

Semi-Parametric Probability-Weighted Moments Estimation Revisited*

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Abstract

In this paper, for heavy-tailed models and through the use of probability weighted moments based on the largest observations, we deal essentially with the semi-parametric estimation of the *Value-at-Risk* at a level p , the size of the loss occurred with a small probability p , as well as the dual problem of estimation of the *probability of exceedance* of a high level x . These estimation procedures depend crucially on the estimation of the *extreme value index*, the primary parameter in *Statistics of Extremes*, also done on the basis of the same weighted moments. Under regular variation conditions on the right-tail of the underlying distribution function F , we prove the consistency and asymptotic normality of the estimators under consideration in this paper, through the usual link of their asymptotic behaviour to the one of the extreme value index estimator they are based on. The performance of these estimators, for finite samples, is illustrated through Monte-Carlo simulations. An adaptive choice of thresholds is put forward. Applications to a real data set in the field of insurance as well as to simulated data are also provided.

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1 Introduction, preliminaries and scope of the article

Let X_1, X_2, \dots, X_n be a set of n independent and identically distributed (i.i.d.), or even possibly weakly dependent and stationary, random variables (r.v.'s), from a population with distribution function (d.f.) F . Let us arrange them in ascending order, to get the order statistics (o.s.'s) $X_{1:n} \leq \dots \leq X_{n:n}$. Suppose that we are interested in the estimation of a *high quantile* of probability $1 - p$, or equivalently, in the estimation of the *Value-at-Risk* (VaR) at a level p , the size of the loss occurred with a small probability p , given by

$$\text{VaR}_p \equiv \chi_{1-p} := F^{\leftarrow}(1 - p) = \inf\{x : F(x) \geq 1 - p\}, \quad (1.1)$$

with the notation F^{\leftarrow} standing thus for the generalized inverse function of F . Moreover, we are also interested in the estimation of the *probability of exceedance* of a high level $x = x_n$,

$$p = p_x := 1 - F(x) =: \bar{F}(x). \quad (1.2)$$

Extreme Value Theory (EVT) provides a great variety of results that enable us to deal with alternative approaches in the statistical analysis of extreme events. Those approaches are essentially based on the well-established limiting results described in the following.

1.1 Main limiting results in EVT

The main limiting result in EVT can be attributed to Gnedenko (1943), who fully characterized the possible non-degenerate limiting distribution of the linearly normalised maximum, $(X_{n:n} - b_n)/a_n$, $a_n > 0$, $b_n \in \mathbb{R}$. Such a limit is of the type of the general *extreme value distribution* (EVD),

$$EV_\gamma(x) := \begin{cases} \exp(-(1 + \gamma x)^{-1/\gamma}), & 1 + \gamma x > 0 & \text{if } \gamma \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R} & \text{if } \gamma = 0. \end{cases} \quad (1.3)$$

When such a non-degenerate limit exists, we say that F belongs to the max-domain of attraction of EV_γ and denote this by $F \in \mathcal{D}_M(EV_\gamma)$. The shape parameter γ is related with the heaviness of the right-tail $\bar{F} = 1 - F$ and it is often called the *extreme value index* (EVI).

Another seminal result in the field of EVT is due to Balkema and de Haan (1974) and Pickands (1975). If we properly scale the excesses over a high threshold u , the limit distribution of those scaled excesses is the *Generalized Pareto distribution* (GPD), strongly related with the d.f. $EV_\gamma(x)$, in (1.3), and defined by,

$$GP_\gamma(x) := 1 + \ln EV_\gamma(x) = \begin{cases} 1 - (1 + \gamma x)^{-1/\gamma}, & 1 + \gamma x > 0, & x > 0 & \text{if } \gamma \neq 0 \\ 1 - \exp(-x), & x > 0 & & \text{if } \gamma = 0 \end{cases} \quad (1.4)$$

(see, for instance, Embrechts *et al.*, 1997, Section 3.4, and Reiss and Thomas, 2007, Section 1.4, for more details).

1.2 Most relevant approaches in the field of Statistics of Univariate Extremes

We shall briefly refer the three most important approaches in the area of *Statistics of Univariate Extremes*: the *block maxima* (BM) method, the *peaks-over-threshold* (POT) or even the *peaks-over-random-threshold* (PORT) methods and the *largest observations* (LOB) method. For a more detailed review, with extensive associated references, see Gomes *et al.* (2008) and Beirlant *et al.* (2012).

- The first method, the BM method, is of a parametric nature: we work with a sample of maxima of adequate blocks of observations, and estimate the parameters $(\lambda, \delta, \gamma)$ of the EVD, $EV_\gamma((x - \lambda)/\delta)$, $\lambda \in \mathbb{R}$, $\delta > 0$, $\gamma \in \mathbb{R}$, with $EV_\gamma(x)$ given in (1.3). This method is known to be possibly inefficient, due to the fact that the loss of information in each block can be catastrophic.
- In the second approach, the POT method, inference is performed through the use of the sample of excesses over a high deterministic threshold u . The limiting d.f. of these excesses is, up to a scale factor, the distribution $GP_\gamma(x)$, in (1.4), and the method can be of a parametric or a semi-parametric nature. Note that the high threshold can also be a random value, leading to the PORT methodology, a terminology recently introduced in Araújo Santos *et al.* (2006).
- The third approach, the LOB method, is the one we shall consider in this paper. It uses the *largest k observations* to make inference about the right tail $\bar{F} = 1 - F$, assuming only that F belongs to a wide sub-domain of $\mathcal{D}_{\mathcal{M}}(EV_\gamma)$.

1.3 Estimators under study

Under the largest observations framework, and whenever dealing with heavy-tailed models, the classical semi-parametric EVI and VaR-estimators are the Hill (Hill, 1975) and Weissman-Hill's estimators (Weissman, 1978), with functional expressions

$$\hat{\gamma}_{k,n}^H := \frac{1}{k} \sum_{i=1}^k (\ln X_{n-i+1:n} - \ln X_{n-k:n}) \quad (1.5)$$

and

$$\hat{Q}_{k,n}^H(p) := X_{n-k:n} c_k^{\hat{\gamma}_{k,n}^H}, \quad c_k \equiv c_k(p) := \frac{k}{np}, \quad k = 1, 2, \dots, n-1, \quad (1.6)$$

respectively, which are pseudo-maximum likelihood estimators, consistent in the whole $\mathcal{D}_{\mathcal{M}}^+ := \mathcal{D}_{\mathcal{M}}(EV_\gamma)_{\gamma>0}$, provided that k is intermediate, i.e. if

$$k = k_n \rightarrow \infty \quad \text{and} \quad k/n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (1.7)$$

In a way dual to (1.6), and given a high level $x = x_n$, the probability $p = p_x$ of exceedance of such a level can be estimated by

$$\hat{p}_{k,n}^H(x) := \left(\frac{k}{n}\right) \tilde{C}_k^{-1/\hat{\gamma}_{k,n}^H}, \quad \tilde{C}_k \equiv \tilde{C}_k(x) := \frac{x}{X_{n-k:n}}, \quad k = 1, 2, \dots, n-1. \quad (1.8)$$

Under further adequate restrictions on k , we can guarantee the asymptotic normality of the estimators $\hat{\gamma}_{k,n}^H$, $\hat{Q}_{k,n}^H(p)$ and $\hat{p}_{k,n}^H(x)$, in (1.5), (1.6) and (1.8), respectively. But most of the times, these estimators exhibit a large variance for small k , a strong bias for moderate k , sample paths with very short stability regions around the target value and a very peaked mean square error (MSE) structure, as a function of k . This has led researchers to the search of alternative estimators, with a smaller MSE.

Since heavy-tailed models only have mean value if $\gamma < 1$, methods based on sample moments have been rarely considered when we work with such a type of distributions. But in many practical fields like in finance or insurance, for example, we usually have a positive EVI smaller than one, and even smaller than $1/2$. In this article, and for the estimation of the above mentioned parameters of extreme events, we now revisit the use of a *probability weighted moments* (PWM) method based on the largest observations, developed in Caeiro and Gomes (2011) for the EVI.

The PWM method is a generalization of the *method of moments*. It also consists in equating sample moments with their corresponding theoretical moments, and then solving those equations in order to obtain estimates of the different parameters under play. The PWM of a r.v. X are defined by

$$M_{p,r,s} := E(X^p(F(X))^r(1-F(X))^s),$$

where p , r and s are any real numbers (Greenwood *et al.*, 1979). When $r = s = 0$, $M_{p,0,0}$ are the usual noncentral moments of order p . Hosking *et al.* (1985) advise the use of $M_{1,r,s}$, because then the relations between parameters and moments have usually a much simpler form. Also, when r and s are integers, $F^r(1-F)^s$ can be written as a linear combination of powers of F or $1-F$. So it is usual to work with the particular case,

$$a_r := M_{1,0,r} = E(X(1-F(X))^r), \quad r \geq 0,$$

and the associated estimator,

$$\hat{a}_r = \frac{1}{n} \sum_{i=1}^{n-r} \frac{(n-1-r)!(n-i)!}{(n-1)!(n-i-r)!} X_{i:n} = \frac{1}{n} \sum_{i=1}^n \frac{(n-i)(n-i-1)\dots(n-i-r+1)}{(n-1)(n-2)\dots(n-r)} X_{i:n}. \quad (1.9)$$

For $\gamma < 1$ and for d.f.'s like the EVD, $EV_\gamma((x-\lambda)/\delta)$, with $EV_\gamma(x)$ given in (1.3), the *Pareto* d.f.,

$$P_\gamma(x; \delta) = 1 - (x/\delta)^{-1/\gamma}, \quad x > \delta, \quad (1.10)$$

and the GPD, $GP_\gamma(x/\delta)$, with $GP_\gamma(x)$ defined in (1.4), the PWM have simple expressions, which allow a straightforward estimation of the EVI, γ . For the EVD, see Hosking *et al.* (1985) and the improved versions in Diebolt *et al.* (2007, 2008). As an example, the Pareto PWM (PPWM) and the generalized Pareto PWM (GPPWM) estimators of γ are valid for $\gamma < 1$, and given by

$$\hat{\gamma}^{PPWM} = 1 - \left(\frac{\hat{a}_1}{\hat{a}_0 - \hat{a}_1} \right) \quad \text{and} \quad \hat{\gamma}^{GPPWM} = 1 - \frac{2\hat{a}_1}{\hat{a}_0 - 2\hat{a}_1}, \quad (1.11)$$

respectively, where \hat{a}_0 and \hat{a}_1 are given in (1.9). The estimator $\hat{\gamma}^{GPPWM}$, in (1.11), was introduced and studied in Hosking and Wallis (1987).

We shall consider in this paper, the PPWM estimators of VaR_p and p_x , the parameters respectively defined in (1.1) and (1.2), associated with the PPWM EVI-estimators studied in Caeiro and Gomes (2011). Those estimators are semi-parametric in nature and, for comparison with the equivalent estimators based on the Hill EVI-estimator, in (1.5), are based on the top $k + 1$ largest o.s.'s, $X_{n-k:n} \leq X_{n-k+1:n} \leq \dots \leq X_{n:n}$. Under such a framework, the estimators \hat{a}_0 and \hat{a}_1 , in (1.9), should be replaced by,

$$\hat{a}_0(k) := \frac{1}{k+1} \sum_{i=1}^{k+1} X_{n-i+1:n} \quad \text{and} \quad \hat{a}_1(k) := \frac{1}{k+1} \sum_{i=1}^{k+1} \frac{i}{k+1} X_{n-i+1:n},$$

respectively. The PPWM EVI, VaR and p -estimators, based on the largest values are

$$\hat{\gamma}_{k,n}^{PPWM} := 1 - \frac{\hat{a}_1(k)}{\hat{a}_0(k) - \hat{a}_1(k)}, \quad (1.12)$$

$$\hat{Q}_{k,n}^{PPWM}(p) := \frac{\hat{a}_0(k) \hat{a}_1(k)}{\hat{a}_0(k) - \hat{a}_1(k)} \left(\frac{k}{np} \right)^{\hat{\gamma}_{k,n}^{PPWM}} \quad (1.13)$$

and

$$\hat{p}_{k,n}^{PPWM}(x) := \left(\frac{k}{n} \right) \left(\frac{x(\hat{a}_0(k) - \hat{a}_1(k))}{\hat{a}_0(k)\hat{a}_1(k)} \right)^{-1/\hat{\gamma}_{k,n}^{PPWM}}, \quad (1.14)$$

respectively, with $k = 1, 2, \dots, n - 1$, and are consistent whenever $\gamma < 1$.

De Haan and Ferreira (2006) considered, also for $\gamma < 1$, the semi-parametric GPPWM EVI-estimator, based on the sample of excesses over the high random level $X_{n-k:n}$, i.e.,

$$\hat{\gamma}_{k,n}^{GPPWM} := 1 - \frac{2\hat{a}_1^*(k)}{\hat{a}_0^*(k) - 2\hat{a}_1^*(k)}, \quad (1.15)$$

with $k = 1, 2, \dots, n - 1$, and $\hat{a}_s^*(k) := \sum_{i=1}^k (i/k)^s (X_{n-i+1:n} - X_{n-k:n})/k$, $s = 0, 1$. For a finite-sample comparison between the PPWM EVI-estimators in (1.12) and the GPPWM EVI-estimators in (1.15), see Caeiro and Gomes (2011).

1.4 Scope of the article

In Section 2, after reviewing a few results already available in the literature, we state a lemma and a theorem related with the asymptotic properties of the PPWM-estimators, defined in (1.13) and (1.14), of the above mentioned parameters of extreme events, the Value-at-Risk at the level p and the probability p_x of exceedance of a high level x , defined in (1.1) and (1.2), respectively. The performance of these estimators, for finite samples, is illustrated, in Section 3, through a Monte-Carlo simulation study. In Section 4, we put forward an adaptive choice of thresholds, again on the basis of bootstrap computer-intensive methods. Applications to a real data set in the field of insurance as well as to a simulated data set are provided in Section 5.

