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Semi-Markov Jump Linear Systems with Bi-boundary Sojourn Time: Anti-modal-asynchrony Control

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Abstract

This paper investigates the problem of control synthesis for a class of discrete-time semi-Markov jump linear systems, in which the sojourn time of each mode is bi-boundary (with upper and lower bound). The system is subject to modal asynchrony, which means that the switchings of the mode-dependent controller to be designed lag behind the ones of the controlled plant, and the lag is mode-dependent. In contrast with the traditional mode-independent lag commonly assumed in the existing studies, not only is the modal lag more practical and general, but also it yields less conservatism of the controller design. By virtue of the semi-Markov kernel approach, the conditions on the existence of the anticipated stabilizing controller capable of overcoming the modal asynchrony are derived. Illustrative examples including a class of VTOL helicopter models are presented to demonstrate the necessity and the validity of the designed *anti-modal-asynchrony* controllers.

Key words: anti-modal-asynchrony control, bi-boundary sojourn time, mean-square stability, semi-Markov jump linear systems

1 Introduction

The past decade has seen increasing usages of semi-Markov jump linear systems (S-MJLSs) in modeling stochastic switching systems instead of MJLSs. Recent examples can be found in communication networks under environment changes [12], robot arm systems with varying payloads [13], and so on. The powerful capability of system modeling of S-MJLSs mainly benefits from the fact that the sojourn time in S-MJLSs is not necessarily subject to exponential distribution unlike MJLSs. Earlier works on S-MJLSs consider that the sojourn time is of *special* distributions, such as phase-type distribution in [6], Weibull distribution in [7], etc., in order to simplify the derivations of the controller design. In [21], the semi-Markov kernel approach is proposed, which greatly impulses the studies on stability analysis and control synthesis for S-MJLSs allowing for any types distribution of sojourn time.

On control synthesis problems of stochastic switching systems, the mode-dependent controller is generally expected as it brings less conservatism than the mode-independent controller, cf., [11, 18, 19]. However, a drawback in the scheme of the mode-dependency approaches is the *asynchrony* which generally means that the switchings of the controller lag behind the ones of the controlled plant [1]. It is actually a common and non-ignorable issue in the area of switching systems, and the lag is mainly caused by the time needed for identifying the activated mode of the plant in the cases of no switching information available *a priori*. Referring to the previous studies [5, 8, 9, 17] in solving the issue of asynchrony, efforts have been devoted to the asynchronously switched control of S-MJLSs, see, e.g., [22]. Unfortunately, the proposed criteria in [22] are not applicable to the case that the sojourn time is with lower bounds. In practice, it is seldom encountered that the mode changes to another one immediately after the last switching. Then, if ignoring the lower bounds of the sojourn time, the resulting conditions on the existence of controller may be rather conservative. It is quite necessary to fill such a void by solving the control synthesis problem for S-MJLSs with bi-boundary sojourn time against asynchronous mode jumps.

Moreover, note that almost all the existing results with respect to asynchronously switched control systems presume that the lags corresponding to different modes of the plant are all identical. Surprisingly, the issue of mode-dependent lag, i.e., the modal asynchrony has never been considered in the field, let alone the complex S-MJLSs involving with non-specific distribution of sojourn time. In fact, it is quite common that the time for identifying different modes of the controlled

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[1] In this paper, we will slightly abuse the notion of controlled plant to mean the real physical plant itself and its surrounding environment.

plant is mode-dependent. Meanwhile, it is worth mentioning that although the considerations of lag mode-independency in [4, 14], etc., facilitate the derivations of the controller design, the simplification leads to great conservatism. Therefore, it is necessary and significant to probe the control synthesis problems for switching systems while taking the modal asynchrony into account. Note that in S-MJLSs, the consideration of bi-boundary sojourn time will inevitably increase the difficulty of derivation for the case of modal asynchrony.

Motivated by the above observations, this study is concerned with the control synthesis for a class of discrete-time S-MJLSs with bi-boundary sojourn time and modal asynchrony. Compared with the previous studies assuming the common lag, the obtained results are more practical and general yielding the less conservatism of the controller design. The remainder of this paper is organized as follows. Section II presents the preliminaries and problem formulation. The detailed derivations of the proposed approach are given in Section III. Two numerical examples are provided to verify the validity and applicability of theoretical findings in Section IV, and Section V concludes this paper.

Notations: The superscripts "T" and "-1" indicate transposition and inverse of a vector or a matrix, respectively; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\|\cdot\|$ the Euclidean vector norm in $\mathbb{R}^n$; $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real numbers and the set of non-negative integers, respectively; $\mathbb{R}_{>0}$ signifies the sets $\{i \in \mathbb{R} | i > 0\}$; $\mathbb{N}_{\geq N}$ and $\mathbb{N}_{[N_1,N_2]}$ denote the sets $\{i \in \mathbb{N} | i \geq N\}$ and $\{i \in \mathbb{N} | N_1 \leq i \leq N_2\}$, respectively. Besides, $\mathbb{E}\{x\}|_y$ stands for the expectation of x conditional on y; the notation $P \succ \mathbf{0}$ ($P \prec \mathbf{0}$) means that P is a positive (negative) definite matrix; $\otimes$ represents the Kronecker product of matrices, $diag\{\cdots\}$ a block-diagonal matrix, and $sign(x)$ the signum function which is defined as $sign(x) \triangleq 1$ for $x > 0$, $sign(x) \triangleq -1$ for $x < 0$, and $sign(x) \triangleq 0$ for $x = 0$. In addition, $sym(A)$ is a shorthand of $A + A^T$. To reduce clutter, the symbol "$*$" is employed as an ellipsis for the terms introduced by symmetry. The symbol $\Theta_a^b(A_i)$ is defined as $\prod_{i=a}^b A_i$ for $a \leq b$, and I for $a > b$. I_n and $\mathbf{0}$ are identity matrix of order n and zero matrix with appropriate dimensions, respectively.

2 Preliminaries and Problem Formulation

Fix the complete probability space (Ψ, $\mathcal{F}$, Pr) and consider the following discrete-time S-MJLS:

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k) \tag{1}$$

where Ψ is the sample space, $\mathcal{F}$ the σ-algebra of the subsets of Ψ, and Pr the probability measure on $\mathcal{F}$; the vectors $x(k) \in \mathbb{R}^{n_x}$ and $u(k) \in \mathbb{R}^{n_u}$ denote the system state and control input, respectively; $\{r(k)\}_{k\in\mathbb{N}}$ stands for a semi-Markov chain that governs the switching among different plant modes and takes the values in the finite set $\mathbb{M} \triangleq \{1, 2, \ldots, M\}$.

Throughout the paper, $\{R_n\}_{n\in\mathbb{N}}$ denotes the index of plant mode at the nth jump, $\{k_n\}_{n\in\mathbb{N}}$ the time at the nth jump with $k_0 = 0$, and $\{S_n\}_{n\in\mathbb{N}}$ the sojourn time between the $(n-1)$th jump and nth jump denoted as $S_n = k_n - k_{n-1}$.

Definition 1 [1] The stochastic process $\{(r(k))\}_{k\in\mathbb{N}}$ is considered as a semi-Markov chain associated with a homogeneous Markov renewal chain $\{(R_n, k_n)\}_{n\in\mathbb{N}}$, if $\forall a, b \in \mathbb{M}$, $a \neq b$, $\forall \tau \in \mathbb{N}_{\geq 1}$, $\forall n, k \in \mathbb{N}$, $\Pr(R_{n+1} = b, S_{n+1} = \tau | R_0, k_0, R_1, k_1, \ldots, R_n = a, k_n) = \Pr(R_{n+1} = b, S_{n+1} = \tau | R_n = a) = \Pr(R_1 = b, S_1 = \tau | R_0 = a)$, and $r(k) = R_{N(k)}$, where $N(k) \triangleq \max\{n \in \mathbb{N} | k_n \leq k\}$.

Given a semi-Markov chain $\{r(k)\}_{k\in\mathbb{N}}$ corresponding to a Markov renewal chain $\{R_n, k_n\}_{n\in\mathbb{N}}$, the transition probability, the probability mass function of sojourn time, and the semi-Markov kernel are, respectively, defined as $\pi_{ab} \triangleq \Pr(R_{n+1} = b | R_n = a)$, $f_{ab}(\tau) \triangleq \Pr(S_{n+1} = \tau | R_{n+1} = b, R_n = a)$, $\theta_{ab}(\tau) \triangleq \Pr(R_{n+1} = b, S_{n+1} = \tau | R_n = a)$, with $\pi_{aa} = f_{aa}(\tau) = \theta_{aa}(\tau) \triangleq 0$, $\forall b \neq a \in \mathbb{M}$, $\forall \tau \in \mathbb{N}_{\geq 1}$. According to the definitions of $f_{ab}(\tau)$, $\theta_{ab}(\tau)$ and π_{ab}, it can be derived that $\theta_{ab}(\tau) = \pi_{ab} f_{ab}(\tau)$, $\forall a, b \in \mathbb{M}$, $\forall \tau \in \mathbb{N}_{\geq 1}$. Now, to precisely present the purposes of this paper, the following stability definition is needed to recall.

Definition 2 [16] Consider a discrete-time switching system $x_{k+1} = f(x_k, r_k)$, where r_k is a certain stochastic process. The system is mean-square stable if for any initial conditions $x(0) \in \mathbb{R}^{n_x}$, $r(0) \in \mathbb{M}$, the following holds:

$$\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|^2\}|_{x(0),r(0)} = 0.$$

Note that throughout the paper, the sojourn time of the ath mode in the stochastic switching system (1) is considered to have both the upper bound $\bar{\tau}_a$ and the lower bound $\underline{\tau}_a$ with $1 \leq \underline{\tau}_a \leq \bar{\tau}_a - 1$. It is straightforward that the demerit of Definition 2 is that it considers the sojourn time to be any length (even infinity) instead of the bi-boundary one. Then, a stability definition in mean-square sense for the underlying S-MJLSs with bi-boundary sojourn time is required.

