SEMI-INNER-PRODUCT SPACES

BY

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In the theory of operators on a Hilbert space, the latter actually does not function as a particular Banach space (whose norm satisfies the parallelogram law), but rather as an inner-product space. It is in terms of the inner-product space structure that most of the terminology and techniques are developed. On the other hand, this type of Hilbert space considerations find no real parallel in the general Banach space setting.

Some time ago, while trying to carry over a Hilbert space argument to a general Banach space situation, we were led to use a suitable mapping from a Banach space into its dual in order to make up for the lack of an inner-product.

Our procedure suggested the existence of a general theory which it seemed should be useful in the study of operator (normed) algebras by providing better insight on known facts, a more adequate language to "classify" special types of operators, as well as new techniques. These ideas evolved into a theory of semi-inner-product spaces which is presented in this paper (together with certain applications)(1).

We shall consider vector spaces on which instead of a bilinear form there is defined a form $[x, y]$ which is linear in one component only, strictly positive, and satisfies a Schwarz inequality. Such a form induces a norm, by setting $\|x\| = ([x, x])^{1/2}$; and for every normed space one can construct at least one such form (and, in general, infinitely many) consistent with the norm in the sense $[x, x] = \|x\|^2$. In such a setting, one can then, for instance, talk about a pseudo quadratic form (we shall use the term "numerical range") of an operator T, i.e., $[Tx, x]$; one can define hermitian operators as those for which $[Tx, x]$ is real; and one can extend the concept of a point state ω to the case of an arbitrary algebra of normed space operators, by defining $\omega(T) = [Tx, x]$, with x fixed.

The important fact is that, roughly speaking, a semi-inner-product still provides one with sufficient structure to obtain certain nontrivial general results.

The definitions and general results are given in parts I and II. Part I centers around the numerical range (pseudo quadratic form) $W(T)$ associated with an operator. In particular, it is shown that, despite the loss of the essen-

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(1) Part of the material in this paper belongs to the author's doctoral thesis written at the University of Chicago, under the guidance of I. Kaplansky. The author was at the time a fellow of the John S. Guggenheim foundation.

tial algebraic properties of a quadratic form, it still holds in the case of complex semi-inner-product spaces, that the norm of an operator is bounded in terms of $W(T)$. Part II deals with states and point states on algebras of operators on a semi-inner-product (normed) space. Here the main result is that the cone of states is the closed convex hull of the point states. This result extends a theorem of Bohnenblust and Karlin [2]—and is actually a noncommutative extension of the representation theorem of F. Riesz.

In part III, by considering normed spaces, and algebras, as semi-inner-product spaces, we give new simple proofs of results by Bohnenblust and Karlin [2] and slightly extend one of these results in connection with the renorming of a linear space. We consider furthermore a "natural" definition of hermiticity for operators on any semi-inner-product space, i.e., $[Tx, x]$ real, which when applied to operators on a normed space is consistent with the different semi-inner-products which this space admits. It turns out that this approach from a completely different point of view supports a definition of hermiticity given by I. Vidav [12], i.e., $\|I+i\alpha T\|=1+o(\alpha)$, α real.

In part IV, we consider $*$ algebras, and derive a very simple (and we believe technically illuminating) new proof of the fact that B^* algebras are C^*. Finally we prove that, the C^* character of a $*$ algebra with identity e (at least within equivalent renorming) does not depend on a global condition connecting the involution and the norm, but rather on "local differential condition" near the identity; i.e., if in a $*$ algebra $\boldsymbol{A}$ we have $\|x^*x\|/\|x\|\,\|x^*\|=1+o(r)$, $r=\|e-x\|$, for small r, then $\boldsymbol{A}$ is (at least within equivalent renorming) a C^* algebra (in particular $\boldsymbol{A}$ is symmetric, and this, again, implies directly that B^* is C^*)(2).

Sides results, certain examples and applications, and a few proofs have been left out in order not to distract the reader from the main issues.

Prerequisites and notations. The reader is assumed to be familiar with the current terminology of functional analysis. The latter as well as the notation is close to that used in [7]. In principle [7], also [1], [5] (for Hilbert space results, and [10] (for terminology and results on C^* algebras) may be used as general references.

Only a few unusual terms (or notations), seem to call for an explanation: Given a set S of numbers, we write $|S|$ for $\sup\{|s| : s\in S\}$. We write $\alpha+\beta S$, α and β numbers, for $\{\alpha+\beta s: s\in S\}$. $S\geqq 0$, means s real and $\geqq 0$ for all $s\in S$ (and of course $S=0$ stands for $S=\{0\}$).

We shall call a subset C of a normed linear space a cone, if it is convex and positive-homogeneous (i.e., if $x\in C$, $\alpha x\in C$ for α real $\geqq 0$). We shall also refer to the set $\{x\in C: \|x\|=1\}$ as the base of the cone C.

(2) In this paper, virtually, only bounded operators on semi-inner-product spaces are considered.

Unbounded operators on a semi-inner-product space have been considered recently in connection with "dissipative" operators on a Banach soace, in a forthcoming paper by R. S. Phillips and the author.

