Semi-primary Lattices and Tableau Algorithms

by

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Abstract

We develop the theory of semi-primary lattices, a class of modular lattices, including abelian subgroup lattices and invariant subspace lattices, in which an integer partition is assigned to every element and every interval. Flags in these lattices give rise to chains of partitions, which may be encoded as tableaux. In certain of these lattices, Steinberg and van Leeuwen respectively have shown that relative positions and cotypes, which describe configurations of elements in flags, are generically computed by the well known Robinson-Schensted and evacuation algorithms on standard tableaux. We explore extensions of this to semi-primary lattices: we consider the nongeneric configurations, leading to nondeterministic variations of the Robinson-Schensted and evacuation tableau games, and consider exact and asymptotic enumeration of the number of ways to achieve certain configurations. We also introduce other configuration questions leading to new tableau games, and develop a number of deterministic and nondeterministic tableau operators that can be combined to describe the generic and degenerate configurations of flags undergoing various transformations.

We also look at similar problems in the class of modular lattices whose complemented intervals have height at most 2, such as Stanley's Fibonacci lattice $Z(r)$. Here the generic relative position is related to Fomin's analogue of the Robinson-Schensted correspondence in $Z(1)$.

Thesis Supervisor: Richard P. Stanley
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Notation

Symbol	**Description**	**Page**
$\star$	a hole in a tableau during a game	72
$\bullet$	in a tableau of shape λ/μ, the "entry" in cells of μ; also, a cell in a Ferrers diagram	20, 22
$\#S$	cardinality of set S	
$\emptyset$	empty partition, composition, sequence, or tableau	20, 21
$\hat{0}$	minimum element of poset	16
$\hat{1}$	maximum element of poset	16
$x \vee y$	least upper bound, or "join," of x and y	16
$x \wedge y$	greatest lower bound, or "meet," of x and y	16
x^*	inclusion of x into the dual lattice	17
$[x, y][k]$	join of all cycles in $[x, y]$ of rank $\leq k$; $x[k] = [\hat{0}, x][k]$	35
$[x, y](k)$	meet of all cocycles in $[x, y]$ of corank $\leq k$; $x(k) = [x, \hat{1}](k)$	35
$\vec{x}[[\lambda]]$	$x_1[\lambda_1] \vee x_2[\lambda_2] \vee \cdots$	51
$\vec{x}((\lambda))$	$x_1(\lambda_1) \wedge x_2(\lambda_2) \vee \cdots$	51
$x \lessdot y$	cover relation: $x < y$ and nothing is between them	16
$x \leq y$	$x \lessdot y$ or $x = y$	16
$i \lessdot j$	$i = j - 1$ in $\mathbb{Z}$	16
$\mu \leq \lambda$	$\mu_i \leq \lambda_i$ for all $i > 0$	21
$\mu \leq_{\boxed{e}} \lambda$	$\mu \leq \lambda$, and fill λ/μ with e	75
$\mu \leq_{\vee} \lambda$	$\mu \leq \lambda$ and λ/μ is a vertical strip	56
$\mu \lessdot_{\lvert r \rvert} \lambda$	$\lambda'_r \gtrdot \mu'_r$ and $\lambda'_i = \mu'_i$ for $i \neq r$	77
$\mu \lessdot_x \lambda$	$\lambda \gtrdot \mu$ and $\lambda = \mu \oplus x$	75
$\lambda \vdash n$	$\sum_i \lambda_i = n$	20
$\mu \sqsubseteq \lambda$	subcolumn partition: $m_i(\mu') \leq m_i(\lambda')$ for $i > 0$	53
λ'	conjugate partition: $\lambda'_j = \#\{\, i \,:\, \lambda_i \geq j \,\}$	20
$\boldsymbol{A} + \boldsymbol{B}$	append $\boldsymbol{B}$ to $\boldsymbol{A}$ rowwise	60
$\lambda + \mu$	$(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$	21
$\lambda \cup \mu$	$(\lambda'_1 + \mu'_1, \lambda'_2 + \mu'_2, \ldots)'$	21
$\lambda \oplus \mu$	union of subsets of $\mathbb{P} \times \mathbb{P}$	21
$\lambda \ominus \mu$	set difference of subsets of $\mathbb{P} \times \mathbb{P}$	21
λ/μ	a skew partition or composition; an ordered pair with $\mu \leq \lambda$	21
$\lvert\lambda\rvert$	$\sum_i \lambda_i$	20
$\binom{n}{k}$	binomial coefficient $n!/(k!(n-k)!)$	
$\langle n \rangle$	$(q^n - 1)/(q - 1) = 1 + q + \cdots + q^{n-1}$	44
$\langle n \rangle!$	$\langle 1 \rangle \cdot \cdots \cdot \langle n \rangle$, $\langle 0 \rangle! = 1$	44
Ax	join of all weak upper covers of x	25
$A_y x$	join of all weak upper covers of x in $[x, y]$	25
$\mathcal{A}_\lambda \mu$	partition with ith part $\min\{\mu_i + 1, \lambda_i\}$	52
ALR	Littlewood-Richardson sequence of an element	63
$\mathrm{arank}[x, y]$	atomic rank of $[x, y]$	30
$\mathcal{C}_\mu \lambda$	partition with ith part $\max\{\lambda_i - 1, \mu_i\}$	52
$\mathcal{C}\lambda$	$(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$ where $\lambda_k > \lambda_{k+1} = 0$	40
Cy	meet of all weak lower covers of y	25

$C_x y$	meet of all weak lower covers of y in $[x, y]$	25
CLR	Littlewood-Richardson sequence of an element	63
$\mathrm{col}(\boldsymbol{P}, k)$	column of entry k in $\boldsymbol{P}$	23
$\mathrm{entry}(\boldsymbol{P}, c)$	entry at cell c of $\boldsymbol{P}$	64
$F^{\boldsymbol{P}}(q)$	number of flags of type $\boldsymbol{P}$	45
$f_\Lambda(q)$	number of flags with itype $f = \Lambda$	89
$\Gamma_{\mathcal{L}}(x)$	complements to x in $\mathcal{L}$	49
$g^\lambda_{\mu\nu}(q)$	Hall polynomial	62
$\mathrm{high}(f)$	high index used in a flag or tableau	16, 23
λ	lowercase greek letters are partitions or compositions	20
Λ	interval type table, meet type table, A-statistics	73, 85, 131, 161, 173, 178
Λ'	transpose, dependent on structure	87, 131
$\mathcal{L}$	lattice	
$L(G)$	lattice of subgroups of G	24
$L(V, N)$	lattice of N-invariant subspaces of V	24
$\mathrm{LR}_*, \mathrm{LR}^*$	extremal Littlewood-Richardson sequences	67
$\mathrm{low}(f)$	low index used in a flag or tableau	16, 23
$M_\lambda(q)$	$\prod_i \langle m_i(\lambda)\rangle!$	46
$m_i(\lambda)$	multiplicity: number of parts of λ of size i	20
$\mathbb{N}$	$\{0, 1, 2, \ldots\}$	
$n(\lambda)$	$\sum_{i>0}(i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}$	21
$n(\lambda/\mu)$	$n(\lambda) - n(\mu)$	21
$n_0(\lambda)$	$\sum_{i>0} \binom{\lambda'_i - \lambda'_{i+1}}{2}$	46
$\mathrm{osh}\, \boldsymbol{P}$	outer shape of tableau $\boldsymbol{P}$	22
$O(q^{-1})$	a function whose product with q is bounded by a constant as $q \to \infty$	
$\mathbb{P}$	$\{1, 2, 3, \ldots\}$	
$\boldsymbol{P}$	uppercase bold italic letters are tableau	22
$\rho(x)$	rank of x	16
$\bar{\rho}(x)$	corank of x	16
$\mathrm{row}(c)$	row of cell c	23
$\mathrm{row}(\boldsymbol{P}, k)$	row of entry k in $\boldsymbol{P}$	23
$S(\lambda)$	straightening: arrange parts of λ in decreasing order	67
$S(\boldsymbol{P})$	straightening: arrange entries of each column in order, and remove gaps	67
$\mathrm{sh}\, \boldsymbol{P}$	skew shape of tableau $\boldsymbol{P}$	22
$\mathrm{cotype}\, x$	$\mathrm{type}[x, \hat{1}]$	34
$\mathrm{Atype}[x, y]$	$(\rho(A_y^{k-1}x, A_y^k x))_{k>0}$	28
$\mathrm{Ctype}[x, y]$	$(\rho(C_x^k y, C_x^{k-1} y))_{k>0}$	28
$\mathrm{ftype}\, f$	$(\mathrm{type}[f_l, f_l], \mathrm{type}[f_l, f_{l+1}], \ldots, \mathrm{type}[f_l, f_h])$	45
$\mathrm{ftype}_x\, f$	$(\mathrm{type}[x, f_l], \mathrm{type}[x, f_{l+1}], \ldots, \mathrm{type}[x, f_h])$	45
$\mathrm{itype}\, f$	$(\mathrm{type}[f_i, f_j])_{0\le i\le j\le n}$	76
$\mathrm{itype}\, \boldsymbol{P}$	$(S(\lambda^{(j)} - \lambda^{(i)}))_{l\le i\le j\le h}$	102
$\mathrm{type}[x, y]$	type of interval in a semi-primary lattice	34
$\mathrm{type}\, x$	$\mathrm{type}[\hat{0}, x]$	34

vlen	vertical strip length	152
$\mathbb{Z}$	$\{0, \pm 1, \pm 2, \ldots\}$	

Fibonacci lattice notation.

$x \wedge y$	greatest lower bound in $Z(r)$	169
$x \vee y$	least upper bound in $Z(r)$	169
$x \leq_F y$	$x \leq y$ in $Fib(r)$	165
$x \leq_Z y$	$x \leq y$ in $Z(r)$	165
$x \lessdot_1 y$	$x = 2^k\alpha$ and $y = 2^k 1_i \alpha$ in $Z(r)$	165
$x \lessdot_2 y$	$x = 2^k 1_i \alpha$ and $y = 2^k 2\alpha$ in $Z(r)$	165
$x \lessdot_{1_*} y$	$y = x1$ in Z_*	172
$x \lessdot_{2_*} y$	$x = \alpha 11^m$ and $y = \alpha 21^m$ in Z_*	172
$\kappa^+(x)$	number of upper covers of x	165
$\kappa^-(x)$	number of lower covers of x	165
μ, ν, τ	decomposition of a pair of words	166
$Fib(r)$	one of the Fibonacci lattices	165
$Z(r)$	one of the Fibonacci lattices	165
Z_*	restriction of $Fib(1)$	172

Operations on flags.

flag g	low(g)	high(g)
$f{=}(f_l, f_{l+1}, \ldots, f_h)$	l	h
$\partial f{=}(f_{l+1}, \ldots, f_h)$	$l+1$	h
$df{=}(f_l, \ldots, f_{h-1})$	l	$h-1$
$Cf{=}(C_{f_l} f_l, \ldots, C_{f_l} f_h)$	l	h
$C_x f{=}(C_x f_l, \ldots, C_x f_h)$	l	h
$Af{=}(A_{f_h} f_l, \ldots, A_{f_h} f_h)$	l	h
$A_x f{=}(A_x f_l, \ldots, A_x f_h)$	l	h
$f^*{=}(f_h^*, \ldots, f_l^*)$	$n-h$	$n-l$
$f \vee x{=}(f_l \vee x, \ldots, f_h \vee x)$	l	h
$f \wedge x{=}(f_l \wedge x, \ldots, f_h \wedge x)$	l	h
$f^s{=}$ saturate f	l	h

Tableau games.

	LR-sequences of adjacent elements	66
$\mathcal{A}$	delete rowends, prepend $\bullet$'s	103
$\mathcal{C}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} Cf$	delete first column	103, 117
$\mathcal{C}: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} C_{\hat{0}} f$		
$\mathcal{C}\mathcal{L}, \mathcal{C}\widetilde{\mathcal{L}}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} Af$		157
$\Delta, \widetilde{\Delta}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} \partial f$	elementary evacuation step	72, 76, 116
$\partial: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} \partial f$	$l+1 \mapsto \bullet$	116
$\partial: \ \operatorname{ftype} f \longmapsto \operatorname{ftype}_{f_l} \partial f$		
$\mathcal{D}_c, \widetilde{\mathcal{D}}_c: \ \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \wedge x)$	Robinson-Schensted deletion step $f_l \leq x \lessdot f_h$ and $c = \lambda/(\operatorname{type}[f_l, x])$	126
$\mathcal{D}_c, \widetilde{\mathcal{D}}_c: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \wedge x)$	$x \lessdot f_h$ and $c = \lambda/\operatorname{type} x$	
$\mathcal{D}^*_c, \widetilde{\mathcal{D}^*_c}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \vee x)$	transposed R-S deletion step $x \gtrdot f_l$ and $c = \lambda/(\operatorname{type}[f_l \vee x, f_h \vee x])$	137
$\mathcal{D}^*_c, \widetilde{\mathcal{D}^*_c}: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_x(f \vee x)$	$x \gtrdot \hat{0}$ and $c = \lambda/\operatorname{type}[x, f_h \vee x]$	
$d: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} df$	remove cells with h	103, 117
$d: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} df$		
$\operatorname{ev}, \widetilde{\operatorname{ev}}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} f^*$	Schützenberger's evacuation	121
$\operatorname{ev}^\lambda, \widetilde{\operatorname{ev}}^\lambda: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}^*} f^*$		121
$\mathcal{I}_c, \widetilde{\mathcal{I}}_c: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \vee x)$	internal insertion $x \gtrdot \hat{0}$ and $c = \operatorname{type} x/\operatorname{type} f_l$	140
$j, \tilde{j}: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype} f$	*jeu de taquin*	115
$j^c, \tilde{j}^c: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_x f$	*jeu de taquin* slide $x \gtrdot \hat{0}$ and $c = \operatorname{type}[\hat{0}, f_l]/\operatorname{type}[x, f_l]$	112
$\mathcal{L}, \widetilde{\mathcal{L}}: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Af$	leftward vertical strip game	145
$\mathcal{R}, \widetilde{\mathcal{R}}: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Cf$	rightward vertical strip game	150
$\mathcal{S}, \widetilde{\mathcal{S}}: \ \operatorname{ftype} f \longmapsto \operatorname{ftype} f^s$	saturate a row and column weak tableau	
$\mathcal{S}, \widetilde{\mathcal{S}}: \ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} f^s$		118
$\mathcal{S}^{-1}: \ \operatorname{ftype} f^s \longmapsto \operatorname{ftype} f$	unsaturate	120
$\mathcal{S}^{-1}: \operatorname{ftype}_{\hat{0}} f^s \longmapsto \operatorname{ftype}_{\hat{0}} f$		

Games $\mathcal{G}$ are deterministic, and follow the same rules as the ordinary game by that name when such a game was already known. Games $\widetilde{\mathcal{G}}$ are nondeterministic, with many possible outcomes, among which is $\mathcal{G}$. Most maps are uniform (see Section 5.3), so composition of flag operators induces composition of the deterministic games. The only exceptions are $\mathcal{S}$ (similar to uniform, see Section 6.8); $\mathcal{D}_c$ and $\widetilde{\mathcal{D}}_c$ (generic but not uniform); $\mathcal{I}_c$ (neither generic nor uniform); and $\mathcal{A}$ and LR-sequence transformations (tableau games describing transformations of a different nature than the others).

