, for each i there exists a continuous function $\omega_i : [0, \infty)$ with $\lim_{t \to 0} \omega_i(0) = 0$ such that

$$f_i(a + t\eta) - f_i(a) = \langle t\eta, \nabla f_i(a) \rangle + t\,\|\eta\|\,\omega_i(t)$$

whenever $t > 0$ and $a + t\eta \in X_0$.

Obviously, there exists $\delta_i > 0$ such that $a + t\eta \in X_0$ and $\langle \eta, \nabla f_i(a) \rangle + \|\eta\|\,\omega_i(t) < 0$ for every $t \in (0, \delta_i)$. Hence $a + t\eta \in X_0$ and

$$f_i(a + t\eta) - f_i(a) < 0$$

for every $i = 1, \ldots, p$ and $t \in (0, \delta'')$, where $\delta'' = \min_{1 \le i \le p} \delta_i$. Taking $\delta = \min\{\delta', \delta''\}$, it follows that for any $t \in (0, \delta)$, $a + t\eta \in X$ and $f_i(a + t\eta) < f_i(a)$ for $i = 1, \ldots, p$, which is absurd.

Now, apply the strict separation theorem for the convex hull $\operatorname{co} Q$ of the compact set $Q = \{\nabla_x g(a, y) \mid y \in J(a)\} \cup \{\nabla f_i(a) \mid i = 1, \ldots, p\}$ and the origin of $\mathbb{R}^n$. If $0 \notin \operatorname{co} Q$ then $\langle \eta, q \rangle < 0$ for some η and for all $q \in \operatorname{co} Q$, so that η would be a solution of the system (16), (17). Because this is not possible, it follows that $0 \in \operatorname{co} Q$, so that 0 is a linear convex combination of at most $n + 1$ points of Q. □

Remark 2. As any point in $\operatorname{co} Q$ may be represented as a convex linear combination of $n + 1$ points in Q, the number of all non-zero components of (λ, μ) in (4) can be limited to at most $n + 1$.

As well as in the classical case, the optimality implies the Kuhn-Tucker conditions provided some constraints qualification are satisfied. Slater's condition cannot play the role of such constraints qualification without additional convexity assumptions on the restriction functions. Following the same line as in the previous section, we will show that the convexity can be replaced by the weaker condition of $(\Phi, \rho_y)^w$-invexity.

An obvious extension of Slater's condition to the infinite case is

$$\text{there exists } x^0 \in X \text{ such that } \sup_{y \in Y} g(x^0, y) < 0.$$

A weaker form of this condition appears in [6]:

$$\text{for each } n+1 \text{ points } y^1,\dots,y^{n+1}\in Y \\ \text{there exists } x\in X \text{ such that } g(x,y^j)<0,\ j=1,\dots,n+1. \tag{18}$$

Theorem 5. *Let a be a weakly efficient solution of (VP). Suppose that* (18) *holds* (*weak Slater's condition*), *and that for each $y \in J(a)$, $g(\cdot\,,y)$ is $(\Phi,\rho_y)^w$-semistrict quasi invex at a, where $\rho_y(x,a)\geq 0$ for every $x\in X$. Then* (4) *and* (5) *hold for some $\lambda=(\lambda_1,\dots,\lambda_p)\in\mathbb{R}^p_+$, $\lambda\neq 0$, $0\leq k\leq n$, $\mu=(\mu_1,\dots,\mu_k)\in\mathbb{R}^k_+$, and $y^j\in Y(a)$, $j=1,\dots,k$.*

Proof. By previous theorem, (4) holds for some $y^1,\dots,y^k\in J(a)$ and non-negative vectors λ and μ. Moreover, we can assume that the number of non-zero components of (λ,μ) is at most $n+1$. All that we need to prove is that at least one λ_i should be positive.