I. Semi-inner-products

1. Semi-inner-product spaces.

Definition 1. Let $\boldsymbol{X}$ be a complex (real) vector space. We shall say that a complex (real) semi-inner-product is defined on $\boldsymbol{X}$, if to any $x, y \in \boldsymbol{X}$ there corresponds a complex (real) number $[x, y]$ and the following properties hold:

$$\text{(i)} \quad [x+y, z] = [x, z] + [y, z]$$

$$[\lambda x, y] = \lambda[x, y] \quad \text{for} \quad x, y, z \in \boldsymbol{X}; \quad \lambda \text{ complex (real)},$$

$$\text{(ii)} \quad [x, x] > 0 \quad \text{for} \quad x \neq 0,$$

$$\text{(iii)} \quad |[x, y]|^2 \leqq [x, x][y, y].$$

We then call $\boldsymbol{X}$ a complex (real) semi-inner-product space (in short s.i.p.s.). The concrete significance of the previous notion is shown by the following

Theorem 2. *A semi-inner-product space is a normed linear space with the norm* $[x, x]^{1/2}$. *Every normed linear space can be made into a semi-inner-product space (in general, in infinitely many different ways).*

Proof. We first show that $\|x\| = [x, x]^{1/2}$ is a norm.

$$\|x+y\|^2 = [x+y, x+y] = [x, x+y] + [y, x+y] \leqq (\|x\| + \|y\|)\|x+y\|,$$

$$\|x+y\| \leqq \|x\| + \|y\|,$$

$$\|\lambda x\|^2 = \lambda[x, \lambda x] \leqq |\lambda| \|x\| \|\lambda x\|,$$

$$\|\lambda x\| \leqq |\lambda| \|x\|. \quad \text{For } \lambda \neq 0, \ \|x\| = \left\| \frac{1}{\lambda} \lambda x \right\| \leqq (1/|\lambda|)\|\lambda x\|.$$

Thus

$$\|\lambda x\| = |\lambda| \|x\|.$$

On the other hand let $\boldsymbol{X}$ be a normed linear space, and $\boldsymbol{X}^*$ its dual. For each $x \in \boldsymbol{X}$, there exists by the Hahn-Banach theorem at least one (and we shall choose exactly one) functional $Wx \in \boldsymbol{X}^*$ such that $(x, Wx) = \|x\|^2$. Given any such mapping W from $\boldsymbol{X}$ into $\boldsymbol{X}^*$ (and there exist in general for a given $\boldsymbol{X}$ infinitely many such mappings), it is at once verified that $[x, y] = (x, Wy)$ defines a semi-inner-product.

Unless stated differently, the topology on a s.i.p.s. will be the one induced by the norm $[x, x]^{1/2}$, and it will be in this sense that we shall refer to "bounded operators."

Two immediate and natural questions are the following: when is a s.i.p.s. a Hilbert space; moreover, when is there a unique semi-inner-product associated to a given normed linear space. The answer is quite elementary; namely we have

THEOREM 3. *A Hilbert space $\boldsymbol{H}$ can be made into a s.i.p.s. in a unique way; a semi-inner-product is an inner-product if and only if the norm it induces verifies the parallelogram law.*

Proof. Given any semi-inner-product on $\boldsymbol{H}$, $[x, y]$ is for fixed $y \neq 0$, a linear bounded functional on $\boldsymbol{H}$, and by a well known theorem there exists $z \in \boldsymbol{H}$, such that $[x, y]=(x, z)$, the latter bracket denoting here the usual inner-product. From this, $\|y\|=\|z\|$, and from $\|y\|^2=(y, z)$ by the strict Schwarz inequality it follows that $z=\lambda y$; but again $(y, \lambda y)=\|y\|^2$ so that $z=y$. If one had a pre-Hilbert space to start with, one would use its completion arriving at the same end result.

In general, one shows easily that a normed linear space can be made into a semi-inner-product space in a unique way if and only if its unit sphere is—what is usually called—smooth (i.e., there is a unique support hyperplane at each point of the unit surface).

2. **The numerical range of an operator.** The notion of a quadratic form associated with a matrix, leads in the theory of operators on a Hilbert space to that of the numerical range $W(T)$ of an operator T, defined by $W(T) = \{(Tx, x) : \|x\|=1\}$. Here we introduce an extension of this concept that will play an important role in our future considerations.

DEFINITION 4. Let $\boldsymbol{X}$ be any s.i.p.s., and T any operator (linear transformation) on $\boldsymbol{X}$. The set of numbers $W(T)=\{[Tx, x]: [x, x]=1\}$ will be called the numerical range of the operator T. That the above concept of a numerical range actually extends the classical one follows at once from Theorem 3.

The following elementary properties are readily verified: Let T, T' be any operators on a s.i.p.s., I the identity operator, and α, β numbers, then

$$|W(T)| \leqq \|T\|,$$
$$W(\alpha T+\beta I) = \alpha W(T)+\beta,$$
$$W(T+T') \subset W(T)+W(T'),$$

hence $|W(\alpha T)| = |\alpha|\,|W(T)|$ and $|W(T+T')| \leqq |W(T)|+|W(T')|$ so that $|W(\ )|$ defines a seminorm. In fact we shall see later that it actually defines a norm.

Next, denote the spectrum of an operator T by $\sigma(T)$, and by $\pi(T)$ its approximate point spectrum [11, p. 231](3). Let ∂ stand for "boundary of." Then we have

THEOREM 4. *Let T be any bounded operator on a s.i.p.s. $\boldsymbol{X}$, then $\pi(T) \subset [W(T)]^-$: In particular $\partial\sigma(T) \subset [W(T)]^-$. $^-$ denotes "closure."*

Proof. If $\lambda \in \pi(T)$, there exist $x_n \in \boldsymbol{X}$, such that $[x_n, x_n]=\|x_n\|^2=1$, and $(\lambda I-T)x_n \to 0$. Now

(3) Another reference is: [6, p. 145].

$$\|(\lambda I - T)x_n\| \geqq |[(\lambda I - T)x_n, x_n]| = |\lambda - [Tx_n, x_n]|.$$

Thus $[Tx_n, x_n] \to \lambda$, and hence $\lambda \in [W(T)]^-$.