Part I. Background

1. Introduction

An **integer partition** of n is a weakly decreasing sequence of nonnegative integers, $\lambda_1 \geq \cdots \geq \lambda_k$, whose sum is n. It may be represented by a diagram consisting of left justified rows of boxes, where the ith row from the top has λ_i boxes. Young's lattice is the set of all partitions ordered by $\mu \leq \lambda$ iff the diagram of μ is contained in the diagram of λ. A **standard tableau** of shape λ is a filling of the boxes of the diagram of λ with the numbers $1, \ldots, n$ so that each row increases from left to right, and each column increases from top to bottom. Robinson, Schensted, and others created an algorithmic bijection between ordered pairs of standard tableaux of the same shape with entries $1, \ldots, n$, and permutations of $1, \ldots, n$. Schützenberger introduced an involution ev on standard tableaux of any given shape, also algorithmic in nature; a good exposition of these algorithms may be found in Sagan [**22**]. The purely algorithmic Robinson-Schensted map was found by Steinberg [**29**] to describe naturally certain properties of the flag variety of all subspaces of V fixed by a given nilpotent transformation N, where V is a vector space over an infinite field. Hesselink [**10**] and van Leeuwen [**31**] found that ev naturally describes other properties in the same flag variety.

To any pair $W_1 \subset W_2$ of N-invariant subspaces, we may associate an integer partition called the **type** of the quotient W_2/W_1, by examining the Jordan canonical form of the action of N on the quotient W_2/W_1; it is a block sum of matrices with 1's on the main diagonal and the diagonal just above it. The lengths of the main diagonals, sorted into decreasing order, form an integer partition, denoted $\mathrm{type}(W_2/W_1)$ or in lattice terms, $\mathrm{type}[W_1, W_2]$. Any subinterval $[W_1', W_2']$ (i.e., W_1', W_2' are N-invariant and $W_1 \subseteq W_1' \subseteq W_2' \subseteq W_2$) has a smaller type in Young's lattice. Given a saturated flag $0 = W_0 \subset W_1 \subset \cdots \subset W_n = V$ of N-invariant subspaces of V, form a chain of increasing partitions in Young's lattice: $\mathrm{type}(W_0/0), \ldots, \mathrm{type}(W_n/0)$. This chain can be encoded as a standard Young tableau by filling the square added in the ith term with i. The tableau so obtained is called the type of the flag. The cotype of the flag is the tableau encoding the chain of partitions $\mathrm{type}(V/W_n), \ldots, \mathrm{type}(V/W_0)$; van Leeuwen showed that, in the Zariski topology, the generic cotype of a flag of type P is $\mathrm{ev}(P)$. The **relative position** of two flags in a modular lattice is a permutation that measures the distance between two flags; the higher the permutation is in the Bruhat order, the farther apart the flags are. Steinberg showed that, in the Zariski topology, the generic relative position of flags of type P and Q is the same permutation that the Robinson-Schensted algorithm associates with the pair (P, Q).

A **semi-primary lattice** is a modular lattice of finite height in which, for every join-irreducible x, the interval $[\hat{0}, x]$ is a chain, and for every meet-irreducible x, the interval $[x, \hat{1}]$ is a chain; see Jónsson and Monk [**13**] (for background on lattices see Stanley [**27**]). A **q-regular semi-primary lattice** is a semi-primary lattice in which every interval of length 2 is either a chain or has exactly $q+1$ atoms. The lattice of N-invariant subspaces of a vector space over any field (not only the complex numbers) is a semi-primary lattice, and if the field is finite of order q, it is q-regular. The lattice of subgroups of a finite abelian p-group is another semi-primary lattice, and is p-regular. As with the first example, an integer partition may be assigned to any interval in the lattice; the quotient of two groups is isomorphic to a product of cyclic p-groups, $\mathbb{Z}/p^{\lambda_1} \times \cdots \times \mathbb{Z}/p^{\lambda_k}$, and if the λ_i are arranged in weakly decreasing order, they form the integer partition corresponding to the type of the quotient. There is a lattice theoretic definition of this partition for all semi-primary lattices, coinciding with the partitions given in these two cases: for an interval $[x, y]$, express y as the join of independent join-irreducibles over x (the meet of any one with the join of the others is x). Sort the ranks (as measured in $[x, y]$) of these irreducibles into weakly decreasing order to obtain a partition. This partition turns out to be independent of what independent join-irreducibles were chosen, and is taken as the definition of $\mathrm{type}[x, y]$.

Lattices and Combinatorial Algorithms. Although objects that are instances of semi-primary lattices have been studied at length, they have not generally been considered as semi-primary lattices, and the results about them do not always immediately apply to other semi-primary lattices. The generic flag cotype and relative positions described previously do not apply if the field is not infinite, because the notion of "genericity" and the proofs of these results are critically dependent on the Zariski topology. We must reformulate the problems and find new methods of proof if we are to solve them in the wider context of semi-primary lattices. We consider probabilistic and enumerative analogues and degenerate cases.

For q-regular semi-primary lattices, a probabilistic analogue of Hesselink and van Leeuwen's result is that all but a fraction $O(q^{-1})$ of the flags of type P have cotype $\mathrm{ev}(P)$ (see Theorem 5.14). An enumerative extension is the following empirical observation, verified for certain cases: the number of flags of type P whose cotype is Q is often a polynomial in q, and the degree of this polynomial is maximized when $Q = \mathrm{ev}(P)$. We describe a refinement of these polynomials that allows us to analyze the degenerate behavior of flags in semi-primary lattices. Denote a saturated flag by $f = (f_0 \lessdot \cdots \lessdot f_n)$. Given partitions $\Lambda = (\lambda^{(ij)})$ (where $0 \leq i \leq j \leq n$) the number of flags in a q-regular semi-primary lattice whose **interval type table** is $\mathrm{type}[f_i, f_j] = \lambda^{(ij)}$ is often a polynomial in q depending only on the partitions Λ, and on no further structure of the lattice; in particular, the polynomials are the same for abelian subgroup lattices and invariant subspace lattices. (In the case of invariant subspaces over an infinite field, the dimension of the closure of the set of flags achieving this type table equals the degree of this polynomial, but any meaning the lower order terms might have is lost.) This polynomial generalizes Hall polynomials (see Macdonald [**16**] for a comprehensive treatment), which may be used to count the number of flags in which only certain intervals' types are specified. These polynomials have been computed for all arrays Λ with $n \leq 8$ (see Appendix A), and for a class of arrays that includes all those occurring for lattices whose type has two columns (see Section 5.6).

The Fibonacci Lattice. Fomin [**4**, **5**] has generalized the Robinson-Schensted algorithm from standard Young tableaux, which encode chains in Young's lattice, to chains in certain other lattices. One such lattice is the **Fibonacci Lattice** $Z(r)$, introduced by Stanley [**28**]. It consists of words from the alphabet $\{1_1, \ldots, 1_r, 2\}$, in which y covers x iff some 1_i in x preceded only by 2's is changed to a 2, or a 1_i is inserted in a position preceded only by 2's. Fomin bijectively associates permutations in S_n to pairs of saturated chains in $Z(1)$ from $\hat{0}$ to equal elements of rank n.

This is a direct analogue of the Robinson-Schensted algorithm in Young's lattice. In Section 9, we introduce **strongly modular lattices**, the class of modular lattices whose complemented intervals all have height at most 2, of which $Z(r)$ is an example. We study this in Section 9. Steinberg's result about the Robinson-Schensted algorithm for invariant subspace lattices over infinite fields has an analogue for strongly modular lattices: the relative position of two flags is bounded above in the Bruhat order by a variation of Fomin's Robinson-Schensted algorithm for $Z(1)$ applied to the flags' types, where the types are chains of integer compositions given by a generalized lattice interval type defined in Section 3.3. For $Z(r)$, the number of pairs of flags of prescribed types in a prescribed relative position is a polynomial in r, whose degree is maximized when the relative position is the same permutation assigned given by the modified Robinson-Schensted algorithm. Fomin does not have an analogue of Schützenberger's evacuation for $Z(r)$, but there is an analogue of the interval type table for strongly modular lattices. In $Z(r)$, the number of ways of achieving a particular interval type table is a polynomial in r, and there is a generic interval type table for flags of a given type.

Overview. Part I includes this introduction, and continues with a thorough grounding in the required background on lattices, partitions, and tableaux.

In Part II, we develop semi-primary lattices, describe their basic properties, representation, enumeration, and prior work on them. In Section 3, we introduce new lattice operators A and C that generalize $\ker N$ and NV in invariant subspace lattices, or $\operatorname{socle} G$ and pG for p-subgroup lattices, to discrete modular lattices whose complemented intervals have finite length. This includes semi-primary lattices and the Fibonacci lattice. Integer compositions, coinciding with the interval type previously defined for semi-primary lattices, can be assigned to any interval in terms of these operators A and C. These generalized interval types, and the new operators, are studied in a general setting.

In Section 4, we develop the fundamentals of semi-primary lattices. We begin in Section 4.1 by developing them essentially as Jónsson and Monk [**13**] did, but with ties to the new lattice operators. We also establish that the type of an interval is larger (in Young's lattice) than the type of any subinterval. In Section 4.2, we introduce several new characterizations of semi-primary lattices: they may be characterized by lattice polynomials, rank equations, and local conditions in the lattice, in much the same way that modular lattices can. They also may be characterized in terms of properties of the generalized lattice type. In Section 4.3, we discuss what is known about representing many semi-primary lattices as submodule lattices.

In Sections 4.4–4.5, we introduce enumeration in semi-primary lattices. The number of ways to achieve various configurations of elements is often a polynomial in the parameter q described previously. In Section 4.6, we completely characterize and enumerate complemented elements in semi-primary lattices, generalizing what was already known for particular cases.

The notion of a basis in a vector space or matroid has a counterpart in semi-primary lattices. In Sections 4.7–4.8, we enumerate such bases, determine when different elements have similar bases, and analyze lattice polynomials whose variables are elements of a basis.

The Littlewood-Richardson sequence of an element is a statistic introduced by Green [**8**] as a means of computing Hall polynomials. In Section 4.9, we describe Hall polynomials, and then in Sections 4.10–4.11, we describe new properties of these statistics: how these statistics change for adjacent elements; using them to determine the most and least likely cotype of an element given its type; and using them to determine the types between certain pairs of elements in a flag.

The type of a flag is the tableau encoding the chain of partitions that are the types of the elements in the flag. In Part III we discuss how various operators transform flag types. First, in Section 5, we look at intervals between elements of a flag. We describe Schützenberger's evacuation algorithm, the connection Hesselink and van Leeuwen established between it and the generic cotype of a flag of specified type in invariant subspace lattices over infinite fields, and then establish the probabilistic counterpart for it. We then describe an algorithm for computing how many flags have particular interval type tables (computer runs of which are summarized in Appendix A), and conclude with a specific formula for this polynomial for a large class of interval type tables, including all those occurring in lattices whose type has two columns.

In Section 6, we consider many more operations on flags. Given a flag of type $\boldsymbol{P}$ and an operator ϕ, the flag $\phi(f)$ can usually have many types, but a particular one is the most common. The ordinary evacuation, *jeu de taquin*, and Robinson-Schensted games are among those that describe the most common types associated with certain operators. Nondeterministic variants of these games describe other possible values of the type: the ordinary rules of these games are modified to allow choices of several moves at each step, one of which is the usual move. We enumerate the number of pairs of flags $(f, \phi(f))$ where f and $\phi(f)$ have prescribed types. We show how composition of flag operators leads to composition of tableaux games, and determine when composition of the deterministic games yields the most likely value of the composition of the corresponding flag operators. All these games are for saturated flags, so in Section 6.8, we describe a canonical way to transform games on standard tableaux to games on row and column weak tableaux, reflecting how the corresponding operators behave on nonsaturated flags. This transformation generalizes that of Knuth [**15**] for the Robinson-Schensted algorithm on semistandard tableaux.

In Section 7, we describe the Robinson-Schensted correspondence and its relation to relative positions of flags. We are not as successful in providing probabilistic and enumerative counterparts of Steinberg's results as we were for Hesselink and van Leeuwen's in Section 5, but we present partial results and conjectures.

In Section 8, we introduce new tableau games, similar in spirit to the Robinson-Schensted and evacuation algorithms. Instead of sliding or bumping numbers entries along a single path as is done in the other games, a vertical strip of holes is formed in a tableau and numbers slide into and out of it. These games and their nondeterministic versions describe how the new operators A and C behave on flags. A connection between these games and evacuation and the *jeu de taquin* is discussed in Section 8.3. A generalization of the interval type table polynomials based on these games is described in Section 8.5.

Finally, in Part IV, we discuss strongly modular lattices, including Fibonacci lattices and semi-primary lattices whose type has two rows. The Fibonacci lattice $Z(r)$

was defined by Stanley [**28**] in terms of its cover relations. In Section 9.1, we give a simple direct test, apparently new, to determine if two elements are comparable, without having to exhibit a chain of elements from one to the other satisfying this cover relation. We then compute the type of any interval in $Z(r)$; our generalized lattice type yields two different integer compositions, whose parts are 1's and 2's, as types of intervals in strongly modular lattices.

While Young's lattice describes the relation between the type of an interval and the type of its subintervals in a semi-primary lattice, $Z(1)$ and a restriction of $Fib(1)$ describe the analogous problem in strongly modular lattices: lowering the upper boundary of an interval causes the type to decrease in one of these, and raising the lower boundary causes the type to decrease in the other. We describe this in Section 9.2. Next, in Section 9.3, we develop interval type tables for strongly modular lattices, by considering these properties of types of subintervals. We determine the "generic" interval type table of a flag given its type in a strongly modular lattice, and we quantify this for $Z(r)$ by computing how many flags achieve each possible interval type table. It is a polynomial in r, and the polynomials for the "generic" interval type tables have the maximal degrees.

We conclude with an analysis of relative positions of flags in strongly modular lattices. A variation of Fomin's version of the Robinson-Schensted game in $Z(1)$ describes the maximum relative position of flags in any strongly modular lattice, and a nondeterministic variant of the game gives the other possible relative positions. As a consequence, the conjectured maximum relative position for semi-primary lattices does in fact hold for two row semi-primary lattices. In $Z(r)$, we compute polynomials for how many pairs of flags have specified types and relative positions, and these polynomials have maximum degree precisely when the relative position is the permutation given by this modified Robinson-Schensted algorithm on the types of the flags.

2. Review of combinatorial objects

2.1. Posets. Our notation and definitions for partially ordered set and lattice notions are those in Stanley [**27**]; consult that or Dilworth [**2**] for further information.

A **partially ordered set** P, or **poset**, is a set (also denoted P by abuse of notation) with a binary relation $\leq$ satisfying three axioms.

Reflexive: For all $x \in P$, we have $x \leq x$.
Antisymmetric: If $x \leq y$ and $y \leq x$, then $x = y$.
Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$.