If it is not this case, then

$$\sum_{j=1}^{k}\mu_j\nabla_x g(a,y^j)=0 \tag{19}$$

and $\mu\geq 0$, $\sum_{j=1}^k\mu_j=1$. Let $x\in X$ satisfy (18) with respect to $y^1,\dots,y^k$. Since $g(x,y^j)-g(a,y^j)=g(x,y^j)<0$ we have $\Phi(x,a,(\nabla_x g(a,y^j),\rho_{y^j}(x,a)))<0$. Hence the quasi-convexity of Φ implies the inequality

$$\Phi\Big(x,a,\Big(\sum_{j=1}^{k}\mu_j\nabla_x g(a,y^j),\sum_{j=1}^{k}\mu_j\rho_{y^j}(x,a)\Big)\Big)<0.$$

But the opposite inequality follows from (4) because $\sum_{j=1}^k\mu_j\rho_{y^j}(x,a)\geq 0$. □

Theorem 6. *Let a be a properly efficient solution of (VP). If all assumptions of one of the last two previous theorems hold, then there exist $\lambda_i>0$, $i=1,\dots,p$, $0\leq k\leq n$, $\mu_j\geq 0$, $j=1,\dots,k$ and $y^j\in J(a)$, $j=1,\dots,k$, verifying* (4).

Proof. Since a is weakly efficient, the previous theorem applies, so that there exist $\lambda^0\in\mathbb{R}^p_+$, $\lambda^0\neq 0$, $k_0\leq n$, $\mu^0\in\mathbb{R}^{k_0}_+$ and $y^{0j}\in J(a)$, $j=1,\dots,k_0$, such that

$$\sum_{i=1}^{p}\lambda_i^0\nabla f_i(a)+\sum_{j=1}^{k_0}\mu_j^0\nabla_x g(a,y^{0j})=0. \tag{20}$$

Set $I(a)=\{i\in\{1,\dots,p\}\mid\lambda_i^0>0\}$ and $\overline{I}(a)=\{1,\dots,p\}\setminus I(a)$. If $\overline{I}(a)=\emptyset$ then $\lambda=\lambda^0$ and $\mu=\mu^0$. Otherwise, $I(a)\neq\emptyset$ and $\overline{I}(a)\neq\emptyset$. We are going to show that an equality similar with (4) is verified for a set of Lagrange multipliers with increasing number of positive λ_i.

It is not hard to verify that a is an efficient solution in X of the multiobjective programming problem with objectives f_i, $i \in \overline{I}(a)$ and $Kf_j + f_i$, $j \in I(a)$, $i \in \overline{I}(a)$, where K is the constant of Definition 3. Then Theorem 5 applies to this problem and we obtain a set of non-negative Lagrange multipliers, λ_i, $i \in \overline{I}(a)$, λ_{ji}, $j \in I(a)$, $i \in \overline{I}(a)$, and μ_j, $j = 1, \ldots, k$, associated to objectives, respectively, to some $y^j \in Y(a)$, $j = 1, \ldots, k$, where $0 \leq k \leq n$ such that

$$\sum_{i \in \overline{I}(a)} \lambda_i \nabla f_i(a) + \sum_{j \in I(a)} \sum_{i \in \overline{I}(a)} \lambda_{ji}(K \nabla f_j(a) + \nabla f_i(a)) + + \sum_{j=1}^{k} \mu_j \nabla_x g(a, y^j) = 0. \tag{21}$$

Moreover, at least one of λ_i, λ_{ji} is not zero. Summing (19) and (20), and setting $\lambda_i^1 = \lambda_i^0 + K \sum_{j \in \overline{I}(a)} \lambda_{ij}$ if $i \in I(a)$ and $\lambda_i^1 = \lambda_i + \sum_{j \in I(a)} \lambda_{ji}$ if $i \in \overline{I}(a)$ we get

$$\sum_{i=1}^{p} \lambda_i^1 \nabla f_i(a) + \sum_{j=1}^{k_0} \mu_j^0 \nabla_x g(a, y^{0j}) + \sum_{j=1}^{k} \mu_j \nabla_x g(a, y^j) = 0. \tag{22}$$

Now, λ_i^1 are positive for all $i \in I(a)$ and for at least one $i \in \overline{I}(a)$. To end the proof we need to show that (21) can be rewritten obtaining a new null linear combination with the same coefficients λ but with at most n nonzero coefficients μ.