2 Asymptotic Behaviour of the Estimators

2.1 Most common first and second-order frameworks for heavy tails

For heavy-tailed models, i.e., models with a positive EVI, we assume that F has a Pareto-type right-tail, i.e., with the notation $a(t) \sim b(t)$ if and only if $a(t)/b(t) \rightarrow 1$, as $t \rightarrow \infty$,

$$\bar{F}(x) = 1 - F(x) \sim (x/C)^{-1/\gamma} \iff U(t) \sim C t^\gamma, \quad C > 0, \gamma > 0, \quad (2.1)$$

where C is a scale parameter and $U(t) := F^{\leftarrow}(1 - 1/t)$, $t > 1$. Note that models with the Pareto-type right-tail, in (2.1), have a regularly varying (RV) right-tail with a negative index of regular variation equal to $-1/\gamma$ (with the notation $\bar{F} \in RV_{-1/\gamma}$), and belong to the max-domain of attraction $\mathcal{D}_{\mathcal{M}}^+$. Indeed, more specifically and for all $x > 0$, we have,

$$F \in \mathcal{D}_{\mathcal{M}}^+ \iff \lim_{t \rightarrow \infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-1/\gamma} (\bar{F} \in RV_{-1/\gamma}) \iff \lim_{t \rightarrow \infty} \frac{U(tx)}{U(t)} = x^\gamma (U \in RV_\gamma), \quad (2.2)$$

i.e. $\bar{F}(x) = x^{-1/\gamma} L_{\bar{F}}(x)$ and $U(t) = x^\gamma L_U(t)$, where $L_{\bar{F}}$ and L_U are both slowly varying functions (i.e. they both belong to RV_0), not necessarily converging to constants, as happens if condition (2.1) holds.

To guarantee the consistency of many semi-parametric estimators, we usually need to assume that k is intermediate, i.e., that k is a sequence of integers in $[1, n[$, such that (1.7) holds. To obtain information on the non-degenerate distributional behaviour of semi-parametric estimators of parameters of extreme events, we often assume a second-order condition, like

$$\lim_{t \rightarrow \infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho} \iff \lim_{t \rightarrow \infty} \frac{\frac{U(tx)}{U(t)} - x^\gamma}{A(t)} = x^\gamma \frac{x^\rho - 1}{\rho}, \quad (2.3)$$

valid for all $x > 0$, where $\rho \leq 0$ is a second-order parameter controlling the speed of convergence of $U(tx)/U(t)$ to x^γ , as $t \rightarrow \infty$. We then say that U is of second-order regular variation, with parameters γ and ρ , and denote such a fact by $U \in 2RV_\gamma^\rho$. If the limits in (2.3) exist, they are necessarily of the above mentioned types and $|A| \in RV_\rho$ (Geluk and de Haan, 1987; see also de Haan and Ferreira, 2006). Moreover, we have

$$U \in 2RV_\gamma^\rho \iff \bar{F} \in 2RV_{-1/\gamma}^{\tilde{\rho}}, \quad \text{with } \tilde{\rho} = \rho/\gamma$$

and a rate function \tilde{A} related with A , in (2.3), through the relation $\tilde{A}(t) = A(1/\bar{F}(t))/\gamma^2$. The validity of condition (2.3), with $\rho < 0$, is equivalent to condition (2.1).

2.2 Auxiliary results on intermediate order statistics

In the sequel, let us denote (Y_1, \dots, Y_n) a random sample of size n from a strict Pareto model, with d.f. $P_\gamma(x; 1)$, $P_\gamma(x; \delta)$ given in (1.10). Let $(Y_{1:n}, \dots, Y_{n:n})$ denote the sample of associated ascending o.s.'s. We first state without proof the following results on the asymptotic behaviour of intermediate o.s.'s (see de Haan and Ferreira, 2006, for details).

Lemma 2.1. *If (1.7) holds, then for the intermediate Pareto o.s. $Y_{n-k:n}$, $k Y_{n-k:n}/n \xrightarrow{P} 1$ and consequently, $Y_{n-k:n} \xrightarrow{P} \infty$, as $n \rightarrow \infty$. Moreover,*

$$B_{k,n} := \sqrt{k}(k Y_{n-k:n}/n - 1) \xrightarrow{d} \mathcal{B}, \quad \text{a standard normal r.v.}, \quad (2.4)$$

and $\text{Cov}(B_{r,n}, B_{s,n}) = \sqrt{rs} (1 - s/n)/(s - 1)$, $r < s$.

Lemma 2.2. *Under the first-order framework in (2.2), let us assume that k is intermediate, i.e., (1.7) holds. Then, for the intermediate o.s. $X_{n-k:n} (\stackrel{d}{=} U(Y_{n-k:n}))$, $X_{n-k:n}/U(n/k) \xrightarrow{P} 1$, and consequently, $X_{n-k:n} \xrightarrow{P} \infty$, as $n \rightarrow \infty$. If we further assume the validity of the second-order framework in (2.3), then, as $n \rightarrow \infty$, the asymptotic distributional representation*

$$X_{n-k:n}/U(n/k) \stackrel{d}{=} 1 + \gamma B_{k,n}/\sqrt{k} + o_p(A(n/k))$$

holds, where $B_{k,n}$ is the asymptotically standard normal sequence of r.v.'s in (2.4).

2.3 Asymptotic behaviour of the EVI-estimators under play

Regarding the Hill estimator, $\hat{\gamma}_{k,n}^H$, in (1.5):

Proposition 2.1 (de Haan and Peng, 1998, Theorem 1). *Under the second-order framework in (2.3), and for intermediate k , i.e. if (1.7) holds, the asymptotic distributional representation*

$$\hat{\gamma}_{k,n}^H \stackrel{d}{=} \gamma + \frac{\sigma_H Z_k^H}{\sqrt{k}} + b_H A(n/k)(1 + o_p(1)), \quad \sigma_H = \gamma, \quad b_H = \frac{1}{1 - \rho},$$

holds, with Z_k^H asymptotically standard normal. Consequently, if we choose k such that $\sqrt{k} A(n/k) \rightarrow \lambda_A$, finite and not necessarily null,

$$\sqrt{k}(\hat{\gamma}_{k,n}^H - \gamma) \xrightarrow[n \rightarrow \infty]{d} N\left(\frac{\lambda_A}{1 - \rho}, \gamma^2\right).$$

We next refer the following results related with the asymptotic behaviour of the PPWM EVI-estimator, $\hat{\gamma}_{k,n}^{PPWM}$, in (1.12):

Proposition 2.2 (Caeiro and Gomes, 2011, Proposition 2.4). *Under the conditions of Proposition 2.1, and with Z_k^{PPWM} asymptotically standard normal, the asymptotic distributional representation*

$$\hat{\gamma}_{k,n}^{PPWM} \stackrel{d}{=} \gamma + \frac{\sigma_{PPWM} Z_k^{PPWM}}{\sqrt{k}} + b_{PPWM} A(n/k)(1 + o_p(1))$$

holds for any $\gamma < 1/2$, as $n \rightarrow \infty$, where

$$\sigma_{PPWM}^2 := \frac{\gamma^2(1 - \gamma)(2 - \gamma)^2}{(1 - 2\gamma)(3 - 2\gamma)} \quad \text{and} \quad b_{PPWM} := \frac{(1 - \gamma)(2 - \gamma)}{(1 - \gamma - \rho)(2 - \gamma - \rho)}. \quad (2.5)$$

Consequently, if we choose k such that $\sqrt{k} A(n/k) \rightarrow \lambda_A$, finite and not necessarily null,

$$\sqrt{k}(\hat{\gamma}_{k,n}^{PPWM} - \gamma) \xrightarrow[n \rightarrow \infty]{d} N(\lambda_A b_{PPWM}, \sigma_{PPWM}^2).$$

Remark 2.1. *It is obvious that $\sigma_H^2 := \gamma^2 < \sigma_{PPWM}^2$, for every $0 < \gamma < 1/2$. On the other hand, $b_{PPWM} < b_H = 1/(1 - \rho)$, unless $\rho = 0$. An asymptotic comparison of H and $PPWM$ EVI-estimators at optimal levels can be seen in Caeiro and Gomes (2011).*

2.4 Main asymptotic results

If we combine the results of Lemma 2.1 with the fact that $X_{n-k:n} \stackrel{d}{=} U(Y_{n-k:n})$ and the results in Drees (1998), we first state, without the need of a proof, the following lemma:

Lemma 2.3. *For intermediate k , i.e., whenever (1.7) holds, and under the second-order framework in (2.3), but with $\rho < 0$, (2.1) holds and, if $p = p_n$ is a sequence of probabilities such that*

$$c_k \equiv c_k(p) = k/(np) \rightarrow \xi \in (0, \infty], \quad \text{as } n \rightarrow \infty, \quad (2.6)$$

then, with

$$C_k \equiv C_k(p) := (pY_{n-k:n})^{-1} = \frac{nc_k}{kY_{n-k:n}},$$

and $\text{VaR}_p = U(1/p)$,

$$\frac{\frac{\text{VaR}_p}{X_{n-k:n}}/C_k^\gamma - 1}{A(n/k)} \xrightarrow{p} \frac{\xi^\rho - 1}{\rho} =: h_\rho(\xi), \quad (2.7)$$

as $n \rightarrow \infty$. By continuity arguments, $h_\rho(\xi) = -1/\rho$, if $\xi = \infty$. If we furthermore assume that $x = x_n$ is a sequence of high levels such that

$$\tilde{C}_k \equiv \tilde{C}_k(x) = x/X_{n-k:n} \xrightarrow{p} \tilde{\xi} \in (0, \infty], \quad \text{as } n \rightarrow \infty, \quad (2.8)$$

and the rate function \tilde{A} satisfies $\gamma^2 \tilde{A}(t) = A(1/\bar{F}(t))$, with A given in (2.3), then, as $n \rightarrow \infty$,

$$\frac{\frac{\bar{F}(x)}{1/Y_{n-k:n}}/\tilde{C}_k^{-1/\gamma} - 1}{\tilde{A}(X_{n-k:n})} \xrightarrow{p} \frac{\tilde{\xi}^{\tilde{\rho}} - 1}{\tilde{\rho}} = h_{\tilde{\rho}}(\tilde{\xi})$$

with $\tilde{\rho} = \rho/\gamma$ and $h_{\tilde{\rho}}(\tilde{\xi})$ defined in (2.7).

Corollary 2.1. *Under the conditions of Lemma 2.3, if $p = p_n$ is a sequence of probabilities and $x = x_n$ a sequence of high thresholds such that (2.6) and (2.8) hold, then, as $n \rightarrow \infty$,*

$$\frac{\tilde{C}_k(\text{VaR}_p)}{c_k^\gamma(p)} = 1 + o_p(1) \quad \text{as well as} \quad \frac{\tilde{C}_k(x)}{c_k^\gamma(\bar{F}(x))} = 1 + o_p(1).$$

Just as happens with the semi-parametric EVI-estimators $\hat{\gamma}_{k,n}^H$ and $\hat{\gamma}_{k,n}^{PPWM}$, in (1.5) and (1.12), respectively, let generally $\hat{\gamma}_{k,n}^\bullet$ be any semi-parametric estimator of γ , such that, with Z_k^\bullet asymptotically standard normal, $\sigma_\bullet > 0$ and $b_\bullet \in \mathbb{R}$,

$$\hat{\gamma}_{k,n}^\bullet \stackrel{d}{=} \gamma + \frac{\sigma_\bullet Z_k^\bullet}{\sqrt{k}} + b_\bullet A(n/k)(1 + o_p(1)), \quad \text{as } n \rightarrow \infty. \quad (2.9)$$

Consequently, if $\sqrt{k}A(n/k) \rightarrow \lambda_A$, finite

$$\sqrt{k}(\hat{\gamma}_{k,n}^\bullet - \gamma) \xrightarrow[n \rightarrow \infty]{d} N(\lambda_A b_\bullet, \sigma_\bullet^2) =: \mathcal{G}_\bullet. \quad (2.10)$$

More generally than Theorem 4.1 in Beirlant *et al.* (2008), and Proposition 5 in Caeiro and Gomes (2009), we now state the following theorem, a generalization to the case $\xi < \infty$ of Theorem 4.3.8 and 4.4.7 in de Haan and Ferreira (2006).

