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Semi-Invariant Submanifolds of a Lorentzian Para-Sasakian Manifold

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Abstract. Recently Matsumoto [1] introduced the idea of Lorentzian para contact structure and studied its several properties. In the present paper we studied the integrability condition of the distribution on semi-invariant submanifolds of LP-Sasakian manifold.

1. Introduction

Let $\overline{M}$ be an n-dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞-vector valued linear function ϕ, a C^∞ vector field ξ, 1-form η and Lorentzian metric g of type $(0, 2)$ such that for each $p \in \overline{M}$, the tensor $g_p : T_p\overline{M} \times T_p\overline{M} \to R$ is a non-degenerate inner product of signature $(-, +, +, \cdots, +)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and R is the real number space, which satisfies

$$\phi^2(X) = X + \eta(X)\,\xi, \tag{1.1}$$
$$\eta(\xi) = -1,$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\,\eta(Y), \tag{1.2}$$
$$g(X, \xi) = \eta(X),$$

for all vector fields X, Y tangent to $\overline{M}$. Such structure (ϕ, ξ, η, g) is termed as *Lorentzian para contact* [1].

In a Lorentzian para-contact structure the following holds

$$\phi\xi = 0, \quad \eta(\phi X) = 0,$$
$$\text{rank } (\phi) = n - 1.$$

A Lorentzian para contact manifold $\overline{M}$ is called *Lorentzian para-Sasakian* (LP-Sasakian) manifold if [1]

$$(\overline{\nabla}_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\,\eta(Y)\xi, \tag{1.3}$$

and from (1.3), we find

$$\overline{\nabla}_X \xi = \phi X \tag{1.4}$$

$\forall\ X, Y$ tangent to $\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection with respect to g.

Let us put

$$\Phi(X, Y) = g(\phi X, Y)$$

then the tensor field Φ is symmetric $(0, 2)$ - tensor field. Thus we have,

$$\Phi(X, Y) = \Phi(Y, X),$$

and

$$\Phi(X, Y) = (\overline{\nabla}_X \eta)(Y).$$

Definition 1.1. *The submanifold M of the LP-Sasakian manifold* $\overline{M}$ *is said to be semi-invariant if it is endowed with the pair of orthogonal distribution* $(D, D^{\perp})$ *satisfying the conditions*

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,

(ii) *the distribution* D *is invariant under* ϕ, *that is*

$$\phi D_x = D_x, \quad \textit{for each} \quad x \in M,$$

(iii) *the distribution* $D^{\perp}$ *is anti-invariant under* ϕ, *that is*

$$\phi D_x^{\perp} \subset T_x M^{\perp}, \quad \textit{for each} \quad x \in M.$$

The distribution D (respectively $D^{\perp}$) is called the *horizontal* (respectively *vertical*) *distribution*. A semi-invariant submanifold M is said to be *invariant* (respectively *anti-invariant*) *submanifold* if we have $D_x^{\perp} = \{0\}$ respectively $(D_x = 0)$ for each $x \in M$. We say that M is a *proper semi-invariant submanifold* if it is a semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

We denote by same symbol g both metrices on $\overline{M}$ and M. The projection morphisms of TM to D and $D^{\perp}$ are denoted by P and Q respectively. For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we have

$$X = PX + QX + \eta(X)\xi \tag{1.5}$$

$$\phi N = BN + CN \tag{1.6}$$

where BN (respectively CN) denotes the tangential (respectively normal) component of ϕN.

The equations of Gauss and Weingarten for the immersion of M in $\overline{M}$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{1.8}$$

for any $X, Y \in \Gamma(TM)$ and $N \in TM^{\perp}$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the linear connection induced by $\overline{\nabla}$ on the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. Also we have

$$g(h(X, Y), N) = g(A_N X, Y) \tag{1.9}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^{\perp})$.

2. Basic Lemmas

For $X, Y \in \Gamma(TM)$, we put

$$u(X, Y) = \nabla_X \phi PY - A_{\phi QY} X. \tag{2.1}$$

We begin with the following lemma.

Lemma 2.1. *Let M be a semi-invariant submanifold of LP-Sasakian manifold $\overline{M}$. Then we have*

$$P(u(X, Y)) = \phi P \nabla_X Y + \eta(Y) PX + 2\eta(X)\eta(Y) P\xi + g(X, Y) P\xi \tag{2.2}$$

$$Q(u(X, Y)) = Bh(X, Y) + \eta(Y) QX + 2\eta(X)\eta(Y)(Q\xi) + g(X, Y) Q\xi \tag{2.3}$$

$$h(X, \phi PY) + \nabla^{\perp}_X \phi QY = \phi Q \nabla_X Y + Ch(X, Y) \tag{2.4}$$

$$\eta(u(X, Y)) = -g(\phi X, \phi Y), \tag{2.5}$$

for all $X, Y \in TM$.