Definition 3 Consider an S-MJLS (1) with the upper bound $\bar{\tau}_a$ and lower bound $\underline{\tau}_a$ of sojourn time for the ath mode, $a \in \mathbb{M}$. The system is mean-square stable if for $u(k) \equiv 0$ and any initial conditions $x(0) \in \mathbb{R}^{n_x}$, $r(0) \in \mathbb{M}$, the following formula holds:

$$\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|^2\}|_{x(0),r(0),S_n\in\mathbb{N}_{[\underline{\tau}_a,\bar{\tau}_a]}}|_{R_n=a} = 0.$$

In this paper, we consider the general and practical phenomenon of modal asynchrony (illustrated in Fig. 1(a)), where the modal lags are defined as d_a, $a \in \mathbb{M}$. Note that although the consideration of lag mode-independency in the previous literature facilitates the derivations, it is incompetent to describe the modal asynchronous switching of controllers accurately, which is shown in Fig. 1(b).

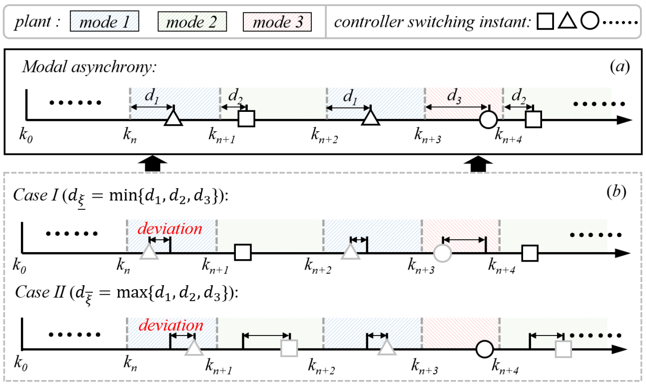


Fig. 1. The illustration of modal asynchrony and its simplications by mode-independent lag $d_{\underline{\xi}}$ and $d_{\bar{\xi}}$, where d_1, d_2, d_3 denote the lags corresponding to mode 1, 2 and 3 of the plant, respectively.

In view of this, a state-feedback controller is employed and formed as

$$u(k) = K_c x(k) \tag{2}$$

where $c \in \mathbb{M}$ is the current controller mode, and K_c the corresponding controller gain. It can be seen that $r(k - d_c) = c$ due to the existence of the modal asynchrony. Also, it is generally supposed that $r(k - d_{r(0)}) = r(0)$, $\forall k \in \mathbb{N}_{[0,d_{r(0)}-1]}$.

The modal lag is considered as $\underline{\tau}_a \le d_a \le \bar{\tau}_a - 1$, $\forall a \in \mathbb{M}$ in this work, where $\bar{\tau}_a$ and $\underline{\tau}_a$ denote the upper and lower bound of sojourn time of the ath mode, respectively. The main reasons for the consideration are twofold. (i) The case of $d_a \ge \bar{\tau}_a$ is rather extreme in asynchronous switching systems. (ii) For the case of $d_a < \underline{\tau}_a$, the criteria of stability analysis and control synthesis can be readily obtained from the results to be derived in the sequel.

Then, combining (1) and (2), the closed-loop S-MJLS can be expressed as

$$x(k+1) = \bar{A}_{r(k),r(k-d_c)} x(k) \tag{3}$$

where $\bar{A}_{r(k),r(k-d_c)} = A_{r(k)} + B_{r(k)} K_{r(k-d_c)}$.

The purpose of this paper is to design anti-modal-asynchrony (AMA) controllers in the form of (2) to guarantee the mean-square stability (MSS) for the closed-loop S-MJLSs (3).

Remark 1 It is worth emphasizing that the studies on the modal asynchrony are significant not only for S-MJLSs considering in this paper but also for any other types of nondeterministic switching systems (e.g., [2, 3]), as the modal asynchronous phenomenon is not seldom encountered in engineering practice of switching systems. Then, if setting $d_a \equiv d_\xi$, $\forall a \in \mathbb{M}$, where d_ξ is a constant, the modal lags will be simplified to the mode-independent one, which means that the modal asynchrony covers the asynchronous switching with mode-independent lag commonly assumed in the existing studies as a special case.

3 Main Results

In this section, the numerically testable stability criteria and the conditions on the existence of AMA controller for the closed-loop S-MJLS (3) with bi-boundary sojourn time will be developed in the presence of modal asynchrony.

Lemma 1 *[22] Consider a discrete-time stochastic switching system $x_{k+1} = f(x_k, r_k)$, where x_k and r_k stand for the system state and the mode index, respectively. The system is mean-square stable, if $\forall n \in \mathbb{N}$, $\forall k \in \mathbb{N}_{[k_n,k_{n+1}-1]}$, $\forall r(k) \in \mathbb{M}$, there exist a set of functions $V(x(k), r(k), k - k_n)$ and three class $\mathcal{K}_\infty$ functions $\kappa_1(\cdot)$, $\kappa_2(\cdot)$ and $\kappa_3(\cdot)$, such that for any initial conditions $x(0) \in \mathbb{R}^{n_x}$, $r(0) \in \mathbb{M}$ and given finite positive constants $\phi_{r(k_n)} > 0$, the following inequalities hold:*

$$\kappa_1(||x(k)||) \le V(x(k), r(k), k - k_n) \le \kappa_2(||x(k)||) \tag{4}$$

$$\begin{aligned} V(x(k), r(k), k - k_n) &\le \phi_{r(k_n)} V(x(k_n), r(k_n), 0) \\ \forall k \in \mathbb{N}_{[k_n+1,k_{n+1}-1]}, &\quad if \quad k_{n+1} - k_n \in \mathbb{N}_{\ge 2} \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbb{E}\{V(x(k_{n+1}), r(k_{n+1}), 0)\}|_{x(k_n),r(k_n)} \\ - V(x(k_n), r(k_n), 0) \le -\kappa_3(||x(k_n)||). \end{aligned} \tag{6}$$

To reduce the conservatism of the derived results, we invoke an elapsed-time-dependent Lyapunov function $V(x(k), r(k), t) \triangleq x^T(k) P_{r(k)}(t) x(k)$ as presented in [15], where $t \triangleq k - k_n$ is the sojourn time that the system has been spent in the current mode $r(k_n)$. Based on Lemma 1, the MSS criteria of S-MJLSs are established in the following theorem.

Theorem 1 *Given finite constants $\underline{\tau}_a \in \mathbb{N}_{\ge 1}$, $\bar{\tau}_a \in \mathbb{N}_{\ge \underline{\tau}_a+1}$, $d_a \in \mathbb{N}_{[\underline{\tau}_a,\bar{\tau}_a-1]}$, $\phi_a \in \mathbb{R}_{>0}$, $a \in \mathbb{M}$, the closed-loop S-MJLS (3) is mean-square stable, if there exist a set of matrices $P_a^{(t)} \succ \mathbf{0}$, where $a \in \mathbb{M}$ and $t \in \mathbb{N}_{[1,\bar{\tau}_a-1]}$, such that $\forall a_0, a_1, \ldots, a_{d_{a_0}} \in \mathbb{M}$, $a_1 \neq a_0$, $\forall t_1 \in \mathbb{N}_{[1,d_{a_0}]}$, $\forall t_2 \in \mathbb{N}_{[d_{a_0}+1,\bar{\tau}_{a_0}-1]}$*

$$[\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i)]^T P_{a_0}^{(t_1)} \Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i) - \phi_{a_0} P_{a_0}^{(0)} \prec \mathbf{0} \tag{7}$$

$$[\tilde{A}_0^{\alpha_2} \Theta_1^{\bar{d}}(\tilde{A}_i)]^T P_{a_0}^{(t_2)} \tilde{A}_0^{\alpha_2} \Theta_1^{\bar{d}}(\tilde{A}_i) - \phi_{a_0} P_{a_0}^{(0)} \prec \mathbf{0} \tag{8}$$

$$\begin{aligned} &\sum_{\tau=\bar{d}+1}^{\bar{\tau}_{a_0}} [\tilde{A}_0^{\tau-\bar{d}} \Theta_1^{\bar{d}}(\tilde{A}_i)]^T \mathcal{P}_{a_0}^{(\tau)} \tilde{A}_0^{\tau-\bar{d}} \Theta_1^{\bar{d}}(\tilde{A}_i) \\ &+ \sum_{\tau=\underline{\tau}_{a_0}}^{\bar{d}} [\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i)]^T \mathcal{P}_{a_0}^{(\tau)} \Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i) - P_{a_0}^{(0)} \prec \mathbf{0} \end{aligned} \tag{9}$$

where $\tilde{A}_i \triangleq \bar{A}_{a_0,a_i}$, $\bar{d} \triangleq d_{a_0}$, $\alpha_1 \triangleq \bar{d} - t_1 + 1$, $\alpha_2 \triangleq t_2 - \bar{d}$, $\mathcal{P}_{a_0}^{(\tau)} \triangleq \sum_{b\in\mathbb{M}} \theta_{a_0 b}(\tau) P_b^{(0)} / \varepsilon_{a_0}$, with $\varepsilon_{a_0} = \sum_{\tau=\underline{\tau}_{a_0}}^{\bar{\tau}_{a_0}} \sum_{b\in\mathbb{M}} \theta_{a_0 b}(\tau)$.

Proof. By considering the Lyapunov function as $V(x(k), r(k), t)$ with $r(k_n) = a$ and $\forall k \in \mathbb{N}_{[k_n, k_{n+1}-1]}$, it holds that

$$\begin{cases} V(x(k), a, t) \geq \inf_{a\in\mathbb{M}, t\in\mathbb{N}_{[\underline{\tau}_a-1, \bar{\tau}_a-1]}}\{\lambda_{min}(P_a^{(t)})\}||x(k)||^2 \\ V(x(k), a, t) \leq \sup_{a\in\mathbb{M}, t\in\mathbb{N}_{[\underline{\tau}_a-1, \bar{\tau}_a-1]}}\{\lambda_{max}(P_a^{(t)})\}||x(k)||^2 \end{cases}$$

where $\lambda_{min}(P_a^{(t)})$ (respectively, $\lambda_{max}(P_a^{(t)})$) denotes the minimum (respectively, maximum) eigenvalue of $P_a^{(t)}$. It is from the construction of the Lyapunov function that (4) in Lemma 1 is satisfied.