Theorem 2.1. *Under the conditions of Lemma 2.3, let us further assume that $p = p_n$ is a sequence of probabilities such that, as $n \rightarrow \infty$,*

$$(A1) \quad c_k \equiv c_k(p) = k/(np) \rightarrow \xi, \quad \xi \in (0, \infty], \quad \ln c_k = o(\sqrt{k}),$$

$$(A2) \quad r_k A(n/k) \rightarrow \eta_A \in \mathbb{R},$$

with

$$r_k = \begin{cases} \sqrt{k} & \text{if } \xi \in (0, \infty) \\ \sqrt{k}/\ln c_k & \text{if } \xi = \infty. \end{cases}$$

Then, with $\text{VaR}_p = F^{\leftarrow}(1-p) = U(1/p)$, $\widehat{Q}_{k,n}^H(p)$, $\widehat{Q}_{k,n}^{PPWM}(p)$, generally denoted $\widehat{Q}_{k,n}^\bullet(p)$, \mathcal{B} and \mathcal{G}_\bullet given in (1.1), (1.6), (1.13), (2.4) and (2.10), respectively, $\rho < 0$ and $h_\rho(\xi)$ given in (2.7),

$$r_k (\ln \widehat{Q}_{k,n}^\bullet(p) - \ln \text{VaR}_p) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}_p = \begin{cases} (\ln \xi) \mathcal{G}_\bullet + \gamma \mathcal{B} - \eta_A h_\rho(\xi) & \text{if } \xi \in (0, \infty) \\ \mathcal{G}_\bullet + \eta_A / \rho & \text{if } \xi = \infty. \end{cases}$$

Indeed, we have the validity of the asymptotic distributional representation

$$\ln \widehat{Q}_{k,n}^\bullet(p) - \ln \text{VaR}_p \stackrel{d}{=} \ln c_k (\hat{\gamma}_{k,n}^\bullet - \gamma) + \gamma \mathcal{B} - A(n/k) h_\rho(\xi) + o_p(A(n/k)).$$

On the other hand, let $x = x_n$ be a sequence of levels such that, as $n \rightarrow \infty$,

$$(B1) \quad \widetilde{C}_k \equiv \widetilde{C}_k(x) := x/X_{n-k:n} \xrightarrow{p} \tilde{\xi} \in (0, \infty], \quad \ln \widetilde{C}_k = o_p(\sqrt{k}),$$

$$(B2) \quad \widetilde{R}_k A(n/k) \xrightarrow{p} \tilde{\eta}_A \in \mathbb{R},$$

with

$$\widetilde{R}_k = \begin{cases} \sqrt{k} & \text{if } \tilde{\xi} \in (0, \infty) \\ \sqrt{k}/\ln \widetilde{C}_k & \text{if } \tilde{\xi} = \infty. \end{cases}$$

Then, similarly as before, with p_x , $\hat{p}_{k,n}^H(x)$ and $\hat{p}_{k,n}^{PPWM}(x)$, generally denoted $\hat{p}_{k,n}^\bullet(x)$, defined in (1.2), (1.8) and (1.14), respectively,

$$\widetilde{R}_k (\ln \hat{p}_{k,n}^\bullet(x) - \ln p_x) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}_x = \begin{cases} (\ln \tilde{\xi}) \mathcal{G}_\bullet / \gamma^2 + \mathcal{B} - \tilde{\eta}_A h_{\tilde{\rho}}(\tilde{\xi}) / \gamma^2 & \text{if } \tilde{\xi} \in (0, \infty) \\ (\mathcal{G}_\bullet + \tilde{\eta}_A / \tilde{\rho}) / \gamma^2 & \text{if } \tilde{\xi} = \infty. \end{cases}$$

If $\tilde{\xi} = \infty$ and $\sqrt{k} A(n/k) \rightarrow \lambda_A$, finite,

$$\frac{\gamma^2 \sqrt{k}}{\ln \widetilde{C}_k} \left(\frac{\hat{p}_{k,n}^\bullet(x)}{p_x} - 1 \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{G}_\bullet. \quad (2.11)$$

Indeed, as $n \rightarrow \infty$, we have the validity of the asymptotic distributional representation,

$$\hat{p}_{k,n}^\bullet(x) \stackrel{d}{=} p \left\{ 1 + \frac{\ln \tilde{C}_k}{\gamma^2} (\hat{\gamma}_{k,n}^\bullet - \gamma)(1 + o_p(1)) + \frac{\mathcal{B}}{\sqrt{k}} + \frac{\tilde{A}(n/k)(1 + o_p(1))}{\tilde{\rho}} \right\},$$

with $\tilde{A}(t) = A(1/\bar{F}(t))/\gamma^2$.

Proof. The proof follows straightforwardly from the results in Lemma 2.3, and is left to the reader. \square

Remark 2.2. With $x = U(1/p)$, and since $\gamma > 0$, we can replace (B1) by (A1). And we have $\tilde{\xi} = \xi^\gamma$. Also, if $\tilde{\xi} = \infty$, we can choose $\tilde{R}_k = \sqrt{k}/(\gamma \ln c_k)$ (or equivalently, $\tilde{C}_k = c_k^\gamma$).

Corollary 2.2 (de Haan and Ferreira, 2006, Theorem 4.3.8). For intermediate k , i.e., whenever (1.7) holds, and under the second-order framework in (2.3), but with $\rho < 0$, if $p = p_n$ is a sequence of probabilities such that (A1) and (A2) hold with $\xi = \infty$, and if $\sqrt{k}A(n/k) \rightarrow \lambda_A$, finite, then,

$$\frac{\sqrt{k}}{\ln c_k} \left(\frac{\hat{Q}_{k,n}^\bullet(p)}{\text{VaR}_p} - 1 \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{G}_\bullet,$$

given in (2.10). We can indeed write the asymptotic distributional representation,

$$\hat{Q}_{k,n}^\bullet(p) \stackrel{d}{=} \text{VaR}_p \left\{ 1 + \ln c_k (\hat{\gamma}_{k,n}^\bullet - \gamma)(1 + o_p(1)) + \frac{\gamma \mathcal{B}}{\sqrt{k}} + \frac{A(n/k)(1 + o_p(1))}{\rho} \right\}.$$

Corollary 2.3 (de Haan and Ferreira, 2006, Theorem 4.4.7). Under the conditions of Corollary 2.2, if we consider that $x = U(1/p)$, then

$$\frac{\gamma \sqrt{k}}{\ln c_k} \left(\frac{\hat{p}_{k,n}^\bullet(x)}{p} - 1 \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{G}_\bullet.$$

Remark 2.3. For $\gamma > 0$, the asymptotic dominant behaviour of $\hat{Q}_{k,n}^\bullet(p)$ and $\hat{p}_{k,n}^\bullet(x)$ is thus fully determined by the asymptotic behaviour of $\hat{\gamma}_{k,n}^\bullet$.

Remark 2.4. In a certain dualistic way, the above conditions on the sequence of probabilities $p = p_x$ and the Value-at-Risk, VaR_p , regarded as a level $\text{VaR}_p = x = x_n$ can be interchanged. From the second half of Corollary 2.1, for instance, when $x = x_n$ is such that $\tilde{C}_k = \tilde{C}_k(x)$ satisfies the conditions of Theorem 2.1, it is readily understood that the sequence $c_k \equiv c_k(\bar{F}(x)) = k/(n \bar{F}(x))$ satisfies the condition $\tilde{C}_k^{1/\gamma}(x)/c_k(\bar{F}(x)) \xrightarrow{p} 1$, as $n \rightarrow \infty$, and consequently, also the conditions on c_k assumed in Theorem 2.1. Alternatively, when $p = p_n$ is such that $c_k = c_k(p)$ satisfies the conditions in Theorem 2.1, from the first half of Corollary 2.1 we see that $\tilde{C}_k(U(1/p))/c_k^\gamma(p) \xrightarrow{p} 1$. Consequently, the conditions on $\tilde{C}_k(U(1/p))$ assumed in Theorem 2.1 also hold.

Remark 2.5. Because of this dualistic interpretation, i.e. more specifically from the fact that $\tilde{C}_k(x)/c_k^\gamma(\bar{F}(x)) \xrightarrow{p} 1$, as $n \rightarrow \infty$, it is also easily understood how, for the exceedance probability estimator, $\hat{p}_{k,n}^\bullet(x)$, and with

$$r_k(x) = \begin{cases} \sqrt{k} & \text{if } \tilde{\xi} \in (0, \infty) \\ \sqrt{k}/\ln c_k(\bar{F}(x)) & \text{if } \tilde{\xi} = \infty, \end{cases}$$

we can state in an equivalent way that, as $n \rightarrow \infty$, $r_k(x)(\ln \hat{p}_{k,n}^\bullet - \ln p) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}_x/\gamma$.

3 Simulation results

In this section, we have implemented a multi-sample Monte Carlo simulation experiment of size 5000×10 , to obtain the distributional behaviour of the new semi-parametric VaR-estimators $\hat{Q}_{k,n}^{PPWM}(p)$, in (1.13), comparatively with the behaviour of the classical estimators $\hat{Q}_{k,n}^H(p)$, in (1.6), for $p = 1/n$ ($\xi = \infty$), $p = n^{-1/(1-2\rho)}/10$ (ξ finite and non-null at optimal levels) and $p = 0.01$ ($\xi = 0$, a value out of the scope of Theorem 2.1), for $n = 50, 100, 200, 500, 1000, 2000, 5000, 10000$ and 20000 , and for the following underlying parents: the Fréchet(γ) parent, with d.f. $F(x) = \exp(-x^{-1/\gamma})$, $x > 0$, $\gamma > 0$ ($\rho = -1$) and the Burr(γ, ρ) parent, with d.f. $F(x) = 1 - (1 + x^{-\rho/\gamma})^{1/\rho}$, $x > 0$. For the same models, we have also dealt with the dual problem of estimation of p (again known to be equal to the same values as before), the probability of exceedance of a high level $x = \text{VaR}_p$, through the new estimator $\hat{p}_{k,n}^{PPWM}(x)$, in (1.14), comparatively with the behaviour of the classical estimators $\hat{p}_{k,n}^H(x)$, in (1.8). We have further considered the non-parametric (NP) Var_p -estimator, $X_{[n(1-p)]+1:n}$, and p_x -estimator, $\hat{p}_x = \{\#X_i > x, 1 \leq i \leq n\}/(n+1)$.

3.1 Simulated mean values and root mean squared errors of the VaR-estimators

To illustrate the finite sample behaviour of the VaR-estimators, we present, in Figures 1 and 2, the simulated mean value (E) and root mean square error (RMSE) patterns of the normalised values of $\hat{Q}_{k,n}^H(p)$ and $\hat{Q}_{k,n}^{PPWM}(p)$, denoted $\bar{Q}^H := \hat{Q}_{k,n}^H(p)/\text{VaR}_p$ and $\bar{Q}^{PPWM} := \hat{Q}_{k,n}^{PPWM}(p)/\text{VaR}_p$, respectively, as functions of k , the number of top o.s. used, for Fréchet(0.25) and Burr(0.25, -0.75) parents, respectively, and sample size $n = 500$. Similar patterns were obtained for the other simulated models and other values of n .

3.1.1 Finite sample behaviour of the VaR-estimators at simulated optimal levels

In Table 1, we present the simulated mean values of the above mentioned normalised VaR_p -estimators, denoted H and PPWM, for the sake of simplicity, at their simulated optimal levels k_0^H and k_0^{PPWM} , respectively, for $p = 1/n$. We also present the simulated mean values of the normalised NP estimator of VaR_p . In Table 2, we present, in the first row, the simulated relative efficiencies (*REFF*) of $\hat{Q}_{k,n}^{PPWM}(p)$, $p = 1/n$, comparatively with the Weissman-Hill estimator, whenever computed at their simulated optimal levels, i.e., the simulated values of

$$REFF_{PPWM|H} := RMSE(\hat{Q}_{n,k_0^H}^H(p))/RMSE(\hat{Q}_{n,k_0^{PPWM}}^{PPWM}(p)).$$

We also present, in the second row of each entry, the equivalent REFF indicator of the NP estimator of VaR_p . In the third row, we present the simulated value of $RMSE(\hat{Q}_{n,k_0^H}^H(p))$, also for $p = 1/n$, so that we can easily recover the RMSE of the other estimators. In all tables the “best” values are written in **bold**.

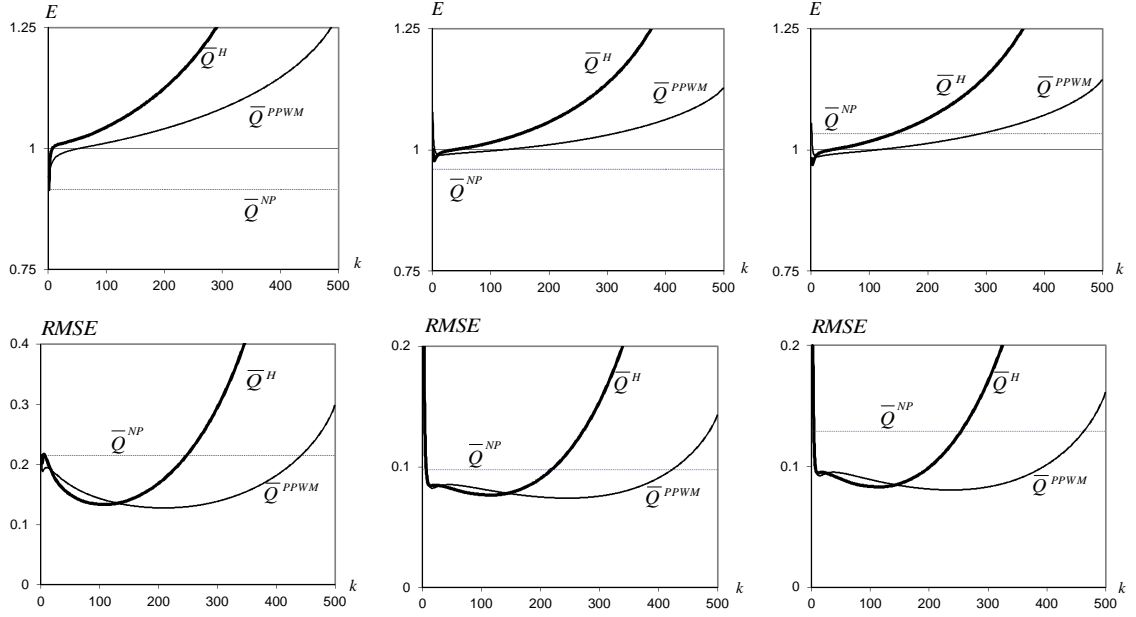


Figure 1: Simulated mean values (*above*) and root mean squared errors (*below*) for the Fréchet(γ) model, $\gamma = 0.25$, $n = 500$ and $p = 1/n$ (*left*), $p = n^{-1/(1-2\rho)}/10$ (*center*) and $p = 0.01$ (*right*).