The notation $x < y$ means $x \leq y$ and $x \neq y$, while $y \geq x$ means $x \leq y$, and $y > x$ means $x < y$. When dealing with multiple posets, or multiple partial orders on the same ground set, use subscripts, such as $\leq_P$, or other special symbols, to specify which order is to be used. The relation $\leq$ is **weak** while $<$ is **strict**. Elements x and y of P are **comparable** if either $x \leq y$ or $y \leq x$, and are **incomparable** otherwise.

Two posets P and Q are **isomorphic** if there is a bijection $\phi : P \to Q$ such that $x \leq y$ in P iff $\phi(x) \leq \phi(y)$ in Q.

An **induced subposet** of P (often abbreviated to **subposet**) is a subset Q of the elements of P with the order, if x and y are in Q then $x \leq y$ in Q iff $x \leq y$ in P. A particular subposet of interest is the **closed interval** $[x, y] = \{ z \in P : x \leq z \leq y \}$ with the order induced by P, defined for all x, y with $x \leq y$ in P.

We say that y **covers** x in P if $x < y$ and no element $z \in P$ satisfies $x < z < y$. Equivalently, we say x **is covered by** y, that x **is a lower cover of** y, and that y **is an upper cover of** x. This may be written as $y \gtrdot x$ or $x \lessdot y$. The notations $y \geq x$ and $x \leq y$ mean that either $x = y$ or $x \lessdot y$. For integers x, y we have $x \lessdot y$ iff $x = y - 1$, and $x \leq y$ iff $x = y$ or $x = y - 1$.

A poset P has a **minimum element** $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $x \geq \hat{0}$ for all $x \in P$. Dually, P has a **maximum element** $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$. When dealing with multiple posets, use subscripts, $\hat{0}_P$ and $\hat{1}_P$, to differentiate these elements. For example, $\hat{0}_{[x,y]} = x$ and $\hat{1}_{[x,y]} = y$. When dealing with subposets Q of P, denote the minimum and maximum elements of P by $\hat{0}$ and $\hat{1}$, and those of Q by $\hat{0}_Q$ and $\hat{1}_Q$. An element x is **minimal** if for no y does $y < x$ hold, and an element x is **maximal** if for no y does $y > x$ hold.

A **flag** in a poset is a sequence of elements $f = (f_l, f_{l+1}, \ldots, f_h)$ with $f_l \leq f_{l+1} \leq \cdots \leq f_h$. To indicate a more specific relationship between consecutive elements in a flag, such as equality, strictly less than, or weakly or strongly covered by, replace the commas with this relationship. A **saturated flag** has the form $f = (f_l \lessdot f_{l+1} \lessdot \cdots \lessdot f_h)$ and a **multisaturated flag** has the form $f = (f_l \leq f_{l+1} \leq \cdots \leq f_h)$. We may need to be concerned with the indices of the elements of a flag, so define $\mathrm{low}(f) = l$ and $\mathrm{high}(f) = h$. Define operators that remove the bottom and top elements of a flag by $\partial f = (f_{l+1}, f_{l+2}, \ldots, f_h)$ and $df = (f_l, f_{l+1}, \ldots, f_{h-1})$. A **chain** is synonymous with a flag.

The **length of a saturated chain** $(f_l, \ldots, f_h)$ is $h - l$. A poset is **graded** if whenever $x \leq y$, all saturated chains from x to y have the same length, which is denoted $\rho(x, y)$. In posets with $\hat{0}$, the **rank of** x is $\rho(x) = \rho(\hat{0}, x)$, and in posets with $\hat{1}$, the **corank of** x is $\bar{\rho}(x) = \rho(x, \hat{1})$. A graded poset has **finite length** if all saturated chains have finite length. This does not require the poset to have a finite number of elements; the subspaces of a finite dimensional vector space, ordered by subspace inclusion, form a poset (actually a modular lattice, defined below) of finite length, but if the field is infinite, there will be an infinite number of subspaces (in dimensions larger than 1). From now on, we will only consider graded posets in which every closed interval has finite length, even if the poset itself is not of finite length.

2.2. Lattices. If x and y are in a poset P, an **upper bound** is some $z \in P$ such that $z \geq x$ and $z \geq y$. A **least upper bound** of x and y is an upper bound z such that every upper bound w satisfies $z \leq w$. In general, neither an upper bound nor a least upper bound exists, but if a least upper bound exists, it is unique, and is denoted $x \vee y$, which is read "x join y." Dually, the greatest lower bound, if it exists, is denoted $x \wedge y$, which is read "x meet y." A lattice is a poset for which every pair of elements has a least upper bound and a greatest lower bound. Alternately, we may define a lattice as a set with a pair of binary operations $\wedge$ and $\vee$, obeying the following laws.

Associative: $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
Commutative: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.
Idempotent: $x \vee x = x \wedge x = x$.
Absorption: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

The partial order description and the binary operation description can be recovered from each other via $x \wedge y = x$ iff $x \vee y = y$ iff $x \leq y$. Because meet and join are associative and commutative, any finite set of elements in a lattice has a meet and a join. If all infinite sets of elements also have meets and joins, a lattice is called **complete**. In particular, all lattices of finite length are complete.

All lattices of finite length have $\hat{0}$ and $\hat{1}$.

The **dual** of a poset $(P, \leq)$ is $(P^*, \leq^*)$, with the same ground set as P but the reverse order. For $x \in P$, let x^* be inclusion of x into P^*. The order is $x^* \leq^* y^*$ iff $y \leq x$. We have $\hat{0}^* = \hat{1}$ when either exists, and $\hat{1}^* = \hat{0}$ when either exists. In a lattice, $x^* \vee^* y^* = (x \wedge y)^*$ and $x^* \wedge^* y^* = (x \vee y)^*$. Applying an operator, definition, or other notion to the dual of a poset is referred to as dualizing that notion, and often, we can dualize theorems by replacing every notion in its statement and proof with the dual notion.

In any lattice, a nonzero element x is **join-irreducible** if it cannot be expressed as $x = y \vee z$ with $y < x$ and $z < x$. The element $\hat{0}$ is not considered to be a join-irreducible. A local condition for an element to be join-irreducible is that it covers precisely one element. **Meet-irreducibles** are defined dually as elements $x \neq \hat{1}$ which cannot be expressed as $x = y \wedge z$ with $y > x$ and $z > x$. A local condition for an element to be meet-irreducible is that it is covered by precisely one element.

A special type of join-irreducible is an **atom**, which is an element that covers $\hat{0}$. The atoms of an interval $[x, y]$ are the elements of the interval that cover x. Dually, a **coatom** is any element covered by $\hat{1}$, and a coatom of an interval $[x, y]$ is any element of the interval that is covered by y. A lattice is **atomic** if every element can be expressed as the join of atoms, and dually, is **coatomic** if every element can be expressed as the meet of coatoms.

A **complement** to an element x in a lattice is some y with $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. While $\hat{0}$ and $\hat{1}$ are always unique complements to each other, it is not always true that elements have complements, or that complements are unique when they do exist. A **complemented lattice** is a lattice in which all elements possess complements.

There are several classes of lattices of interest in combinatorics. An **upper semimodular** lattice is a lattice in which all closed intervals have finite length, and the following local condition is satisfied: if x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y. If a lattice is of finite length, it is upper semimodular iff it is graded and its rank function satisfies $\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$ for all x, y in the lattice. Lower semimodular lattices are defined dually by inverting the meets, joins, and inequalities.

A **modular** lattice is a lattice that is both upper and lower semimodular. That is, $x \vee y$ covers both x and y iff both x and y cover $x \wedge y$. A lattice of finite length is modular iff it is graded and its rank function satisfies $\rho(x)+\rho(y) = \rho(x \wedge y)+\rho(x \vee y)$. In addition to this local condition and the rank equation, there is also a lattice identity that applies to all lattices without any finiteness restrictions: a lattice is modular iff $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

A **distributive** lattice is a lattice in which all x, y, z satisfy either of the equivalent equations $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Every distributive lattice is modular, but not conversely. For example, the lattice of all subsets of a set, ordered by set inclusion, has union and intersection as meet and join, and is distributive. However, the lattice of all subgroups of an abelian group, ordered by subgroup inclusion, is modular but usually not distributive.

We hope to add semi-primary lattices to this list. We will show that in addition to the classical definitions of semi-primary lattices, they, too, may be defined by local conditions, rank equations, and lattice identities similar to the ones used for the preceding classes of lattices.

2.3. Properties of modular lattices. The following facts about modular lattices will be frequently used.

Lemma 2.1. *For any x, y in a modular lattice $\mathcal{L}$, there is an isomorphism from $[x \wedge y, y]$ to $[x, x \vee y]$ preserving meets and joins. The intervals are said to be* ***perspective****. The isomorphism is given by $z \mapsto z \vee x$, and its inverse is $z \mapsto z \wedge y$. Conversely, any lattice of finite length in which these maps are inverse for all x, y is modular.*

Proof. Let x, y be in a modular lattice $\mathcal{L}$. For any $z \in [x \wedge y, y]$, the composition of the two maps given is $(z \vee x) \wedge y$. By the modular law, this equals $z \vee (x \wedge y)$, which is z, since everything in this interval is at least $x \wedge y$. Similarly, for $z \in [x, x \vee y]$, the composition of the maps in the other order is the identity.

For the detailed proof of the converse, consult any reference on lattice theory, such as [**2**, pp. 22–23] (which actually proves it in a class of lattices called **compactly generated lattices** that is broader than finite length lattices). □

Lemma 2.2. *Let $[x, y]$ be an interval of finite length in a modular lattice $\mathcal{L}$. For any $z \in \mathcal{L}$ such that x, y, z lie in an interval of finite length, $\rho(x \wedge z, y \wedge z) \leq \rho(x, y)$, and if equality holds, $[x \wedge z, y \wedge z]$ and $[x, y]$ are perspective. Dually, the same statements apply with meets replaced by joins.*

Proof. Let $w = y \wedge z$, so that $[x \wedge z, y \wedge z] = [x \wedge w, w] \cong [x, x \vee w]$. Then $x \vee w \leq y$ because $x, w \leq y$, so the interval $[x, x \vee w]$ is contained in $[x, y]$ and hence has weakly smaller length. If the lengths are the same then $x \vee w = y$ so $[x \wedge z, y \wedge z]$ and $[x, y]$ are the perspective pair $[x \wedge w, w] \cong [x, x \vee w]$.

The dual statements are proved similarly. □

Lemma 2.3. *In a modular lattice, if $y \gtrdot x$, then for any z such that x, y, z are contained in an interval of finite length, either*

(1) *$y \wedge z \gtrdot x \wedge z$ and $y \vee z = x \vee z$, or*
(2) *$y \wedge z = x \wedge z$ and $y \vee z \gtrdot x \vee z$.*

Proof. All ranks are relative to the minimal element of a finite length interval containing x, y, z. By the modular law,

$$\begin{array}{rcccl} \rho(y \wedge z) & + & \rho(y \vee z) & = & \rho(y) + \rho(z) \\ \rho(x \wedge z) & + & \rho(x \vee z) & = & \rho(x) + \rho(z) \\ \text{so} \quad (\rho(y \wedge z) - \rho(x \wedge z)) & + & (\rho(y \vee z) - \rho(x \vee z)) & = & 1. \end{array}$$

The two parenthesized expressions are nonnegative, so one is 0 and the other is 1. □

Lemma 2.4. *A closed interval in a modular lattice is itself a modular lattice.*

Proof. The equation defining a modular lattice,

$$x \vee (y \wedge z) = (x \vee y) \wedge z \text{ when } x \leq z,$$

is obeyed for x, y, z in any subinterval of the lattice, because the values of $\wedge$ and $\vee$ do not change. $\square$

Theorem 2.5. *The following are equivalent for modular lattices $\mathcal{L}$ of finite length.*

(1) $\hat{1}$ *is the join of atoms.*
(2) $\hat{0}$ *is the meet of coatoms.*
(3) $\mathcal{L}$ *is atomic.*
(4) $\mathcal{L}$ *is coatomic.*
(5) $\mathcal{L}$ *is complemented.*

Proof. Let $x \in \mathcal{L}$. Any set $\{a_1, \ldots, a_n\}$ of atoms of minimal cardinality such that $a_1 \vee \cdots \vee a_n = x$ has $n = \rho(x)$, because in the successive joins $\hat{0} \lessdot a_1 \leq a_1 \vee a_2 \leq \cdots \leq a_1 \vee \cdots \vee a_n = x$, if any $\leq$ is $=$, we can eliminate the atom introduced in the join to its right without affecting the value of any joins. Such a set of S atoms is called an atomic basis of x. If $Y \subseteq S$ and y is the join of all elements of Y, then $\rho(y) = \#Y$, for if it is smaller we can again eliminate atoms from S without affecting its join. If Y and Z are subsets of S with joins y and z, then $\rho(y \vee z) = \#(Y \cup Z)$ so $\rho(y \wedge z) = \rho(y) + \rho(z) - \rho(y \vee z) = \#(Y \cap Z)$, and $y \wedge z$ is the join of the atoms of $Y \cap Z$ (for it is bounded below by that join, and has the same rank).

(1) ⇒ (2): Let $a_1, \ldots, a_n$ be an atomic basis of $\hat{1}$. Then $a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n$ (with $i = 1, \ldots, n$) are coatoms whose meet is $\hat{0}$.

(1) ⇒ (3): Assume (3) fails, and let $x > \hat{0}$ be a minimal element not expressible as the join of atoms. If x covers at least two elements, it is the join of the atoms of which those two elements are joins, so x can only cover one element, and thus is a join-irreducible; denote the unique lower cover y.

Let $a_1, \ldots, a_k$ be an atomic basis of y, and $b_1, \ldots, b_n$ be an atomic basis of $\hat{1}$. Let $y_m = y \vee b_1 \vee \cdots \vee b_m$, so that $y = y_0 \leq \cdots \leq y_n = \hat{1}$. Extend the sequence $a_1, \ldots, a_k$ to an atomic basis $a_1, \ldots, a_n$ of $\hat{1}$ by including those b_m with $y_{m-1} \lessdot y_m$. Let $z = a_{k+1} \vee \cdots \vee a_n$, so $y \wedge z = \hat{0}$.

Since $x \vee z \geq y \vee z = \hat{1}$, in fact $x \vee z = y \vee z$. Since $x \gtrdot y$, it follows that $x \wedge z \gtrdot y \wedge z = \hat{0}$ by Lemma 2.3. Since the only element less than or equal to x not also less than or equal to y is x itself, $x = x \wedge z$ and $y = y \wedge z = \hat{0}$. Thus, x is an atom, contradicting the assumption that it is not a join of atoms.

(3) ⇒ (5): The maximum element $\hat{1}$ has complement $\hat{0}$. Now let $x < \hat{1}$ have an atomic basis $a_1, a_2, \ldots, a_k$. Choose atoms $a_{k+1}, \ldots, a_m$ with m maximal so that $x \lessdot x \vee a_{k+1} \lessdot \cdots x \vee a_k \vee \cdots \vee a_m$, and call the final join y. If $y \neq \hat{1}$, then since the lattice is atomic, $\hat{1}$ is the join of all atoms of the lattice, so there is some atom $a_{m+1} \not\leq y$ extending the sequence, contradicting maximality of m. Thus, $y = \hat{1}$, $m = n$, and x has complement $a_{k+1} \vee \cdots \vee a_n$.