It is well known that $\partial\sigma(T) \subset \pi(T)$, thus in particular $\partial\sigma(T) \subset [W(T)]^-$. Also notice that consequently one always has $|\sigma(T)| \leqq |W(T)|$.

Deeper and more important is the consideration of $\phi(x, y) = [Tx, y]$ as compared with $\hat{\phi}(x) = [Tx, x]$, where T is a bounded operator on any s.i.p.s. $\boldsymbol{X}$; $x, y \in \boldsymbol{X}$. When $\boldsymbol{X}$ is a complex Hilbert space, ϕ is sesquilinear, $\hat{\phi}$ is the associated quadratic form;

$$\|\phi\| = \sup_{\|x\|=1;\|y\|=1} |\phi(x, y)| = \|T\|,$$

and $\|\hat{\phi}\| = |W(T)|$. In this case it is well known that if $\hat{\phi}(x) \equiv 0$, then $T = 0$ and moreover $\|T\| \leqq 2|W(T)|$. If the inner-product is replaced by a semi-inner-product, though the algebraic connection between ϕ and $\hat{\phi}$ is lost, one still obtains a very similar result.

THEOREM 5. *If $\boldsymbol{X}$ is any complex s.i.p.s., and T any bounded operator on $\boldsymbol{X}$, then $\|T\| \leqq 4|W(T)|$. In particular if $[Tx, x] \equiv 0$, then $T = 0$.*

Proof. Since $|\sigma(T)| \leqq |W(T)|$, the operator valued function of a complex variable $F(\lambda) = (I + \lambda T)^{-1}$ is defined and analytical for $|\lambda| < 1/|W(T)|$, and a fortiori for $|\lambda| \leqq R = 1/2|W(T)|$.

For $x \in \boldsymbol{X}$, $\|x\| = 1$; $\|x + \lambda Tx\| \geqq |[(I + \lambda T)x, x]| = |1 + \lambda[Tx, x]| \geqq 1/2$, if $|\lambda| \leqq R$. Hence for all $x \in \boldsymbol{X}$, $\|(I + \lambda T)x\| \geqq \|x\|/2$ if $|\lambda| \leqq R$, thus also $\|F(\lambda)\| \leqq 2$ for $|\lambda| \leqq R$.

On the other hand, $F(\lambda) = I - T\lambda + T^2\lambda^2 \cdots$, and the Cauchy estimates [7, p. 97] applied to the coefficient of λ give $\|T\| \leqq 2/R = 4|W(T)|$.

Notice that for Theorem 5 it is essential that $\boldsymbol{X}$ be complex. The theorem fails to hold for real spaces, even when the space is the finite dimensional Hilbert space R_n; for instance let

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{on } R_2.$$

Consider next the case when the s.i.p.s. $\boldsymbol{A}$ is an algebra. In this case each $a \in \boldsymbol{A}$ defines an operator (a) on $\boldsymbol{A}$, by setting $(a)x = ax$ for all $x \in \boldsymbol{A}$; and this leads us to define a numerical range of a.

DEFINITION 6. Given a s.i.p.s. $\boldsymbol{A}$ which is also an algebra, the numerical range of any $a \in \boldsymbol{A}$ is the set of numbers $W(a) = \{[ax, x]: [x, x] = 1\}$.

We shall be particularly interested in the case in which $[xy, xy] \leqq [x, x][y, y]$ (i.e., when we have a Banach algebra or rather a "normed algebra"). In this case, if $\boldsymbol{A}$ has an identity, the algebra $\boldsymbol{A}$ is isomorphic and isometric to $(\boldsymbol{A}) = \{(a): a \in \boldsymbol{A}\}$, and the results of this section carry over without change.

3. **Example and comments.** As an example of the use of the previous methods—although in principle the discussion of applications is relegated to

later sections—we mention here a simple "Hilbert space like" proof of the fact that in the algebra $\boldsymbol{B}(\boldsymbol{X})$ of bounded operators on any complex Banach space $\boldsymbol{X}$, the identity I is an extreme point of the unit sphere: Consider $\boldsymbol{X}$ as a s.i.p.s.; then all we have to prove is that for any $T\in\boldsymbol{B}(\boldsymbol{X})$, $\|I+T\|=\|I-T\|=1$ implies $T=0$. Define $D=\{\lambda \text{ complex}: |\lambda|\leqq 1\}$. Then $|W(T\pm I)|\leqq 1$ gives $\{W(T)\pm 1\}\subset D$. It follows $W(T)\subset(D+1)\cap(D-1)=0$; thus from Theorem 5, $T=0$.

At this point, a few comments on the previous section seem appropriate. For one thing, the reader might wonder about the convexity of $W(T)$ in the general situation. The answer is no; i.e., $W(T)$ is not necessarily convex in the general case. This will become clear later.

Particular propositions involving the numerical range of Hilbert space operators, which seem to have no special interest or application in a more general context, have not been considered here. Yet one may point out that the existence of an inner-product is often not essential for such properties to hold. As an example let us consider

PROPOSITION 7. *Let $\boldsymbol{X}$ be a s.i.p.s. such that the unit sphere in the induced norm is uniformly convex. Let T be a bounded operator on $\boldsymbol{X}$. Then $\{\lambda$ complex: $|\lambda|=\|T\|\}\cap[W(T)]^-\subset\partial\sigma(T)$: in particular, if $|W(T)|=\|T\|$ then $|\sigma(T)|=\|T\|$.*

Proof. Suppose $\lambda_0\in\{\lambda: |\lambda|=\|T\|\}\cap[W(T)]^-$. We may without loss of generality suppose that $\|T\|=1$, and $\lambda_0=1$. Hence there exist $x_n\in\boldsymbol{X}$, $\|x_n\|=1$, such that $[Tx_n, x_n]\to 1$, and we have then: $[(x_n+Tx_n)/2, x_n]\to 1$; $1\geqq\|(x_n+Tx_n)/2\|\geqq|[(x_n+Tx_n)/2, x_n]|$. Hence $\|x_n+Tx_n/2\|\to 1$. From the uniform convexity it follows that $\|x_n-Tx_n\|=\|(I-T)x_n\|\to 0$, hence $1\in\sigma(T)$.