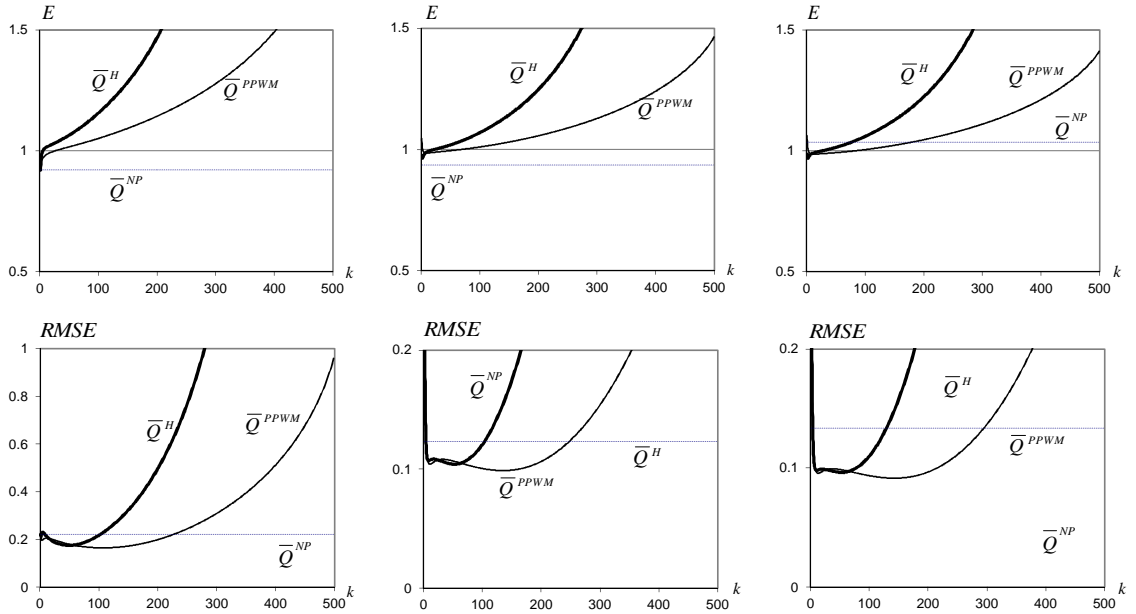


Figure 2: Simulated mean values (*above*) and root mean squared errors (*below*) for the Burr(γ, ρ) model, with $(\gamma, \rho) = (0.25, -0.75)$, $n = 500$ and $p = 1/n$ (*left*), $p = n^{-1/(1-2\rho)}/10$ (*center*) and $p = 0.01$ (*right*).

Table 1: Simulated mean values of the normalised NP and semi-parametric VaR_p-estimators under consideration, at their simulated optimal levels, for $p = 1/n$.

	n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	H	0.921	0.931	0.938	0.945	0.948	0.950	0.980	1.083	1.087
	PPWM	0.824	0.985	1.036	1.032	1.040	1.050	1.062	1.068	1.072
	NP	0.864	0.874	0.882	0.892	0.896	0.899	0.901	0.905	0.907
Burr(0.5, -0.5)	H	0.932	0.928	0.932	1.017	1.175	1.164	1.148	1.137	1.125
	PPWM	0.770	0.857	1.034	1.079	1.049	1.005	1.031	1.019	1.104
	NP	0.873	0.878	0.880	0.886	0.885	0.886	0.883	0.884	0.885
Burr(0.25, -0.75)	H	0.959	0.962	1.058	1.066	1.062	1.057	1.050	1.046	1.042
	PPWM	1.013	1.041	1.053	1.057	1.057	1.055	1.049	1.045	1.040
	NP	0.914	0.917	0.918	0.920	0.920	0.919	0.918	0.919	0.919
Fréchet(0.25)	H	1.001	1.044	1.050	1.050	1.045	1.042	1.036	1.030	1.027
	PPWM	1.031	1.038	1.042	1.044	1.043	1.041	1.035	1.031	1.027
	NP	0.916	0.913	0.912	0.915	0.917	0.919	0.920	0.919	0.919
Burr(0.25, -1.5)	H	1.035	1.051	1.052	1.045	1.040	1.034	1.028	1.023	1.020
	PPWM	1.033	1.039	1.044	1.044	1.040	1.036	1.029	1.024	1.021
	NP	0.919	0.918	0.919	0.921	0.920	0.920	0.918	0.919	0.919
Burr(0.75, -1.5)	H	0.990	1.247	1.220	1.168	1.144	1.117	1.090	1.075	1.063
	PPWM	0.724	0.769	0.823	0.900	0.962	1.024	1.109	1.069	0.896
	NP	0.900	0.897	0.898	0.913	0.911	0.912	0.903	0.904	0.904

Table 2: Simulated values of the indicators $REFF_{PPWM|H}$ (first row), $REFF_{NP|H}$ (second row) and $RMSE_H := RMSE(\hat{Q}_{n,k_0}^H(p))$ (third row) for the VaR_p estimation, $p = 1/n$.

	n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	PPWM H	1.381	1.999	1.299	1.207	1.181	1.181	1.219	1.180	1.158
	NP H	0.931	0.933	0.947	0.947	0.951	0.949	0.960	0.905	0.859
	RMSE _H	0.372	0.340	0.316	0.292	0.279	0.268	0.254	0.235	0.217
Burr(0.5, -0.5)	PPWM H	1.659	2.181	2.535	1.516	1.310	1.144	0.977	0.873	0.800
	NP H	0.965	0.963	0.992	0.972	0.903	0.790	0.688	0.610	0.544
	RMSE _H	0.537	0.516	0.505	0.485	0.444	0.391	0.329	0.290	0.255
Burr(0.25, -0.75)	PPWM H	1.107	1.121	1.102	1.067	1.046	1.032	1.027	1.024	1.021
	NP H	0.952	0.953	0.914	0.790	0.701	0.614	0.514	0.447	0.388
	RMSE _H	0.220	0.217	0.204	0.175	0.155	0.136	0.113	0.098	0.085
Fréchet(0.25)	PPWM H	1.118	1.091	1.064	1.041	1.019	1.005	0.998	0.996	0.995
	NP H	0.944	0.857	0.762	0.623	0.526	0.448	0.363	0.306	0.257
	RMSE _H	0.204	0.182	0.160	0.133	0.115	0.098	0.079	0.067	0.057
Burr(0.25, -1.5)	PPWM H	1.103	1.071	1.029	1.001	0.992	0.980	0.976	0.974	0.971
	NP H	0.928	0.828	0.726	0.585	0.496	0.412	0.324	0.268	0.221
	RMSE _H	0.200	0.178	0.156	0.128	0.109	0.091	0.071	0.058	0.048
Burr(0.75, -1.5)	PPWM H	2.209	2.176	1.945	1.586	1.289	0.995	0.679	0.491	0.393
	NP H	0.962	0.861	0.740	0.522	0.421	0.320	0.261	0.211	0.178
	RMSE _H	0.810	0.729	0.603	0.466	0.383	0.309	0.234	0.189	0.154

Tables 3 and 4 are similar to Tables 1 and 2, respectively, but for $p = n^{-1/(1-2\rho)}/10$. Finally, Tables 5 and Table 6 are similar to Tables 1 and 2, respectively, but for a fixed value $p = 0.01$.

Table 3: Simulated mean values of the normalised NP and semi-parametric VaR $_p$ -estimators under consideration, at their simulated optimal levels, for $p = n^{-1/(1-2\rho)}/10$.

	n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25,-0.2)	H	0.995	0.944	0.919	0.902	0.899	0.899	0.948	1.064	1.061
	PPWM	1.176	1.039	1.031	1.041	1.051	1.056	1.057	1.055	1.053
	NP	0.523	0.594	0.660	0.744	0.804	0.862	0.793	0.848	0.903
Burr(0.5,-0.5)	H	0.887	0.928	0.941	0.975	0.982	1.051	1.043	1.038	1.030
	PPWM	0.790	0.857	0.946	1.075	1.059	1.022	0.965	0.974	0.981
	NP	0.715	0.878	0.773	0.818	0.856	0.920	0.924	0.987	0.962
Burr(0.25,-0.75)	H	0.958	0.971	0.993	1.021	1.018	1.015	1.012	1.010	1.009
	PPWM	1.020	1.028	1.027	1.023	1.020	1.017	1.013	1.011	1.009
	NP	0.799	0.897	0.915	0.936	0.961	0.980	0.988	0.988	0.992
Fréchet(0.25)	H	0.955	1.022	1.024	1.020	1.016	1.013	1.010	1.008	1.007
	PPWM	1.024	1.023	1.021	1.018	1.016	1.014	1.011	1.009	1.007
	NP	0.866	0.891	0.938	0.960	0.990	0.992	0.991	0.995	0.998
Burr(0.25,-1.5)	H	0.977	1.020	1.017	1.014	1.010	1.008	1.006	1.005	1.004
	PPWM	1.012	1.018	1.014	1.013	1.011	1.009	1.006	1.005	1.004
	NP	0.942	0.922	0.956	0.982	0.992	0.996	0.996	0.999	0.998
Burr(0.75,-1.5)	H	0.984	0.995	1.039	1.037	1.030	1.023	1.017	1.013	1.010
	PPWM	0.777	0.844	0.891	0.936	0.957	0.972	0.984	0.989	0.993
	NP	0.909	0.818	0.899	0.964	0.987	0.995	0.992	0.999	0.996

Table 4: Simulated values of the indicators $REFF_{PPWM|H}$ (first row), $REFF_{NP|H}$ (second row) and $RMSE_0 := RMSE(\hat{Q}_{n,k_0}^H(p))$ (third row) for the VaR $_p$ estimation, $p = n^{-1/(1-2\rho)}/10$.

	n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25 - 0.2)	PPWM H	2.547	1.627	1.354	1.171	1.138	1.186	1.214	1.173	1.150
	NP H	1.657	1.297	1.135	1.002	0.976	0.984	0.913	0.894	0.873
	RMSE $_H$	0.875	0.606	0.468	0.352	0.307	0.283	0.237	0.206	0.179
Burr(0.5, -0.5)	PPWM H	1.963	2.181	2.283	1.922	1.276	1.125	1.023	0.990	0.961
	NP H	1.149	0.963	1.051	0.989	0.956	0.911	0.858	0.814	0.815
	RMSE $_H$	0.608	0.516	0.405	0.314	0.261	0.213	0.161	0.132	0.108
Burr(0.25, -0.75)	PPWM H	1.099	1.089	1.079	1.052	1.033	1.023	1.017	1.011	1.006
	NP H	0.868	0.925	0.897	0.846	0.834	0.818	0.800	0.780	0.776
	RMSE $_H$	0.214	0.174	0.141	0.104	0.083	0.066	0.049	0.039	0.032
Fréchet(0.25)	PPWM H	1.093	1.080	1.053	1.028	1.010	0.997	0.989	0.987	0.985
	NP H	0.901	0.839	0.827	0.784	0.771	0.760	0.748	0.740	0.736
	RMSE $_H$	0.177	0.137	0.106	0.076	0.060	0.047	0.034	0.027	0.021
Burr(0.25, -1.5)	PPWM H	1.071	1.054	1.019	0.994	0.987	0.978	0.973	0.970	0.966
	NP H	0.910	0.848	0.832	0.821	0.810	0.802	0.795	0.785	0.781
	RMSE $_H$	0.149	0.114	0.087	0.062	0.047	0.036	0.025	0.020	0.015
Burr(0.75, -1.5)	PPWM H	1.522	1.323	1.207	1.083	1.028	0.988	0.957	0.940	0.928
	NP H	0.939	1.021	0.944	0.870	0.839	0.818	0.813	0.794	0.791
	RMSE $_H$	0.512	0.379	0.288	0.196	0.148	0.112	0.078	0.059	0.046

Table 5: Simulated mean values of the normalised semi-parametric and NP VaR_p-estimators under consideration, at their simulated optimal levels, for $p = 0.01$.

		n									
		n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	H		0.893	0.931	0.955	0.972	0.976	0.989	0.996	0.998	0.999
	PPWM		0.975	0.985	0.990	0.993	0.994	0.995	0.998	0.999	0.999
	NP		0.638	1.400	1.162	1.059	1.029	1.015	1.006	1.003	1.001
Burr(0.5, -0.5)	H		0.955	0.928	0.964	0.986	0.988	0.991	0.996	0.999	1.000
	PPWM		0.841	0.857	0.879	0.902	0.914	0.973	0.984	0.989	0.992
	NP		0.589	1.870	1.285	1.095	1.046	1.023	1.009	1.004	1.002
Burr(0.25, -0.75)	H		0.932	0.962	1.004	1.017	1.010	1.006	1.003	1.002	1.001
	PPWM		0.944	1.041	1.032	1.020	1.013	1.009	1.004	1.002	1.001
	NP		0.763	1.234	1.096	1.035	1.017	1.009	1.003	1.002	1.001
Fréchet(0.25)	H		1.071	1.044	1.035	1.023	1.016	1.011	1.007	1.004	1.003
	PPWM		1.050	1.038	1.030	1.021	1.016	1.012	1.007	1.005	1.003
	NP		0.769	1.193	1.086	1.033	1.016	1.008	1.003	1.002	1.001
Burr(0.25, -1.5)	H		1.070	1.051	1.036	1.023	1.016	1.011	1.007	1.005	1.003
	PPWM		1.052	1.039	1.032	1.022	1.016	1.012	1.007	1.005	1.003
	NP		0.773	1.212	1.093	1.034	1.017	1.009	1.003	1.002	1.001
Burr(0.75, -1.5)	H		1.459	1.247	1.142	1.072	1.050	1.032	1.020	1.013	1.009
	PPWM		0.736	0.769	0.799	0.842	0.934	0.962	0.982	0.989	0.994
	NP		0.535	2.938	1.510	1.157	1.074	1.036	1.014	1.007	1.003

Table 6: Simulated values of the indicators $REFF_{PPWM|H}$ (first row), $REFF_{NP|H}$ (second row) and $RMSE_0 := RMSE(\widehat{Q}_{n,k_H}^H(p))$ (third row) for the VaR_p estimation, $p = 0.01$.