For $r(k_n) = a_0$, it can be obtained from (7) and (8) that

$$\begin{aligned} & x(k_n)^T[(\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i))^T P_{a_0}^{(t_1)}\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i) - \phi_{a_0}P_{a_0}^{(0)}]x(k_n) \\ =& (\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i)x(k_n))^T P_{a_0}^{(t_1)}\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i)x(k_n) \\ & - \phi_{a_0}x(k_n)^T P_{a_0}^{(0)}x(k_n) \\ =& V(x(k), a_0, t_1) - \phi_{a_0}V(x(k_n), a_0, 0) \prec \mathbf{0} \end{aligned}$$

$$\begin{aligned} & x(k_n)^T[(\tilde{A}_0^{\alpha_2}\Theta_1^{\bar{d}}(\tilde{A}_i))^T P_{a_0}^{(t_2)}\tilde{A}_0^{\alpha_2}\Theta_1^{\bar{d}}(\tilde{A}_i) - \phi_{a_0}P_{a_0}^{(0)}]x(k_n) \\ =& (\tilde{A}_0^{\alpha_2}\Theta_1^{\bar{d}}(\tilde{A}_i)x(k_n))^T P_{a_0}^{(t_2)}\tilde{A}_0^{\alpha_2}\Theta_1^{\bar{d}}(\tilde{A}_i)x(k_n) \\ & - \phi_{a_0}x(k_n)^T P_{a_0}^{(0)}x(k_n) \\ =& V(x(k), a_0, t_2) - \phi_{a_0}V(x(k_n), a_0, 0) \prec \mathbf{0} \end{aligned}$$

which ensure (5) in Lemma 1. In addition, let $\Omega \triangleq \tilde{A}_0^{\tau-\bar{d}}\Theta_1^{\bar{d}}(\tilde{A}_i)$, $r(k_n) = a_0$, $r(k_{n+1}) = b$, and one knows from (9) that

$$\begin{aligned} & x(k_n)^T\left[\sum_{\tau=\underline{\tau}_{a_0}}^{\bar{d}}(\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i))^T\mathcal{P}_{a_0}^{(\tau)}\,\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i)\right. \\ & \left. + \sum_{\tau=\bar{d}+1}^{\bar{\tau}_{a_0}}\Omega^T\mathcal{P}_{a_0}^{(\tau)}\Omega - P_{a_0}^{(0)}\right]x(k_n) \\ =& \mathbb{E}\{V(x(k_{n+1}), b, 0)\}|_{x(k_n), a_0} - V(x(k_n), a_0, 0). \end{aligned}$$

Thus, one has

$$\begin{aligned} & \mathbb{E}\{V(x(k_{n+1}), b, 0)\}|_{x(k_n), a_0} \\ & \qquad - V(x(k_n), a_0, 0) \leq -\lambda_3||x(k_n)||^2 \end{aligned}$$

where

$$\begin{aligned} \lambda_3 \triangleq \inf_{a_0, a_1, \ldots, a_{\bar{d}}\in\mathbb{M}} & \left\{\lambda_{min}\left[-\sum_{\tau=\underline{\tau}_{a_0}}^{\bar{d}}(\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i))^T\right.\right. \\ & \left.\left.\times\mathcal{P}_{a_0}^{(\tau)}\Theta_{\alpha_1}^{\bar{d}}(\tilde{A}_i) - \sum_{\tau=\bar{d}+1}^{\bar{\tau}_{a_0}}\Omega^T\mathcal{P}_{a_0}^{(\tau)}\Omega + P_{a_0}^{(0)}\right]\right\} \end{aligned}$$

and $\underline{\tau}_{a_0}$, $\bar{\tau}_{a_0}$ denote the upper and lower bound of sojourn time at mode a_0, respectively. In conclusion, MSS of the underlying S-MJLS (3) is ensured by Lemma 1. □

Then, to derive the existence conditions of AMA controller, certain techniques eliminating the existence of the power of A_i are further employed in Theorem 2.

Theorem 2 *Given finite constants $\underline{\tau}_a \in \mathbb{N}_{\geq 1}$, $\bar{\tau}_a \in \mathbb{N}_{\geq\underline{\tau}_a+1}$, $d_a \in \mathbb{N}_{[\underline{\tau}_a, \bar{\tau}_a-1]}$, $\phi_a \in \mathbb{R}_{>0}$, $a \in \mathbb{M}$ with $\bar{\tau}_a > \underline{\tau}_a$, the closed-loop S-MJLS (3) is mean-square stable, if there exist sets of matrices $Q_a^{(t,\kappa)} \succ \mathbf{0}$ and $\mathcal{Q}_a^{(\tau,\beta)} \succ \mathbf{0}$ with $a \in \mathbb{M}$, $t \in \mathbb{N}_{[0,\bar{\tau}_a-1]}$, $\kappa \in \mathbb{N}_{[0,t]}$, $\tau \in \mathbb{N}_{[\underline{\tau}_a, \bar{\tau}_a]}$, $\beta \in \mathbb{N}_{[0,\bar{\tau}_a]}$, such that $\forall a_0, a_1, \ldots, a_{d_{a_0}} \in \mathbb{M}$, $a_1 \neq a_0$, $\forall t_1 \in \mathbb{N}_{[1,d_{a_0}]}$, $\forall t_2 \in \mathbb{N}_{[d_{a_0}+1,\bar{\tau}_{a_0}-1]}$, $\forall\kappa_1 \in \mathbb{N}_{[0,t_1-1]}$, $\forall\kappa_2 \in \mathbb{N}_{[0,d_{a_0}-1]}$, $\forall\kappa_3 \in \mathbb{N}_{[d_{a_0},t_2-1]}$, $\forall\beta_1 \in \mathbb{N}_{[\underline{\tau}_{a_0}-1,d_{a_0}-1]}$, $\forall\beta_2 \in \mathbb{N}_{[d_{a_0},\bar{\tau}_{a_0}-1]}$, $\forall\beta_3 \in \mathbb{N}_{[0,\underline{\tau}_{a_0}-2]}$, $\forall l \in \mathbb{N}_{[1,2]}$, the following hold:*

$$\begin{bmatrix} -Q_{a_0}^{(t_l,\kappa_l+1)} & Q_{a_0}^{(t_l,\kappa_l+1)}\tilde{A}_{\bar{d}-\kappa_l} \\ * & -Q_{a_0}^{(t_l,\kappa_l)} \end{bmatrix} \prec \mathbf{0} \tag{10}$$

$$Q_{a_0}^{(t_l,0)} - \phi_{a_0}Q_{a_0}^{(0,0)} \prec \mathbf{0} \tag{11}$$

$$\begin{bmatrix} -Q_{a_0}^{(t_2,\kappa_3+1)} & Q_{a_0}^{(t_2,\kappa_3+1)}\tilde{A}_0 \\ * & -Q_{a_0}^{(t_2,\kappa_3)} \end{bmatrix} \prec \mathbf{0} \tag{12}$$

$$\begin{bmatrix} -\bar{\mathcal{Q}}_{a_0}^{(\beta_1)} & * & * \\ \bar{Q}(I_M \otimes \tilde{A}_{\bar{d}-\beta_1})\Xi_{a_0}(\beta_1+1) & -\bar{Q} & * \\ \bar{\mathcal{Q}}_{a_0}^{(\beta_1+1)}\tilde{A}_{\bar{d}-\beta_1} & 0 & -\bar{\mathcal{Q}}_{a_0}^{(\beta_1+1)} \end{bmatrix} \prec \mathbf{0} \tag{13}$$

$$\begin{bmatrix} -\bar{\mathcal{Q}}_{a_0}^{(\beta_2)} & * & * \\ \bar{Q}(I_M \otimes \tilde{A}_0)\Xi_{a_0}(\beta_2+1) & -\bar{Q} & * \\ \bar{\mathcal{Q}}_{a_0}^{(\beta_2+1)}\tilde{A}_0\mathcal{I}_{a_0}(\beta_2+1) & 0 & -\bar{\mathcal{Q}}_{a_0}^{(\beta_2+1)} \end{bmatrix} \prec \mathbf{0} \tag{14}$$

$$\tilde{\mathcal{Q}}_{a_0}^{(0)} - Q_{a_0}^{(0,0)} \prec \mathbf{0} \tag{15}$$

$$\begin{bmatrix} -\tilde{\mathcal{Q}}_{a_0}^{(\beta_3+1)} & \tilde{\mathcal{Q}}_{a_0}^{(\beta_3+1)}\tilde{A}_{\bar{d}-\beta_3} \\ * & -\tilde{\mathcal{Q}}(\beta_3) \end{bmatrix} \prec \mathbf{0} \tag{16}$$

where

$$\bar{Q} \triangleq diag\left\{Q_1^{(0,0)}, Q_2^{(0,0)}, Q_3^{(0,0)}, \ldots, Q_M^{(0,0)}\right\}$$

$$\Xi_{a_0}(\beta) \triangleq [\eta_{a_0 1}(\beta), \eta_{a_0 2}(\beta), \ldots, \eta_{a_0 M}(\beta)]^T$$

$$\mathcal{I}_{a_0}(\beta) \triangleq sign(\bar{\tau}_{a_0} - \beta)I_{n_x}, \quad \forall\beta \in \mathbb{N}_{[\bar{d}+1,\bar{\tau}_{a_0}]}$$

and $\tilde{A}_i \triangleq \bar{A}_{a_0,a_i}$, $\bar{d} \triangleq d_{a_0}$, $\tilde{\mathcal{Q}}_{a_0}^{(\beta)} \triangleq \sum_{\tau=\underline{\tau}_{a_0}}^{\bar{\tau}_{a_0}}\mathcal{Q}_{a_0}^{(\tau,\beta)}$, $\bar{\mathcal{Q}}_{a_0}^{(\beta)} \triangleq \sum_{\tau=\beta+1}^{\bar{\tau}_{a_0}}\mathcal{Q}_{a_0}^{(\tau,\beta)}$, $\eta_{a_0 b}(\beta) \triangleq \sqrt{\theta_{a_0 b}(\beta)/\varepsilon_{a_0}}I_{n_x}$, $\varepsilon_{a_0} \triangleq \sum_{\tau=\underline{\tau}_{a_0}}^{\bar{\tau}_{a_0}}\sum_{b\in\mathbb{M}}\theta_{a_0 b}(\tau)$.