In the case of $\boldsymbol{X}$ being a Hilbert space, this proposition is often used as an elementary starting point in the spectral theory of hermitian operators. From $(Tx, Tx)=(T^2x, x)$ one has $|W(T^2)|=\|T^2\|=\|T\|^2$ hence $|\sigma(T^2)|=|\sigma(T)|^2=\|T\|^2$.

II. STATES AND POINT STATES

4. **Definitions.** Consider any real or complex s.i.p.s. $\boldsymbol{X}$. Let $\boldsymbol{B}(\boldsymbol{X})$ be the normed algebra of all bounded operators on $\boldsymbol{X}$. Let $\boldsymbol{A}$ be any normed subalgebra of $\boldsymbol{B}$, containing I.

DEFINITION 8. A bounded linear functional ω on $\boldsymbol{A}$ will be called a state, if $\|\omega\|=\omega(I)$. If $\|\omega\|=1$, we shall call ω a normalized state.

DEFINITION 9. A state of the form $\omega(T)=[Tx, x]$, where x is a fixed vector, will be called a point state.

It is clear that a point state on $\boldsymbol{A}$, is a point state on its completion A^-, and that the states of A^- are formed exactly by all the extensions of the states of $\boldsymbol{A}$. Furthermore, we shall need the following simple facts, the proofs of which are left to the reader.

PROPOSITION 10. *Let Ω be the set of all states on $\boldsymbol{A}$, and Ω_0 the set of all normalized states. Then Ω is a cone of weakly compact convex base Ω_0.*

5. **States as generated by point states.** In the present theory the following result, which extends a theorem of H. F. Bohnenblust and S. Karlin covering the case of C^* algebras [2, p. 228], is important.

THEOREM 11. *Let $\boldsymbol{X}$ be any real or complex s.i.p.s., $\boldsymbol{A}$ any normed algebra of bounded operators on $\boldsymbol{X}$, containing I. Let us denote by Ω the cone of states on $\boldsymbol{A}$, by π the set of all point states. Let Ω_0 denote the base of Ω, and π_0 the set of all normalized point states. Then the weakly closed convex hull of π_0 is Ω_0; similarly Ω is the weakly closed convex hull of π.*

Proof. We first suppose that $\boldsymbol{X}$ and $\boldsymbol{A}$ are both complete. We recall that if $T\in\boldsymbol{A}$, the limit, $\lim_{\alpha\to 0^+}(\|I+\alpha T\|-1)/\alpha=\delta(T)$ (α real) always exists (the existence of this Gâteaux differential follows, for instance, from the fact that $\|I+\alpha T\|$ is a convex function of α).

The first part of the proof will consist in the establishing of a relation between $\delta(T)$ and $W(T)$, namely we shall show that $\delta(T)=\sup \operatorname{Re} W(T)$. For $x\in\boldsymbol{X}$, $\|x\|=1$, $\|(I+\alpha T)x\|\geqq|[x+\alpha Tx,\ x]|=|1+\alpha[Tx,\ x]|$ $=(1+2\alpha\operatorname{Re}[Tx,\ x]+\alpha^2|[Tx,\ x]|^2)^{1/2}\geqq(1+2\alpha\inf\operatorname{Re}W(T))^{1/2}$. Here Re stands for "real part of"; and $\inf\operatorname{Re}W(T)=\inf\operatorname{Re}\{\lambda:\lambda\in W(T)\}$. Now, for α small $F(\alpha)=(I+\alpha T)^{-1}$ exists, and from the above inequality it follows

$$\|(I+\alpha T)x\|\geqq(1+2\alpha\inf\operatorname{Re}W(T))^{1/2}\|x\|\qquad\text{for all } x\in\boldsymbol{X},$$

$$\|F(\alpha)\|\leqq\frac{1}{(1+2\alpha\inf\operatorname{Re}W(T))^{1/2}}.$$

On the other hand, one verifies easily that $F(\alpha)=1-\alpha T+\alpha^2T^2F(\alpha)$ and it follows that

$$\begin{aligned}\delta(-T)&=\lim_{\alpha\to 0^+}\frac{\|I-\alpha T\|-1}{\alpha}=\lim_{\alpha\to 0^+}\frac{\|F(\alpha)\|-1}{\alpha}\\&\leqq\lim_{\alpha\to 0^+}\frac{1}{\alpha}\left(\frac{1}{(1+2\alpha\inf\operatorname{Re}W(T))^{1/2}}-1\right)\\&=-\inf\operatorname{Re}W(T).\end{aligned}$$

Next, replace T by $-T$, and notice that $-\inf\operatorname{Re}W(-T)=\sup\operatorname{Re}W(T)$; it follows that $\delta(T)\leqq\sup\operatorname{Re}W(T)$.