Proof. By using the decompositions (1.5), (1.6), (1.7), (1.8) in (1.3), we obtain (2.2), (2.3), (2.4) and (2.5) respectively.

Lemma 2.2. *Let M be a semi-invariant submanifold of LP-Sasakian manifold* $\overline{M}$, *then we have*

$$\nabla_X \xi = \phi X, \; h(X, \xi) = 0, \; \textit{for any } X \in \Gamma(D); \tag{2.6}$$
$$\nabla_Y \xi = 0, \; h(Y, \xi) = \phi Y, \; \textit{for any } Y \in \Gamma(D^\perp); \tag{2.7}$$
$$\nabla_\xi \xi = 0, \; h(\xi, \xi) = 0. \tag{2.8}$$

Proof. In consequence of (1.4) and (1.5), we obtain

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

or

$$\nabla_X \xi + h(X, \xi) = \phi PX + \phi QX. \tag{2.9}$$

Thus, $(2.6) - (2.8)$ follows from (2.9).

Lemma 2.3. *Let M be a semi-invariant submanifold of LP-Sasakian manifold* $\overline{M}$. *Then we have*

$$A_{\phi X} Y = -A_{\phi Y} X, \tag{2.10}$$

for all $X, Y \in \Gamma(D^\perp)$.

Proof. By using (1.2), (1.7), (1.9), we get

$$\begin{aligned} g(A_{\phi X} Y, Z) &= g(h(Y, Z), \phi X) = g(\overline{\nabla}_Z Y, \phi X) \\ &= g(\phi \overline{\nabla}_Z Y, X) = g(\overline{\nabla}_Z \phi Y, X) \\ &= -g(\phi Y, \overline{\nabla}_Z X) = -g(\phi Y, h(X, Z)) \\ &= -g(A_{\phi Y} X, Z), \end{aligned}$$

for all $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(TM)$, which proves (2.10).

Lemma 2.4. *Let M be a semi-invariant submanifold of LP-Sasakian manifold* $\overline{M}$. *Then we find*

$$\nabla_\xi U \in \Gamma(D), \quad \textit{for any } U \in \Gamma(D),$$
$$\nabla_\xi V \in \Gamma(D^\perp), \quad \textit{for any } V \in \Gamma(D^\perp).$$

The proof is obvious.

Lemma 2.5. *Let* M *be a semi-invariant submanifold of LP-Sasakian manifold* $\overline{M}$. *Then we obtain*

$$[X, \xi] \in \Gamma(D), \quad \textit{for any} \quad X \in \Gamma(D), \tag{2.11}$$

$$[Y, \xi] \in \Gamma(D^{\perp}), \quad \textit{for any} \quad Y \in \Gamma(D^{\perp}). \tag{2.12}$$

The proof follows from Lemma 2.4.

3. Integrability of distribution on a semi-invariant submanifold in a LP-Sasakian manifold

Theorem 3.1. *Let M be a semi-invariant submanifold in LP-Sasakian manifold* $\overline{M}$. *Then the distribution D is integrable if and only if*

$$h(X, \phi Y) = h(Y, \phi X).$$

Proof. We have from (2.6)

$$\begin{aligned} g([X, Y], \xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= -g(Y, \phi X) + g(X, \phi Y) \\ &= 0, \end{aligned}$$

for all $X, Y \in \Gamma(D)$.

In consequence of (2.4), we find

$$h(X, \phi Y) - h(Y, \phi X) = \phi Q[X, Y],$$

which proves the theorem.

Corollary 3.1. *The distribution* $D \oplus \{\xi\}$ *is integrable if and only if* $h(X, \phi Y) = h(Y, \phi X)$ *is satisfied.*

Proof follows from Theorem 3.1 and (2.11).

Theorem 3.2. *Let M be semi-invariant submanifold in LP-Sasakian manifold* $\overline{M}$. *Then the distribution* $D^{\perp}$ *is never integrable.*

Proof. From (2.1), we have for $X, Y \in \Gamma(D^{\perp})$

$$u(X, Y) = -A_{\phi Y} X.$$

Operating ϕ in (2.2) and using (1.1), we get

$$P\nabla_X Y = \phi P(A_{\phi Y} X), \tag{3.1}$$

for any $X, Y \in \Gamma(D^\perp)$. By virtue of Lemma 2.3, (3.1) reduces to

$$P([X, Y]) = 2\phi P(A_{\phi Y} X),$$

which proves the statement.

Corollary 3.2. *The distribution on* $D^\perp \oplus \{\xi\}$ *is never integrable.*

Proof follows from Theorem 3.2 and (2.12).

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