Proof. Performing the congruence transformation to (10) via $diag\{(Q_{a_0}^{(t_l,\kappa_l+1)})^{-1}, I_{n_x}\}$ and by the Schur complement, it yields that

$$\tilde{A}_{\bar{d}-\kappa_l}^T Q_{a_0}^{(t_l,\kappa_l+1)}\tilde{A}_{\bar{d}-\kappa_l} - Q_{a_0}^{(t_l,\kappa_l)} \prec \mathbf{0}.$$

By considering the case of $l=1$, one has

$$\sum_{\kappa_1=0}^{t_1-1}(\Theta_{\bar{d}-\kappa_1+1}^{\bar{d}}(\tilde{A}_i))^T \times(\tilde{A}_{\bar{d}-\kappa_1}^T Q_{a_0}^{(t_1,\kappa_1+1)}\tilde{A}_{\bar{d}-\kappa_1}-Q_{a_0}^{(t_1,\kappa_1)})\Theta_{\bar{d}-\kappa_1+1}^{\bar{d}}(\tilde{A}_i)\prec \mathbf{0}$$

which is equivalent to

$$(\Theta_{\bar{d}-t_1+1}^{\bar{d}}(\tilde{A}_i))^T Q_{a_0}^{(t_1,t_1)}\Theta_{\bar{d}-t_1+1}^{\bar{d}}(\tilde{A}_i)-Q_{a_0}^{(t_1,0)}\prec \mathbf{0}.$$

Combining (11) and the above inequality, we have

$$(\Theta_{\bar{d}-t_1+1}^{\bar{d}}(\tilde{A}_i))^T Q_{a_0}^{(t_1,t_1)}\Theta_{\bar{d}-t_1+1}^{\bar{d}}(\tilde{A}_i)-\phi_{a_0}Q_{a_0}^{(0,0)}\prec \mathbf{0}.$$

By setting $P_{a_0}^{(t)}\triangleq Q_{a_0}^{(t,t)}$, the inequality (7) in Theorem 1 is satisfied. Besides, by considering $l=2$ with the same techniques as the case of $l=1$, it holds that

$$(\Theta_1^{\bar{d}}(\tilde{A}_i))^T Q_{a_0}^{(t_2,\bar{d})}\Theta_1^{\bar{d}}(\tilde{A}_i)-Q_{a_0}^{(t_2,0)}\prec \mathbf{0}. \quad (17)$$

Then, performing a congruence transformation to (12) via $diag\{(Q_{a_0}^{(t_2,\kappa_3+1)})^{-1},I_{n_x}\}$, and by the Schur complement, the following inequality holds

$$\tilde{A}_0^T Q_{a_0}^{(t_2,\kappa_3+1)}\tilde{A}_0-Q_{a_0}^{(t_2,\kappa_3)}\prec \mathbf{0}.$$

Summarizing from $\kappa_3=\bar{d}$ to $\kappa_3=t_2-1$, it implies that

$$\sum_{\kappa_3=\bar{d}}^{t_2-1}(\tilde{A}_0^{\kappa_3-\bar{d}})^T(\tilde{A}_0^T Q_{a_0}^{(t_2,\kappa_3+1)}\tilde{A}_0-Q_{a_0}^{(t_2,\kappa_3)})\tilde{A}_0^{\kappa_3-\bar{d}}\prec \mathbf{0}$$

which is equivalent to

$$(\tilde{A}_0^{\alpha_2})^T Q_{a_0}^{(t_2,t_2)}\tilde{A}_0^{\alpha_2}-Q_{a_0}^{(t_2,\bar{d})}\prec \mathbf{0}.$$

With the aid of (11), (17) and the above inequality, it can be verified that $(\tilde{A}_0^{\alpha_2}\Theta_1^{\bar{d}}(\tilde{A}_i))^T Q_{a_0}^{(t_2,t_2)}\tilde{A}_0^{\alpha_2}\Theta_1^{\bar{d}}(\tilde{A}_i)-\phi_{a_0}Q_{a_0}^{(0,0)}\prec \mathbf{0}$. Further, (8) can be arrived at by setting the matrices $P_{a_0}^{(t)}\triangleq Q_{a_0}^{(t,t)}$.

In addition, by bearing in mind $\mathcal{Q}_{a_0}^{(\beta,\beta)}\triangleq\sum_{b\in\mathbb{M}}\theta_{a_0 b}(\beta)\times Q_b^{(0,0)}/\varepsilon_{a_0}$, performing congruence transformations to (13) via $diag\{I_{n_x},\bar{Q}^{-1},(\bar{\mathcal{Q}}_{a_0}^{(\beta_1+1)})^{-1}\}$ and (14) via $diag\{I_{n_x},\bar{Q}^{-1},(\bar{\mathcal{Q}}_{a_0}^{(\beta_2+1)})^{-1}\}$, as well as applying the Schur complement, the following inequalities hold

$$\tilde{A}_{\bar{d}-\beta_1}^T(\bar{\mathcal{Q}}_{a_0}^{(\beta_1+1)}+\mathcal{Q}_{a_0}^{(\beta_1+1,\beta_1+1)})\tilde{A}_{\bar{d}-\beta_1}-\bar{\mathcal{Q}}_{a_0}^{(\beta_1)}\prec \mathbf{0}$$

$$\tilde{A}_0^T(\mathcal{I}_{a_0}(\beta_2+1)\bar{\mathcal{Q}}_{a_0}^{(\beta_2+1)}+\mathcal{Q}_{a_0}^{(\beta_2+1,\beta_2+1)})\tilde{A}_0-\bar{\mathcal{Q}}_{a_0}^{(\beta_2)}\prec \mathbf{0}.$$

Then, taking the summations from $\underline{\tau}_{a_0}-1$ to $\bar{d}-1$ with respect to β_1, and from $\bar{d}$ to $\bar{\tau}_{a_0}-1$ with respect to β_2, respectively, it follows that

$$\sum_{\beta_1=\underline{\tau}_{a_0}-1}^{\bar{d}-1}(\Theta_{\bar{d}-\beta_1}^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\beta_1+1,\beta_1+1)}\Theta_{\bar{d}-\beta_1}^{\bar{d}}(\tilde{A}_i) -\sum_{\tau=\underline{\tau}_{a_0}}^{\bar{\tau}_{a_0}}(\Theta_{\bar{d}-\underline{\tau}_{a_0}+2}^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\tau,\underline{\tau}_{a_0}-1)}\Theta_{\bar{d}-\underline{\tau}_{a_0}+2}^{\bar{d}}(\tilde{A}_i) +\sum_{\tau=\bar{d}+1}^{\bar{\tau}_{a_0}}(\Theta_1^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\tau,\bar{d})}\Theta_1^{\bar{d}}(\tilde{A}_i)\prec \mathbf{0}$$

$$\sum_{\beta_2=\bar{d}}^{\bar{\tau}_{a_0}-1}(\tilde{A}_0^{\beta_2-\bar{d}+1})^T\mathcal{Q}_{a_0}^{(\beta_2+1,\beta_2+1)}\tilde{A}_0^{\beta_2-\bar{d}+1} -\sum_{\tau=\bar{d}+1}^{\bar{\tau}_{a_0}}Q_{a_0}^{(\tau,\bar{d})}\prec \mathbf{0}.$$

Combining the above two inequalities, we have

$$\sum_{\beta_1=\underline{\tau}_{a_0}-1}^{\bar{d}-1}(\Theta_{\bar{d}-\beta_1}^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\beta_1+1,\beta_1+1)}\Theta_{\bar{d}-\beta_1}^{\bar{d}}(\tilde{A}_i) -\sum_{\tau=\underline{\tau}_{a_0}}^{\bar{\tau}_{a_0}}(\Theta_{\bar{d}-\underline{\tau}_{a_0}+2}^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\tau,\underline{\tau}_{a_0}-1)}\Theta_{\bar{d}-\underline{\tau}_{a_0}+2}^{\bar{d}}(\tilde{A}_i) +\sum_{\beta_2=\bar{d}}^{\bar{\tau}_{a_0}-1}(\tilde{A}_0^{\beta_2-\bar{d}+1}\Theta_1^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\beta_2+1,\beta_2+1)} \times\tilde{A}_0^{\beta_2-\bar{d}+1}\Theta_1^{\bar{d}}(\tilde{A}_i)\prec \mathbf{0}. \quad (18)$$

By performing the congruence transformation to (16) via $diag\{(\tilde{\mathcal{Q}}_{a_0}^{(\beta_3+1)})^{-1},I_{n_x}\}$ and applying the Schur complement, it yields that $\tilde{A}_{\bar{d}-\beta_3}^T\tilde{\mathcal{Q}}_{a_0}^{(\beta_3+1)}\tilde{A}_{\bar{d}-\beta_3}-\tilde{\mathcal{Q}}_{a_0}^{(\beta_3)}\prec \mathbf{0}$. Then, summarizing from $\beta_3=0$ to $\beta_3=\underline{\tau}_{a_0}-2$, it can be obtained that

$$\sum_{\beta_3=0}^{\underline{\tau}_{a_0}-2}(\Theta_{\bar{d}-\beta_3+1}^{\bar{d}}(\tilde{A}_i))^T \times(\tilde{A}_{\bar{d}-\beta_3}^T\tilde{\mathcal{Q}}_{a_0}^{(\beta_3+1)}\tilde{A}_{\bar{d}-\beta_3}-\tilde{\mathcal{Q}}_{a_0}^{(\beta_3)})\Theta_{\bar{d}-\beta_3+1}^{\bar{d}}(\tilde{A}_i)\prec \mathbf{0}$$

which is equivalent to $(\Theta_{\bar{d}-\underline{\tau}_{a_0}+2}^{\bar{d}}(\tilde{A}_i))^T\tilde{\mathcal{Q}}_{a_0}^{(\underline{\tau}_{a_0}-1)}\times\Theta_{\bar{d}-\underline{\tau}_{a_0}+2}^{\bar{d}}(\tilde{A}_i)-\tilde{\mathcal{Q}}_{a_0}^{(0)}\prec \mathbf{0}$. Combining (15), (18) and above inequality, it yields that

$$\sum_{\beta_1=\underline{\tau}_{a_0}-1}^{\bar{d}-1}(\Theta_{\bar{d}-\beta_1}^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\beta_1+1,\beta_1+1)}\Theta_{\bar{d}-\beta_1}^{\bar{d}}(\tilde{A}_i) +\sum_{\beta_2=\bar{d}}^{\bar{\tau}_{a_0}-1}(\tilde{A}_0^{\beta_2-\bar{d}+1}\Theta_1^{\bar{d}}(\tilde{A}_i))^T\mathcal{Q}_{a_0}^{(\beta_2+1,\beta_2+1)} \times\tilde{A}_0^{\beta_2-\bar{d}+1}\Theta_1^{\bar{d}}(\tilde{A}_i)-Q_{a_0}^{(0,0)}\prec \mathbf{0}.$$

By setting $\mathcal{P}_{a_0}^{(t)}\triangleq\mathcal{Q}_{a_0}^{(t,t)}$, (9) holds, then the MSS for S-MJLSs (3) can be concluded by Theorem 1. □

Now, based on Theorem 2, we are in a position to give the existence conditions of AMA controller for the underlying S-MJLS (3).

