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SEMI-PRECONVEX SETS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concept of the semi-preconvex set on preconvexity spaces. We study some properties for the semi-preconvex set. Also we introduce the concepts of the sc-convex function and s^*c-convex function. Finally, we characterize sc-convex functions, s^*-convex functions and semi-preconvex sets by using the co-convexity hull and the convexity hull.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set X and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [4] yields a topological space. The author introduced the concepts of the co-convexity hull and co-convex sets on preconvexity spaces in [3]. In particular, we showed that the complement of a co-convex set is a convex set and the union of co-convex sets is a co-convex set. And we characterized c-convex functions and c-concave functions by using the co-convexity hull and the convexity hull.

In this paper, we introduce the semi-preconvex set defined by the co-convexity hull on a preconvexity space and study some basic properties. And we introduce the concepts of sc-convex functions and s^*c-convex functions which are defined by the semi-preconvex sets. In particular, the sc-convex function is a generalized c-convex function.

Finally, some properties of sc-convex functions, s^*c-convex functions and semi-preconvex sets are discussed.

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2. Preliminaries

Definition 2.1 ([1]). Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:

(1) If $A \subset B$, then $A\sigma B$.
(2) If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.
(3) If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
(4) If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair (X, σ) is called a preconvexity space. Let (X, σ) be a preconvexity space and $A \subset X$. $G(A) = \{x \in X : x\sigma A\}$ is called the convexity hull of a subset A. A is called convex [1] if $G(A) = A$.

$I_\sigma(A) = \{x \in A : x\not\sigma(X - A)\}$ (simply, $I(A)$) is called the co-convexity hull [3] of a subset A. And A is called a co-convex set if $I(A) = A$ [3]. Let $\mathcal{I}(X) = \{A \subset X : I(A) = A\}$ and $\mathcal{G}(X) = \{A \subset X : G(A) = A\}$.

Theorem 2.2 ([3]). *Let (X, σ) be a preconvexity space and $A \subset X$. Then*

(1) $I(A) = X - G(X - A)$.
(2) $G(A) = X - I(X - A)$.

Theorem 2.3 ([1], [3]). *For a preconvexity space (X, σ),*

(1) $G(\emptyset) = \emptyset$, $I(X) = X$.
(2) $A \subset G(A)$, $I(A) \subset A$ *for all* $A \subset X$.
(3) *If* $A \subset B$, *then* $G(A) \subset G(B)$, $I(A) \subset I(B)$.
(4) $G(G(A)) = G(A)$, $I(I(A)) = I(A)$ *for* $A \subset X$.

Theorem 2.4 ([1], [3]). *Let σ be a preconvexity on X and $A, B \subset X$. Then*

(1) $A\sigma B$ *if and only if* $A \subset G(B)$ *if and only if* $I(X - B) \subset X - A$.
(2) $A\sigma B$ *if and only if* $G(A)\sigma G(B)$ *if and only if* $I(X - B)\sigma I(X - A)$.

We recall that the notions of c-convex function and c-concave function: Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be c-concave [2] if for $C, D \subset Y$ whenever $C\mu D$, $f^{-1}(C)\sigma f^{-1}(D)$. A function $f : X \to Y$ is said to be c-convex [1] if $A\sigma B$ implies $f(A)\mu f(B)$. And f is c-convex if and only if for each $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$ [3].

3. Semi-preconvex sets

Definition 3.1. Let (X, σ) be a preconvexity space and $A \subset X$. A is called a semi-preconvex set if $A\sigma I(A)$. And A is called a cosemi-preconvex set if the complement of A is a semi-preconvex set.

Let $\mathcal{S}_\sigma(X)$ (resp., $\mathcal{SC}_\sigma(X)$) denote the set of all semi-preconvex sets (resp., cosemi-preconvex sets) in a preconvexity space (X, σ).

From Theorem 2.2 and Theorem 2.4, we get the following theorem.

Theorem 3.2. *Let (X, σ) be a preconvexity space and $A \subset X$. Then*

(1) *A is a semi-preconvex set if and only if* $A \subset G(I(A))$.
(2) *A is a cosemi-preconvex set if and only if* $I(G(A)) \subset A$.

Theorem 3.3. *Every co-convex set is a semi-preconvex set in a preconvexity space* (X, σ).

Proof. Let A be a co-convex set; then by the concept of co-convex sets, $A = I(A)$. By Definition 2.1, $A\sigma I(A)$. □

Theorem 3.4. *Every convex set is a cosemi-preconvex set in a preconvexity space* (X, σ).