The reversed inequality is easily obtained, since

$$\operatorname{Re}[Tx, x] = \lim_{\alpha\to 0^+} \frac{1}{\alpha}((1 + 2\alpha \operatorname{Re}[Tx, x] + \alpha^2 |[Tx, x]|^2)^{1/2} - 1)$$
$$\leqq \lim_{\alpha\to 0^+} \frac{\|I + \alpha T\| - 1}{\alpha}.$$

Next we use the following general lemma on Banach spaces (a proof can be found in [2]). Let $\boldsymbol{X}$ be any Banach space, $\boldsymbol{X}^*$ its dual. Let C be a weakly compact convex subset of $\boldsymbol{X}^*$ normalized at a point (i.e., there exists $x_0 \in \boldsymbol{X}$ such that $(x_0, c) = 1$ for all $c \in C$), and S a subset of C such that $\sup_{s\in S}|(x, s)| = \max_{c\in C}|(x, c)|$ for all $x \in \boldsymbol{X}$. Then the weak closure of S contains the extreme points of C.

Now for any state $\omega \in \Omega_0$, and any $T \in \boldsymbol{A}$:

$$\omega(T) = \frac{\omega(I + \alpha T) - 1}{\alpha} \leqq \frac{\|I + \alpha T\| - 1}{\alpha} \qquad \text{for } \alpha > 0;$$

hence

$$\max_{\omega\in\Omega_0} |(T, \omega)| = \sup_{0\leqq\theta\leqq 2\pi} \delta(e^{i\theta}T) = \max_{0\leqq\theta\leqq 2\pi} \operatorname{Re} W(e^{i\theta}T) = \max_{0\leqq\theta\leqq 2\pi} \operatorname{Re}(e^{i\theta}W(T))$$
$$= |W(T)| = \sup_{\|x\|=1} |[Tx, x]| = \sup_{\omega\in\pi_0} |(\omega, T)|.$$

Hence the above lemma applies, since Ω_0 is convex weakly compact and normalized at I. The rest of our statement now follows from the Krein-Milman theorem.

When neither $\boldsymbol{X}$ nor $\boldsymbol{A}$ are complete, the same result is obtained by modifying the proof slightly. Ω_0 is still compact. Although $F(\alpha)$ may not exist, this is no serious difficulty since actually all one needs is a partial sum of $I - \alpha T + \alpha^2 T^2 \cdots$.

For the sake of ready reference, we list separately the formula obtained in the proof above.

LEMMA 12. *For any bounded operator T on a s.i.p.s.*, $\lim_{\alpha\to 0^+}(\|I + \alpha T\| - 1)/\alpha = \sup \operatorname{Re} W(T)$.

Again one can apply the previous considerations to the case in which our s.i.p.s. is an algebra $\boldsymbol{A}$, and in particular a normed algebra. Explicitly one is led to

DEFINITION 8′. Let $\boldsymbol{A}$ be an algebra with identity e, which is also a s.i.p.s., a linear bounded functional ω on $\boldsymbol{A}$ is called a state if $\|\omega\| = \omega(e)$. A state of norm 1, is said "normalized." A state ω of the form $\omega(y) = [yx, x]$ where x is a fixed element of $\boldsymbol{A}$, is called a point state.

Thus, let $\boldsymbol{A}$ be an algebra, as well as a s.i.p.s., such that the induced norm makes it a normed algebra (i.e., $[xy, xy] \leqq [x, x][y, y]$ holds for $x, y \in \boldsymbol{A}$).

Consider the algebra $\boldsymbol{B}(\boldsymbol{A})$ of all bounded operators on $\boldsymbol{A}$, and the algebra $(\boldsymbol{A})$ of right-multiplication operators on $\boldsymbol{A}$ (i.e., $(a)\in(\boldsymbol{A})$ defined by $(a)x=ax$ for $x\in\boldsymbol{A}$). If ω is a state on $\boldsymbol{A}$, then it induces a state $\bar{\omega}$ on $\boldsymbol{B}(\boldsymbol{A})$, by defining $\bar{\omega}(T)=\omega(Te)$ for $T\in\boldsymbol{B}(\boldsymbol{A})$. Conversely, given a state $\bar{\omega}$ on $\boldsymbol{B}(\boldsymbol{A})$, it induces a state ω on $\boldsymbol{A}$, by $\omega(a)=\bar{\omega}((a))$, $(a)\in(\boldsymbol{A})$. This corresponds between states on $\boldsymbol{A}$ and $\boldsymbol{B}(\boldsymbol{A})$ is one-one, affine and isometric. A similar situation holds for the point states. Hence the previous theorem can be applied directly.

THEOREM 13. *Let $\boldsymbol{A}$ be an algebra, as well as a s.i.p.s. such that the induced norm makes it into a normed algebra (i.e., $[xy, xy]\leq[x, x][y, y]$ holds for $x, y\in\boldsymbol{A}$). Let Ω be the cone of states on $\boldsymbol{A}$, Ω_0 its base (i.e., the set of all normalized states); π and π_0 the set of all point states and all normalized point states respectively. Then the weakly closed convex hull of π_0 is Ω_0 and the weakly closed convex hull of π is Ω.*

6. **Banach space operators.** We have already seen that a Banach space $\boldsymbol{X}$ can always be made into a s.i.p.s., by constructing an isometric mapping W from $\boldsymbol{X}$ into $\boldsymbol{X}^*$ such that $(x, Wx)=\|x\|^2$, and setting $[x, y]=(x, Wy)$. Let T be an operator on $\boldsymbol{X}$; then for each W, considered as an operator on the corresponding s.i.p.s., T has a numerical range which we shall call the "W determination" of the numerical range of the Banach space operator T. Naturally this leads to the question of the relation between different determinations of the numerical range.

Theorem 11 provides a reasonably good answer, since it shows that for any operator T on a s.i.p.s. the convex hull of $W(T)$ is the set $\{\omega(T):\omega\in\Omega_0\}$, Ω_0 being the set of normalized states; and the latter is essentially the same for all W. In conclusion

THEOREM 14. *All determinations of the numerical range of an operator have the same convex hull.*

COROLLARY. *If the numerical range of a Banach space operator is real valued for any determination, then it is real valued for all determinations.*

Similar considerations concerning determinations of the numerical range arise for Banach algebras; the situation is analogous to the one considered above and requires no further discussion.