		n								
		50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	PPWM H	1.144	1.091	1.059	1.030	1.010	0.996	0.986	0.984	0.980
	NP H	0.939	0.360	0.529	0.644	0.707	0.749	0.785	0.803	0.818
	RMSE _H	0.487	0.340	0.243	0.154	0.110	0.078	0.049	0.035	0.025
Burr(0.5, -0.5)	PPWM H	3.129	2.181	1.768	1.396	1.167	1.013	0.987	0.969	0.954
	NP H	1.666	0.168	0.445	0.669	0.751	0.797	0.825	0.831	0.836
	RMSE _H	0.917	0.516	0.345	0.213	0.150	0.105	0.066	0.047	0.033
Burr(0.25, -0.75)	PPWM H	1.173	1.121	1.087	1.050	1.029	1.018	1.011	1.006	1.002
	NP H	1.038	0.378	0.579	0.722	0.774	0.807	0.829	0.832	0.837
	RMSE _H	0.309	0.217	0.154	0.096	0.068	0.048	0.030	0.022	0.015
Fréchet(0.25)	PPWM H	1.144	1.091	1.059	1.030	1.010	0.996	0.986	0.984	0.980
	NP H	0.939	0.360	0.529	0.644	0.707	0.749	0.785	0.803	0.818
	RMSE _H	0.268	0.182	0.129	0.083	0.060	0.043	0.028	0.020	0.015
Burr(0.25, -1.5)	PPWM H	1.117	1.071	1.026	0.998	0.989	0.979	0.973	0.970	0.966
	NP H	0.918	0.351	0.496	0.628	0.687	0.727	0.762	0.778	0.794
	RMSE _H	0.259	0.178	0.126	0.081	0.059	0.042	0.027	0.020	0.014
Burr(0.75, -1.5)	PPWM H	3.384	2.176	1.566	1.104	1.007	0.961	0.943	0.940	0.944
	NP H	1.948	0.065	0.313	0.553	0.655	0.715	0.764	0.781	0.798
	RMSE _H	1.326	0.729	0.455	0.269	0.188	0.132	0.084	0.059	0.042

Some remarks:

1. Regarding mean values at optimal levels, and whenever $\xi = \infty$, the PPWM VaR $_p$ estimator performs better than the classical Weissman-Hill estimator, in a large variety of situations, beating always the NP-estimator, unless we work with a Burr(0.75, -1.5) model, for which the PPWM estimators are not valid. If ξ is finite the outperformance of the PPWM VaR $_p$ -estimator becomes clear when $\gamma + \rho > 0$, and for small values of n whenever $\gamma + \rho \leq 0$. For large n and $\gamma + \rho < 0$, the NP-estimator appears then as an interesting alternative.
2. Regarding RMSE, or equivalently REFF-indicators, the PPWM VaR $_p$ estimators performs better than the Weissman-Hill quantile estimator at optimal levels, for all n , provided that $\gamma + \rho > 0$ and $\xi = \infty$ or ξ finite and non-null. But in this region of the (γ, ρ) -plane, even for finite ξ (possibly null), the H (the best for large n) and the PPWM (the best for small n) estimators are never beaten by the NP estimator.
3. The new PPWM quantile estimator can be used as an alternative to the Weissman-Hill estimator, specially for small to moderate sample sizes, but also for large samples, in most situations. All the simulated results suggest that its RMSE is never much bigger than the one of Weissman-Hill's estimator RMSE and it has often a smaller bias than the Weissman-Hill estimator. At their optimal levels and for small sample sizes, the PPWM VaR $_p$ estimator is usually much more efficient than Weissman-Hill's estimator.
4. The asymptotic properties of the new PPWM quantile estimators do not hold for the Burr model with $(\gamma, \rho) = (0.75, -1.5)$. But even in this example, the PPWM estimators are more efficient than the Weissman-Hill estimators at their simulated optimal levels, for small up to moderate sample sizes.

3.2 Simulated mean values and root mean squared errors of the p_x -estimators

Figures 3 and 4 are equivalent to Figures 1 and 2, respectively, but for the estimation of $p = p_x = 1 - F(x) = 1/n$, i.e. $x = (-\ln(1 - 1/n))^{-\gamma}$ for the Fréchet(γ) parent, and $x = (n^{-\rho} - 1)^{-\gamma/\rho}$ for the Burr(γ, ρ) parent. In Figure 3 and Figure 4, we have considered a sample size $n = 500$, again from Fréchet(0.25) and Burr(0.5, -0.5) parents, respectively.

3.2.1 Finite sample behaviour of the p_x -estimators at simulated optimal levels

Tables 7 and 8 are similar to Tables 1 and 2, respectively, for the estimation of p_x , with $x = F^{\leftarrow}(1 - 1/n)$. Tables 9 and 10 are equivalent to Tables 7 and 8, respectively, now for the estimation of p_x , with $x = F^{\leftarrow}(1 - n^{-1/(1-2\rho)}/10)$. Also, Tables 11 and 12 are equivalent to Tables 7 and 8, respectively, for the estimation of p_x , with $x = F^{\leftarrow}(1 - 0.01)$. We have decided to work with the normalised semi-parametric estimators $\bar{p}^H = \hat{p}_{n,k_0}^H(x)/p_x$ and $\bar{p}^{PPWM} = \hat{p}_{n,k_0}^{PPWM}(x)/p_x$, as well as with the normalised NP estimator of p_x . For the sake of simplicity, these estimators are again denoted respectively H, PPWM and NP.

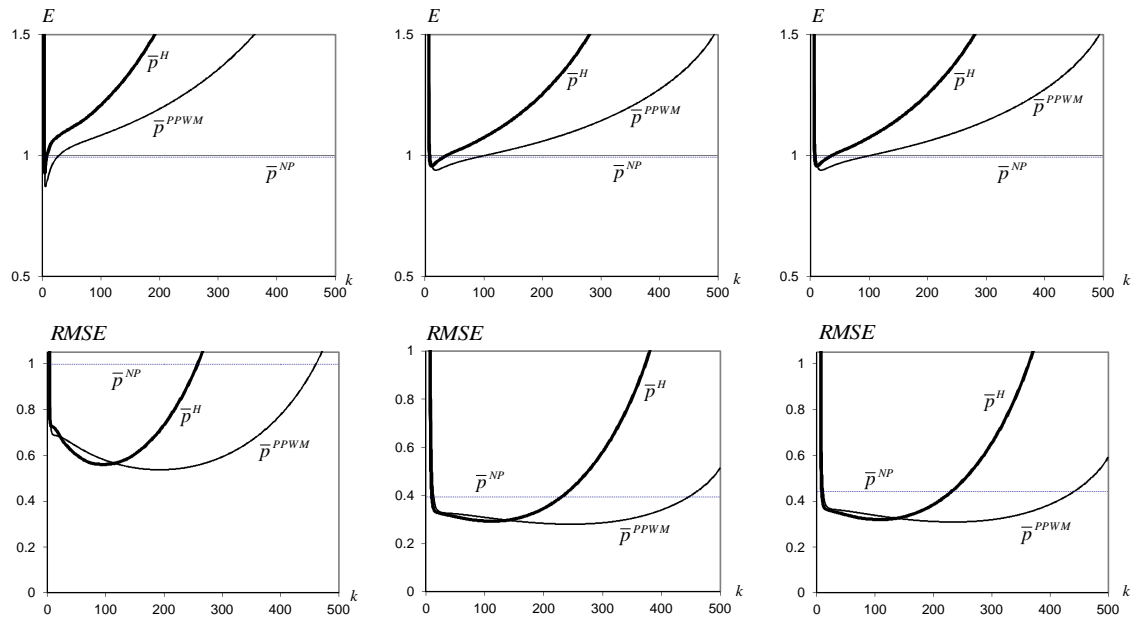


Figure 3: Simulated mean values (above) and root mean squared errors (below) for the Fréchet(γ) model, with $\gamma = 0.25$, $n = 500$ and $p = 1/n$ (left), $p = n^{-1/(1-2\rho)}/10$ (center) and $p = 0.01$ (right).

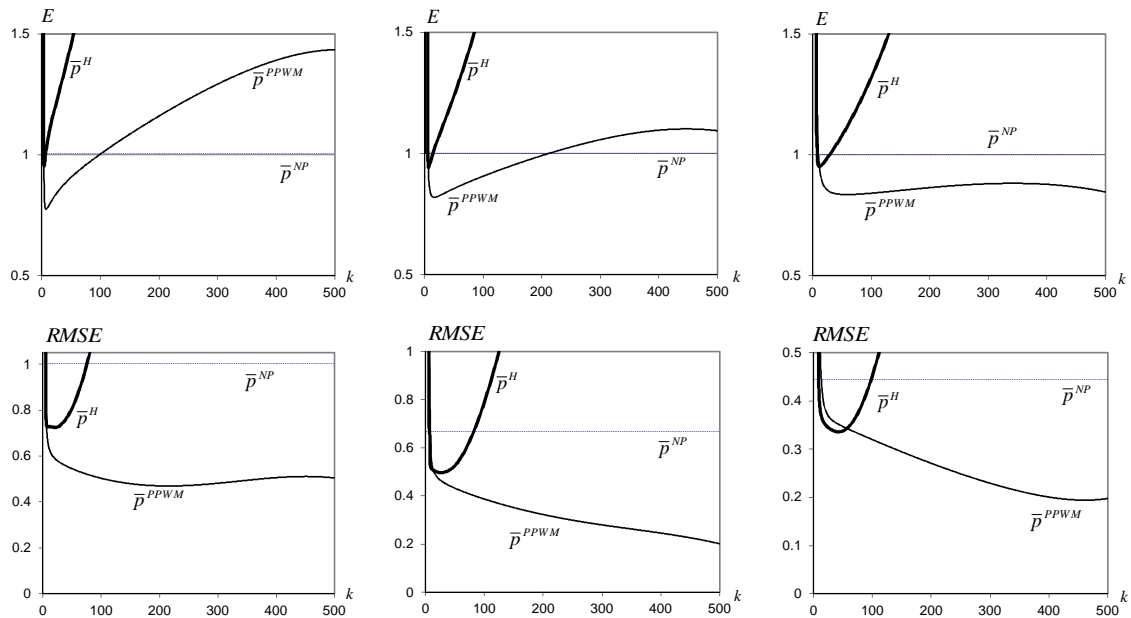


Figure 4: Simulated mean values (above) and root mean squared errors (below) for the Burr(γ, ρ) model, with $(\gamma, \rho) = (0.5, -0.5)$, $n = 500$ and $p = 1/n$ (left), $p = n^{-1/(1-2\rho)}/10$ (center) and $p = 0.01$ (right).

Table 7: Simulated mean values of the normalised NP and semi-parametric p_x -estimators under consideration, at their simulated optimal levels, for $x = F^{\leftarrow}(1 - 1/n)$.

		n									
		n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	H		1.046	1.044	1.032	1.027	1.025	1.023	1.031	1.189	1.261
	PPWM		0.737	0.984	1.156	1.122	1.123	1.134	1.159	1.183	1.212
	NP		0.981	0.996	1.001	1.005	1.004	0.997	0.994	0.996	1.003
Burr(0.5, -0.5)	H		1.039	1.040	1.027	1.154	1.230	1.256	1.252	1.252	1.237
	PPWM		0.671	0.823	1.039	1.182	1.139	1.127	1.138	1.152	1.176
	NP		0.981	0.996	1.001	1.005	1.004	0.997	0.994	0.996	1.003
Burr(0.25, -0.75)	H		1.037	1.053	1.176	1.230	1.237	1.236	1.211	1.197	1.180
	PPWM		1.154	1.146	1.170	1.209	1.229	1.228	1.211	1.194	1.175
	NP		0.981	0.996	1.001	1.005	1.004	0.997	0.994	0.996	1.003
Fréchet(0.25)	H		1.069	1.171	1.192	1.196	1.185	1.177	1.150	1.130	1.117
	PPWM		1.125	1.136	1.160	1.184	1.185	1.176	1.154	1.132	1.114
	NP		0.968	0.969	0.970	0.992	0.990	1.002	0.999	0.998	0.999
Burr(0.25, -1.5)	H		1.132	1.176	1.197	1.185	1.169	1.145	1.116	1.098	1.083
	PPWM		1.145	1.153	1.176	1.187	1.176	1.156	1.126	1.104	1.087
	NP		0.977	0.989	0.997	1.004	1.004	0.997	0.994	0.996	1.003
Burr(0.75, -1.5)	H		1.132	1.176	1.197	1.185	1.169	1.145	1.116	1.098	1.083
	PPWM		0.648	0.704	0.772	0.867	0.944	1.022	1.129	1.187	1.091
	NP		0.977	0.989	0.997	1.004	1.004	0.997	0.994	0.996	1.003

Table 8: Simulated values of the indicators $REFF_{PPWM|H}$ (first row), $REFF_{NP|H}$ (second row) and $RMSE_H := RMSE(\hat{p}_{n,k_0}^H(x))$ (third row) for the p_x -estimation, with $x = F^{\leftarrow}(1 - 1/n)$.