Proof. Let A be a convex set; then $G(A) = A$. Thus $IG(A) \subset G(A) = A$. □

Theorem 3.5. *In a preconvexity space* (X, σ), X *and* $\emptyset$ *are both semi-preconvex sets and cosemi-preconvex sets.*

Proof. Since X and $\emptyset$ are both co-convex sets and convex sets, we get the result. □

Theorem 3.6. *In a preconvexity space* (X, σ), *the arbitrary union of semi-preconvex sets is a semi-preconvex set.*

Proof. Let $\mathbf{A} = \{A_\alpha : A_\alpha \text{ is a semi-preconvex set}\} \subset \mathbf{S}_\sigma(\mathbf{X})$. We show that $\cup\mathbf{A}\sigma I(\cup\mathbf{A})$. For Definition 2.1(3), let $x \in \cup\mathbf{A}$; then there exists a semi-preconvex set A_α containing x. Since $A_\alpha \sigma I(A_\alpha)$, from Definition 2.1(4), it follows $x\sigma I(A_\alpha)$. Since $A_\alpha \subset \cup\mathbf{A}$, $I(A_\alpha) \subset I(\cup\mathbf{A})$ and the transitive property gives $x\sigma I(\cup\mathbf{A})$. Finally, we get $\cup\mathbf{A}\sigma I(\cup\mathbf{A})$ by Definition 2.1(3). □

Theorem 3.7. *In a preconvexity space* (X, σ), *the arbitrary intersection of cosemi-preconvex sets is a cosemi-preconvex set.*

Proof. See Theorem 3.6. □

Definition 3.8. Let (X, σ) be a preconvexity space and $A \subset X$.
(1) $SG(A) = \cap\{F : A \subset F, F^c \in S_\sigma(X)\}$.
(2) $SI(A) = \cup\{U : U \subset A, U \in S_\sigma(X)\}$.

From Theorem 3.3, Theorem 3.6, Theorem 3.7, and Definition 3.8, we get the following theorem:

Theorem 3.9. *Let* (X, σ) *be a preconvexity space and* $A, B \subset X$.
(1) $I(A) \subset SI(A) \subset A$.
(2) $A \subset SG(A) \subset G(A)$.
(3) *A is semi-preconvex if and only if* $A = SI(X)$.
(4) *A is cosemi-preconvex if and only if* $A = SC(X)$.

Theorem 3.10. *Let* (X, σ) *be a preconvexity space and* $A, B \subset X$.
(1) $SI(X) = X$.
(2) $SI(A) \subset A$.

(3) *If* $A \subset B$, *then* $SI(A) \subset SI(B)$.
(4) $SI(SI(A)) = SI(A)$.

Proof. (1), (2) and (3) are obvious.
(4) Since $SI(A) \subset A$, $SI(SI(A)) \subset SI(A)$ by (3).
For the converse, let $x \in SI(A)$; then since $x \in SI(A) \subset SI(A)$ and $SI(A)$ is a semi-preconvex set, by Definition 3.8(2), we get $x \in SI(SI(A))$. □

From Theorem 3.5, Theorem 3.7, Definition 3.8, and Theorem 3.9, we have the following theorem:

Theorem 3.11. *Let* (X, σ) *be a preconvexity space and* $A, B \subseteq X$.
(1) $SG(\emptyset) = \emptyset$.
(2) $A \subset SG(A)$.
(3) *If* $A \subset B$, *then* $SG(A) \subset SG(B)$.
(4) $SG(SG(A)) = SG(A)$.

4. *sc*-convex functions and s^*c-convex functions

Definition 4.1. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be *sc*-convex if for each $A \in \mathcal{I}(Y)$, $f^{-1}(A) \in \mathcal{S}_\sigma(X)$.

Every *c*-convex function is *sc*-convex but the converse is not always true as the following example:

Example 4.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Define $A\sigma B$ to mean $A \subset cl(B)$, the closure of B in X. Then σ is a preconvexity on X. In the preconvexity space (X, σ), $\mathcal{G}(X) = \{\emptyset, X, \{b, c\}\}$, $\mathcal{I}(X) = \{\emptyset, X, \{a\}\}$ and $\mathcal{S}_\sigma(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as the following: $f(a) = a, f(b) = a, f(c) = c$. Then f is *sc*-convex but it is not *c*-convex because for the co-convex set $\{a\}$, $f^{-1}(\{a\}) = \{a, b\}$ is semi-preconvex but not co-convex.

Theorem 4.3. *Let* $f : X \to Y$ *be a function on two preconvexity spaces* (X, σ) *and* (Y, μ). *Then* f *is sc-convex if and only if for each*

$$A \subset Y,\ f^{-1}(I_\mu(A))\sigma I_\sigma(f^{-1}(A)).$$

Proof. Let f be *sc*-convex and $A \subset Y$; then since $I_\mu(A) \subset A$, by Theorem 2.3(3), we get $I_\sigma(f^{-1}(I_\mu(A))) \subset I_\sigma(f^{-1}(A))$. Since $I_\mu(A) \in \mathcal{I}(Y)$ and f is *sc*-convex, $f^{-1}(I_\mu(A))\sigma I_\sigma(f^{-1}(I_\mu(A)))$. The transitive property gives $f^{-1}(I_\mu(A))\sigma I_\sigma(f^{-1}(A))$.
For the converse, let $A \in \mathcal{I}(Y)$; then since $A = I_\mu(A)$,

$$f^{-1}(A) = f^{-1}(I_\mu(A))\sigma I_\sigma(f^{-1}(A)).$$

Thus $f^{-1}(A) \in \mathcal{S}_\sigma(X)$. □

Theorem 4.4. *Let* $f : X \to Y$ *be a function on two preconvexity spaces* (X, σ) *and* (Y, μ). *Then the following things are equivalent:*