Note first that the vector $q = \sum_{i=1}^{p} \lambda_i^1 \nabla f_i(a)$ cannot be 0. Otherwise, the same reasoning as in the proof of Theorem 5 leads to a contradiction. Then, the equality (21) becomes

$$\alpha_0 d + \sum_{j=1}^{k_0+k} \alpha_j \nabla_x g(a, y^j) = 0,$$

where $\alpha_0 = \frac{1}{1+\sum_{j=1}^{k_0} \mu_j^0 + \sum_{j=1}^{k} \mu_j}$ and $\alpha_j = \begin{cases} \alpha_0 \mu_j^0 & \text{if } j = 1, \ldots, k_0, \\ \alpha_0 \mu_{j-k_0} & \text{if } j = k_0+1, \ldots, k_0+k. \end{cases}$

This means that $0 \in \operatorname{co} Q_0$, where $Q_0 = \{q, \nabla_x g(a, y^{0j}),\ j = 1, \ldots, k_0, \nabla_x g(a, y^j),\ j = 1, \ldots, k\}$. Hence, 0 can be written as a convex combination of at most $n + 1$ vectors of Q_0. Obviously, the coefficient of d should be positive, otherwise a linear combination of $\nabla_x g(a, y^j)$ would be zero. Thus, there exist $k_1 \leq n$ distinct points, say $y^{11}, \ldots, y^{1k_1}$, among $y^{01}, \ldots, y^{0k_0}, y^1, \ldots, y^k$

such that

$$\beta_0 q + \sum_{j=1}^{k_1} \beta_j \nabla_x g(a, y^{1j}) = 0$$

for some $\beta_0, \beta_1, \ldots, \beta_{k_1} > 0$, with $\sum_{j=0}^{k_1} \beta_j = 1$. Dividing with β_0 and denoting $\mu_j^1 = \frac{\beta_j}{\beta_0}$, $j = 1, \ldots, k_1$, the above equality becomes

$$\sum_{i=1}^{p} \lambda_i^1 \nabla f_i(a) + \sum_{j=1}^{k_1} \mu_j^1 \nabla_x g(a, y^{1j}) = 0. \quad \square$$

DUALITY

For the case when Y is infinite, we introduce two duals of Mond-Weir type. The second one reduces in the finite case to the most commonly used formulation of the Mond-Weir dual, but the first one seems to be more interesting in the general case, because it allows us to prove the basic duality property under weaker invexity conditions.

$$(MWD_1): \quad \max \Big\{ f(u) \mid u \in X_0,\ v = (v_1, \ldots, v_p) \in \mathbb{R}_+^p,\ \sum_{i=1}^{p} v_i = 1,$$
$$0 \le k \le n,\ y^1, \ldots, y^k \in J(u),\ w = (w_1, \ldots, w_k) \in \mathbb{R}_+^k,$$
$$\sum_{i=1}^{p} v_i \nabla f_i(u) + \sum_{j=1}^{k} w_j \nabla_x g(u, y^j) = 0 \Big\},$$

$$(MWD_2): \quad \max \Big\{ f(u) \mid u \in X_0,\ v = (v_1, \ldots, v_p) \in \mathbb{R}_+^p,\ \sum_{i=1}^{p} v_i = 1,$$
$$k \in \mathbb{N},\ y^1, \ldots, y^k \in Y,\ w = (w_1, \ldots, w_k) \in \mathbb{R}_+^k,$$
$$\sum_{i=1}^{p} v_i \nabla f_i(u) + \sum_{j=1}^{k} w_j \nabla_x g(u, y^j) = 0,\ \sum_{j=1}^{k} w_j g(u, y^j) \ge 0 \Big\}.$$

The sets of feasible solutions of (MWD_1) and (MWD_2) will be denoted by U_1, respectively, U_2, but for both the problems a feasible solution will be represented as $(u, v, (k, y^1, \ldots, y^k), w)$.