		n									
		n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	PPWM H		1.809	3.054	1.584	1.325	1.252	1.206	1.181	1.162	1.148
	NP H		0.659	0.675	0.690	0.696	0.705	0.711	0.716	0.709	0.698
	RMSE _H		0.635	0.665	0.685	0.697	0.710	0.711	0.710	0.703	0.696
Burr(0.5, -0.5)	PPWM H		1.638	2.391	2.935	1.538	1.323	1.183	1.038	0.938	0.861
	NP H		0.698	0.710	0.723	0.721	0.702	0.668	0.605	0.553	0.498
	RMSE _H		0.673	0.700	0.718	0.722	0.707	0.668	0.603	0.549	0.497
Burr(0.25, -0.75)	PPWM H		1.187	1.128	1.097	1.065	1.042	1.024	1.017	1.015	1.016
	NP H		0.719	0.726	0.727	0.683	0.630	0.570	0.485	0.423	0.367
	RMSE _H		0.693	0.715	0.722	0.683	0.634	0.570	0.482	0.420	0.366
Fréchet(0.25)	PPWM H		1.122	1.098	1.070	1.038	1.011	0.998	0.990	0.989	0.991
	NP H		0.730	0.700	0.655	0.560	0.494	0.423	0.343	0.288	0.241
	RMSE _H		0.710	0.688	0.641	0.558	0.488	0.423	0.341	0.287	0.241
Burr(0.25, -1.5)	PPWM H		1.105	1.076	1.034	0.994	0.983	0.968	0.966	0.966	0.966
	NP H		0.729	0.696	0.641	0.541	0.464	0.390	0.305	0.250	0.203
	RMSE _H		0.703	0.684	0.635	0.542	0.467	0.390	0.303	0.248	0.203
Burr(0.75, -1.5)	PPWM H		1.505	1.624	1.657	1.539	1.341	1.085	0.760	0.550	0.420
	NP H		0.729	0.696	0.641	0.541	0.464	0.390	0.305	0.250	0.203
	RMSE _H		0.703	0.684	0.635	0.542	0.467	0.390	0.303	0.248	0.203

Table 9: Simulated mean values of the normalised NP and semi-parametric p_x -estimators under consideration, at their simulated optimal levels, for $x = F^{\leftarrow}(1 - n^{-1/(1-2\rho)}/10)$.

		n								
		50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	H	1.197	1.138	1.078	1.037	1.028	1.024	1.053	1.137	1.168
	PPWM	1.236	1.237	1.168	1.134	1.133	1.140	1.148	1.150	1.154
	NP	0.990	0.995	1.004	1.002	1.002	0.996	0.995	0.996	0.998
Burr(0.5, -0.5)	H	1.044	1.040	1.058	1.084	1.100	1.086	1.078	1.070	1.059
	PPWM	0.727	0.823	0.930	1.093	1.234	1.089	1.065	1.056	1.050
	NP	0.981	0.996	1.003	1.001	1.003	1.002	1.002	1.001	1.001
Burr(0.25, -0.75)	H	1.050	1.077	1.079	1.078	1.073	1.058	1.046	1.039	1.036
	PPWM	1.156	1.123	1.108	1.091	1.078	1.066	1.053	1.044	1.037
	NP	0.980	0.990	0.999	1.003	1.002	1.001	1.001	1.000	1.001
Fréchet(0.25)	H	1.088	1.083	1.085	1.077	1.062	1.052	1.041	1.032	1.026
	PPWM	1.123	1.098	1.085	1.073	1.063	1.054	1.042	1.034	1.028
	NP	0.973	0.983	0.988	0.994	0.995	0.999	1.001	1.001	1.000
Burr(0.25, -1.5)	H	1.083	1.079	1.069	1.060	1.047	1.032	1.024	1.019	1.014
	PPWM	1.136	1.097	1.077	1.058	1.047	1.037	1.026	1.020	1.016
	NP	0.977	0.989	0.994	1.000	1.000	1.001	1.001	1.000	1.000
Burr(0.75, -1.5)	H	1.083	1.079	1.069	1.060	1.047	1.032	1.024	1.019	1.014
	PPWM	0.641	0.691	0.741	0.874	0.919	0.949	0.972	0.982	0.988
	NP	0.977	0.989	0.994	1.000	1.000	1.001	1.001	1.000	1.000

Table 10: Simulated values of the indicators $REFF_{PPWM|H}$ (first row), $REFF_{NP|H}$ (second row) and $RMSE_H := RMSE(\hat{p}_{n,k_0}^H(x))$ (third row) for the p_x -estimation, with $x = F^{\leftarrow}(1 - n^{-1/(1-2\rho)}/10)$.

		n								
n		50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	PPWM H	2.541	1.585	1.387	1.278	1.233	1.202	1.181	1.158	1.140
	NP H	0.656	0.665	0.667	0.676	0.695	0.707	0.720	0.719	0.719
	RMSE _H	1.164	1.078	0.983	0.883	0.821	0.754	0.670	0.605	0.549
Burr(0.5, -0.5)	PPWM H	2.085	2.391	2.795	2.471	1.491	1.223	1.057	0.966	0.904
	NP H	0.689	0.710	0.728	0.746	0.750	0.749	0.744	0.748	0.748
	RMSE _H	0.797	0.700	0.612	0.497	0.423	0.356	0.281	0.236	0.199
Burr(0.25, -0.75)	PPWM H	1.193	1.146	1.106	1.069	1.047	1.035	1.024	1.017	1.011
	NP H	0.721	0.733	0.748	0.757	0.756	0.765	0.763	0.759	0.759
	RMSE _H	0.678	0.573	0.479	0.370	0.303	0.247	0.187	0.151	0.123
Fréchet(0.25)	PPWM H	1.129	1.097	1.069	1.037	1.016	1.002	0.992	0.990	0.988
	NP H	0.738	0.741	0.747	0.745	0.741	0.741	0.735	0.728	0.726
	RMSE _H	0.617	0.495	0.395	0.292	0.232	0.184	0.136	0.107	0.085
Burr(0.25, -1.5)	PPWM H	1.136	1.089	1.047	1.011	0.999	0.989	0.979	0.975	0.970
	NP H	0.742	0.753	0.762	0.775	0.777	0.775	0.779	0.775	0.775
	RMSE _H	0.521	0.412	0.324	0.234	0.183	0.141	0.100	0.077	0.060
Burr(0.75, -1.5)	PPWM H	1.170	1.054	0.944	0.853	0.838	0.834	0.838	0.843	0.850
	NP H	0.742	0.753	0.762	0.775	0.777	0.775	0.779	0.775	0.775
	RMSE _H	0.521	0.412	0.324	0.234	0.183	0.141	0.100	0.077	0.060

Table 11: Simulated mean values of the normalised NP and semi-parametric p_x -estimators under consideration, at their simulated optimal levels, for $x = F^{\leftarrow}(1 - 0.01)$.

		n									
		n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25, -0.2)	H		1.093	1.044	1.037	1.031	1.019	1.010	1.005	1.003	1.001
	PPWM		0.975	0.984	0.989	0.992	0.992	0.993	0.996	0.998	0.999
	NP		0.986	0.996	1.001	1.002	1.002	1.001	1.001	1.000	1.000
Burr(0.5, -0.5)	H		1.067	1.040	1.065	1.051	1.038	1.024	1.011	1.007	1.003
	PPWM		0.803	0.823	0.836	0.863	0.878	0.889	0.896	0.976	0.984
	NP		0.986	0.996	1.001	1.002	1.002	1.001	1.001	1.000	1.000
Burr(0.25, -0.75)	H		1.052	1.053	1.092	1.062	1.048	1.035	1.017	1.010	1.006
	PPWM		1.132	1.146	1.120	1.081	1.057	1.038	1.020	1.012	1.007
	NP		0.986	0.996	1.001	1.002	1.002	1.001	1.001	1.000	1.000
Fréchet(0.25)	H		0.990	1.171	1.127	1.089	1.062	1.044	1.029	1.018	1.013
	PPWM		1.152	1.136	1.112	1.084	1.063	1.047	1.030	1.020	1.013
	NP		0.948	0.969	0.982	0.994	0.995	0.999	1.000	1.000	1.000
Burr(0.25, -1.5)	H		1.108	1.176	1.132	1.089	1.063	1.042	1.028	1.019	1.013
	PPWM		1.131	1.153	1.125	1.090	1.066	1.047	1.030	1.020	1.014
	NP		0.974	0.989	0.999	1.002	1.002	1.001	1.001	1.000	1.000
Burr(0.75, -1.5)	H		1.108	1.176	1.132	1.089	1.063	1.042	1.028	1.019	1.013
	PPWM		0.660	0.704	0.744	0.784	0.806	0.927	0.968	0.982	0.990
	NP		0.974	0.989	0.999	1.002	1.002	1.001	1.001	1.000	1.000

Table 12: Simulated values of the indicators $REFF_{PPWM|H}$ (first row), $REFF_{NP|H}$ (second row) and $RMSE_H := RMSE(\hat{p}_{n,k_0}^H(x))$ (third row) for the p_x -estimation, with $x = F^{\leftarrow}(1 - 0.01)$.

		n									
		n	50	100	200	500	1000	2000	5000	10000	20000
Burr(0.25 - 0.2)	PPWM H		2.976	3.054	3.122	3.155	3.163	3.176	3.163	3.144	3.155
	NP H		0.654	0.675	0.695	0.714	0.721	0.725	0.730	0.726	0.727
	RMSE _H		0.904	0.665	0.489	0.317	0.227	0.161	0.102	0.072	0.051
Burr(0.5, -0.5)	PPWM H		2.578	2.391	2.125	1.728	1.447	1.191	0.881	0.835	0.832
	NP H		0.680	0.710	0.733	0.756	0.766	0.774	0.783	0.780	0.783
	RMSE _H		0.940	0.700	0.516	0.336	0.241	0.172	0.110	0.078	0.055
Burr(0.25, -0.75)	PPWM H		1.137	1.128	1.101	1.073	1.056	1.047	1.040	1.035	1.032
	NP H		0.692	0.726	0.743	0.764	0.778	0.789	0.802	0.802	0.808
	RMSE _H		0.957	0.715	0.523	0.339	0.245	0.176	0.112	0.080	0.057
Fréchet(0.25)	PPWM H		1.108	1.098	1.068	1.037	1.016	1.002	0.993	0.991	0.987
	NP H		0.693	0.700	0.706	0.725	0.741	0.757	0.777	0.794	0.805
	RMSE _H		0.950	0.688	0.492	0.320	0.232	0.169	0.110	0.079	0.057
Burr(0.25, -1.5)	PPWM H		1.108	1.076	1.036	1.004	0.995	0.985	0.979	0.975	970
	NP H		0.695	0.696	0.698	0.717	0.732	0.745	0.765	0.775	789
	RMSE _H		0.957	0.684	0.490	0.318	0.231	0.166	0.107	0.077	0.055
Burr(0.75, -1.5)	PPWM H		1.934	1.624	1.353	1.055	0.866	0.802	0.821	0.843	868
	NP H		0.695	0.696	0.698	0.717	0.732	0.745	0.765	0.775	789
	RMSE _H		0.957	0.684	0.490	0.318	0.231	0.166	0.107	0.077	0.055

Despite of the fact that there is not a clear agreement between the behaviour of the PPWM p_x and VaR $_p$ -estimators, the final conclusions related with RMSEs do not differ too much from the ones we have previously drawn for the PPWM VaR $_p$ -estimator. In what concerns the pattern of mean values, and for ξ finite, the NP-estimator outperforms the semi-parametric ones for all simulated models and for almost all values of n . Just as before, the new quantile estimator \hat{p}_x^{PPWM} can be used as an alternative to \hat{p}_x^H , specially for small to moderate sample sizes, but also for large samples, in most situations. All the simulated results suggest that the RMSE of \hat{p}_x^{PPWM} is never much bigger than the one of \hat{p}_x^H . At their optimal levels and for small sample sizes, \hat{p}_x^{PPWM} is usually much more efficient than \hat{p}_x^H . The variance of the NP-estimator is terribly high. Regarding RMSE, the NP-estimator can never beat the semi-parametric ones, at optimal levels, even when ξ is finite.

4 The bootstrap methodology in the estimation of optimal thresholds

We now put forward an adaptive choice of thresholds, again on the basis of bootstrap computer-intensive methods. Indeed, due to the specificity of these PPWM estimators, and contrarily to what happens with the Hill estimators, in (1.5), the most common estimators of a positive EVI, a direct estimation of the optimal sample fraction (OSF), done on the basis of estimates of scale and shape second-order parameters, is problematic. The use of bootstrap computer intensive methods helps us to provide an adaptive choice of the optimal number of o.s.'s to be used in the estimation, and will be the topic of discussion in this section.

Again with $\hat{\gamma}_{k,n}^\bullet$ denoting either the Hill or the PPWM EVI-estimators, in (1.5) and (1.12), respectively, let us use the notation

$$k_0 \equiv k_0^\bullet(n) := \arg \min_k MSE(\hat{\gamma}_{k,n}^\bullet).$$

Under a semi-parametric framework, and under the validity of the second-order condition, in (2.3), with $\rho < 0$, let us parameterize the function $A(\cdot)$ as $A(t) = \gamma\beta t^\rho$, where $\beta \neq 0$ and $\rho < 0$ are generalized scale and shape second-order parameters. Then, with \mathbb{E} denoting the mean value operator, a possible substitute for the MSE of any classical EVI-estimator $\hat{\gamma}_{k,n}^\bullet$ is, cf. equation (2.9),

$$AMSE(\hat{\gamma}_{k,n}^\bullet) := \mathbb{E}(\sigma_\bullet Z_k^\bullet / \sqrt{k} + b_\bullet A(n/k))^2 = \sigma_\bullet^2/k + b_\bullet^2 \gamma^2 \beta^2 (n/k)^{2\rho},$$

depending on n and k , and with $AMSE$ standing for *asymptotic mean squared error*. We get (Dekkers and de Haan, 1993),

$$k_{0|\bullet}(n) := \arg \min_k AMSE(\hat{\gamma}_{k,n}^\bullet) = ((-2\rho) b_\bullet^2 \gamma^2 \beta^2 n^{2\rho} / \sigma_\bullet^2)^{-1/(1-4\rho)} = k_0^\bullet(n)(1 + o(1)).$$

For the Hill estimator, we have, in (2.9), $\sigma_H = \gamma$ and $b_H = 1/(1 - \rho)$. Consequently, with $(\hat{\beta}, \hat{\rho})$ any consistent estimator of the vector (β, ρ) of second-order parameters, we have an asymptotic justification for the estimator

$$\hat{k}_0^H := [((1 - \hat{\rho})^2 n^{-2\hat{\rho}} / (-2\hat{\rho}\hat{\beta}^2))^{1/(1-2\hat{\rho})}], \quad (4.1)$$

where, as usual, $[x]$ denotes the integer part of x . Moreover, provided that $\sqrt{k} (n/k)^\rho \rightarrow \lambda$, finite, and with $b_{k,n,\rho} = 1 + \beta(n/k)^\rho/(1 - \rho)$, $\sqrt{k}\{\hat{\gamma}_{k,n}^H/\gamma - b_{k,n,\rho}\}$ is approximately $\mathcal{N}(0, 1)$. We may then get approximate $100(1 - \alpha)\%$ confidence intervals (CI's) for γ ,

$$\left(\hat{\gamma}_{k,n}^H / (b_{k,n,\rho} + \xi_{1-\alpha/2}/\sqrt{k}), \hat{\gamma}_{k,n}^H / (b_{k,n,\rho} - \xi_{1-\alpha/2}/\sqrt{k}) \right), \quad (4.2)$$

where ξ_p is the p -quantile of a $\mathcal{N}(0, 1)$ d.f. If $\lambda = 0$, we may replace in (4.2) the bias summand $\beta(n/k)^\rho/(1 - \rho)$ by 0, i.e., $b_{k,n,\rho} = 1$.