7. **Remarks on $C(\boldsymbol{K})$ as a s.i.p.s.** Let us now briefly consider the case when $\boldsymbol{X}=C(\boldsymbol{K})$, the algebra of all continuous complex valued functions on the compact Hausdorff space $\boldsymbol{K}$, normed with the "sup" norm.

THEOREM 15. *There exists a semi-inner-product on $C(\boldsymbol{K})$ which is consistent with the "sup" norm and such that for $f\in C(\boldsymbol{K})$, $W(f)=\{f(\xi):\xi\in\boldsymbol{K}\}$.*

Proof. For $\xi\in\boldsymbol{K}$, there exists $x_\xi\in C(\boldsymbol{K})$ such that $x_\xi(\xi)=1$ and $\|x\|=1$. For such x_ξ, we define $[z, x_\xi]=z(\xi)$. Now, if $x\in C(\boldsymbol{K})-\{x_\xi\}$, pick $\xi\in\boldsymbol{K}$ such that $|x(\xi)|=\|x\|$, and define $[z, x]=\bar{x}(\xi)z(\xi)$. In this fashion a semi-inner-product on $C(\boldsymbol{K})$ is obtained; and for $\|x\|=1, f\in C(\boldsymbol{K})$, $[fx, x]=(fx)(\xi)\bar{x}(\xi)=f(\xi)$. Hence $W(f)\subset\{f(\xi):\xi\in K\}$. But since $[fx_\xi, x_\xi]=f(\xi)$, the reversed inclusion also holds; this completes the proof. From this we see at once that the numerical range is not convex in general.

A certain similarity between the role played by point states on noncommutative operator algebras and homomorphisms on commutative ones can now be easily illustrated. Let $\boldsymbol{B}$ denote a subalgebra of $C(\boldsymbol{K})$, and let $\boldsymbol{A}$ denote the algebra of multiplication operators generated by $\boldsymbol{B}$ itself and 1. It is immediate that a point state on $\boldsymbol{A}$ is a homomorphism and application of Theorem 11 (with $\boldsymbol{X}=\boldsymbol{B}$, and $\boldsymbol{A}$ the algebra defined above) then easily yields the Šilov boundary theorem.

III. Normed algebras

8. **On a result of H. F. Bohnenblust and S. Karlin.** Let $\boldsymbol{A}$ be a normed algebra with identity e. A point u on the surface of the unit sphere S of $\boldsymbol{A}$ will be called a vertex, if the intersection of all the support hyperplanes to S at u is precisely u. This amounts to saying that u is a vertex if whenever $(a, a^*)=0$ for $a\in\boldsymbol{A}$ and all $a^*\in\boldsymbol{A}^*$ such that $(u, a^*)=1$ and $\|a^*\|=1$, then $a=0$. A vertex is always an extreme point. Bohnenblust and Karlin have shown [2] the following:

Theorem 16. *Let $\boldsymbol{A}$ be any Banach algebra with identity e, then e is a vertex of the unit sphere S of $\boldsymbol{A}$.*

Consideration of $\boldsymbol{A}$ as a s.i.p.s. leads us to a new proof, namely: If in the above definition of a vertex we take $u=e$, then a^* becomes a normalized state. Considering, in particular, normalized point states a^* of the form $a^*=[\cdot x, x]$ with $[x, x]=1$, we see that the hypothesis $(a, a^*)=0$, implies that the numerical range $W(a)=0$; hence $a=0$ and the theorem is proved.

Bearing in mind the problem of renorming a given normed algebra we shall follow these considerations a little further and introduce the following:

Definition 17. Let u be a point on the surface of the unit sphere S of a normed space $\boldsymbol{X}$. We shall say that u is a point of local uniform convexity (in short l.u.c.), if $(x_n+y_n)/2\to u$ with $x_n\in S$, $y_n\in S$, $(n=1, 2, \cdots)$, implies $x_n\to u$ and $y_n\to u$ (or equivalently $\|x_n-y_n\|\to 0$). A point of l.u.c. of S is of course not always a vertex, but clearly it is an extreme point of S.

Theorem 18. *Let $\boldsymbol{A}$ be any normed algebra with identity e, then e is a point of l.u.c. of the unit sphere S of $\boldsymbol{A}$.*

Proof. We may consider $\boldsymbol{A}$ as a s.i.p.s., and use as before the notation $W(\ )$ for the numerical range. Use will be made of the following elementary fact: if α, β, λ, are complex numbers laying in the (closed) unit disc, and if $\lambda=(\alpha+\beta)/2$, and $|1-\lambda|\leqq 1/2$, then $|1-\alpha|\leqq(2|1-\lambda|)^{1/2}$ and $|1-\beta|$

$\leqq (2|1-\lambda|)^{1/2}$. Now suppose $a, b \in \boldsymbol{A}$, $\|a\| = \|b\| = 1$. We have $|W(e-(a+b)/2)| = |1 - W((a+b)/2)| \leqq \|e-(a+b)/2\|$.

But since for $x \in \boldsymbol{A}$, $[x, x] = 1$, one has: $[((a+b)/2)x, x] = ([ax, x] + [bx, x])/2$ and since $[((a+b)/2)x, x]$, $[ax, x]$, $[bx, x]$ are in the unit disc, it follows that $|1 - W(a)| = |W(e-a)| \leqq (2\|e-(a+b)/2\|)^{1/2}$, whenever $\|e-(a+b)/2\| \leqq 1/2$.