(5) ⇒ (1): Suppose $\mathcal{L}$ is complemented and has length n. Form a maximal sequence of atoms $a_1, a_2, \ldots, a_k$ so that $a_1 \lessdot a_1 \vee a_2 \lessdot a_1 \vee a_2 \vee a_3 \lessdot \cdots$. Since the lattice has length n, we must have $k \leq n$. Suppose $k < n$. Let $x = a_1 \vee \cdots \vee a_k$, and choose a complement x' of x. Then x' has rank $n-k \geq 1$, so there is an atom $a_{k+1} \leq x'$. Then $x \wedge a_{k+1} \leq x \wedge x' = \hat{0}$, so $x \wedge a_{k+1} = \hat{0}$ and

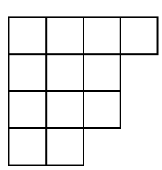
$\rho(x \vee a_{k+1}) = \rho(x) + \rho(a_{k+1}) - \rho(x \wedge a_{k+1}) = k + 1 - 0 = k + 1$, contradicting maximality of the sequence. Thus $k = n$, so $\hat{1}$ is the join of atoms.

(2) ⇒ (4) ⇒ (5) ⇒ (2) ⇒ (1): Dualize (1) ⇒ (3) ⇒ (5) ⇒ (1) ⇒ (2). □

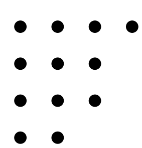
2.4. Compositions, partitions, and tableaux. A **(strong) composition** of a positive integer n is a sequence of positive integers whose sum is n. For example, the compositions of 3 are $(1, 1, 1)$, $(1, 2)$, $(2, 1)$, and (3). A **weak composition** may have parts that are 0, and so we may insert any number of zeros into these sequences.

A **partition** of a nonnegative integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of weakly decreasing nonnegative integers whose sum is n. We write $\lambda \vdash n$ or $n = |\lambda| = \sum_i \lambda_i$. For example, the partitions of 3 are $(1, 1, 1)$, $(2, 1)$, and (3). For convenience, we allow both strong compositions and partitions to have any number, finite or infinite, of zeros appended to them, and we consider two strong compositions or partitions with the same sequence of nonzero entries but differing in the number of zeros at the end to be equivalent; for example, $(4, 3, 3, 2) = (4, 3, 3, 2, 0) = (4, 3, 3, 2, 0, 0) = \cdots$. In instances where all entries are single digits, we may also drop parentheses and commas, yielding 4332 in this example. An exponent indicates that an entry is repeated that number of times consecutively, so that $(4, 3^2, 2, 1^0) = (4, 3, 3, 2)$. The number of parts of λ equal to i is called the **multiplicity** of i, and is denoted $m_i(\lambda)$, or m_i when the partition λ is known from context. The nonzero terms are called the **parts** of λ, and the number of nonzero terms is called the **length** of λ. For $\lambda = (4, 3, 3, 2)$, we have length 4; multiplicities $m_2 = m_4 = 1$, $m_3 = 2$, and $m_i = 0$ otherwise; and $|\lambda| = 12$. We apply the same notational conventions concerning parentheses, commas, exponents, trailing zeroes, and the definition of length to arbitrary sequences and to strong compositions. The unique partition of 0, strong composition of 0, and empty sequence, are denoted $\emptyset$.

The **Young diagram** of a composition or partition λ of n is a left-justified array of n squares (also called **cells**) with λ_i squares in the ith row, while the **Ferrers diagram** has dots in place of the squares. It may be viewed as a subset of $\mathbb{P} \times \mathbb{P}$ (where $\mathbb{P}$ is the set of positive integers) using matrix coordinates, where (i, j) is the cell at the ith row from the top, jth column from the left; the diagram consists of all $(i, j) \in \mathbb{P} \times \mathbb{P}$ with $j \le \lambda_i$. For example, the Young diagram and Ferrers diagram of $(4, 3, 3, 2)$ are respectively



and



The same symbol λ will be used to denote both a sequence of integers and its corresponding diagram.

The **conjugate** λ' of a partition λ is the partition obtained by reflecting its diagram about the main diagonal. For example, the conjugate of $(4, 3, 3, 2)$ is $(4, 4, 3, 1)$. Clearly $\lambda'' = \lambda$. While λ_i is the number of squares in the ith row of λ, the number of squares in the jth column is λ'_j. The rows that have squares in the jth column are precisely those whose length is at least j, so $\lambda'_j = \#\{\, i : \lambda_i \ge j \,\}$ (where $\#S$ is the cardinality of set S). Hence, $m_j(\lambda) = \lambda'_j - \lambda'_{j+1}$. Note that λ'_1 is the length of λ.

We define a partial order on compositions and partitions by $\mu \le \nu$ iff $\mu_i \le \nu_i$ for all i, or equivalently, if the the cells of the diagram of μ are a subset of the cells of the diagram of ν. We have $(\mu \vee \nu)_i = \max\{\mu_i, \nu_i\}$ and $(\mu \wedge \nu)_i = \min\{\mu_i, \nu_i\}$. In the case of partitions, this partial order forms a lattice called **Young's lattice**. From the diagram interpretation of the order, it is clear that in Young's lattice, $\mu \le \nu$ iff $\mu' \le \nu'$.

The sum of two partitions or compositions μ and ν is $\mu + \nu = (\mu_1 + \nu_1, \mu_2 + \nu_2, \ldots)$. It is obtained by appending the rows of ν to those of μ. The difference of two compositions, $\mu - \nu = (\mu_1 - \nu_1, \mu_2 - \nu_2, \ldots)$, is well defined when $\nu \le \mu$. The union $\mu \cup \nu$ of two partitions is obtained by adding together the multiplicities of the parts, so $m_i(\mu \cup \nu) = m_i(\mu) + m_i(\nu)$; equivalently, it is obtained by appending columns of ν to the bottoms of those of μ, so for partitions we obtain $\mu \cup \nu = (\mu' + \nu')'$. Beware that this is completely different from the join $\mu \vee \nu$. The concatenation of two strong compositions or finite sequences $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_m)$, is $xy = (x_1, \ldots, x_k, y_1, \ldots, y_m)$.

For $\lambda, \mu \subset \mathbb{P} \times \mathbb{P}$, we define $\lambda \oplus \mu = \{\, c : c \in \lambda \text{ or } c \in \mu \,\}$ as the ordinary set union of these, and $\lambda \ominus \mu = \{\, c : c \in \lambda \text{ and } c \notin \mu \,\}$ as the ordinary set difference. Typically, λ will be a partition or composition and μ will be a single cell.

An **inner corner** of a partition λ is a cell of λ, which if removed would yield a valid partition, and an **outer corner** is a cell not in λ, which if added to λ would yield a valid partition. The inner and outer corners of $(4, 3, 3, 2)$ are marked i and o below.

$$
\begin{array}{|c|c|c|c|l}
\hline
\; & \; & \; & i & \multicolumn{1}{|l}{o} \\
\hline
\; & \; & \; & \multicolumn{1}{|l}{o} & \\
\cline{1-3}
\; & \; & i & & \\
\cline{1-3}
\; & i & \multicolumn{1}{|l}{o} & & \\
\cline{1-2}
\multicolumn{1}{l}{o} & & & & \\
\end{array}
$$

For a partition λ, we define

$$
n(\lambda) = \sum_{i \ge 1} (i - 1)\lambda_i = \sum_{i \ge 1} \binom{\lambda'_i}{2}
$$

as the number obtained by filling each square in the top row of λ with 0, each square in the next row with 1, and so on, and then adding together all the numbers. The middle expression is obtained by adding the numbers in each row and then adding the row totals, while the right expression is obtained by adding the numbers in each column, and then adding the column totals.

For partitions $\mu \le \lambda$, we define a **skew partition** or **skew shape** λ/μ. It is simply a pair of partitions. Its diagram is $\{\, (i, j) \in \mathbb{P} \times \mathbb{P} : \mu_i < j \le \lambda_i \,\}$. The diagram is ambiguous because different pairs of partitions may yield the same diagram; for example, λ/λ is always empty. The skew partition $\lambda/\emptyset$ is identified with the ordinary partition λ. For a skew partition λ/μ, define $n(\lambda/\mu) = n(\lambda) - n(\mu)$.

A **tableau** of (partition or composition) shape λ is obtained from the Ferrers diagram of λ by replacing the dots with numbers, or from the Young diagram by writing numbers in the boxes. Occasionally we may fill the squares with other symbols. The tableau of shape $\emptyset$ is denoted $\emptyset$. A skew tableau is similarly defined by filling in a skew partition. The plural of "tableau" is "tableaux." Denote tableaux with uppercase bold italic letters, $\boldsymbol{P}$, and denote the shape λ or skew shape λ/μ by $\operatorname{sh}\boldsymbol{P}$. Also denote the outer shape λ by $\operatorname{osh}\boldsymbol{P}$.

A **row strict** tableau has the numbers strictly increasing from left to right on each row: if the entry at cell (i,j) is a_{ij} then $a_{ij} < a_{i,j+1}$, provided cells $(i,j),(i,j+1) \in \lambda/\mu$. A **row weak** tableau is similar with $<$ replaced by $\leq$, and **column strict** and **column weak** are similar but for columns. A **standard Young tableau** is row and column strict with all numerical entries distinct; if the entries aren't specified, assume $1,\ldots,n$, where n is the number of cells in the diagram. A **semistandard tableau** is row weak and column strict. Here is a standard tableau and a semistandard tableau:

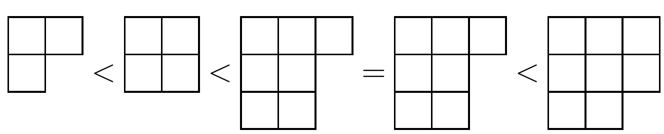
$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\ \hline 3&7&8\\ \cline{1-3} 4\\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&2&3\\ \cline{1-3} 3&4\\ \cline{1-2}\end{array}$$

The **hook length** of cell (i,j) in a partition λ is $h(i,j) = (\lambda_i - j) + (\lambda'_j - i) + 1$, which is the number of cells of λ weakly right of (i,j) on the same row, or weakly below it in the same column. The number of standard tableaux of shape λ is denoted f^λ, and may be computed by the well known **Hook Formula**, introduced by Frame, Robinson, and Thrall [**7**, pp. 317–318]:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

See also Sagan [**22**, pp. 91-97] for a survey of derivations of this formula. The number of standard skew tableaux of shape λ/μ is denoted $f^{\lambda/\mu}$, and there is no similarly elegant formula known for it.

A finite or infinite chain of partitions $\lambda^{(l)} \leq \lambda^{(l+1)} \leq \cdots$ may be represented by a row and column weak skew tableau where the cells of $\lambda^{(k)}/\lambda^{(k-1)}$ are filled with k, for each $k > l$. The cells of the inner partition $\lambda^{(l)}$ are filled with $\bullet$ in the Ferrers diagram, and either omitted or filled with $\bullet$ in the Young diagram. Whichever representation is used, the entries in the cells of λl are considered to be $\bullet$. For example, the chain $\lambda^{(3)},\ldots,\lambda^{(7)}$



is represented by the Young tableau or the Ferrers tableau, respectively,

$$\begin{array}{ccc} & & \boxed{5}\\ & \boxed{4} & \boxed{7}\\ \boxed{5} & \boxed{5} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \bullet & \bullet & 5\\ \bullet & 4 & 7\\ 5 & 5 & \end{array}$$

A standard skew tableau on entries $l+1, l+2, \ldots, h$ is uniquely associated with a saturated chain of partitions $\lambda^{(l)} \lessdot \cdots \lessdot \lambda^{(h)}$ and conversely.

We will often view tableaux and chains of partitions interchangeably, writing $\boldsymbol{P} = (\lambda^{(l)},\ldots,\lambda^{(h)})$, because almost all the tableaux we consider will arise from such chains. This is a skew tableau of shape $\lambda^{(h)}/\lambda^{(l)}$. There is a minor ambiguity in this representation. If $\lambda^{(l)} = \lambda^{(l+1)}$ (respectively, $\lambda^{(h-1)} = \lambda^{(h)}$), the number $l+1$ (respectively, h) will not appear in $\boldsymbol{P}$, so the tableau representation of a chain in Young's lattice does not unambiguously determine the upper and lower indices used in the chain.

When this is an issue, we define $\mathrm{low}(\boldsymbol{P}) = l$ and $\mathrm{high}(\boldsymbol{P}) = h$ (or $\mathrm{high}(\boldsymbol{P}) = \infty$ if the indices are unbounded).

In a tableau $\boldsymbol{P}$ where e occurs just once, let $\mathrm{row}(\boldsymbol{P}, e)$ be the number of the row containing e. Alternately, if $\boldsymbol{P} = (\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$, and $\lambda^{(e-1)} \lessdot \lambda^{(e)}$, define $\mathrm{row}(\boldsymbol{P}, e)$ as the unique value r for which $\lambda_r^{(e-1)} \lessdot \lambda_r^{(e)}$. Also define $\mathrm{row}_k(\boldsymbol{P}, e) = \lambda_{k+r}^{(e)\prime}$, which is the length of the column k columns right of e once all entries larger than e are removed. Similarly, define $\mathrm{col}(\boldsymbol{P}, e)$ to be the number of the column containing e, and $\mathrm{col}_k(\boldsymbol{P}, e)$ to be the length of the row k rows below e when all entries larger than e are removed. For example, in the tableau

$$\boldsymbol{P} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 4 & 9 & 10 \\ \cline{1-3}\end{array}$$

where $\lambda^{(8)} = (4, 3, 1) = (3, 2, 2, 1)'$, we have $\mathrm{row}(\boldsymbol{P}, 8) = \mathrm{row}_0(\boldsymbol{P}, 8) = 2$, and $\mathrm{row}_{-2}(\boldsymbol{P}, 8) = 3$; $\mathrm{row}_{-1}(\boldsymbol{P}, 8) = 2$; $\mathrm{row}_1(\boldsymbol{P}, 8) = 1$; and $\mathrm{row}_k(\boldsymbol{P}, 8) = 0$ for $k > 1$. Also, $\mathrm{col}(\boldsymbol{P}, 8) = 3$; the values $\mathrm{col}_k(\boldsymbol{P}, 8)$ for $k = -1, 0, 1$ are $4, 3, 1$; and $\mathrm{col}_k(\boldsymbol{P}, 8) = 0$ for $k > 1$.

Part II. Semi-primary lattices

3. Lattice types

3.1. Subgroup lattices, subspace lattices, and the Fibonacci lattice. We now introduce examples of the lattices that will be studied in depth in this manuscript.

(1) **Subgroup lattice.** Let G be a finite abelian group. The collection $L(G)$ of subgroups of G, with the order $H \leq K$ iff H is a subgroup of K, forms a modular lattice. The meet of two subgroups is their intersection, and the join is the group generated by the two.

Any finite abelian p-group is isomorphic to a product of cyclic p-groups,

$$\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

Sort the λ_i into weakly decreasing order to form a partition called the **type of** G**.** Any quotient K/H of finite abelian p-groups is itself a finite abelian p-group, and the interval $[H, K]$ of $L(G)$ is isomorphic to $L(K/H)$. The type of the interval $[H, K]$ is the type of the group K/H, and this is less than the type of G in Young's lattice. The length of an interval of type μ is $|\mu|$.