Theorem 3 *Given finite constants* $\underline{\tau}_a \in \mathbb{N}_{\geq 1}$, $\bar{\tau}_a \in \mathbb{N}_{\geq \underline{\tau}_a+1}$, $d_a \in \mathbb{N}_{[\underline{\tau}_a, \bar{\tau}_a-1]}$, $\phi_a \in \mathbb{R}_{>0}$, $a \in \mathbb{M}$, *the closed-loop S-MJLS (3) is mean-square stable, if there exist a set of matrices* $U_a^{(t,\kappa)} \succ \mathbf{0}$, $\mathcal{U}_a^{(\tau,\beta)} \succ \mathbf{0}$, X_a *and* Y_a *with* $a \in \mathbb{M}$, $t \in \mathbb{N}_{[0,\bar{\tau}_a-1]}$, $\kappa \in \mathbb{N}_{[0,t]}$, $\tau \in \mathbb{N}_{[\underline{\tau}_a,\bar{\tau}_a]}$, $\beta \in \mathbb{N}_{[0,\bar{\tau}_a]}$, *such that* $\forall a_0, a_1, \ldots, a_{d_{a_0}} \in \mathbb{M}$, $a_1 \neq a_0$, $\forall t_1 \in \mathbb{N}_{[1,d_{a_0}]}$, $\forall t_2 \in \mathbb{N}_{[d_{a_0}+1,\bar{\tau}_{a_0}-1]}$, $\forall \kappa_1 \in \mathbb{N}_{[0,t_1-1]}$, $\forall \kappa_2 \in \mathbb{N}_{[0,d_{a_0}-1]}$, $\forall \kappa_3 \in \mathbb{N}_{[d_{a_0},t_2-1]}$, $\forall \beta_1 \in \mathbb{N}_{[\underline{\tau}_{a_0}-1,d_{a_0}-1]}$, $\forall \beta_2 \in \mathbb{N}_{[d_{a_0},\bar{\tau}_{a_0}-1]}$, $\forall \beta_3 \in \mathbb{N}_{[0,\underline{\tau}_{a_0}-2]}$, $\forall l \in \mathbb{N}_{[1,2]}$, *the following hold:*

$$\begin{bmatrix} U_{a_0}^{(t_l,\kappa_l+1)} - sym(X_{a_{\bar{d}-\kappa_l-1}}) & \tilde{\mathcal{A}}_{\bar{d}-\kappa_l} \\ * & -U_{a_0}^{(t_l,\kappa_l)} \end{bmatrix} \prec \mathbf{0} \tag{19}$$

$$U_{a_0}^{(t_l,0)} - \phi_{a_0} U_{a_0}^{(0,0)} \prec \mathbf{0} \tag{20}$$

$$\begin{bmatrix} U_{a_0}^{(t_2,\kappa_3+1)} - sym(X_{a_0}) & \tilde{\mathcal{A}}_0 \\ * & -U_{a_0}^{(t_2,\kappa_3)} \end{bmatrix} \prec \mathbf{0} \tag{21}$$

$$\begin{bmatrix} -\bar{\mathcal{U}}_{a_0}^{(\beta_1)} & * & * \\ (I_M \otimes \tilde{\mathcal{A}}_{\bar{d}-\beta_1})\Xi_{a_0}(\beta_1+1) & \bar{V} & * \\ \tilde{\mathcal{A}}_{\bar{d}-\beta_1} & 0 & \bar{\mathcal{V}}_{a_0}^{(\beta_1+1)} \end{bmatrix} \prec \mathbf{0} \tag{22}$$

$$\begin{bmatrix} -\bar{\mathcal{U}}_{a_0}^{(\beta_2)} & * & * \\ (I_M \otimes \tilde{\mathcal{A}}_0)\Xi_{a_0}(\beta_2+1) & \bar{V} & * \\ \tilde{\mathcal{A}}_0 \mathcal{I}_{a_0}(\beta_2+1) & 0 & \bar{\mathcal{V}}_{a_0}^{(\beta_2+1)} \end{bmatrix} \prec \mathbf{0} \tag{23}$$

$$\tilde{\mathcal{U}}_{a_0}^{(0)} - U_{a_0}^{(0,0)} \prec \mathbf{0} \tag{24}$$

$$\begin{bmatrix} \tilde{\mathcal{U}}_{a_0}^{(\beta_3+1)} - sym(X_{a_{\bar{d}-\beta_3-1}}) & \tilde{\mathcal{A}}_{\bar{d}-\beta_3} \\ * & -\tilde{\mathcal{U}}_{a_0}^{(\beta_3)} \end{bmatrix} \prec \mathbf{0} \tag{25}$$

where

$$\bar{\mathcal{V}}_{a_0}^{(\beta)} \triangleq \begin{cases} \bar{\mathcal{U}}_{a_0}^{(\beta)} - sym(X_{a_{\bar{d}-\beta}}) & \beta \in \mathbb{N}_{[0,\bar{d}-1]} \\ \bar{\mathcal{U}}_{a_0}^{(\beta)} - sym(X_{a_0}) & \beta \in \mathbb{N}_{[\bar{d},\bar{\tau}_{a_0}]} \end{cases}$$

$$\bar{\mathcal{U}}_{a_0}^{(\beta)} \triangleq \sum_{\tau=\beta+1}^{\bar{\tau}_{a_0}} \mathcal{U}_{a_0}^{(\tau,\beta)}, \quad \tilde{\mathcal{U}}_{a_0}^{(\beta)} \triangleq \sum_{\tau=\underline{\tau}_{a_0}}^{\bar{\tau}_{a_0}} \mathcal{U}_{a_0}^{(\tau,\beta)}$$

$$\mathcal{X} \triangleq diag\{X_1, X_2, X_3, \ldots, X_M\}$$

$$\bar{U} \triangleq diag\left\{U_1^{(0,0)}, U_2^{(0,0)}, U_3^{(0,0)}, \ldots, U_M^{(0,0)}\right\}$$

$\tilde{\mathcal{A}}_i \triangleq A_{a_0}X_{a_i} + B_{a_0}Y_{a_i}$, $\bar{V} \triangleq \bar{U} - sym(\mathcal{X})$, *and* $\Xi_{a_0}(\beta)$, $\mathcal{I}_{a_0}(\beta)$ *are defined in Theorem 2. Further, the admissible gains of controllers are given by* $K_a \triangleq Y_a X_a^{-1}$, $a \in \mathbb{M}$.

Proof. Note that $\forall a, b \in \mathbb{M}$, $\forall t \in \mathbb{N}_{[\underline{\tau}_{a_0}-1,\bar{\tau}_{a_0}-1]}$, $\forall \kappa \in \mathbb{N}_{[0,t]}$, $\forall \beta \in \mathbb{N}_{[0,\bar{\tau}_{a_0}]}$, the following inequalities

$$[U_a^{(t,\kappa)} - X_b](U_a^{(t,\kappa)})^{-1}[U_a^{(t,\kappa)} - X_b]^T \succ \mathbf{0}$$

$$[\bar{U} - \mathcal{X}]\bar{U}^{-1}[\bar{U} - \mathcal{X}]^T \succ \mathbf{0}$$

$$[\bar{\mathcal{U}}_a^{(\beta)} - X_b](\bar{\mathcal{U}}_a^{(\beta)})^{-1}[\bar{\mathcal{U}}_a^{(\beta)} - X_b]^T \succ \mathbf{0}$$

$$[\tilde{\mathcal{U}}_a^{(\beta)} - X_b](\tilde{\mathcal{U}}_a^{(\beta)})^{-1}[\tilde{\mathcal{U}}_a^{(\beta)} - X_b]^T \succ \mathbf{0}$$

ensure

$$U_a^{(t,\kappa)} - X_b - X_b^T + X_b(U_a^{(t,\kappa)})^{-1}X_b^T \succ \mathbf{0}$$

$$\bar{U} - \mathcal{X} - \mathcal{X}^T + \mathcal{X}\bar{U}^{-1}\mathcal{X}^T \succ \mathbf{0}$$

$$\bar{\mathcal{U}}_a^{(\beta)} - X_b - X_b^T + X_b(\bar{\mathcal{U}}_a^{(\beta)})^{-1}X_b^T \succ \mathbf{0}$$

$$\tilde{\mathcal{U}}_a^{(\beta)} - X_b - X_b^T + X_b(\tilde{\mathcal{U}}_a^{(\beta)})^{-1}X_b^T \succ \mathbf{0}.$$

Also, by setting

$$U_{a_0}^{(t,\kappa)} \triangleq \begin{cases} X_{a_{\bar{d}-\kappa}}^T Q_{a_0}^{(t,\kappa)} X_{a_{\bar{d}-\kappa}} & \kappa \in \mathbb{N}_{[0,\bar{d}-1]} \\ X_{a_0}^T Q_{a_0}^{(t,\kappa)} X_{a_0} & \kappa \in \mathbb{N}_{[\bar{d},\bar{\tau}_{a_0}]} \end{cases} \tag{26}$$

$$\mathcal{U}_{a_0}^{(\tau,\beta)} \triangleq \begin{cases} X_{a_{\bar{d}-\beta}}^T \mathcal{Q}_{a_0}^{(\tau,\beta)} X_{a_{\bar{d}-\beta}} & \beta \in \mathbb{N}_{[0,\bar{d}-1]} \\ X_{a_0}^T \mathcal{Q}_{a_0}^{(\tau,\beta)} X_{a_0} & \beta \in \mathbb{N}_{[\bar{d},\bar{\tau}_{a_0}]} \end{cases} \tag{27}$$

and it follows from (19) that

$$\begin{bmatrix} (Q_{a_0}^{(t_l,\kappa_l+1)})^{-1} & \tilde{A}_{\bar{d}-\kappa_l} X_{a_{\bar{d}-\kappa_l}} \\ * & -X_{a_{\bar{d}-\kappa_l}}^T Q_{a_0}^{(t_l,\kappa_l)} X_{a_{\bar{d}-\kappa_l}} \end{bmatrix} \prec \mathbf{0}.$$