The same does not happen with the PPWM EVI-estimators, with an asymptotic variance (σ_{PPWM}^2) and a dominant component of bias (b_{PPWM}) dependent on γ (see equation (2.5)). In this situation, it is sensible to use the bootstrap methodology for the adaptive PPWM EVI-estimation. Just as in Gomes and Oliveira (2001), for the estimation of γ through the Hill estimator, and in Gomes *et al.* (2011, 2012), for adaptive reduced-bias estimation, let us more generally consider the auxiliary statistics,

$$T_{k,n}^\bullet := \hat{\gamma}_{[k/2],n}^\bullet - \hat{\gamma}_{k,n}^\bullet, \quad k = 2, \dots, n-1, \quad (4.3)$$

which converge in probability to the known value zero, for any intermediate k , enabling thus easily the simulation of their MSE through the non-central moment of order two. On the basis of results similar to the ones in Gomes *et al.* (2000) and Gomes and Oliveira (2001), we can get, for the auxiliary statistic $T_{k,n}$, in (4.3), the asymptotic distributional representation,

$$T_{k,n} \equiv T_{k,n}^\bullet \stackrel{d}{=} \sigma_\bullet Q_k^\bullet / \sqrt{k} + b_\bullet (2^\rho - 1) A(n/k) + o_p(A(n/k)),$$

with Q_k asymptotically standard normal, and $(b_\bullet, \sigma_\bullet)$ given in (2.9). The $AMSE$ of $T_{k,n}$ is thus minimal at a level $k_{0|T}(n)$ such that $\sqrt{k} A(n/k) \rightarrow \lambda'_A \neq 0$, i.e. a level such that

$$k_{0|\bullet}(n) = k_{0|T}(n) (1 - 2^\rho)^{\frac{1}{1-2^\rho}} (1 + o(1)).$$

Then (see the above mentioned papers for further details), given the sample $\underline{X}_n = (X_1, \dots, X_n)$ from an unknown model F , for any $n_1 = o(n)$, and a bootstrap sample $\underline{X}_{n_1}^* = (X_1^*, \dots, X_{n_1}^*)$, denote T_{k_1, n_1}^* , $1 < k_1 < n_1$, the corresponding bootstrap auxiliary statistic. With the obvious notation $k_{0|T}^*(n_1) = \arg \min_{k_1} AMSE(T_{k_1, n_1}^*)$,

$$k_{0|T}^*(n_1)/k_{0|T}(n) = (n_1/n)^{-\frac{2^\rho}{1-2^\rho}} (1 + o(1)).$$

Consequently, for another sample size $n_2 = n_1^2/n$, we have $(k_{0|T}^*(n_1))^2/k_{0|T}^*(n_2) = k_{0|T}(n)(1 + o(1))$, as $n \rightarrow \infty$. On the basis of the estimation of $k_{0|T}^*$, we are now able to estimate $k_{0|T}$, and next $k_0^\bullet(n)$. With $\hat{k}_{0|T}^*$ denoting the sample counterpart of $k_{0|T}^*$, we can build the k_0 -estimate,

$$\hat{k}_0^{\bullet*} \equiv \hat{k}_0^{\bullet*}(n; n_1) := \min \left(n-1, \left[\frac{(1-2^\rho)^{\frac{1}{1-2^\rho}} (\hat{k}_{0|T}^*(n_1))^2}{\hat{k}_{0|T}^*([n_1^2/n] + 1)} \right] + 1 \right), \quad (4.4)$$

and the γ -estimate

$$\hat{\gamma}^{\bullet\bullet} \equiv \hat{\gamma}^{\bullet\bullet}(n; n_1) := \hat{\gamma}_{\hat{k}_0^{\bullet\bullet}(n; n_1), n}^{\bullet} \quad (4.5)$$

Note also that, with $c_k(p)$ defined in (1.6), on the basis of (2.2), for the estimation of VaR_p , and on the basis of (2.11) with \tilde{c}_k the observed value of \tilde{C}_k , in (1.8), for the estimation of $p_x = \bar{F}(x)$, we get

$$k_{0|p}^{\bullet} := \arg \min_k \text{AMSE}(\hat{Q}_{k,n}^{\bullet}(p)) = \arg \min_k (\ln c_k)^2 \text{AMSE}(\hat{\gamma}_{k,n}^{\bullet}),$$

and

$$k_{0|x}^{\bullet} := \arg \min_k \text{AMSE}(\hat{p}_{k,n}^{\bullet}(x)) = \arg \min_k (\ln \tilde{c}_k)^2 \text{AMSE}(\hat{\gamma}_{k,n}^{\bullet}).$$

A few practical questions, some of them with answers out of the scope of this paper, may be raised under the set-up developed: How does the asymptotic method work for moderate sample sizes? What is the type of the sample path of the new estimator for different values of n_1 ? Is the method strongly dependent on the choice of n_1 ? Although aware of the theoretical need to have $n_1 = o(n)$, what happens if we choose $n_1 = n$? Answers to these questions are expected not to be a long way from the ones given in previous papers (see Hall, 1990; Draisma *et al.*, 1999; Danielsson *et al.* 2001; Gomes and Oliveira, 2001; Gomes *et al.*, 2009, 2011, 2012), and some of them will be given in Section 5 of this article, on the basis of the analysis of real and simulated data.

4.1 An algorithm for adaptive estimation of parameters

The estimates $\hat{\rho}$, of the second-order parameter ρ , are the ones already used in previous papers. See for instance the algorithm provided in Gomes and Pestana (2007). Now, with \bullet denoting either H or *PPWM*, the algorithm is the following:

1. Given an observed sample (x_1, x_2, \dots, x_n) , compute the estimates $\hat{\gamma}_{k,n}^{\bullet}$, $\hat{Q}_{k,n}^{\bullet}(p)$ and $\hat{p}_{k,n}^{\bullet}(x)$, $k = 1, 2, \dots, n-1$.
2. Compute, for the tuning parameters $\tau = 0$ and $\tau = 1$, the observed values of the most simple ρ -estimators introduced and studied in Fraga Alves *et al.* (2003), and also used in the algorithm of Gomes and Pestana (2007). Such estimators depend on the statistics

$$V_{k,n}^{(\tau)} := \begin{cases} \frac{(M_{k,n}^{(1)}) - (M_{k,n}^{(2)}/2)^{1/2}}{(M_{k,n}^{(2)}/2)^{1/2} - (M_{k,n}^{(3)}/6)^{1/3}} & \text{if } \tau = 1 \\ \frac{\ln(M_{k,n}^{(1)}) - \frac{1}{2} \ln(M_{k,n}^{(2)}/2)}{\frac{1}{2} \ln(M_{k,n}^{(2)}/2) - \frac{1}{3} \ln(M_{k,n}^{(3)}/6)} & \text{if } \tau = 0, \end{cases}$$

where

$$M_{k,n}^{(j)} := \frac{1}{k} \sum_{i=1}^k (\ln X_{n-i+1:n} - \ln X_{n-k:n})^j, \quad j \geq 1,$$

$M_{k,n}^{(1)} \equiv H(k)$, in (1.5). They have the functional form

$$\hat{\rho}_\tau(k) := \min \left(0, \frac{3(V_{k,n}^{(\tau)} - 1)}{V_{k,n}^{(\tau)} - 3} \right).$$

3. Consider $\{\hat{\rho}_\tau(k)\}_{k \in \mathcal{K}}$, with $\mathcal{K} = ([n^{0.995}], [n^{0.999}])$, compute their median, denoted η_τ , and compute $I_\tau := \sum_{k \in \mathcal{K}} (\hat{\rho}_\tau(k) - \eta_\tau)^2$, $\tau = 0, 1$. Next choose the *tuning parameter* $\tau^* = 0$ if $I_0 \leq I_1$; otherwise, choose $\tau^* = 1$.
4. Work with $\hat{\rho} \equiv \hat{\rho}_{\tau^*} = \hat{\rho}_{\tau^*}(k_1)$, $k_1 = [n^{0.999}]$.
5. Next, consider a sub-sample size $n_1 = o(n)$, and $n_2 = [n_1^2/n] + 1$.
6. For l from 1 until $B = 250$, generate independently B bootstrap samples $(x_1^*, \dots, x_{n_2}^*)$ and $(x_{n_2+1}^*, \dots, x_{n_1}^*)$, of sizes n_2 and n_1 , respectively, from the empirical d.f. $F_n^*(x) = \frac{1}{n} \sum_{i=1}^n I_{\{X_i \leq x\}}$ associated with the observed sample (x_1, \dots, x_n) .
7. Denoting $T_{k,n}^* \equiv T_{k,n}^{\bullet*}$ the bootstrap counterpart of $T_{k,n}^\bullet$, defined in (4.3), obtain $t_{k,n_1,l}^*$, $1 < k < n_1$, $t_{k,n_2,l}^*$, $1 < k < n_2$, $1 \leq l \leq B$, the observed values of the statistic T_{k,n_i}^* , $i = 1, 2$. For $k = 2, \dots, n_i - 1$, compute

$$MSE_1^*(n_i, k) = \frac{1}{B} \sum_{l=1}^B (t_{k,n_i,l}^*)^2, \quad i = 1, 2,$$

as well as

$$MSE_p^*(n_i, k) = (\ln c_k)^2 MSE_1^*(n_i, k), \quad MSE_x^*(n_i, k) = (\ln \hat{c}_k^2) MSE_1^*(n_i, k), \quad i = 1, 2.$$

8. Obtain, for $i = 1, 2$,

$$\hat{k}_{0|T}^*(n_i) := \arg \min_{1 < k < n_i} MSE_1^*(n_i, k), \quad (4.6)$$

$$\hat{k}_{0|p}^*(n_i) := \arg \min_{1 < k < n_i} MSE_p^*(n_i, k), \quad (4.7)$$

$$\hat{k}_{0|x}^*(n_i) := \arg \min_{1 < k < n_i} MSE_x^*(n_i, k), \quad (4.8)$$

and return to Step 6. whenever $\hat{k}_{0|\bullet}^*(n_2) \geq \hat{k}_{0|\bullet}^*(n_1)$.

9. On the basis of (4.6), compute the threshold estimate $\hat{k}_0^{\bullet*} \equiv \hat{k}_0^{\bullet*}(n; n_1)$, in (4.4), for the adaptive EVI-estimation. Compute also the threshold estimates, $\hat{k}_{0|p}^{\bullet*} \equiv \hat{k}_{0|p}^{\bullet*}(n; n_1)$ and $\hat{k}_{0|x}^{\bullet*} \equiv \hat{k}_{0|x}^{\bullet*}(n; n_1)$, on the basis of an equation similar to (4.4), but with $\hat{k}_{0|T}^*$ replaced by $\hat{k}_{0|p}^*$ and $\hat{k}_{0|x}^*$, in (4.7) and (4.8), according as we are interested in the adaptive VaR_p-estimation or the p_x -estimation, respectively. Return also to Step 6. whenever these k -estimates are equal to $n - 1$.
10. Obtain the adaptive EVI-estimate, $\hat{\gamma}_*^\bullet \equiv \hat{\gamma}_*^\bullet(n, n_1) := \hat{\gamma}_{\hat{k}_0^{\bullet*}, n}^\bullet$, already provided in (4.5).

11. Finally, compute the adaptive VaR_p - and p_x -estimates,

$$\widehat{Q}_p^{\bullet\bullet} \equiv \widehat{Q}_{p|\hat{\gamma}}^{\bullet\bullet} := \widehat{Q}_{\hat{k}_{0|p}^{\bullet\bullet}, n}^{\bullet\bullet}, \quad \widehat{p}_x^{\bullet\bullet} \equiv \widehat{p}_{x|\hat{\gamma}}^{\bullet\bullet} := \widehat{p}_{\hat{k}_{0|x}^{\bullet\bullet}, n}^{\bullet\bullet},$$

with $\hat{k}_{0|p}^{\bullet\bullet}$ and $\hat{k}_{0|x}^{\bullet\bullet}$ obtained in Step 9.

Remarks:

- (i) If there are negative elements in the sample, n should be replaced by $n^+ = \sum_{i=1}^n I_{[X_i > 0]}$ (the number of positive values in the sample). Analogously for n_1 and for n_2 .
- (ii) The Monte-Carlo procedure in Steps 6. to 11. of the Algorithm can be replicated r times if we want to associate standard errors to the OSF's estimates and to the γ , VaR_p and p_x bootstrap estimates. The value of B can also be adequately chosen.