(2) **Invariant subspace lattice.** Let V be a finite dimensional vector space, and N be a nilpotent transformation, that is, a linear transformation such that for all vectors $\vec{v} \in V$, we have $N^k\vec{v} = 0$ for sufficiently large k. A subspace W of V is N-invariant iff NW is a subspace of W. The collection $L(V, N)$ of all N-invariant subspaces of V, ordered by subspace inclusion, forms a modular lattice. The meet of two N-invariant subspaces is their intersection, and the join is their span.

Because N is nilpotent, all the roots of its characteristic equation are 0. The Jordan canonical form of N is a matrix that is the block sum of blocks with the characteristic root 0 on the diagonal; 1 just above each entry on the main diagonal; and 0 everywhere else. For example,

$$\left[\begin{array}{cccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

is the sum of two such blocks, one of length 4 and the other of length 3. Different bases of V might put these blocks in a different order, but the partition formed by sorting these lengths into weakly decreasing order is an invariant of N, called the **type of** N. This is the type of the lattice $L(V, N)$.

If $W \leq X$ are two N-invariant subspaces of V, their quotient X/W has an action induced by N. The lattice $L(X/W, N)$ is isomorphic to the interval $[W, X]$ in $L(V, N)$. The type of the interval $[W, X]$ is the type of the lattice $L(X/W, N)$, and is a subpartition of the type of $L(V, N)$ in Young's lattice. The length of an interval of type μ is $|\mu|$.

Both $L(G)$ and $L(V, N)$ are instances of semi-primary lattices. We will prove facts about semi-primary lattices through latticial means, without assuming they have a represention as a subgroup lattice or a subspace lattice. Such proofs are called **synthetic**. The lattice structure admits no knowledge of the elements of the group G, or the elements of the space V; only the subgroups and subspaces are known. We will sometimes provide alternate coordinate-based, or **analytic**, proofs where such elements are known. The analytic proofs will be based on the space $L(V, N)$. When the type of $L(V, N)$ is λ, we can choose a basis of V with elements

$$e_{ij} \text{ with } (i, j) \in \lambda, \text{ such that } Ne_{i1} = 0 \text{ and } Ne_{ij} = e_{i,j-1} \text{ when } j > 1.$$

Since N is linear, this defines the action N on the whole space.

(3) **The Fibonacci Lattice $Z(r)$ (see [28]).** Let $r \in \mathbb{P}$. Form a poset $Z(r)$ on the set of all words from the alphabet $\{1_1, \ldots, 1_r, 2\}$, in which v covers u, denoted $v \gtrdot_Z u$, if u is obtained from v by either deleting the leftmost 1 of v (with any subscript) or by changing a 2 of v preceeded only by 2's to a 1 (with any subscript). So in $Z(2)$, we have $221_2 21_1 \gtrdot_Z 2221_1$, $1_1 21_2 21_1$, $1_2 21_2 21_1$, $21_1 1_2 21_1$, and $21_2 1_2 21_1$.

This forms a modular lattice, which will be studied in Section 9.1. The rank of an element in the lattice is the sum of the digits in the word, ignoring the subscripts. Every complemented interval in this lattice has length at most 2: we see that for any word x, all upper covers of x are also lower covers of $2x$, so all atomic intervals have length at most 2, and a modular interval of finite length is atomic iff it is complemented.

3.2. New lattice operations A and C.

Definition 3.1. Let $\mathcal{L}$ be a lattice and $x \in \mathcal{L}$. Let

$$Ax = \bigvee_{y \gtrdot x} y$$

be the join of all elements covering x, or x if x is maximal, and let

$$Cx = \bigwedge_{y \lessdot x} y$$

be the meet of all elements covered by x, or x if x is minimal. If $x \leq z$ are both in $\mathcal{L}$, let

$$A_z x = \bigvee_{y:\, x \lessdot y \leq z} y$$

be the join of all atoms in the interval $[x, z]$, and

$$C_x z = \bigwedge_{y:\, x \leq y \lessdot z} y$$

be the meet of all coatoms in $[x, z]$. These are well defined in any discrete lattice where all complemented intervals have finite length. The operators A and C are dual to each other.

We also define iterates of A and C: $A_y^0 x = x$ and $A_y^{k+1} x = A_y(A_y^k x)$, and similarly for the unary form of A, and for both forms of C.

This coincides with the **Frattini element** of a complete lattice [**32**, p. 214], which is $C\hat{1}$ in our notation. However, we will consider properties of A and C as binary operators, and the restriction that all complemented intervals have finite length will yield certain properties not present in other complete lattices.

We now consider our three sample lattices.

(1) **Subgroup lattice.** Let

$$G = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

The atoms of $L(G)$ are nonzero subgroups of G with no proper subgroups; they are isomorphic to $\mathbb{Z}/pZ$. The **socle** of G is the maximum elementary subgroup, that is, the unique subgroup isomorphic to one of the form $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}\times\cdots$. It is the join of all the atomic subgroups. Explicitly, it is

$$\mathrm{socle}(G) = p^{\lambda_1-1}\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times p^{\lambda_k-1}\mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

We have $A\hat{0}$ is the socle of G, and for any subgroups $H \le K$ of G, we have $A_K H$ is the image in $L(G)$ of the socle of K/H under the natural isomorphism from $L(K/H)$ to $[H, K]$. Thus,

$$A^r_K H = \{\, g \in H \,:\, p^r g \in K \,\}.$$

The coatoms of $L(G)$ are those K for which $G/K \cong \mathbb{Z}/p\mathbb{Z}$. Their meet is $CG = pG$. For any subgroups $H \le K$ of G, we have $C_H K$ is the image in $L(G)$ of pK/H under the natural isomorphism from $L(K/H)$ to $[H, K]$. So

$$C^r_H K = H + p^r K.$$

(2) **Invariant subspace lattice.** The atoms of $L(V, N)$ are the nonzero N-invariant subspaces of V with no nonzero proper subspaces that are N-invariant. If $W \in L(V, N)$ and $W \ne 0$, then NW is a proper subspace of W because the chain $W \supseteq NW \supseteq N^2W \supseteq \cdots$ is 0 after a finite number of steps, and once a $\supseteq$ is $=$, so are all further ones to the right. Thus, if W is atomic, it is a subspace of the kernel of N. The atoms of $L(V, N)$ are 1-dimensional subspaces of V spanned by some vector from the kernel of N, and the join of all the atoms is the kernel of N.

Let $W \le X$ be N-invariant subspaces of V. Then $A_X W$ is the image in $L(V, N)$ of the kernel of N on X/W, and

$$A^r_X W = \{\, \vec{v} \in W \,:\, N^r\vec{v} \in X \,\}.$$

The coatoms of $L(V, N)$ are maximal proper subspaces of V. Their meet is $CV = NV$. For N-invariant subspaces $W \le X$ of V, we have $C_W X = W + NX$ is the image in $L(V, N)$ of NX/W under the natural isomorphism from $L(X/W)$ to $[W, X]$, and

$$C^r_W X = W + N^r X.$$

(3) **The Fibonacci Lattice $Z(r)$.** The operators A and C have no counterpart in the prior theory of these lattices. Their values will be derived in full in Section 9.1. We derive the values of the unary operators Ax and Cx here. For any word x, if we insert a 1_i in the leading sequence of 2's, we can then change it to a 2, while if we change the first 1_i to a 2, we can then insert a 1_i

following that change. So all upper covers of x are lower covers of $2x$. As long as either $r > 1$ or $x > \hat{0}$, there is more than one upper cover, and so the join of all upper covers of x is $Ax = 2x$. When $r = 1$ and $x = \hat{0}$, the only upper cover of x is 1_1, so $Ax = 1_1$.

Now consider Cx. If x has the form $1_i y$, then the only lower cover is y, so $Cx = y$. If x has the form $2y$, we see that y is a lower cover of all lower covers of x, for we either delete the first 1_i and then change the 2 preceding it to 1_i, or we change some initial 2 to a 1_i and then delete that 1_i, to obtain y. There is always more than one lower cover unless $r = 1$ and $x = 2$, so $Cx = y$ unless $r = 1$ and $x = 2$, when $Cx = 1$. In short, Cx is obtained from x by deleting the first digit, with the exceptions $C\emptyset = \emptyset$ for all r, and $C2 = 1$ for $r = 1$.

We will compute $A_y x$ and $C_x y$ in Section 9.1.

When we develop the theory of semi-primary lattices, the elements of a group G or a vector space V will not be available; only the lattice structure of $L(G)$ or $L(V, N)$ will be known. By examining only the lattice structure and the operators A and C, rather than exhibiting elements of a group or a vector space, the results will be applicable to any semi-primary lattice, and not just ones with particular representations.

Theorem 3.2. *If $\mathcal{L}$ is upper semimodular and $x \leq x'$, then $Ax \leq Ax'$. If $\mathcal{L}$ is lower semimodular and $x \leq x'$, then $Cx \leq Cx'$.*

Proof. It suffices to prove these for $x \lessdot x'$, and since the proofs are the same but with all lattice operations dualized, only the first will be shown. If $y \gtrdot x$, then by upper semimodularity, $y \vee x' \gtrdot x'$. Since Ax is the join of all $y \gtrdot x$, which includes x', it's also the join of all $y \vee x'$ with $y \gtrdot x$. But Ax' is the join of all these and possibly more, so Ax' is at least as large as Ax. □

Theorem 3.3. *Consider a modular lattice of finite length.*

(1) $C_x A x = A_x C x = x$.
(2) $A^k C^k x \geq x$ *and* $C^k A^k x \leq x$ *for all nonnegative integers* k.
(3) $A^k C^k A^k x = A^k x$ *and* $C^k A^k C^k x = C^k x$.
(4) *If* $x \leq y$ *then* $A_y x = Ax \wedge y$ *and* $C_x y = Cy \vee x$.
(5) $A(x \wedge y) = Ax \wedge Ay$ *and* $C(x \vee y) = Cx \vee Cy$.
(6) $C^k x \leq y$ *iff* $x \leq A^k y$.

Note. Items (4) and (5) will be used extensively, and so should be ingrained now as basic arithmetic.

Proof.

(1) The interval $[x, Ax]$ is atomic, hence coatomic by Theorem 2.5, so the meet $C_x A x$ of all its coatoms is x. The dual argument is similar.
(2) First, $A^0 C^0 x = x$ and $ACx \geq A_x C x = x$. Next, let k be a positive integer, and assume that the theorem holds for smaller nonnegative k. Then $AC(C^{k-1}x) \geq C^{k-1}x$ so $A^{k-1}(AC(C^{k-1}x)) \geq A^{k-1}(C^{k-1}x)$, or $A^k C^k x \geq A^{k-1} C^{k-1} x \geq x$. Similarly $C^k A^k x \leq x$.
(3) On the one hand, $C^k A^k x \leq x$ by (2), so $A^k(C^k A^k x) \leq A^k x$ by Theorem 3.2. On the other hand, $A^k C^k (A^k x) \geq (A^k x)$ by (2). So in fact $A^k C^k A^k x = A^k x$. The other equality is proven dually.

(4) Since $A_y x$ is a join of a subset of the elements of which Ax is a join, $A_y x \leq Ax$, and since all the elements of which $A_y x$ is a join are less than or equal to y, also $A_y x \leq y$. Thus $A_y x \leq Ax \wedge y$.

Conversely, the interval $[x, Ax]$ is atomic and hence coatomic (by Theorem 2.5), and the meet of any of its coatoms with y is either $Ax \wedge y$ or a coatom of $[x, Ax \wedge y]$ (by Lemma 2.3). So

$$x \leq C_x(Ax \wedge y) \leq \bigwedge_{v\colon x \leq v \lessdot Ax} (v \wedge y) = C_x Ax \wedge y = x \wedge y = x.$$

So $[x, Ax \wedge y]$ is coatomic, hence atomic, so $Ax \wedge y$ is the join of some of the elements that cover x but are less than y, and hence is less than or equal to the join $A_y x$ of all elements that cover x and are less than or equal to y.

(5) Since $x \wedge y \leq x, y$, it follows that $A(x \wedge y) \leq Ax, Ay$ and hence $A(x \wedge y) \leq Ax \wedge Ay$.

Now we show that $[x \wedge y, Ax \wedge Ay]$ is coatomic. The intervals $[x, Ax]$ and $[y, Ay]$ are atomic and hence coatomic. The meet of any coatom of $[x, Ax]$ with Ay, and the meet of any coatom of $[y, Ay]$ with Ax, is either $Ax \wedge Ay$ or is covered by $Ax \wedge Ay$. So in $(Ay \wedge C_x Ax) \wedge (Ax \wedge C_y Ay) = x \wedge y$, the left side can be expressed as the meet of coatoms of $[x \wedge y, Ax \wedge Ay]$, whence this interval is coatomic, and hence atomic. Thus $Ax \wedge Ay \leq A(x \wedge y)$; the reverse inequality was already shown, so $Ax \wedge Ay = A(x \wedge y)$.

(6) If $C^k x \leq y$ then $x \leq A^k(C^k x) \leq A^k y$; the first inequality is by (2), and the second is by Theorem 3.2. Dually, if $A^k y \geq x$ then $y \geq C^k(A^k y) \geq x$. □

Corollary 3.4 (Monotonicity of A, C lattice polynomials). *In modular lattices of finite length, all lattice polynomials constructed from variables, meets, joins, and the unary or binary forms of the operators A and C, weakly increase as the variables weakly increase.*

This extends the monotonicity of lattice polynomials constructed from variables, meets, and joins in arbitrary lattices.

3.3. Interval types in modular lattices. To each finite length interval $[x, y]$ in a lattice, we associate two strong compositions. Let

$$\begin{aligned} \mathrm{Atype}[x, y] &\stackrel{\text{def}}{=} \big(\rho(x, A_y x), \rho(A_y x, A_y^2 x), \rho(A_y^2 x, A_y^3 x), \ldots\big) \\ \mathrm{Ctype}[x, y] &\stackrel{\text{def}}{=} \big(\rho(C_x y, y), \rho(C_x^2 y, C_x y), \rho(C_x^3 y, C_x^2 y), \ldots\big) \end{aligned}$$

We consider evaluating these in our three example lattices.

(1) **Subgroup lattice.** We compute $\mathrm{Ctype}[0, G]$, where

$$G = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

The rank of G in the lattice is $\lambda_1 + \ldots + \lambda_k$, and

$$C^r G = p^r \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times p^r \mathbb{Z}/p^{\lambda_k}\mathbb{Z}.$$

When $r \geq \lambda_i$, the factor $p^r\mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ is 0, and when $r < \lambda_i$, it has rank $\lambda_i - r$, so the rank of C^rG is

$$\sum_{i=1}^{k} \max\{\lambda_i - r, 0\} = \sum_{j>r} \lambda'_j.$$

The successive differences in ranks as r increases are thus $\lambda'_1, \lambda'_2, \ldots$. So $\mathrm{Ctype}[0, G] = \lambda'$.