The next two theorems concern (MWD_1).

THEOREM 7. *Let $x \in X$ and $(u, v, (k, y^1, \ldots, y^k), w) \in U_1$. Assume that each f_i, $i = 1, \ldots, p$ is $(\Phi, \rho_i)^w$-semistrict quasi invex at u and each $g(\cdot, y^j)$, $j = 1, \ldots, k$, is $(\Phi, \rho_i)^w$-quasi invex at u, where $\sum_{i=1}^{p} v_i \rho_i(x, u) + \sum_{j=1}^{k} w_j \rho_{y^j}(x, u) \ge 0$. Then the relation $f(u) > f(x)$ is impossible.*

Proof. Suppose the contrary, $f(u) > f(x)$. Set $\lambda_i = \frac{v_i}{1+\sum_{j=1}^{k} w_j}$, $i = 1,\dots,p$, $\mu_j = \frac{w_j}{1+\sum_{j=1}^{k} w_j}$, $j = 1,\dots,k$. Then

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(u) + \sum_{j=1}^{k} \mu_j \nabla_x g(u, y^j) = 0$$

and

$$\sum_{i=1}^{p} \lambda_i \rho_i(x,u) + \sum_{j=1}^{k} \mu_j \rho_{y^j}(x,u) \geq 0,$$

that imply the inequality

$$\Phi\Big(x, u, \Big(\sum_{i=1}^{p} \lambda_i \nabla f_i(u) + \sum_{j=1}^{k} \mu_j \nabla_x g(u, y^j), \sum_{i=1}^{p} \lambda_i \rho_i(x,u) + \sum_{j=1}^{k} \mu_j \rho_{y^j}(x,u)\Big)\Big) \geq 0.$$

On the other hand, since each f_i is $(\Phi, \rho_i)^w$-semistrict quasi invex, the inequality $f_i(x) < f_i(u)$ implies $\Phi(x, u, (\nabla f_i(u), \rho_i(u))) < 0$, and since each $g(\cdot\,, y_j)$ is $(\Phi, \rho_i)^w$-quasi invex, the inequality $g(x, y^j) - g(u, y^j) = g(x, y^j) \leq 0$ implies $\Phi(x, u, (\nabla_x g(u, y^j), \rho_{y^j}(x,u))) \leq 0$. Since some λ_i are positive, the strict quasiconvexity of Φ give us the inequality

$$\Phi\Big(x, u, \Big(\sum_{i=1}^{p} \lambda_i \nabla f_i(u) + \sum_{j=1}^{k} \mu_j \nabla_x g(u, y^j), \sum_{i=1}^{p} \lambda_i \rho_i(x,u) + \sum_{j=1}^{k} \mu_j \rho_{y^j}(x,u)\Big)\Big) < 0$$

contradicting the above inequality. □

THEOREM 8. *Assume that g satisfies* (11) *and* (12). *Assume also that the weak Slater's condition* (18) *holds. Let a be a weakly efficient solution of (VP). If each f_i, $i = 1,\dots,p$, is $(\Phi, \rho_i)^w$-semistrict quasi invex on X_0, where $\rho_i(a,u) \geq 0$ for every $u \in X_0$, and each $g(\cdot\,, y)$, $y \in Y$ is $(\Phi, \rho_y)^w$-strict quasi invex on X_0, where $\min\{\rho_y(x,a), \rho_y(a,u)\} \geq 0$ for every $(x,u) \in X \times X_0$, then there exist $\overline{v} \in \mathbb{R}^p_+$, $\overline{k} \leq n$, $\overline{y}^1,\dots,\overline{y}^{\overline{k}} \in J(a)$ and $\overline{w} \in \mathbb{R}^k_+$ such that $(a, \overline{v}, (\overline{k}, \overline{y}^1,\dots,\overline{y}^{\overline{k}}), \overline{w})$ is a weakly efficient solution of (MWD_1).*