5 Applications to simulated and real data

5.1 A simulated data set

In order to have an indication on the way the algorithm in Section 4.1 performs, and to further motivate its use, prior to its validation through a surely highly time-consuming simulation, a topic out of the scope of this paper, we first apply it to an arbitrarily simulated sample, with size $n = 500$, from a Fréchet model, with $\gamma = 0.25$. The algorithm here presented led to the ρ -estimate $\hat{\rho} \equiv \hat{\rho}_0 = -1.173$, obtained at the level $k_1 = [n^{0.999}] = 496$. The associated β -estimate is $\hat{\beta} \equiv \hat{\beta}_0 = 0.901$. Then, we got $\hat{k}_0^H = 102$, with \hat{k}_0^H provided in (4.1), and an associated EVI estimate equal to 0.285, clearly over-estimating the true value $\gamma = 0.25$. The associated approximate 95% confidence interval, in (4.2), is (0.2266, 0.3277), with a size 0.1011.

The application of the algorithm presented in Section 4.1 of this paper, with a sub-sample size $n_1 = [n^{0.955}] = 378$, and $B = 250$ bootstrap generations, led us to $\hat{k}_0^{PPWM*} = 52$ and to the adaptive PPWM EVI-estimate $PPWM^* \equiv \hat{\gamma}_*^{PPWM} = 0.254$. This same algorithm applied to the Hill estimators leads us to $\hat{k}_0^{H*} = 167$ and to the adaptive Hill EVI-estimate $H^* \equiv \hat{\gamma}_*^H = 0.284$. These values are pictured in Figure 5, where we also present the estimates under study as a function of k . The most adequate estimate seems neatly to be the one associated with the PPWM methodology, particularly due to the smoothness of the stability region of the estimates as a function of k . Similar conclusions can be drawn for the VaR_p and p_x -estimates.

For the estimation of $\text{VaR}_{1/(2n)} = \text{VaR}_{0.001}$, equal to 5.623 for this Fréchet sample, and through the PPWM estimators, we were led to a choice of k equal to 52 and to the $\text{VaR}_{0.001}$ estimate, $\hat{Q}_{0.001} = 5.987$, slightly overestimating the true value of $\text{VaR}_{0.001} = 5.623$. For the estimation of $p = 1/(2n) = 0.001$, the PPWM methodology led us to a k -value equal to 51 and $\hat{p} = 0.0013$, quite close to p . Regarding the classical estimators of VaR_p and p , we were led to a k -value equal to 167, in both cases, and to the estimates 5.638 and 0.002, respectively.

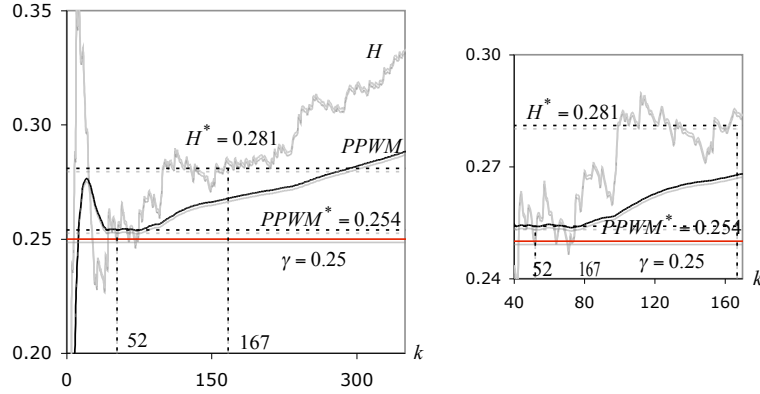


Figure 5: Non-adaptive (as a function of k) and bootstrap adaptive Hill (H) and PPWM EVI-estimates, for a Fréchet sample of size $n = 500$ and $\gamma = 0.25$ (left), and a zoomed figure for $40 \leq k \leq 170$ (right).

5.1.1 Resistance of the methodology to changes in the sub-sample size n_1

In Figure 6, we picture at the left, as a function of the sub-sample size n_1 , ranging from $n_1 = \lceil n^{0.9} \rceil = 268$ until $n_1 = \lceil n^{0.9999} \rceil = 499$, the bootstrap EVI-estimates associated with the Hill and the PPWM estimators, in (1.5) and (1.12), respectively. The pictures in the center and the right are similar to the picture at the left, but related with the corresponding adaptive bootstrap estimation of $\text{VaR}_{0.001} = 5.623$ and $p = 0.001$, respectively.

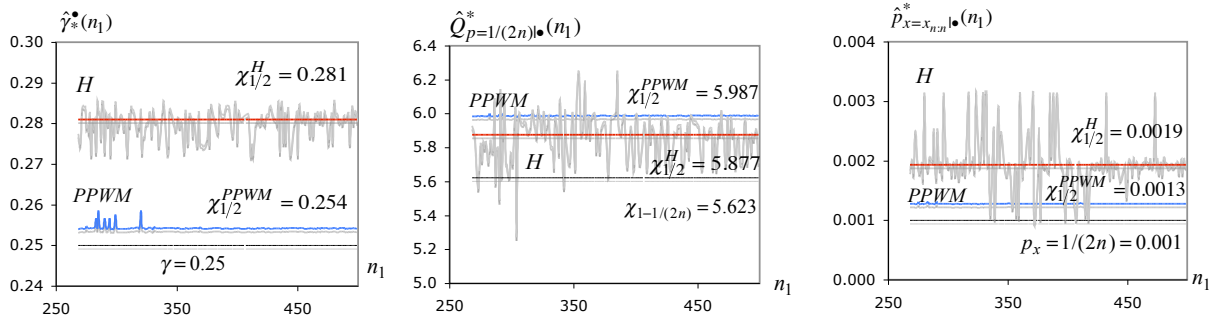


Figure 6: Bootstrap Hill (H) and PPWM EVI-estimates of γ (left), $\text{VaR}_{0.001} = 5.623$ (center) and $p = 1/(2n) = 0.001$ (right), as functions of n_1 .

The low sensitivity of the PPWM estimates to changes of the subsample size n_1 seems also to be a point in favour of these new estimators. Whereas the bootstrap PPWM EVI, VaR_p and p -estimates lie in the intervals $(0.254, 0.259)$, $(5.977, 6.002)$ and $(0.00127, 0.00131)$, respectively, the bootstrap Hill-type EVI, VaR_p and p -estimates lie in the intervals $(0.272, 0.285)$, $(5.273, 6.254)$ and $(0.0010, 0.0029)$, respectively. Note however that the reasonably high volatility of the bootstrap Hill VaR_p and p_x -estimates as a function of n_1 can lead us to estimates closer to the target for some of the values of the sub-sample size n_1 .

5.2 A case study in the field of insurance

We shall next consider an illustration of the performance of the adaptive PPWM EVI-estimates under study, comparatively with the same methodology applied to the Hill EVI-estimates, again through the analysis of automobile claim amounts exceeding 1,200,000 Euro over the period 1988-2001, gathered from several European insurance companies co-operating with the same re-insurer (Secura Belgian Re). This data set was already studied in Beirlant *et al.* (2004), Vandewalle and Beirlant (2006) and Beirlant *et al.* (2008) as an example to excess-of-loss reinsurance rating and heavy-tailed distributions in car insurance. See also Gomes *et al.* (2009). A preliminary graphical analysis of the data, x_i , $1 \leq i \leq n$, $n = 371$, leads us to an immediate conclusion that data have been censored to the left and that the right-tail of the underlying model is quite heavy. The sample paths of the ρ -estimates associated with $\tau = 0$ and $\tau = 1$ lead us to choose, on the basis of any stability criterion for large k , the estimate associated with $\tau = 0$. The algorithm here presented led us to the ρ -estimate $\hat{\rho}_0 = -0.74$, obtained at the level $k_1 = \lceil n^{0.999} \rceil = 368$. The associated β -estimate, based on the β -estimator in Gomes and Martins (2002), was $\hat{\beta}_0 = 0.80$. Then, we got the estimate $\hat{k}_0^H = 55$, with \hat{k}_0^H provided in (4.1), and an associated adaptive EVI estimate equal to 0.291. The associated approximate 95% confidence interval, in (4.2), is (0.2115, 0.3432), with a size 0.1317.

The application of the algorithm presented in Section 4.1 of this paper, with a sub-sample size $n_1 = \lceil n^{0.955} \rceil = 284$, and $B = 250$ bootstrap generations, led us to $\hat{k}_0^{PPWM*} = 58$ and to the adaptive PPWM EVI-estimate $PPWM^* \equiv \hat{\gamma}_*^{PPWM} = 0.272$. This same algorithm applied to the Hill estimates leads us to $\hat{k}_0^{H*} = 52$ and to the adaptive Hill EVI-estimate $H^* \equiv \hat{\gamma}_*^H = 0.299$. These values are pictured in Figure 7, where we also present the EVI-estimates under study as a function of k . Again, similar results have been obtained for the Var_p and p_x -estimates.

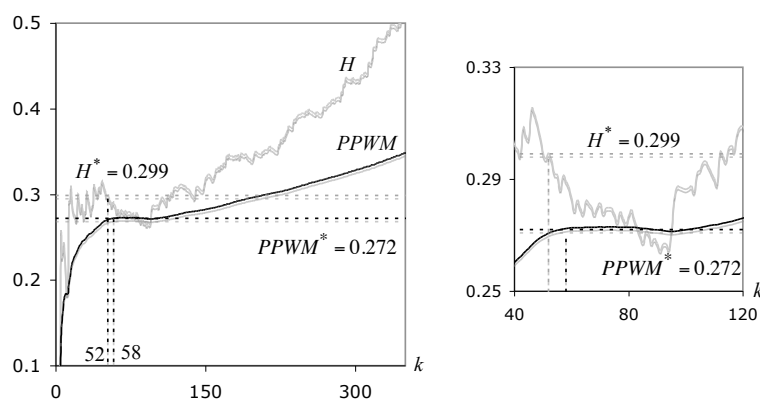


Figure 7: Hill (H) and PPWM EVI-estimates for the SECURA data, as a function of k (left), and a zoomed figure for $40 \leq k \leq 120$ (right).

For the estimation of $VaR_{1/(2n)} = VaR_{0.0013}$, and through the PPWM estimators, we were led to a choice of k equal to 59 and to the VaR estimate, $\hat{Q}_{1/(2n)} = 10,658,360$. For the estimation of $p = \mathbb{P}(X > X_{n:n}) = 1/(n+1) = 0.0027$, the PPWM methodology led us to a k -value also equal to 59

and $\hat{p} = 0.0040$, reasonably above the true value of p . Regarding the classical estimators of VaR_p and p , we were led to k -values respectively equal to 52 and 53, and to the estimates 11, 356, 460 and 0.0053, respectively. The overestimation of the p -estimate is now more flagrant.

Figure 8 is similar to Figure 6, but for the insurance data.

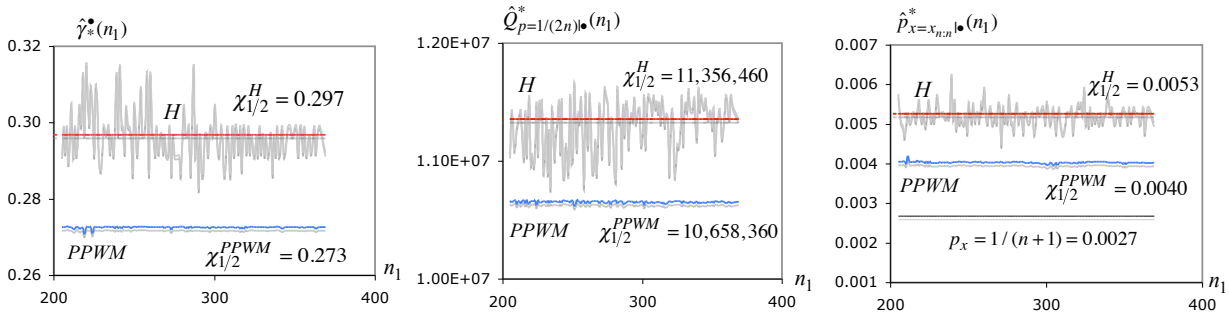


Figure 8: Bootstrap Hill (H) and PPWM EVI-estimates of γ (left), $\text{Var}_{1/(2n)}$ (center) and $p = 1/(n+1) = 0.0027$ (right), as functions of n_1 .

The bootstrap PPWM EVI, VaR_p and p -estimates are indeed quite stable as a function of the sub-sample size n_1 . The bootstrap PPWM EVI-estimates vary from a minimum value equal to 0.271 until 0.273, with a median equal to 0.273, equal to the value obtained for the bootstrap EVI estimate associated to the arbitrarily chosen sub-sample size $n_1 = [n^{0.955}] = 284$. We can indeed guarantee the two decimal figures, i.e. the EVI-estimate 0.27. Up to three decimal figures, all bootstrap p_x -estimates were equal to 0.004. Regarding the bootstrap VaR_p -estimates, we got values between a minimum equal to 10618780 and a maximum equal to 10673260, with a median 10658360. Again, the low sensitivity of the PPWM estimates to changes of the subsample size n_1 seems again to be a point in favour of these new estimators. Whereas the bootstrap PPWM EVI, VaR_p and p -estimates lie in the intervals (0.271, 0.273), (10618780, 10673260) and (0.0040, 0.0042), respectively, the bootstrap Hill-type EVI, VaR_p and p -estimates lie in the intervals (0.283, 0.315), (10764560, 11660710) and (0.0045, 0.0063), respectively. As already detected in previous papers, and in the most diversified comparisons, the Hill estimates are clearly over-estimating the true value of the EVI. As mentioned before, the most adequate estimate seems neatly to be the one associated with the PPWM methodology, particularly due to the smoothness of the stability region of the estimates as a function of k .

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