Now compute $\mathrm{Atype}[0, G]$. We have

$$A^r 0 = \{\, g \in H \,:\, p^r g = 0 \,\} = p^{\max\{\lambda_1 - r, 0\}}\mathbb{Z}/p^{\lambda_1} \times \cdots \times p^{\max\{\lambda_k - r, 0\}}\mathbb{Z}/p^{\lambda_k}.$$

The successive differences in ranks are $\lambda'_1, \lambda'_2, \ldots$ so $\mathrm{Atype}[0, G] = \lambda'$.

(2) **Invariant subspace lattice.** Consider the lattice $L(V, N)$ of type λ. Choose a basis of V,

$$e_{ij} \text{ with } (i, j) \in \lambda, \text{ such that } Ne_{i1} = 0 \text{ and } Ne_{ij} = e_{i,j-1} \text{ when } j > 1.$$

We have

$$C^rV = N^rV = \operatorname{span}\{\, e_{ij} \,:\, (i, j) \in \text{ and } (i, j + r) \in \lambda \,\},$$

which has rank $\lambda'_{r+1} + \lambda'_{r+2} + \cdots$. Thus, the successive differences in ranks again yield the partition $\mathrm{Ctype}[0, V] = \lambda'$.

Now

$$A^rV = \ker N^r = \operatorname{span}\{\, e_{ij} \,:\, (i, j) \in \lambda \text{ and } j \leq r \,\}$$

has rank $\lambda'_1 + \cdots + \lambda'_r$. Thus, the successive differences in ranks yet again yield the same partition, $\mathrm{Atype}[0, V] = \lambda'$.

We will later see that in all semi-primary lattices, Atype and Ctype are the same, and are the conjugate partition of the natural notion of type. This is unique to semi-primary lattices.

(3) **The Fibonacci Lattice $Z(r)$.** We will fully compute Atype and Ctype in $Z(r)$ in Section 9.1; for now we consider an example. Let $x = 221_2 21_1$ in $Z(2)$. We compute $\mathrm{Ctype}[\hat{0}, x]$. We showed before that when $r > 1$, Cx is evaluated by stripping off the first digit of x, so the successive drops in rank are $2, 2, 1, 2, 1$, and $\mathrm{Ctype}[\hat{0}, x] = (2, 2, 1, 2, 1)$. Now observe that $A^3\hat{0} = 222 \leq x$, and that the interval $[222, x]$ is the chain $\{222 \lessdot 2221_1 \lessdot 221_2 21_1\}$. So $\mathrm{Atype}[\hat{0}, x] = (2, 2, 2, 1, 1)$, which is different from $\mathrm{Ctype}[\hat{0}, x]$.

Theorem 3.5. *The sequences* $\mathrm{Atype}[x, y]$ *and* $\mathrm{Ctype}[x, y]$ *are strong compositions. In a modular lattice, they both have the same number of parts.*

Proof. The first is a strong composition because if r is the minimum nonnegative integer with $A_y^r x = y$, then for $0 < i \leq r$, the ith part of $\mathrm{Atype}[x, y]$ is nonzero as $A_y(A_y^{i-1}x) > A_y^{i-1}x$, while for $i > r$, the ith part is 0. Similarly, if s is the minimum nonnegative integer with $C_x^s y = x$, then $\mathrm{Ctype}[x, y]$ is a strong composition with s parts.

By Theorem 3.3(2), we have $y \leq A_y^s C_x^s y$. Also $y \geq A_y^s C_x^s y$, since A_y yields elements at most y. Thus, $y = A_y^s C_x^s y = A_y^s x$, so $s \geq r$. By a dual argument, $s \leq r$, so $s = r$. $\square$

The number of parts of $\mathrm{Atype}[x, y]$, or equivalently of $\mathrm{Ctype}[x, y]$, is the **atomic rank** of the interval $[x, y]$, and will be denoted $\mathrm{arank}[x, y]$. It is also the minimum nonnegative r satisfying any of the following equivalent conditions: $A^r_y x = y$; $A^r x \geq y$; $C^r_x y = x$; and $C^r y \leq x$. In a modular lattice with $\hat{0}$, define $\mathrm{arank}\, x = \mathrm{arank}[\hat{0}, x]$, and in a finite height modular lattice $\mathcal{L}$, define $\mathrm{arank}\, \mathcal{L} = \mathrm{arank}[\hat{0}, \hat{1}]$. The function $\mathrm{arank}[x, y]$ weakly increases as x decreases or y increases, and when the decrease in x or increase in y is by a single cover relation, the function stays the same or goes up by 1. Also see [**32**, p. 215], which defines the ascending and descending Loewy series in a complete modular lattice, which in our notation are $\hat{0} < A\hat{0} < \cdots < A^r\hat{0} = \hat{1}$ and $\hat{0} = C^s\hat{1} < \cdots < C\hat{1} < \hat{1}$, and shows $r = s$.

Theorem 3.6. *In a modular lattice, if $x \leq y \leq z$ then $\mathrm{Atype}[x, y] \leq \mathrm{Atype}[x, z]$ and $\mathrm{Ctype}[y, z] \leq \mathrm{Ctype}[x, z]$ as compositions.*

Proof. The ith part of $\mathrm{Atype}[x, y]$ is the length of the interval $[A^{i-1}_y x, A^i_y x]$. The ith part of $\mathrm{Atype}[x, v]$ is the length of the interval $[A^{i-1}_v x, A^i_v x] = [v \wedge A^{i-1}_y x, v \wedge A^i_y x]$, and is hence weakly smaller by Lemma 2.2. □

When $x \leq y \leq z$, the relationship between $\mathrm{Atype}[x, z]$ and $\mathrm{Atype}[y, z]$, or between $\mathrm{Ctype}[x, z]$ and $\mathrm{Ctype}[x, y]$, is not as nicely described, in general, but we describe it here for completeness. In the theorem, let $|\mu_{\leq k}|$ denote $\mu_1 + \mu_2 + \cdots + \mu_k$.

Theorem 3.7. *Let $x \lessdot y \leq z$, and set $\mu = \mathrm{Atype}[x, z]$ and $\nu = \mathrm{Atype}[y, z]$. Dually, let $x \leq y \lessdot z$, and set $\mu = \mathrm{Ctype}[x, z]$ and $\nu = \mathrm{Ctype}[x, y]$. In each case, for all $i \geq 0$ we have $|\mu_{\leq i}| \leq 1 + |\nu_{\leq i}| \leq |\mu_{\leq i+1}|$. If either inequality is equality for a particular i, it is equality for all larger i.*

Proof. We have $x \lessdot y$ so $A^0_z x \leq A^0_z y \leq A^1_z x$. Apply A^i_z to obtain $A^i_z x \leq A^i_z y \leq A^{i+1}_z x$. If either inequality is equality for some i, it is equality for all larger i, as we are applying a power of A_z to two equal elements. Take the ranks of these elements in the interval $[x, z]$ to obtain $\rho(x, A^i_z x) \leq \rho(x, A^i_z y) = 1 + \rho(y, A^i_z y) \leq \rho(x, A^{i+1}_z x)$. Finally, express this in terms of μ and ν as $|\mu_{\leq i}| \leq 1 + |\nu_{\leq i}| \leq |\mu_{\leq i+1}|$. □

Theorem 3.8. *In a graded lattice, let $\mathrm{Atype}[a, b] = \lambda = (\lambda_1, \lambda_2, \ldots)$. For each k, the element $A^k_b a$ is the unique element c of $[a, b]$ with $\mathrm{Atype}[a, c] = (\lambda_1, \ldots, \lambda_k)$. Dually, if $\mathrm{Ctype}[a, b] = (\lambda_1, \lambda_2, \ldots)$, then for each k, the unique element c of $[a, b]$ with $\mathrm{Ctype}[c, b] = (\lambda_1, \ldots, \lambda_k)$ is $C^k_a b$.*

Proof. We prove the first statement; the second statement may be proved dually. Let $c \in [a, b]$ with $\mathrm{Atype}[a, c] = (\lambda_1, \ldots, \lambda_k)$. The ith part of $\mathrm{Atype}[a, c]$ is $\rho(A^{i-1}_c a, A^i_c a)$, and of $\mathrm{Atype}[a, b]$ is $\rho(A^{i-1}_b a, A^i_b a)$. We have $A^0_b a = A^0_c a = a$. We have $A^1_b a \geq A^1_c a$, since the right side is the join of a subset of the elements of which the left side is a join. Since the first part of $\mathrm{Atype}[a, b]$ and $\mathrm{Atype}[a, c]$ are equal, $\rho(A^1_b a) = \rho(A^1_c a)$, so in fact $A^1_b a = A^1_c a$. By iterating this, $A^i_b a = A^i_c a$ for $i = 0, \ldots, k$, since the first k parts of $\mathrm{Atype}[a, b]$ and $\mathrm{Atype}[a, c]$ agree. For further parts of $\mathrm{Atype}[a, c]$ to be 0 requires that $A^{i-1}_c a = A^i_c a$ when $i > k$, which requires $A^i_c a = c$ for $i \geq k$. So $c = A^k_c a = A^k_b a$. □

Another way to compute Atype and Ctype is the following.

Theorem 3.9. *Let $\mathcal{L}$ be a modular lattice of finite height. Let $x \in \mathcal{L}$. Let $\lambda = \operatorname{Atype}\mathcal{L}$ and $\nu = \operatorname{Atype}[\hat{0}, x]$. Let $\mu_j = \rho(x \vee A^{j-1}\hat{0}, x \vee A^j\hat{0})$. Then $\mu + \nu = \lambda$.*

Dually, let $\lambda = \operatorname{Ctype}\mathcal{L}$ and $\nu = \operatorname{Ctype}[x, \hat{1}]$. Let $\mu_j = \rho(x \wedge C^{j-1}\hat{1}, x \wedge C^j\hat{1})$. Then $\mu + \nu = \lambda$.

Proof. We prove the statement for Atype. By the modular rank equation,

$$\rho(x \vee A^j\hat{0}) = \rho(x) + \rho(A^j\hat{0}) - \rho(x \wedge A^j\hat{0}) = \rho(x) + \rho(A^j\hat{0}) - \rho(A^j_x a).$$

Subtracting the like equation for $j - 1$ from this yields

$$\mu_j = \rho(A^{j-1}\hat{0}, A^j\hat{0}) - \rho(A^{j-1}_x\hat{0}, A^j_x\hat{0}) = \lambda_j - \nu_j. \quad \square$$

Corollary 3.10. *Let $\mathcal{L}$ be a finite height modular lattice. Let $x \in \mathcal{L}$. Let $\lambda = \operatorname{Atype}\mathcal{L}$ and $\tilde{\nu} = \operatorname{Atype}[\hat{0}, x]$. Then*

$$\operatorname{Atype}[\hat{0}, x \vee A^k\hat{0}] = (\lambda_1, \ldots, \lambda_k, \tilde{\nu}_{k+1}, \tilde{\nu}_{k+2}, \ldots).$$

Dually, let $x \in \mathcal{L}$. Let $\lambda = \operatorname{Ctype}\mathcal{L}$ and $\tilde{\nu} = \operatorname{Atype}[x, \hat{1}]$. Then

$$\operatorname{Ctype}[x \wedge A^k_a b, x] = (\lambda_1, \ldots, \lambda_k, \tilde{\nu}_{k+1}, \tilde{\nu}_{k+2}, \ldots).$$

Proof. We prove the statement for Atype. Let $z = x \vee A^k\hat{0}$. Let $\nu = \operatorname{Atype}[\hat{0}, z]$ and for all $j \geq 1$ let $\mu_j = \rho(z \vee A^{j-1}\hat{0}, z \vee A^j\hat{0})$. When $j \leq k$,

$$z \vee A^j\hat{0} = x \vee A^k\hat{0} \vee A^j\hat{0} = x \vee A^k\hat{0}$$

so $\mu_1 = \cdots = \mu_j = 0$. Thus the first k parts of ν equal the first k parts of λ. When $j \geq k$ we have $z \vee A^j\hat{0} = x \vee A^j\hat{0}$, so that the remaining parts of ν equal the remaining parts of $\tilde{\nu}$. $\square$

4. Semi-primary lattices

4.1. Basic concepts.

Definition 4.1. An element c of a finite length lattice is called a **cycle** if $[\hat{0}, c]$ is a chain and a **cocycle** if $[c, \hat{1}]$ is a chain [**13**, 4.1]. An r-cycle is a cycle of rank r, and an r-cocycle is a cocycle of corank r.

Definition 4.2. A lattice is **semi-primary** if it is modular of finite length, all join-irreducibles are cycles, and all meet-irreducibles are cocycles [**13**, 4.2].

We consider our three sample lattices.

(1) **Subgroup lattice.** Consider $L(G)$. If a subgroup H of G is the product of two or more nonzero groups, then H is the join of those smaller groups, and hence is not a join-irreducible. So the join-irreducibles are the nonzero cyclic subgroups of G, that is, those isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$ for some $r > 0$. The subgroups of this group form a chain

$$p^r\mathbb{Z}/p^r\mathbb{Z} \lessdot p^{r-1}\mathbb{Z}/p^r\mathbb{Z} \lessdot \cdots \lessdot \mathbb{Z}/p^r\mathbb{Z},$$

so any join-irreducible in $L(G)$ is a cycle.

Dually, the meet-irreducibles are those proper subgroups H of G for which G/H is a cyclic group, and a similar argument applies. Thus, $L(G)$ is a semi-primary lattice.

(2) **Invariant subspace lattice.** Consider $L(V, N)$. For any nonzero vector $\vec{v} \in V$, let $S(\vec{v}) = \operatorname{span}\{\vec{v}, N\vec{v}, N^2\vec{v}, \ldots\}$. We see that for any N-invariant subspace W of V, if $\vec{v} \in W$ then $S(\vec{v}) \leq W$. If we cannot express W as $W = S(\vec{v})$, we must choose several vectors and express W as the span of the spaces $S(\vec{v_1}), S(\vec{v_2}), \ldots$. Each of these spaces is a proper N-invariant subspace of V, so W is not join-irreducible. Hence, the join-irreducibles are those spaces of the form $S(\vec{v})$. The subspaces of $S(\vec{v})$ form the chain

$$\hat{0} = S(N^r\vec{v}) \lessdot S(N^{r-1}\vec{v}) \lessdot \cdots \lessdot S(\vec{v})$$

for some r. Thus the join-irreducibles of $L(V, N)$ are cycles. By a dual argument, the meet-irreducibles are cocycles.

(3) **The Fibonacci Lattice $Z(r)$.** These are not semi-primary. The only lower cover of 122 is 22, so 122 is a join-irreducible. However, the interval $[\emptyset, 122]$ has incomparable elements 12 and 21, so it is not a chain.

The lattice $Z(r)$ has infinite length, but even the finite length closed intervals in it are not semi-primary, except in trivial cases.

Now we develop the theory of semi-primary lattices. The development is based on that of Jónsson and Monk [**13**], but we introduce the operators A and C; the partitions Atype and Ctype; and we show the connection the types have with Young's lattice.