Now from Theorem 5 we see that $\|e-a\| \leqq 4(2\|e-(a+b)/2\|)^{1/2}$, whenever $\|e-(a+b)/2\| \leqq 1/2$; same for $\|e-b\|$. This completes the proof.

The identity of a normed algebra is thus a vertex as well as a point of l.u.c. under any admissible norm. An example showing that a vertex is not always a point of l.u.c. was suggested to us by V. L. Klee, Jr. An elaboration of his idea enables one to show that given a normed linear space $\boldsymbol{X}$—rather than a normed algebra—and an arbitrary point u on the surface of the unit sphere, then a renorming such that u is a vertex but not a point of l.u.c. of the new unit sphere exists, provided $\boldsymbol{X}$ is infinite dimensional. For expository reasons we omit the proof of this fact.

9. **Hermitian elements. Vidav's definition.** The question of extending the notion of hermiticity to operators on any Banach space—or elements of any Banach algebra seems to have no obvious answer.

I. Vidav [12], in a paper to which we shall make further reference later on, introduces the following notion of hermiticity: an element h of a Banach algebra $\boldsymbol{A}$ with identity e will be called hermitian if $\|e+i\alpha h\| = 1 + o(\alpha)$ for α real. It is interesting to notice that the present theory supports this definition as being a "natural" one. In fact, if T is any operator on a Hilbert space, then T is hermitian if and only if (Tx, x) is real for all x, in other words $W(T)$ is real. It is "natural" then, to call an operator on a s.i.p.s. hermitian if and only if $W(T)$ is real. Similarly, for any operator T on a Banach space; here we have, in principle, many determinations for $W(T)$, but the corollary of Theorem 14 shows that this does not lead to ambiguity. In consequence T is hermitian if and only if sup Re $W(iT) =$ sup Re $W(-iT) = 0$, and by Lemma 12, this is equivalent to $\|I + i\alpha T\| = 1 + o(\alpha)$, i.e., to hermiticity in the sense of Vidav's definition.

At this point our circle of ideas leads us to a new proof of a result due to Kadison [8] (another proof is given by Bohnenblust and Karlin [2]), namely:

THEOREM. *In a Banach algebra with identity, there is at most one involution which makes it a C^* algebra.*

Proof. By a well-known result of Gelfand and Naimark [4], a given C^* algebra $\boldsymbol{A}$ can be mapped isomorphically and isometrically onto an algebra $\boldsymbol{A}'$ of operators on a Hilbert space. An element in $\boldsymbol{A}$ is self-adjoint if and only if its image in $\boldsymbol{A}'$ is hermitian (i.e., has a real numerical range). But the above map takes the states on $\boldsymbol{A}$ onto the states on $\boldsymbol{A}'$, and since the closed convex hull of any element (in $\boldsymbol{A}$ or $\boldsymbol{A}'$) is determined by the cone of states (of $\boldsymbol{A}$

or A'), we conclude that the self-adjoint elements of A are the elements having real numerical range and are determined by the states (hence by the norm). Thus the involution itself is determined by the norm.

10. **States and positive functionals.** A functional p on a C^* algebra A, is called positive if $p(x^*x) \geqq 0$ for all $x \in A$. We shall suppose A has an identity e; then it is well known that the positivity of p implies that $p(x) \leqq p(e)\|x\|$, whence a positive functional is a state. The converse is also true though less trivial. For the commutative case it follows easily from the Gelfand representation and the Riesz theorem; for the general case it was proved more recently in [2]. This result follows now immediately from Theorem 11. In fact, as before A can be represented on an algebra A' of operators on a Hilbert space H. A state on A is taken into a state on A', and is positive if and only if its image is positive. On the other hand for a point state on A' we have $\omega(T^*T) = (T^*Tx, x) = \|Tx\|^2 \geqq 0$, $x \in H$. The desired conclusion follows now from Theorem 11. However for more general $*$ algebras (see next section for this terminology) states and positive functionals in the above sense are diverging concepts. As for the connection in general between states and functionals positive in the $*$ sense, we mention without proof, the following result:

THEOREM 19. *If A is a $*$ algebra with identity for which the cone of states coincides with the cone of positive functionals, then (within equivalent renorming) A is a C^* algebra.*

IV. $*$ ALGEBRAS

11. **Terminology.** We shall call an algebra A, a $*$ algebra, if there is defined on it an involution $x \rightarrow x^*$ with the usual algebraic properties, i.e.,

$$(x^*)^* = x,$$
$$(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*,$$
$$(xy)^* = y^* x^*.$$

Notice that no continuity of the involution is assumed.

A $*$ algebra A (with identity e) is called symmetric if $e + x^*x$ has an inverse for every $x \in A$. A B^* algebra is a $*$ algebra in which the relation $\|x^*x\| = \|x\|^2$ holds for all x. A C^* algebra is thus a symmetric B^* algebra.

The study of the representations of C^* algebras goes back to Gelfand and Naimark [4]. They also conjectured that symmetry was a superfluous assumption, i.e., that B^* algebras are automatically C^*.

Fukamiya proved the conjecture implicitly in 1952 [3], the implication being noticed first by I. Kaplansky who then communicated an elaborated version. (See for instance [9, appendix].)

12. **On a new proof that B^* algebras are C^*.** The results on s.i.p.s. provide a rather simple and "natural" proof of the fact that B^* algebras are C^*. We limit ourselves to establish Fukamiya's lemma, and start with:

LEMMA 20. *If $\boldsymbol{A}$ is any B^* algebra with unit e, and x is an element of $\boldsymbol{A}$ such that $x^*=x$, then the numerical range $W(x)$ of x is real and has same l.u.b. and g.l.b. as the spectrum $\sigma(x)$ of x.*

Proof. If $x^*=x$, then $\|e+i\alpha x\|^2=\|(e+i\alpha x)(e-i\alpha x)\|=\|e+\alpha^2x^2\|=1+o(\alpha)$, for α real. Hence directly from Lemma 12—or else by observing that x is "hermitian"—it follows that Re $W(ix)=0$, i.e., $W(x)$ is real valued.