Employing the congruence transformation to the above inequality via $diag\{Q_{a_0}^{(t_l,\kappa_l+1)}, (X_{a_{\bar{d}-\kappa_l}})^{-1}\}$, it gives rise to (10). Furthermore, replacing $U_{a_0}^{(t,\kappa)}$, $\mathcal{U}_{a_0}^{(\tau,\beta)}$ in (20) by (26)-(27), one has $X_{a_{\bar{d}}}^T(Q_{a_0}^{(t_l,0)} - \phi_{a_0} Q_{a_0}^{(0,0)})X_{a_{\bar{d}}} \prec \mathbf{0}$, which ensures (11). Similarly, we can rewrite (21) as

$$\begin{bmatrix} -(Q_{a_0}^{(t_2,\kappa_3+1)})^{-1} & \tilde{A}_0 X_{a_0} \\ * & -X_{a_0}^T Q_{a_0}^{(t_2,\kappa_3)} X_{a_0} \end{bmatrix} \prec \mathbf{0}.$$

Then, taking the congruence transformation to the above inequality via $diag\{Q_{a_0}^{(t_2,\kappa_3+1)}, X_{a_0}^{-1}\}$, we can obtain (12). Also, it follows from (22) that

$$\begin{bmatrix} -X_{a_{\bar{d}-\beta_1}}^T \bar{\mathcal{Q}}_{a_0}^{(\beta_1)} X_{a_{\bar{d}-\beta_1}} & * & * \\ (I_M \otimes \tilde{\mathcal{A}}_{\bar{d}-\beta_1})\Xi_{a_0}(\beta_1+1) & -\bar{Q}^{-1} & * \\ \tilde{\mathcal{A}}_{\bar{d}-\beta_1} & 0 & -(\bar{\mathcal{Q}}_{a_0}^{(\beta_1+1)})^{-1} \end{bmatrix} \prec \mathbf{0}.$$

By performing a congruence transformation to the above inequality via $diag\{X_{a_{\bar{d}-\beta_1}}^{-1}, \bar{Q}, I_{n_x}\}$, one can get (13). Then, replacing $U_{a_0}^{(t,\kappa)}$, $\mathcal{U}_{a_0}^{(\tau,\beta)}$ by (26)-(27), it can be obtained from (23) that

$$\begin{bmatrix} -X_{a_0}^T \bar{\mathcal{U}}_{a_0}^{(\beta_2)} X_{a_0} & * & * \\ (I_M \otimes \tilde{\mathcal{A}}_0)\Xi_{a_0}(\beta_2+1) & -\bar{Q}^{-1} & * \\ \tilde{\mathcal{A}}_0 \mathcal{I}_{a_0}(\beta_2+1) & 0 & -(\bar{\mathcal{Q}}_{a_0}^{(\beta_2+1)})^{-1} \end{bmatrix} \prec \mathbf{0}.$$

Also, employing the congruence transformation with $diag\{X_{a_0}^{-1}, \bar{Q}, I_{n_x}\}$, it yields that the above inequality is equivalent to (14). Using similar techniques, (24) is equivalent to $(X_{a_{\bar{d}}}^T)^{-1}\tilde{\mathcal{U}}_{a_0}^{(0)}X_{a_{\bar{d}}}^{-1}-(X_{a_{\bar{d}}}^T)^{-1}U_{a_0}^{(0,0)}X_{a_{\bar{d}}}^{-1}\prec\mathbf{0}$ which guarantees (15).

Similarly, replacing $U_{a_0}^{(t,\kappa)}$, $\mathcal{U}_{a_0}^{(\tau,\beta)}$ in (25) by (26)-(27), as well as performing a congruence transformation via $diag\{\tilde{\mathcal{Q}}_{a_0}^{(\beta_3+1)}, X_{a_{\bar{d}-\beta_3}}^{-1}\}$, (16) can be derived. This completes the proof. □

Remark 2 Note that the modal lag is considered as $d_a \geq \underline{\tau}_a$, $a\in\mathbb{M}$ in the above results, where two cases are considered according to whether the sojourn time is shorter than the lag. Then, if the modal lag is considered as $d_a<\underline{\tau}_a$, the system will reduce to only one case, and the corresponding results can be readily obtained from Theorem 3 by taking no account of inequality (22) and replacing $\beta_2\in\mathbb{N}_{[d_{a_0},\bar{\tau}_{a_0}-1]}$ with $\beta_2\in\mathbb{N}_{[\underline{\tau}_{a_0}-1,\bar{\tau}_{a_0}-1]}$. The proof can be obtained in a similar vein to the one for Theorem 3 and omitted here.

In the absence of the lower bounds of sojourn time, i.e., $\underline{\tau}_a\equiv 1$, $\forall a\in\mathbb{M}$, Theorem 3 will reduce to Corollary 1.

Corollary 1 *Given finite constants* $\bar{\tau}_a\in\mathbb{N}_{\geq 1}$, $\phi_a\in\mathbb{R}_{>0}$, $d_a\in\mathbb{N}_{[1,\bar{\tau}_a-1]}$, $a\in\mathbb{M}$, *the closed-loop S-MJLS (3) is mean-square stable, if there exist a set of matrices* $U_a^{(t,\kappa)}\succ\mathbf{0}$, $\mathcal{U}_a^{(\tau,\beta)}\succ\mathbf{0}$, X_a *and* Y_a *with* $a\in\mathbb{M}$, $t\in\mathbb{N}_{[0,\bar{\tau}_a-1]}$, $\kappa\in\mathbb{N}_{[0,t]}$, $\tau\in\mathbb{N}_{[1,\bar{\tau}_a]}$, $\beta\in\mathbb{N}_{[0,\bar{\tau}_a]}$, *such that* $\forall a_0,a_1,\ldots,a_{d_{a_0}}\in\mathbb{M}$, $a_1\neq a_0$, $\forall t_1\in\mathbb{N}_{[1,d_{a_0}]}$, $\forall t_2\in\mathbb{N}_{[d_{a_0}+1,\bar{\tau}_{a_0}-1]}$, $\forall\kappa_1\in\mathbb{N}_{[0,t_1-1]}$, $\forall\kappa_2\in\mathbb{N}_{[0,d_{a_0}-1]}$, $\forall\kappa_3\in\mathbb{N}_{[d_{a_0},t_2-1]}$, $\forall\beta_1\in\mathbb{N}_{[0,d_{a_0}-1]}$, $\forall\beta_2\in\mathbb{N}_{[d_{a_0},\bar{\tau}_{a_0}-1]}$, $\forall l\in\mathbb{N}_{[1,2]}$, *the following hold:*

$$\begin{bmatrix} U_{a_0}^{(t_l,\kappa_l+1)}-sym(X_{a_{\bar{d}-\kappa_l-1}}) & \tilde{\mathcal{A}}_{\bar{d}-\kappa_1}\\ * & -U_{a_0}^{(t_l,\kappa_l)}\end{bmatrix}\prec\mathbf{0}$$

$$U_{a_0}^{(t_l,0)}-\phi_{a_0}U_{a_0}^{(0,0)}\prec\mathbf{0}$$

$$\begin{bmatrix} U_{a_0}^{(t_2,\kappa_3+1)}-sym(X_{a_0}) & \tilde{\mathcal{A}}_0\\ * & -U_{a_0}^{(t_2,\kappa_3)}\end{bmatrix}\prec\mathbf{0}$$

$$\begin{bmatrix} -\bar{\mathcal{U}}_{a_0}^{(\beta_1)} & * & *\\ (I_M\otimes\tilde{\mathcal{A}}_{\bar{d}-\beta_1})\Xi_{a_0}(\beta_1+1) & \bar{V} & *\\ \tilde{\mathcal{A}}_{\bar{d}-\beta_1} & 0 & \bar{\mathcal{V}}_{a_0}^{(\beta_1+1)}\end{bmatrix}\prec\mathbf{0}$$

$$\begin{bmatrix} -\bar{\mathcal{U}}_{a_0}^{(\beta_2)} & * & *\\ (I_M\otimes\tilde{\mathcal{A}}_0)\Xi_{a_0}(\beta_2+1) & \bar{V} & *\\ \tilde{\mathcal{A}}_0\mathcal{I}_{a_0}(\beta_2+1) & 0 & \bar{\mathcal{V}}_{a_0}^{(\beta_2+1)}\end{bmatrix}\prec\mathbf{0}$$

$$\tilde{\mathcal{U}}_{a_0}^{(0)}-U_{a_0}^{(0,0)}\prec\mathbf{0}$$

where $\tilde{\mathcal{A}}_i$, $\bar{\mathcal{U}}_{a_0}^{(\beta)}$, $\bar{V}$, $\bar{\mathcal{V}}_{a_0}^{(\beta)}$, $\tilde{\mathcal{U}}_{a_0}^{(\beta)}$ *are defined in Theorem 3, and* $\Xi_{a_0}(\beta)$, $\mathcal{I}_{a_0}(\beta)$ *are defined in Theorem 2. Then, the admissible gains of controllers are given by* $K_a\triangleq Y_aX_a^{-1}$, $a\in\mathbb{M}$.

In addition, note that the criteria obtained by Theorem 3 cover the case of mode-independent lag. Then, if $d_a\equiv d_\xi$, $\forall a\in\mathbb{M}$, where d_ξ is a constant, i.e., replaying the modal lag by the mode-independent lag, one has the following corollary.