(1) *f is sc-convex.*
(2) *$f^{-1}(I_\mu(B)) \subset G_\sigma(I_\sigma(f^{-1}(B)))$ for all $B \subset Y$.*
(3) *$I_\sigma(G_\sigma(f^{-1}(B))) \subset f^{-1}(G_\mu(B))$ for all $B \subset Y$.*
(4) *$f(I_\sigma(G_\sigma(A))) \subset G_\mu(f(A))$ for all $A \subset X$.*
(5) *For each $U \in \mathcal{G}(Y)$, $f^{-1}(U) \in \mathcal{SC}_\sigma(X)$.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 4.3 and Theorem 2.4, we get the result.

(2) $\Leftrightarrow$ (3) Let $B \subset Y$; then $f^{-1}(I_\mu(Y - B)) \subset G_\sigma(I_\sigma(f^{-1}(Y - B)))$. By Theorem 2.2, we get $f^{-1}(I_\mu(Y - B)) = X - f^{-1}(G_\mu(B))$ and $G_\sigma(I_\sigma(f^{-1}(Y - B))) = X - I_\sigma(G_\sigma(f^{-1}(B)))$. Consequently, we get (3).

Similarly, we get the converse relation.

(3) $\Leftrightarrow$ (4) Let $A \subset X$; then since $f(A) \subset Y$, (4) is obtained by (3).

The converse is obvious.

(5) $\Leftrightarrow$ (1) It is obvious. □

From Theorem 3.9 and Theorem 4.4, we get the following:

Corollary 4.5. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following things are equivalent:*

(1) *f is sc-convex.*
(2) *$f^{-1}(I_\mu(B)) \subset SI(f^{-1}(B))$ for all $B \subset Y$.*
(3) *$SC(f^{-1}(B)) \subset f^{-1}(G_\mu(B))$ for all $B \subset Y$.*
(4) *$f(SC(A)) \subset G_\mu(f(A))$ for all $A \subset X$.*

Definition 4.6. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be s^*c-convex if for each $A \in \mathcal{S}_\mu(Y)$, $f^{-1}(A) \in \mathcal{S}_\sigma(X)$.

Every s^*c-convex function is sc-convex but the converse is not always true as the following example:

Example 4.7. In Example 4.2, consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as the following: $f(a) = c, f(b) = b, f(c) = c$. Then f is sc-convex but it is not s^*c-convex because for a semi-preconvex set $\{a, b\}$, $f^{-1}(\{a, b\}) = \{b\}$ is not semi-preconvex.

Theorem 4.8. *Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is s^*c-convex if and only if for $A \subset Y$ whenever $A\mu I_\mu(A)$, $f^{-1}(A)\sigma I_\sigma(f^{-1}(A))$.*

Proof. From Theorem 3.2, it is obvious. □

Theorem 4.9. *Let a function $f : X \to Y$ be c-concave on two preconvexity spaces (X, σ) and (Y, μ). Then if f is sc-convex, then it is s^*c-convex.*

Proof. Suppose f is c-concave and sc-convex. Let $A \in \mathcal{S}_\mu(Y)$; then $A\mu I_\mu(A)$. By hypothesis and Theorem 4.3, $f^{-1}(A)\sigma f^{-1}(I_\mu(A))\sigma I_\sigma(f^{-1}(A))$. Thus from Theorem 4.8, f is s^*c-convex. □

From Theorem 4.4 and Corollary 4.5, we get the following results:

Theorem 4.10. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following things are equivalent:*

(1) *f is s^*c-convex*
(2) *$f(SC(A)) \subset SC(f(A))$ for all $A \subset X$.*
(3) *$SC(f^{-1}(B)) \subset f^{-1}(SC(B))$ for all $B \subset Y$.*
(4) *$f^{-1}(SI(B)) \subset SI(f^{-1}(B))$ for all $B \subset Y$.*
(5) *For each $U \in SC_{\mu}(Y)$, $f^{-1}(U) \in SC_{\sigma}(X)$.*

We get the following implications:

$$c - \text{convex} \Rightarrow sc - \text{convex} \Leftarrow s^*c - \text{convex}$$

References

[1] M. D. Guay, *An introduction to preconvexity spaces*, Acta Math. Hungar. **105** (2004), no. 3, 241–248.
[2] W. K. Min, *Some results on preconvexity spaces*, Bull. Korean Math. Soc. **45** (2008), no. 1, 39–44.
[3] ———, *A note on preconvexity spaces*, Honam Math. J. **29** (2007), no. 4, 589–595.
[4] S. A. Naimpally and B. D. Warrack, *Proximity spaces*, Cambridge University Press (1970).

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