On the other hand, using Theorem 1 and the fact that $\|x^2\|=\|x\|^2$, we have, $\|x\|=|\sigma(x)|\leqq|W(x)|\leqq\|x\|$. The fact that $|\sigma(x)|=|W(x)|$ when applied to $\alpha e+x$ for large real values of α and $-\alpha$, together with $\sigma(\alpha e+x)=\alpha+\sigma(x)$ and $W(\alpha e+x)=\alpha+W(x)$, shows at once that $\sigma(x)$ and $W(x)$ have the same upper and lower bounds.

LEMMA (FUKAMIYA). *If $\boldsymbol{A}$ is a B^* algebra and x, y are elements of $\boldsymbol{A}$ such that $x^*=x$, $y^*=y$, $\sigma(x)\geqq 0$, $\sigma(y)\geqq 0$, then also $\sigma(x+y)\geqq 0$.*

Proof. Let π_0 denote the set of all normalized point states of $\boldsymbol{A}$. By Lemma 20, for any $\omega\in\pi_0$, $\omega(x)\geqq 0$, $\omega(y)\geqq 0$; hence g.l.b. $\sigma(x+y)=$ g.l.b. $\omega(x+y)=$ g.l.b. $(\omega(x)+\omega(y))\geqq 0$.

The reader will of course notice the relation between this proof and the manner in which one disposes of the commutative case by means of the Gelfand theory.

13. **C^* as a local "differential condition."** The preceding facts suggest, and we shall indeed prove that, at least within equivalent renorming, the C^* character of a * algebra (again assumed to have an identity e) depends only on a suitable local "differential condition" connecting the involution and the norm, near the identity; and hence the C^* character of a * algebra is preserved by mapping (in particular renormings) behaving suitably near the identity.

In the proof we make use of a result of Vidav [12], which we state here for convenience:

Suppose $\boldsymbol{A}$ is any Banach algebra with identity e. Let H be the set of all hermitian elements of $\boldsymbol{A}$ (i.e., $H=\{h\in A:\|e+i\alpha h\|=1+o(\alpha)\}$). We assume that: (a) Every $a\in\boldsymbol{A}$, has a representation $a=u+iv$, $u\in H$, $v\in H$: (b) If $h\in H$, then there is a representation $h^2=u+iv$ such that $u\in H$, $v\in H$ and $uv=vu$. Then $\boldsymbol{A}$ is (within equivalent renorming) a C^* algebra.

THEOREM 21. *Suppose $\boldsymbol{A}$ is a * algebra with identity e, such that the condition $\|x^*x\|/\|x\|\,\|x^*\|=1+o(r)$, $r=\|e-x\|$, holds near e. Then $\boldsymbol{A}$ is (within equivalence) a C^* algebra.*

Proof. Let H be the set of all hermitian elements of $\boldsymbol{A}$, and S the set of all self-adjoint elements in $\boldsymbol{A}$.

Now observe that what we know about S, is precisely what we would need to know about H in order to be able to use Vidav's result (i.e., for $a\in\boldsymbol{A}$,

$a=u+iv$ where $u\in S$, $v\in S$; furthermore, if $u\in S$, $u^2\in S$). Thus our problems will be solved if we show that $H=S$.

Suppose therefore, that $u\in S$, $u\neq 0$. We have:

$$\frac{\|e+\alpha^2u^2\|}{\|e+i\alpha u\|\,\|e-i\alpha u\|}=1+o(r), \qquad \alpha>0,$$

and $r=\alpha\|u\|$, so that $\|e+\alpha^2x^2\|/1+o(r)=1+o(\alpha)$. It follows that, denoting the numerical range of u by $W(u)$, and using Lemma 12:

$$\sup \operatorname{Re} W(iu)=\lim_{\alpha\to 0^+}\frac{\|e+i\alpha u\|-1}{\alpha}=\lim_{\alpha\to 0^+}\frac{(1+o(\alpha))/\|e-i\alpha u\|-1}{\alpha}$$

$$=-\lim_{\alpha\to 0^+}\frac{\|e-i\alpha u\|-1}{\alpha}=-\sup Re\,W(-iu)=\inf \operatorname{Re}W(iu).$$

But it is clear that the spectrum $\sigma(u)$ of u is symmetric about the real axis. Hence (from Theorem 4) sup Re $W(iu)\geqq 0$, inf Re $W(iu)\leqq 0$. It follows that $W(u)$ is real valued, whence $u\in H$.

Conversely, suppose $h\in H$, and write $h=u+iv$ with u, $v\in S$. Let π_0, be the set of all normalized point states. Since $h\in H$, $W(h)$ is real valued, and since we have just shown that $S\subset H$, also $\omega(u)$ and $\omega(v)$ are real valued, $\omega\in\pi_0$; hence for any $\omega\in\pi_0$, $\omega(v)=0$; from this, by Theorem 5, $v=0$, i.e., $h=u\in S$, and the proof is complete.

As an immediate consequence we have

COROLLARY. *If* **A** *is a* $*$ *algebra with identity* e, *such that* $\|x^*x\|/\|x\|\,\|x^*\|=1+o(r)$, $r=\|e-x\|$, *holds near* e, *then* **A** *is symmetric.*

Hence we obtain again, now directly and without using Fukamiya's lemma:

COROLLARY. *B^* algebras are C^* algebras.*

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