Theorem 4.3. *Every closed interval in a semi-primary lattice $\mathcal{L}$ is semi-primary [**13**, 4.3–4.4].*

Proof. Every interval in a modular lattice is modular. Let $[a, b]$ be a closed interval and c a join-irreducible of the induced subposet $[a, b]$ of $\mathcal{L}$. We must show $[a, c]$ is a chain. We do not consider a to be join-irreducible in $[a, c]$, so $c > a$, and there is a unique coatom v of $[a, c]$. Let z be a join-irreducible of $\mathcal{L}$ with $z \leq c$ but $z \not\leq v$; since every element is the join of join-irreducibles and $c \gtrdot v$, such a join-irreducible exists. Then $a \vee z$ is in $[a, c]$ but not $[a, v]$; these intervals only differ in the element c, so $a \vee z = c$. So $[a, c] = [a, a \vee z] \cong [a \wedge z, z]$, and since z is a join-irreducible of $\mathcal{L}$ and hence a cycle, the last interval is a chain. Thus, c is a cycle in $[a, b]$. The dual argument shows that the meet-irreducibles of $[a, b]$ are cocycles. $\square$

Recall that one way of defining $\operatorname{arank} \mathcal{L}$ is as the minimum r for which $A^r\hat{0} = \hat{1}$.

Theorem 4.4. *The maximum of $\rho(c)$ as c ranges over the cycles of a semi-primary lattice $\mathcal{L}$ is* $\operatorname{arank} \mathcal{L}$ *(cf. [**13**, 4.6]).*

Proof. Let $r = \operatorname{arank} \mathcal{L}$. If $\mathcal{L}$ is a chain then the maximum rank cycle is $\hat{1}$ and indeed $\operatorname{arank} \mathcal{L} = r = \rho(\hat{1})$. So assume $\mathcal{L}$ is not a chain.

For any cycle c, we have $c = c \wedge \hat{1} = c \wedge A^r\hat{0} = A_c^r\hat{0}$, so all cycles have $\rho(c) \leq r$. For any coatom v of $\mathcal{L}$, we have $\operatorname{arank} v \leq r$. If some coatom has $\operatorname{arank} v = r$, then any r-cycle in $[\hat{0}, v]$ is also an r-cycle in $\mathcal{L}$, so the upper bound is achieved. So assume all coatoms have $\operatorname{arank} vler - 1$. Then $v \leq A^{r-1}\hat{0}$ for all coatoms, so $A^{r-1}\hat{0}$ is weakly greater than the join of all coatoms. Since $\mathcal{L}$ is not a chain, $\hat{1}$ is not a join-irreducible, so there are multiple coatoms, and the join of all coatoms is $\hat{1}$. Thus $\hat{1} \leq A^{r-1}\hat{0}$, so $\operatorname{arank} \mathcal{L} \leq r - 1$, contradicting $\operatorname{arank} \mathcal{L} = r$. $\square$

Theorem 4.5. *If $\mathcal{L}$ is a semi-primary lattice and c is a cycle of maximum possible rank $r = \operatorname{arank} \mathcal{L}$, then c has a complement in $\mathcal{L}$. Any such complement is a cocycle of corank r (cf. [**13**, 4.8]).*

Note. See Theorem 4.44 for an enumeration of complemented cycles of each rank; Theorem 4.43 for a complete characterization of complemented elements; and Theorem 4.47 for an enumeration of the number of complements an element has.

Proof. We induct on intervals ordered by inclusion. The trivial case $c = \hat{1}$ has complement $\hat{0}$. Assume $c \neq \hat{1}$. There must be an atom v of $\mathcal{L}$ besides the one in $[\hat{0}, c]$, because if that is the only atom, then $\hat{0}$ is a meet-irreducible and hence a cocyle, so that $[\hat{0}, \hat{1}]$ is a chain, so that the maximum cycle is $c = \hat{1}$. The element $c \vee v$ is an r-cycle in $[v, \hat{1}]$ because $[v, c \vee v] \cong [c \wedge v, c] = [\hat{0}, c]$, so $c \vee v$ is a maximal cycle in $[v, \hat{1}]$ (where maximality follows from the preceding theorem and monotonicity of arank on intervals). By induction, the current theorem applies to this smaller interval, whence there is $c' \in [v, \hat{1}]$ with $(c \vee v) \vee c' = \hat{1}$ and $(c \vee v) \wedge c' = v$. So $c \vee c' = c \vee (v \vee c') = \hat{1}$, and $v \vee (c \wedge c') = v$, so $c \wedge c' \leq v$, so $c \wedge c' \leq c \wedge v = \hat{0}$. Thus c' is a complement to c in $\mathcal{L}$.

For any complement c' of c, we have $[c', \hat{1}] = [c', c' \vee c] \cong [c' \wedge c, c] = [\hat{0}, c]$, so c' is a cocycle of corank r. □

Definition 4.6. A sequence $x_1, \ldots, x_k$ of elements of a finite length modular lattice are **(join) independent** if any of the following equivalent conditions hold [**13**, §2].

(1) $\rho(x_1 \vee \cdots \vee x_k) = \rho(x_1) + \cdots + \rho(x_k)$.
(2) $(x_1 \vee \cdots \vee x_{i-1}) \wedge x_i = \hat{0}$ for $i = 2, \ldots, k$.
(3) For any subsets I and J of $\{1, \ldots, n\}$,
$$\left(\bigvee_{i \in I} x_i \right) \wedge \left(\bigvee_{i \in J} x_i \right) = \left(\bigvee_{i \in I \cap J} x_i \right),$$
so in fact, $\{x_1, \ldots, x_k\}$ generates a Boolean subalgebra of the lattice.

There is also a dual notion of meet independence.

Theorem 4.7. *Every element b of a semi-primary lattice $\mathcal{L}$ is the join of independent cycles [**13**, 4.9]. If $b = x_1 \vee \cdots \vee x_k$ is such a join and the x_i are arranged so that $\rho(x_1) \geq \cdots \geq \rho(x_k)$, then the partition $(\rho(x_1), \ldots, \rho(x_k))$ is $(\mathrm{Ctype}[\hat{0}, b])'$.*

Note. See Theorem 4.48 for an enumeration of the ways to choose $x_1, \ldots, x_k$.

Proof. We induct on $\rho(b)$. If $b = \hat{0}$ the join is empty. By the last two theorems, we may choose a cycle x_1 of maximum rank in b, and then a complement y_1 to it in b; since $\rho(y_1) = \rho(b) - \rho(x_1) < \rho(b)$, we inductively have y_1 as the join of independent cycles $x_2, \ldots, x_k$, and so b is the join of independent cycles $x_1, \ldots, x_k$.

Given $b = x_1 \vee \cdots \vee x_k$ with the x_i independent cycles, for any $r \geq 0$, the cycles $C^r x_1, \ldots, C^r x_k$ are independent, so $C^r b = C^r x_1 \vee \cdots \vee C^r x_k$ has rank $\rho(C^r b) = \rho(C^r x_1) + \cdots + \rho(C^r x_k)$. Since $\rho(C^r x_i) = \max\{\rho(x_i) - r, 0\}$, we have for $r > 0$ that $\rho(C^{r-1} b) - \rho(C^r b)$ is the number of i with $\rho(x_i) = r$. □

Definition 4.8. Although the independent cycles in the last theorem are not uniquely determined by b, the partition determined by their ranks is unique. This partition is called the **type of** b, denoted $\operatorname{type} b$. The **type of a semi-primary lattice** is $\operatorname{type} \mathcal{L} = \operatorname{type} \hat{1}$. The **type of an interval** $[a, b]$, denoted $\operatorname{type}[a, b]$, is the type of the semi-primary lattice $[a, b]$, so it may be expressed as the partition of decreasing ranks of independent join-irreducibles in $[a, b]$ whose join is b, and as $(\operatorname{Ctype}[a, b])'$. The **cotype of** b is $\operatorname{cotype} b = \operatorname{type}[b, \hat{1}]$.

An alternative approach to this last theorem is taken in [**13**, pp. 97,103]. They do not compute anything analogous to $\operatorname{type} b = (\operatorname{Ctype}[\hat{0}, b])'$. Instead, they apply a theorem of Ore [**19**]: If an element b of a finite length modular lattice has two representations as a join of independent join-irreducibles,

$$b = x_1 \vee \cdots \vee x_k = y_1 \vee \cdots \vee y_m,$$

then $k = m$ and there is a permutation σ of $1, \ldots, k$ such that

$$b = y_{\sigma(1)} \vee \cdots \vee y_{\sigma(i)} \vee x_{i+1} \vee \cdots \vee x_k$$

for $i = 1, \ldots, k$. Thus, x_i and $y_{\sigma(i)}$ have the same rank for each i. A complete proof of Ore's theorem in this form can be found in [**2**, pp. 58–60]. If we apply this theorem to a semi-primary lattice, we see that any two expressions of b as the join of independent join-irreducibles have the same number of cycles and the same multiset of ranks of these cycles, so that $\operatorname{type} b = (\rho(x_1), \ldots, \rho(x_k))$ is well defined.

Theorem 4.9. *The type of a semi-primary lattice $\mathcal{L}$ and the type of its dual are equal [**13**, 4.11].*

Proof. Let $\hat{1} = x_1 \vee \cdots \vee x_k$ be the join of independent join-irreducibles with $\rho(x_k) \geq \cdots \geq \rho(x_1)$. The x_i's are independent, so lattice polynomials in them form a boolean algebra. The complement of x_i is $y_i = \vee_{j \neq i} x_i$, the y_i's are independent in the dual lattice to $\mathcal{L}$, and their meet is $\hat{0}$. Further, $[y_i, \hat{1}] = [y_i, x_i \vee y_i] \cong [x_i \wedge y_i, x_i] = [\hat{0}, x_i]$, so y_i is a meet-irreducible in $\mathcal{L}$ whose corank is the rank of x_i. Thus, the partition $(\bar{\rho}(y_1), \ldots, \bar{\rho}(y_k))$ giving the type of the dual of $\mathcal{L}$, and the partition $(\rho(x_1), \ldots, \rho(x_k))$ giving the type of $\mathcal{L}$, are the same. $\square$

Note. A semi-primary lattice need not be isomorphic to its dual. See Figure 1 for a minimal example of an irregular (Definition 4.24) semi-primary lattice that's not self-dual. For regular semi-primary lattices, the smallest examples are provided by non-Desarguean projective planes. There are four projective planes of order 9, two of which are self-dual, and the other two of which are dual to each other; see Stevenson [**30**, p. 80]. Thus, there are two 9-regular semi-primary lattices of type $(1, 1, 1)$ that are not self-dual.

Corollary 4.10. *The type of an interval $[a, b]$ of a semi-primary lattice may be computed as any of the following.*

1. $(\rho(a, x_1), \ldots, \rho(a, x_k))$, *where $x_1, \ldots, x_k$ are independent join-irreducibles of weakly decreasing rank in $[a, b]$ whose join is b;*
2. $(\rho(y_1, b), \ldots, \rho(y_k, b))$, *where $y_1, \ldots, y_k$ are independent meet-irreducibles of weakly decreasing corank in $[a, b]$ whose meet is a;*
3. $(\operatorname{Ctype}[a, b])'$;

(4) $(\mathrm{Atype}[a,b])'$.

Proof. (1) and (3) are equal by Theorem 4.7, while (2) and (4) are equal by a dual argument, and (1) and (2) are equal by Theorem 4.9. $\square$

Inaba [**12**, Theorem 34] proved that if $a \geq b$, the number of parts in type a is weakly larger than in type b; that type a is lexicographically weakly larger than type b; and for each $k > 0$, the number of parts of size at least k in type a weakly exceeds that in type b. This is a cryptic description of Young's lattice. It generalizes to intervals as follows.

Theorem 4.11. *Let $a \leq a' \leq b' \leq b$. Then* $\mathrm{type}[a,b] \geq \mathrm{type}[a',b']$ *in Young's lattice.*

Proof. Both types are partitions in Young's lattice. We have

$$(\mathrm{type}[a,b])' = \mathrm{Atype}[a,b] \geq \mathrm{Atype}[a,b'] = (\mathrm{type}[a,b'])',$$

where the equalities hold by the preceding theorem and the inequality holds by Theorem 3.6 (considering the partitions to be compositions, but noting the order is the same). Similarly,

$$(\mathrm{type}[a,b'])' = \mathrm{Ctype}[a,b'] \geq \mathrm{Ctype}[a',b'] = (\mathrm{type}[a',b'])'.$$

Thus $(\mathrm{type}[a,b])' \geq (\mathrm{type}[a',b'])'$, so $\mathrm{type}[a,b] \geq \mathrm{type}[a',b']$. $\square$

Definition 4.12. For an interval $[a,b]$ in a semi-primary lattice, let $[a,b][k]$ be the join of all cycles of rank at most k, and $[a,b](k)$ be the meet of all cocycles of corank at most k. This extends the notation $x[k] = [\hat{0},x][k]$ defined in [**13**, 4.5]. We also define $x(k) = [x,\hat{1}](k)$.

For any partition λ and $k \geq 0$, let

$$\lambda[k] = (\min\{\lambda_1,k\}, \min\{\lambda_2,k\}, \ldots) = (\lambda'_1,\ldots,\lambda'_k)'.$$

Theorem 4.13. *Let $\mathcal{L}$ be a semi-primary lattice of type λ. There is a unique element of type $\lambda[k]$, and it may be expressed as $\hat{1}[k]$ and $A^k\hat{0}$. In an interval $[a,b]$ of type λ, there is a unique element z with* $\mathrm{type}[a,z] = \lambda[k]$*, and it may be expressed $[a,b][k] = A^k_b a$. Dually, $[a,b](k) = C^k_a b$ is the unique element of cotype $\lambda[k]$ in $[a,b]$.*

Proof. We prove the first sentence, and the others follow by applying it to intervals, which are semi-primary, and by dualizing. By Theorem 3.8, $A^k\hat{0}$ is the unique element with this type. Suppose c is a cycle of rank at most k. Then $c = A^k_c\hat{0} = c \wedge A^k\hat{0}$, so $c \leq A^k\hat{0}$, so the join of all such cycles satisfies $\hat{1}[k] \leq A^k\hat{0}$, whence $\mathrm{type}\,\hat{1}[k] \leq \lambda[k]$. Conversely, given any decomposition of $\hat{1}$ into independent cycles, $\hat{1} = x_1 \vee \cdots \vee x_m$, we have that $x_1[k],\ldots,x_m[k]$ are independent cycles of rank at most k, whose ranks are precisely the row lengths of $\lambda[k]$. Thus, their join has type $\lambda[k]$; this join is a lower bound of $\hat{1}[k]$ because it is a join of only some of the cycles of rank at most k, so $\lambda[k] \leq \mathrm{type}\,\hat{1}[k]$, whence in fact $\mathrm{type}\,\hat{1}[k] = \lambda[k]$. Since there is a unique element of this type, $\hat{1}[k] = A^k\hat{0}$. $\square$

Corollary 4.14. *If a and b are independent elements in a semi-primary lattice, then $(a \vee b)[k] = a[k] \vee b[k]$.*

Proof. (cf. [**13**, 4.13]) Since a and b are independent, $\operatorname{type}(a \vee b) = (\operatorname{type} a) \cup (\operatorname{type} b)$, and

$$\begin{aligned}\operatorname{type}\Big((a \vee b)[k]\Big) &= \Big((\operatorname{type} a) \cup (\operatorname{type} b)\Big)\,[k] \;=\; (\operatorname{type} a)[k] \cup (\operatorname{type} b)[k] \\ &= \Big(\operatorname{type} a[k]\Big) \cup \Big(\operatorname{type} b[k]\Big).\end{aligned}$$

Since $a[k] \leq a$ and $b[k] \leq b$, the elements $a[k], b[k]$ are independent, so the type of $a[k] \vee b[k]$ is precisely the last expression in the above equation. By the previous theorem, $(a \vee b)[k]$ is the unique element of $[\hat{0}, a \vee b]$ whose type is the first expression in this equation. Since $a[k] \vee b[k]$ is in this interval, it must equal $(a \vee b)[k]$. □

4.2. Alternate characterizations of semi-primary lattices. We present some new characterizations of semi-primary lattices. Theorem 4.15(1)–(3) is the classical definition of semi-primary lattices, and all the other characterizations in this section are new. In particular, Theorem 4.15(4) is a local characterization of semi-primary lattices; Theorem 4.16 is a rank equation characterization; and Theorem 4.17 is analogous to interval perspectivity in modular lattices, and is essentially a lattice equation characterization of semi-primary lattices.