Proof. By Theorem 5 there exist $\lambda = (\lambda_1,\dots,\lambda_p) \in \mathbb{R}^p_+$, $\lambda \neq 0$, $\overline{k} \leq n$, $\overline{y}^1,\dots,\overline{y}^{\overline{k}} \in J(a)$ and $\mu = (\mu_1,\dots,\mu_{\overline{k}}) \in \mathbb{R}^{\overline{k}}_+$ verifying (4). Set $\overline{v}_i = \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i}$, $i = 1,\dots,p$, and $\overline{w}_j = \frac{\mu_j}{\sum_{i=1}^{p} \lambda_i}$, $j = 1,\dots,\overline{k}$, we obtain a feasible solution $(a, \overline{v}, (\overline{k}, \overline{y}^1,\dots,\overline{y}^{\overline{k}}), \overline{w})$ of (MWD_1). Moreover, this solution is weakly efficient. Otherwise, $f(u) > f(a)$ for some $(u, v, (k, y^1,\dots,y^k), w) \in U_1$, that contradicts Theorem 7. □

Now, we will refer to the dual (MWD_2). The first two theorems show that the weak duality property hold under both $(\Phi, \rho_i)^w$ and (Φ, ρ_i) invexity.

THEOREM 9. *Let* $x \in X$ *and* $(u, v, (k, y^1, \dots, y^k), w) \in U_2$. *Assume that each* f_i, $i = 1, \dots, p$, *is* $(\Phi, \rho_i)^w$*-semistrict quasi invex at* u *and* $\sum_{j=1}^k \mu_j g(\cdot, y^j)$ *is* $(\Phi, \rho_0)^w$*-quasi invex at* u, *where* $\sum_{i=1}^p v_i \rho_i(x, u) + \rho_0(x, u) \geq 0$. *Then the relation* $f(u) > f(x)$ *is impossible.*

Proof. Set $\lambda_i = \frac{v_1}{1+\sum_{j=1}^k w_j}$, $i = 1, \dots, p$ and $\lambda_{n+1} = \frac{\sum_{j=1}^k w_j}{1+\sum_{j=1}^k w_j}$. Obviously, $\sum_{i=1}^p \lambda_i \nabla f_i(u) + \lambda_{n+1} \sum_{j=1}^k w_j \nabla_x g(u, y^j) = 0$ and $\sum_{i=1}^p \lambda_i \rho_i(x, u) + \lambda_{n+1} \rho_0(x, u) \geq 0$, and then

$$\Phi\Big(x, u, \Big(\sum_{i=1}^p \lambda_i \nabla f_i(u) + \lambda_{n+1} \sum_{j=1}^k w_j \nabla_x g(u, y^j), \sum_{i=1}^p \lambda_i \rho_i(x, u) + \\ + \lambda_{n+1} \rho_0(x, u)\Big)\Big) \geq 0.$$

Now, by way of contradiction, suppose that $f(u) > f(x)$. Then,

$$\Phi(x, u, (\nabla f_i(u), \rho_i(u))) < 0 \text{ for all } i = 1, \dots, p$$

since f_i are (Φ, ρ_i)-semistrict quasi invex at u, and

$$\Phi\Big(x, u, \Big(\sum_{j=1}^k w_j \nabla_x g(u, y^j), \rho_0(x, u)\Big)\Big) \leq 0$$

since $\sum_{j=1}^k w_j g(x, y^j) - \sum_{j=1}^k w_j g(u, y^j) \leq 0$ and $\sum_{j=1}^k w_j g(\cdot, y^j)$ is $(\Phi, \rho_0)^w$-quasi invex at u. Hence, at least one λ_i is positive. It follows that

$$\Phi\Big(x, u, \Big(\sum_{i=1}^p \lambda_i \nabla f_i(u) + \lambda_{n+1} \sum_{j=1}^k w_j \nabla_x g(u, y^j), \sum_{i=1}^p \lambda_i \rho_i(x, u) + \\ + \lambda_{n+1} \rho_0(x, u)\Big)\Big) < 0. \quad \square$$

Remark 3. Theorems 7 and 9 are not direct comparable because the $(\Phi, \rho_0)^w$-quasi invexity of $\sum_{j=1}^k \mu_j g(\cdot, y^j)$ is not implied nor implies the $(\Phi, \rho_0)^w$ -quasi invexity of all $g(\cdot, y^j)$.