Corollary 2 *Given finite constants* $\underline{\tau}_a\in\mathbb{N}_{\geq 1}$, $\bar{\tau}_a\in\mathbb{N}_{\geq\underline{\tau}_a+1}$, $\phi_a\in\mathbb{R}_{>0}$, $a\in\mathbb{M}$, *the closed-loop S-MJLS (3) is mean-square stable, if there exist a set of matrices* $U_a^{(t,\kappa)}\succ\mathbf{0}$, $\mathcal{U}_a^{(\tau,\beta)}\succ\mathbf{0}$, X_a *and* Y_a *with* $a\in\mathbb{M}$, $t\in\mathbb{N}_{[0,\bar{\tau}_a-1]}$, $\kappa\in\mathbb{N}_{[0,t]}$, $\tau\in\mathbb{N}_{[\underline{\tau}_a,\bar{\tau}_a]}$, $\beta\in\mathbb{N}_{[0,\bar{\tau}_a]}$ *such that* $\forall a_0,a_1,\ldots,a_{d_\xi}\in\mathbb{M}$, $a_1\neq a_0$, $\forall t_1\in\mathbb{N}_{[1,d_\xi]}$, $\forall t_2\in\mathbb{N}_{[d_\xi+1,\bar{\tau}_{a_0}-1]}$, $\forall\kappa_1\in\mathbb{N}_{[0,t_1-1]}$, $\forall\kappa_2\in\mathbb{N}_{[0,d_\xi-1]}$, $\forall\kappa_3\in\mathbb{N}_{[d_\xi,t_2-1]}$, $\forall\beta_1\in\mathbb{N}_{[\underline{\tau}_{a_0}-1,d_\xi-1]}$, $\forall\beta_2\in\mathbb{N}_{[d_\xi,\bar{\tau}_{a_0}-1]}$, $\forall\beta_3\in\mathbb{N}_{[0,\underline{\tau}_{a_0}-2]}$, $\forall l\in\mathbb{N}_{[1,2]}$, *the following hold:*

$$\begin{bmatrix} U_{a_0}^{(t_l,\kappa_l+1)}-sym(X_{a_{d_\xi-\kappa_l-1}}) & \tilde{\mathcal{A}}_{d_\xi-\kappa_1}\\ * & -U_{a_0}^{(t_l,\kappa_l)}\end{bmatrix}\prec\mathbf{0}$$

$$\begin{bmatrix} U_{a_0}^{(t_2,\kappa_3+1)}-sym(X_{a_0}) & \tilde{\mathcal{A}}_0\\ * & -U_{a_0}^{(t_2,\kappa_3)}\end{bmatrix}\prec\mathbf{0}$$

$$U_{a_0}^{(t_l,0)}-\phi_{a_0}U_{a_0}^{(0,0)}\prec\mathbf{0}$$

$$\begin{bmatrix} -\bar{\mathcal{U}}_{a_0}^{(\beta_1)} & * & *\\ (I_M\otimes\tilde{\mathcal{A}}_{d_\xi-\beta_1})\Xi_{a_0}(\beta_1+1) & \bar{V} & *\\ \tilde{\mathcal{A}}_{d_\xi-\beta_1} & 0 & \bar{\mathcal{V}}_{a_0}^{(\beta_1+1)}\end{bmatrix}\prec\mathbf{0}$$

$$\begin{bmatrix} -\bar{\mathcal{U}}_{a_0}^{(\beta_2)} & * & *\\ (I_M\otimes\tilde{\mathcal{A}}_0)\Xi_{a_0}(\beta_2+1) & \bar{V} & *\\ \tilde{\mathcal{A}}_0\mathcal{I}_{a_0}(\beta_2+1) & 0 & \bar{\mathcal{V}}_{a_0}^{(\beta_2+1)}\end{bmatrix}\prec\mathbf{0}$$

$$\tilde{\mathcal{U}}_{a_0}^{(0)}-U_{a_0}^{(0,0)}\prec\mathbf{0}$$

$$\begin{bmatrix} \tilde{\mathcal{U}}_{a_0}^{(\beta_3+1)}-sym(X_{a_{d_\xi-\beta_3-1}}) & \tilde{\mathcal{A}}_{d_\xi-\beta_3}\\ * & -\tilde{\mathcal{U}}_{a_0}^{(\beta_3)}\end{bmatrix}\prec\mathbf{0}$$

where $\bar{\mathcal{V}}_{a_0}^{(\beta)}\triangleq\bar{\mathcal{U}}_{a_0}^{(\beta)}-sym(X_{a_{d_\xi-\beta}})$ *if* $\beta\in\mathbb{N}_{[0,d_\xi]}$ *and* $\bar{\mathcal{V}}_{a_0}^{(\beta)}\triangleq\bar{\mathcal{U}}_{a_0}^{(\beta)}-sym(X_{a_0})$ *if* $\beta\in\mathbb{N}_{[d_\xi+1,\bar{\tau}_{a_0}]}$. *Besides,* $\tilde{\mathcal{A}}_i$, $\bar{\mathcal{U}}_{a_0}^{(\beta)}$, $\bar{V}$, $\tilde{\mathcal{U}}_{a_0}^{(\beta)}$ *are defined in Theorem 3, and* $\Xi_{a_0}(\beta)$, $\mathcal{I}_{a_0}(\beta)$ *are defined in Theorem 2. Then, the admissible gains of controllers are given by* $K_a\triangleq Y_aX_a^{-1}$, $a\in\mathbb{M}$.

Remark 3 In the above obtained results, the mode-dependent controllers are employed to stabilize the closed-loop system. Note that the asynchronous situation will vanish if replacing mode-dependent controllers by mode-independent ones. By setting $X_a=X$, $Y_a=Y$, the gains of mode-independent controllers can be given by $K\triangleq YX^{-1}$, and one can refer to [22] for the corresponding conditions on the existence of the controllers.

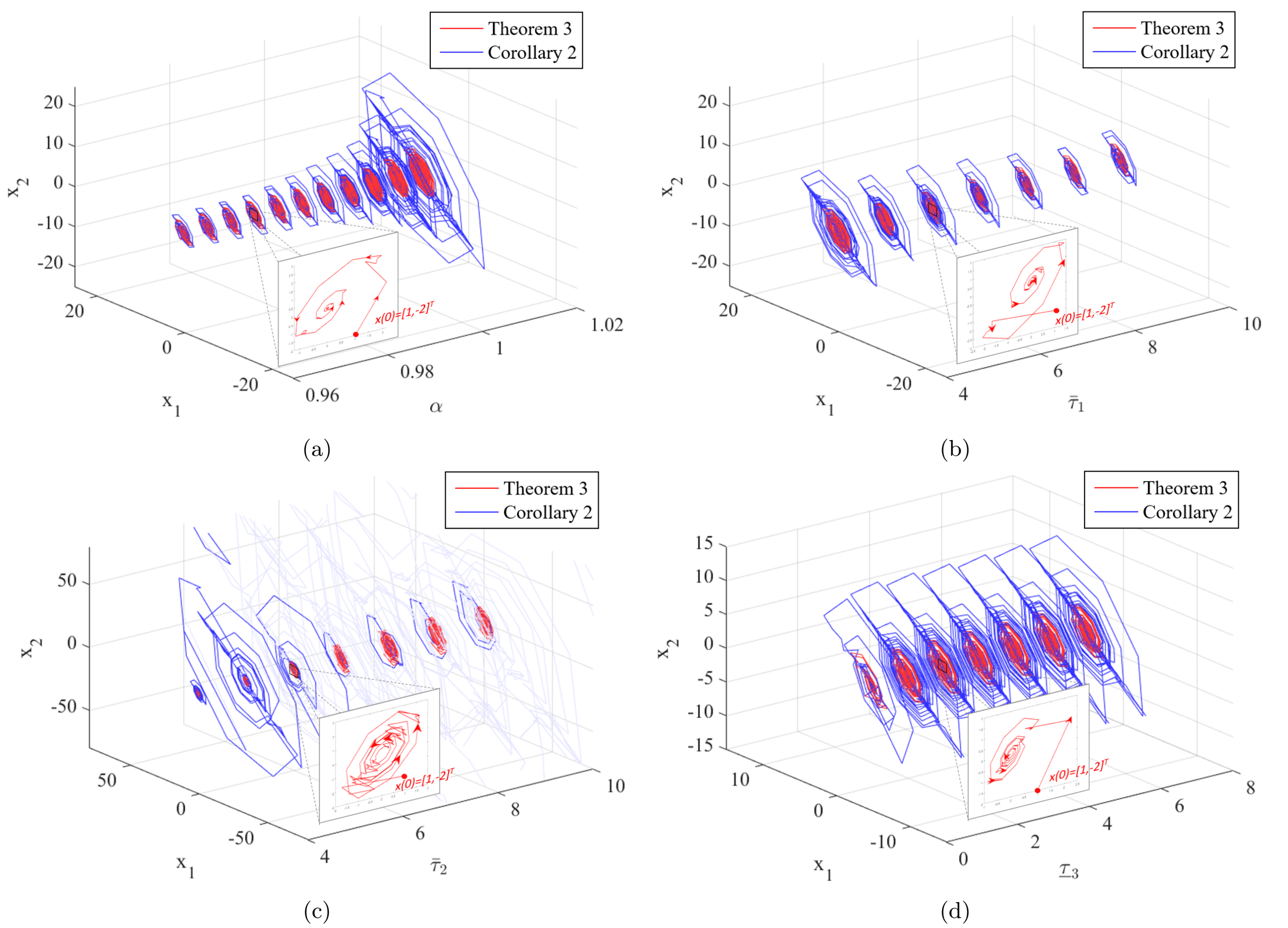


Fig. 2. State responses of the system by two classes of controllers designed based on Theorem 3 and Corollary 2, respectively. The diverging state responses by Corollary 2 are shown as gray-blue curves in (c).

4 Illustrative Examples

Example 1 *(Numerical Example)* Consider an unforced S-MJLS in the form of (3) with three modes:

$$\begin{cases} A_1 = \alpha \begin{bmatrix} 1.14 & -0.52 \\ 0.52 & -0.94 \end{bmatrix}, & B_1 = \begin{bmatrix} 0.1 \\ -5.0 \end{bmatrix} \\ A_2 = \alpha \begin{bmatrix} 0.66 & 0.83 \\ 1.04 & 0.42 \end{bmatrix}, & B_2 = \begin{bmatrix} 1.4 \\ 0.5 \end{bmatrix} \\ A_3 = \alpha \begin{bmatrix} -0.33 & 0.51 \\ 0.11 & 0.92 \end{bmatrix}, & B_3 = \begin{bmatrix} -1.5 \\ 0.1 \end{bmatrix} \end{cases}$$

where $\alpha > 0$ is a given constant, and the switching among the three modes is governed by a semi-Markov chain, where the semi-Markov kernel is given as

$$\theta_{12}(\tau) = 0.7 \cdot \frac{0.7^{\tau-1} \cdot 0.3^{11-\tau} \cdot 10!}{(11-\tau)!(\tau-1)!}$$

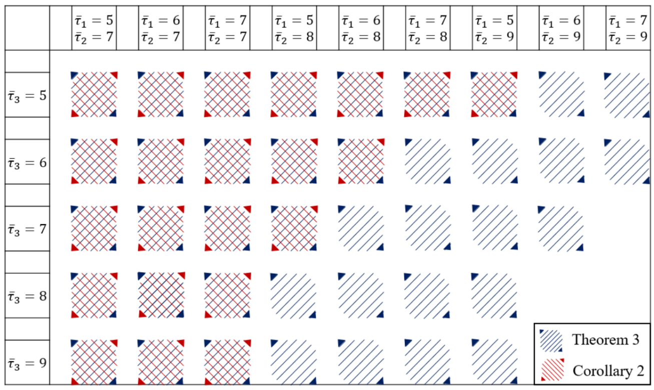


Fig. 3. The regions corresponding to the case whether feasible solutions of the controller exist by Theorem 3 and Corollary 2, respectively, for different ϕ_1, ϕ_2 and ϕ_3.

$$\theta_{13}(\tau) = 0.3 \cdot \frac{0.2^{\tau-1} \cdot 0.8^{11-\tau} \cdot 10!}{(11-\tau)!(\tau-1)!}$$

$$\theta_{21}(\tau) = 0.8 \cdot (0.7^{(\tau-1)^{1.6}} - 0.7^{\tau^{1.6}})$$

$$\theta_{23}(\tau) = 0.2 \cdot \frac{0.5^{10} \cdot 10!}{(12-\tau)!(\tau-2)!}$$

$$\theta_{31}(\tau) = 0.6 \cdot 0.4^{\tau}, \ \theta_{32}(\tau) = 0.4 \cdot (0.8^{(\tau-1)^2} - 0.8^{\tau^2})$$
$$\theta_{11}(\tau) = \theta_{22}(\tau) = \theta_{33}(\tau) = 0.$$

The purpose of the example is to show the necessity of considering the modal lag and designing AMA controllers. Here, the modal lags are set as $d_1 = 4$, $d_2 = 4$, $d_3 = 1$.