THEOREM 10. *Let* $x \in X$ *and* $(u, v, (k, y^1, \dots, y^k), w) \in U_2$. *Assume that each* f_i, $i = 1, \dots, p$, *is* (Φ, ρ_i)*-semistrict quasi invex at* u *and each* $g(\cdot, y^j)$, $j = 1, \dots, k$, *is* (Φ, ρ_{y^j})*-invex at* u, *where* $\sum_{i=1}^p v_i \rho_i(x, u) + \sum_{j=1}^k w_j \rho_{y^j}(x, u) \geq 0$. *Then the relation* $f(u) > f(x)$ *is impossible.*

Proof. Suppose that $f(u) > f(x)$. Then the semistrict quasi invexity of f_i and the convexity of Φ imply the inequalities $\Phi(x, u, (\nabla f_i(u), \rho_i(u))) < 0$, $i = 1, \ldots, p$, hence

$$\Phi\Big(x, u, \Big(\sum_{i=1}^{p} v_i \nabla f_i(u), \sum_{i=1}^{p} v_i \rho_i(x, u)\Big)\Big) \leq$$
$$\leq \sum_{i=1}^{p} v_i \Phi(x, u, (\nabla f_i(u), \rho_i(x, u))) < 0.$$

If $w = 0$, a contradiction easily follows from the equality $\sum_{i=1}^{p} v_i \nabla f_i(u) = 0$ and the inequality $\sum_{i=1}^{p} v_i \rho_i(x, u) \geq 0$. If $w \neq 0$, take $w'_j = \frac{w_j}{\sum_{j=1}^{k} w_j}$, $j = 1, \ldots, k$, and observe that $\sum_{j=1}^{k} w'_j g(x, y^j) - \sum_{j=1}^{k} w'_j g(u, y^j) \leq 0$ and that $\sum_{j=1}^{k} w'_j g(\cdot\,, y^j)$ is (Φ, ρ_0)-invex, where $\rho_0(x, u) = \sum_{j=1}^{k} w'_j \rho_{y^j}(x, u)$. Then, continue as in the proof of Theorem 7 with $w = w'$ and use the convexity of Φ. $\square$

Finally, we obtain a direct duality result for (MWD_2). It is similar with those established in Theorem 8 but the $(\Phi, \rho_0)^w$-invexity is replaced by the stronger condition of (Φ, ρ_0)-invexity.

THEOREM 11. *Assume that g satisfies* (11) *and* (12). *Assume also that the weak Slater's condition* (18) *holds. Let a be a weakly efficient solution of (VP). If each f_i, $i = 1, \ldots, p$, is (Φ, ρ_i)-semistrict quasi invex on X_0, where $\rho_i(a, u) \geq 0$ for every $u \in X_0$, and each $g(\cdot\,, y)$, $y \in Y$ is (Φ, ρ_y)-invex on X_0, where* $\min\{\rho_y(x, a), \rho_y(a, u)\} \geq 0$ *for every $(x, u) \in X \times X_0$, , then there exist $v \in \mathbb{R}^p_+$, $k \in N$,$y^1, \ldots, y^k \in Y$ and $w \in \mathbb{R}^k_+$ such that $(a, v, (k, y^1, \ldots, y^k), w)$ is a weakly efficient solution of (MWD_2).*

Proof. Apply Theorem 5 and use the previous result as in the proof of Theorem 8. $\square$

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