First, applying two classes of controllers obtained by Theorem 3 and Corollary 2 with $d_{\underline{\xi}} \triangleq \min\{d_1, d_2, d_3\}$, respectively, the state responses of the closed-loop system for different α, $\bar{\tau}_1$, $\bar{\tau}_2$, $\underline{\tau}_3$ are shown in Fig. 2. It can be seen that the controllers designed by Theorem 3 are effective despite modal asynchronous switching. However, the state responses of the closed-loop systems by the controllers designed using Corollary 2 are either diverging or with large overshoots, which means that considering $d_{\underline{\xi}}$ instead of modal lags is not qualified to find a stabilizing controller for S-MJLS in the case of modal asynchrony.

Then, turning to the case of taking the maximum of modal lags into consideration, we would check whether it is suitable for stabilizing controller design. Given different ϕ_1, ϕ_2 and ϕ_3, the controller can be obtained by employing Theorem 3 and Corollary 2 with $d_{\bar{\xi}} \triangleq \max\{d_1, d_2, d_3\}$, respectively, and Fig. 3 shows the regions corresponding to the case whether a feasible solution of the controller exists. It can be clearly seen that the region determined by Theorem 3 (considering the modal lag) completely covers that by Corollary 2 (considering the common lag) as a subset, i.e., the latter is more conservative.

In sum, one can conclude that the traditional methodology established on the simplification with mode-independent lag is either incompetent or conservative in handling the underlying systems with modal lag, showing the necessity of considering the modal asynchrony and significance of the AMA controller.

Example 2 *(Illustrative Example of a Class of VTOL Helicopter Models)* In this example, our aim is to apply the AMA controller to a VTOL helicopter model as illustrated in Fig. 4, and demonstrate the validity of the developed theoretical results against the modal asynchrony for S-MJLS with the bi-boundary sojourn time. As shown in [10] and [20], the discrete-time linearized model of the VTOL helicopter can be described as

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k)$$

where $r(k)$ stands for a semi-Markov chain with three subsystems corresponding to the airspeeds of 135(nominal value), 60 and 170 knots, respectively.

The state variables $x(k) = [x_1(k), x_2(k), x_3(k), x_4(k)]^T$ are taken as the horizontal velocity $x_1(k)$, the vertical velocity $x_2(k)$, the pitch rate $x_3(k)$, the pitch angle $x_4(k)$, and the controller inputs $u(k) = [u_1(k), u_2(k)]^T$ are taken as the collective pitch control $u_1(k)$ and the longitudinal cyclic pitch control $u_2(k)$. Given the sampling time $\mathbb{T}$, the system matrices are expressed as

Table 1
Parameters in different modes.

Mode	I (135 knots)	II (60 knots)	III (170 knots)
$a_{32}(r(k))$	0.3681	0.0664	0.5407
$a_{34}(r(k))$	1.4200	0.1198	2.5460
$b_{21}(r(k))$	3.5446	0.9775	5.1120

$$A_{r(k)} = \begin{bmatrix} 1-0.037\mathbb{T} & 0.0271\mathbb{T} & 0.0188\mathbb{T} & -0.4555\mathbb{T} \\ 0.0482\mathbb{T} & 1-1.01\mathbb{T} & 0.0024\mathbb{T} & -4.0208\mathbb{T} \\ 0.1002\mathbb{T} & a_{32}(r(k))\mathbb{T} & 1-0.7\mathbb{T} & a_{34}(r(k))\mathbb{T} \\ 0 & 0 & \mathbb{T} & 1 \end{bmatrix}$$

$$B_{r(k)} = \begin{bmatrix} 0.4422\mathbb{T} & 0.1761\mathbb{T} \\ b_{21}(r(k))\mathbb{T} & -7.5922\mathbb{T} \\ -5.52\mathbb{T} & 4.49\mathbb{T} \\ 0 & 0 \end{bmatrix}$$

where the values of parameters $a_{32}(r(k))$, $a_{34}(r(k))$ and $b_{21}(r(k))$ are shown in Table 1. The semi-Markov kernel is designed as

$$\theta_{12}(\tau) = 0.74 \cdot \frac{0.6^{\tau} \cdot 0.4^{8-\tau} \cdot 8!}{(8-\tau)!\tau!}$$
$$\theta_{13}(\tau) = 0.26 \cdot \frac{0.4^{\tau} \cdot 0.6^{8-\tau} \cdot 8!}{(8-\tau)!\tau!}$$
$$\theta_{21}(\tau) = 0.9^{(\tau-1)^2} - 0.9^{\tau^2}$$
$$\theta_{31}(\tau) = 0.4^{(\tau-1)^{1.3}} - 0.4^{\tau^{1.3}}$$
$$\theta_{11}(\tau) = \theta_{22}(\tau) = \theta_{23}(\tau) = \theta_{32}(\tau) = \theta_{33}(\tau) = 0.$$

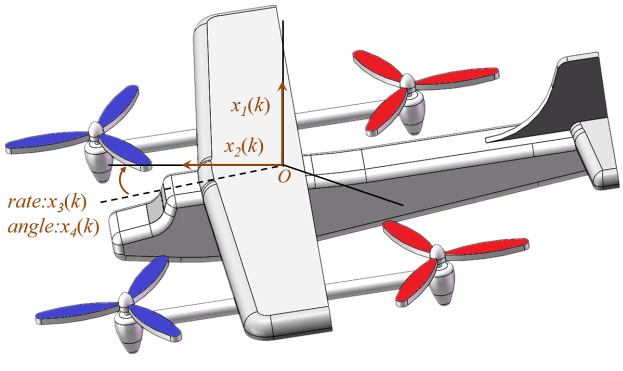
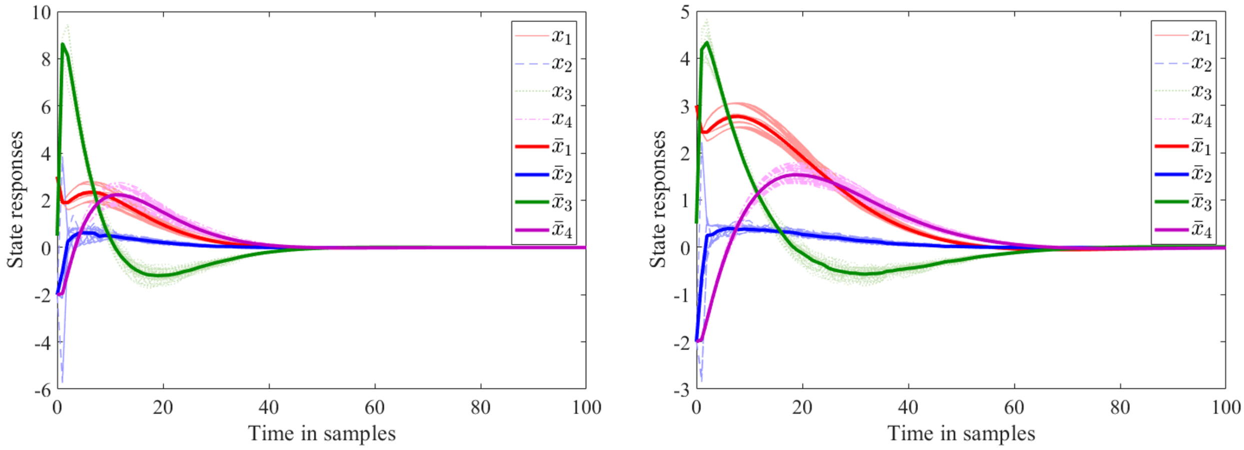


Fig. 4. VTOL helicopter model.

Here, we set $\mathbb{T} = 0.1$, $\bar{\tau}_1 = 6$, $\bar{\tau}_2 = 5$, $\bar{\tau}_3 = 7$, $d_1 = 4$, $d_2 = 4$, $d_3 = 2$. By Theorem 1, it can be checked that the open-loop system is not mean-square stable. We consider the modal asynchrony and design the AMA controller based on Theorem 3 (for brevity, the controller gains are omitted). For different lower bounds of sojourn time, 50 realizations of the state responses of the closed-loop systems are shown in Fig. 5. As seen from Fig. 5, the designed AMA controller is effective against the stochastic variations of the airspeeds, despite the existence of the



(a) (b)

Fig. 5. State responses of the closed-loop system for different lower bounds of sojourn time. (a) $\underline{\tau}_1 = 1$, $\underline{\tau}_2 = 1$, $\underline{\tau}_3 = 1$. (b) $\underline{\tau}_1 = 3$, $\underline{\tau}_2 = 2$, $\underline{\tau}_3 = 3$.

modal asynchronous phenomena and the bi-boundary sojourn time.

5 Conclusions

In this paper, the issues of control synthesis for a class of discrete-time S-MJLSs are investigated, in which the sojourn time is with upper and lower bounds. Besides, a practical and general phenomenon of the modal asynchrony is considered. By employing the semi-Markov kernel approach, numerically testable stability criteria are obtained, based on which the conditions on the existence of AMA controllers capable of overcoming the modal asynchrony are derived. Compared with the traditional studies based on the assumption of mode-independent lag, the derived results are of less conservatism. Two illustrative examples are provided to demonstrate the necessity of considering the modal lag and the validity of the AMA controllers.

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