Theorem 4.15. *Let $\mathcal{L}$ be a modular lattice of finite length. Then the following are equivalent.*

(1) *$\mathcal{L}$ is semi-primary.*
(2) *Every element is the join of cycles and the meet of cocycles.*
(3) *Every join-irreducible is a cycle and every meet-irreducible is a cocycle.*
(4) *Whenever x covers precisely one element, that element is either $\hat{0}$ or in turn covers precisely one element. Whenever x is covered by precisely one element, that element is either $\hat{1}$ or is covered by precisely one element.*
(5) *For every interval $[x, y]$, we have* $\operatorname{Atype}[x, y] = \operatorname{Ctype}[x, y]$.
(6) *For every interval $[x, y]$, both* $\operatorname{Atype}[x, y]$ *and* $\operatorname{Ctype}[x, y]$ *are partitions.*
(7) *Whenever $x \lessdot y$, we have $Cx \leq Cy$ and $Ax \leq Ay$.*

Proof.

(1)⇔(2): This is the definition of a semi-primary lattice.

(2)⇔(3): In any lattice, every element is the join of join-irreducibles and the meet of meet-irreducibles, and all cycles are join-irreducibles and all cocycles are meet-irreducibles.

(3)⇔(4): The elements with unique lower covers are precisely the join-irreducibles, and the elements with unique upper covers are precisely the meet-irreducibles.

(1)⇒(5): This is Theorem 4.9.

(5)⇒(4): Suppose x is a join-irreducible. Then $\operatorname{Ctype}[\hat{0}, x]$ begins with a 1, so by (5), $\operatorname{Atype}[\hat{0}, x]$ does too. Let $v = Cx$. If $v = \hat{0}$, then x is an atom and hence a cycle. Otherwise, $\hat{0} \leq A_v\hat{0} \leq A_x\hat{0}$, but $A_x\hat{0}$ is an atom because $\operatorname{Atype}[\hat{0}, x]$ begins with a 1, and $A_v\hat{0}$ is nonzero because $v > \hat{0}$. So $A_v\hat{0} = A_x\hat{0}$, so $\operatorname{Atype}[\hat{0}, v]$ begins with a 1, so $\operatorname{Ctype}[\hat{0}, v]$ begins with a 1, so v is also a join-irreducible. Thus, (4) holds.

(1)⇒(6): This follows from Theorem 4.7.

(6)⇒(3): Let x be a join-irreducible. Then $\mathrm{Ctype}[\hat{0}, x]$ begins with 1, and the entries are weakly decreasing, so in fact, all the nonzero entries are 1. So $[\hat{0}, x]$ is a chain, whence x is a cycle. Similarly, meet-irreducibles are cocycles, so $\mathcal{L}$ is semi-primary.

(1)⇒(7): Since $\mathrm{Ctype}[\hat{0}, x] \leq \mathrm{Ctype}[\hat{0}, y]$ in Young's lattice (by Theorem 4.11) and the right side has one more square than the left, in fact the two partitions are equal in all but one part, in which the second partition exceeds the first by 1. Now $Cx \leq Cy$ by Theorem 3.2, so if the partitions are equal in the first part, $\rho(Cx) \lessdot \rho(Cy)$ and $Cx \lessdot Cy$, while if the partitions differ by 1 in the first part, $Cx = Cy$. By a dual argument, $Ax \leq Ay$.

(7)⇒(4): Let y be a join-irreducible, so $Cy \lessdot y$. Then $C^2y \leq Cy$, so Cy is a join-irreducible or $\hat{0}$. Dually, the unique element covering a meet-irreducible is also a meet-irreducible or $\hat{1}$. □

Note. (4) characterizes semi-primary lattices by a local condition similar to the local condition that can be used to describe modular and semimodular lattices. While the preceding characterizations are in terms of cycles and cocycles, and hence require there to be a $\hat{0}$ and $\hat{1}$ in the lattice, this characterization does not require either, and so may possibly be used to extend the definition of semi-primary lattices to discrete modular lattices of infinite length.

Theorem 4.16. *Let $\mathcal{L}$ be a finite length graded lattice. Then $\mathcal{L}$ is semi-primary iff the* ***semi-primary equation***

$$\rho(A^k x \wedge y) + \rho(x \vee C^k y) = \rho(x) + \rho(y)$$

holds for all nonnegative integers k and all x, y in $\mathcal{L}$, iff it holds for $k = 0, 1$.

Proof. Let $\mathcal{L}$ be semi-primary. If $x \leq y$, this equation is equivalent to the fact that $\mathrm{Atype}[x, y] = \mathrm{Ctype}[x, y]$, since the sum of the first k parts of $\mathrm{Atype}[x, y]$ is $\rho(A^k x \wedge y) - \rho(x)$ and the sum of the first k parts of $\mathrm{Ctype}[x, y]$ is $\rho(y) - \rho(x \vee C^k y)$. For arbitrary x and y, let $x' = x \wedge y \leq y$, and apply this equation to x', y to get

$$\rho(A^k x' \wedge y) + \rho(x' \vee C^k y) = \rho(x') + \rho(y).$$

The left-hand side is

$$\begin{aligned} \rho(A^k x \wedge A^k y \wedge y) + \rho((y \wedge x) \vee C^k y) &= \\ \rho(A^k x \wedge y) + \rho(y \wedge (x \vee C^k y)) &= \\ \rho(A^k x \wedge y) + \rho(x \vee C^k y) - \rho(y \vee x \vee C^k y) + \rho(y) &= \\ \rho(A^k x \wedge y) + \rho(x \vee C^k y) + (\rho(y) - \rho(x \vee y)) &= \\ \rho(A^k x \wedge y) + \rho(x \vee C^k y) + (\rho(x \wedge y) - \rho(x)) & \end{aligned}$$

Equate it with the right side $\rho(x \wedge y) + \rho(y)$ and rearrange the terms to obtain the semi-primary equation.

Now suppose $\mathcal{L}$ is a finite length graded lattice that satisfies the semi-primary equation for $k = 0, 1$. The $k = 0$ case is precisely the rank identity for modular lattices, so $\mathcal{L}$ is modular. Let $x = \hat{0}$ and let y be a join-irreducible. Then the $k = 1$ equation yields $\rho(A\hat{0} \wedge y) = \rho(y) - \rho(Cy) = 1$, so that $[\hat{0}, y]$ has a unique atom, v. Now y is join-irreducible in $\mathcal{L}$, hence in the sublattice $[v, y]$ as well (or $y = v$), so inducting on the length of the lattice, we have that $[v, y]$ is a chain, so $[\hat{0}, y]$ is too. Meet-irreducibles are cocycles by a dual argument, so $\mathcal{L}$ is semi-primary. $\square$

One way to characterize modular lattices is by perspectivity of intervals, introduced in Lemma 2.1. A lattice is modular iff for each pair of elements x, y, the functions ϕ and ψ below are lattice isomorphisms

$$\begin{array}{rcl} [x, x \vee y] & \longrightarrow & [x \wedge y, y] \\ t & \xmapsto{\phi} & t \wedge y \\ x \vee t' & \xleftarrow{\psi} & t' \end{array}$$

iff the functions ϕ' and ψ' obtained by swapping all occurrences of x and y are lattice isomorphisms. There is a partial analogue for semi-primary lattices, which provides an algebraic counterpart to the semi-primary equation.

Theorem 4.17. *Consider*

$$\begin{array}{rcl} I_k = [x, x \vee C^k y] & \longrightarrow & [A^k x \wedge y, y] = J_k \\ t & \xmapsto{\phi_k} & A^k t \wedge y \\ x \vee C^k t' & \xleftarrow{\psi_k} & t' \end{array}$$

A graded lattice of finite length is semi-primary iff ϕ_k and ψ_k are inverse lattice isomorphisms for all x, y in the lattice and all $k \geq 0$, iff they are inverse for $k = 0, 1$.

Proof. Suppose these are inverse for $k = 0, 1$. The $k = 0$ maps are precisely the characterization of modular lattices in terms of perspective intervals, so the lattice is modular. Now let y be a join-irreducible and $x = \hat{0}$. We must show that Cy is a join-irreducible (or $\hat{0}$) as well. By the $k = 1$ equation, $[\hat{0}, Cy]$ and $[A_y\hat{0}, y]$ are isomorphic. Since y is join-irreducible in $[\hat{0}, y]$, it also is join-irreducible (or minimal) in the subinterval $[A_y\hat{0}, y]$, whence Cy is join-irreducible (or $\hat{0}$) in $[\hat{0}, Cy]$ and hence in the whole lattice. By a dual argument, the unique element covering each meet-irreducible is a meet-irreducible or $\hat{1}$, so by Theorem 4.15(4), the lattice is semi-primary.

Conversely, suppose a lattice is semi-primary. The maps ϕ_k and ψ_k are order-preserving in any modular lattice. For any $t \in I_k$ we have $C^k(A^k t \wedge y) \leq C^k A^k t \leq t$ and $x \leq t$, so that $\psi_k(\phi_k(t)) = x \vee C^k(A^k t \wedge y) \leq t$. If $t, u \in I_k$ and $t \lessdot u$ then $\phi_k(t) \leq \phi_k(u)$ by Theorem 4.15(7) and Lemma 2.3; since ϕ_k maps the endpoints of I_k to those of J_k and both I_k and J_k have the same length by the semi-primary equation, in fact $\phi_k(t) \lessdot \phi_k(u)$. Similarly, for $t', u' \in J_k$ with $t' \lessdot u'$, we have $\psi_k(t') \lessdot \psi_k(u')$. So ϕ_k and ψ_k preserve the rank of an element within the interval. Thus for $t \in I_1$, both sides of $\psi_k(\phi_k(t)) \leq t$ have the same rank, and so are equal. Similarly, $\phi_k(\psi_k(t')) = t'$ when $t' \in J_k$, so ψ_k and ϕ_k are inverses. $\square$

The analogue with modular lattices is not complete. The transposed pair of intervals to consider is $I'_k = [x \vee C^k y, y]$ and $J'_k = [x, A^k x \wedge y]$. Unless $x \leq y$, these intervals are both empty, since they are subintervals of $[x, y]$. When $x \leq y$ does hold, the intervals do not need to be isomorphic, though they both have the same type because $\text{Atype}[x, y] = \text{Ctype}[x, y]$. Figure 1 is the Hasse diagram of a minimal example with these intervals not isomorphic, with $k = 1$. The vertices are the elements of the lattice, and the edges are the cover relations in the lattice.

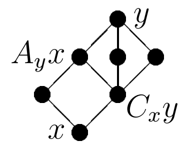


FIGURE 1. An irregular, non self-dual, semi-primary lattice.

Note. Theorem 4.17 can be transformed to equations satisfied by all x, y, t in a semi-primary lattice by projecting any t into the intervals I_k or J_k, as follows. A lattice of finite length is semi-primary iff for all x, y, t,

$$\begin{aligned} \psi_k(\phi_k(x \vee (t \wedge (x \vee C^k y)))) &= x \vee (t \wedge (x \vee C^k y)) \\ \phi_k(\psi_k(y \wedge (t' \vee (A^k x \wedge y)))) &= y \wedge (t' \vee (A^k x \wedge y)) \end{aligned}$$

hold for all nonnegative k, iff they hold for $k = 0, 1$.

Definition 4.18. If P and Q are graded posets, a map $\phi : P \to Q$ is called a **growth** if $x \leq y \Rightarrow \phi(x) \leq \phi(y)$.

Theorem 4.19. *Let $\mathcal{L}$ be a semi-primary lattice. Let $\phi(t)$ be a lattice polynomial constructed from meets, joins, A, C, elements of $\mathcal{L}$, and one occurrence of the variable t. Then ϕ is a growth on $\mathcal{L}$. Further, if intervals $[x, y]$ and $[\phi(x), \phi(y)]$ have the same length, they are isomorphic.*

Proof. For any $x \in \mathcal{L}$, both $t \wedge x$ and $t \vee x$ are growths in t by Lemma 2.2, and if they preserve interval lengths, they induce isomorphisms by Lemma 2.3.

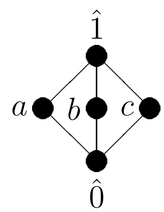
The function At is a growth by Theorem 4.15(7). Suppose $[a, b]$ and $[Aa, Ab]$ have the same length. Let $c = Ab$ and $\lambda = \text{type}[a, c]$. Then $\rho(a, Aa) = \rho(a, Aa \wedge c) = \lambda'_1 = \rho(a \vee Cc, c)$ so $\rho(a, a \vee Cc) = \rho(Aa, c) = \rho(Aa, Ab)$. Also $a \vee Cc = a \vee CAb \leq b$ so $[a, a \vee Cc] \subseteq [a, b]$. Since these have equal lengths, in fact $a \vee Cc = b$. Finally, $[a, b] = [a, a \vee Cc] \cong [Aa \wedge c, c] = [Aa, Ab]$. The proof for Ct is similar.

Finally, all lattice polynomials of the form in the theorem are compositions of the forms just considered, and hence are growths that induce isomorphisms when they preserve interval lengths. $\square$

Note. Isomorphisms of the form in the above theorem may be useful in determining conditions under which intervals of the same type in a lattice are in fact isomorphic. Also see [**1**] for a study of rank and order preserving maps from certain semi-primary lattices of one type to another type.

Example 4.20. Note that not all lattice polynomials are growths, and not all growths expressible as lattice polynomials induce isomorphisms when they preserve an interval's length.

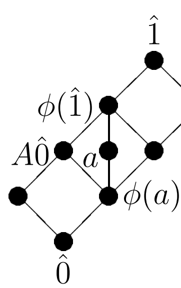
(1) S.V. Fomin (personal communication) notes that all nondistributive modular lattices contain a sublattice isomorphic to



The polynomial $\phi(t) = (t \vee b) \wedge (t \vee c)$ has values $\phi(\hat{0}) = \hat{0}$ and $\phi(a) = \hat{1}$, so it is not a growth. Thus, for all nondistributive modular lattices, some lattice polynomial is not a growth.

(2) In any semi-primary lattice $\mathcal{L}$, choose $x \in \mathcal{L}$ and consider the polynomial $\phi(t) = t \wedge (x \vee Ct)$. It monotonically increases in t. We have $\rho(\phi(t)) = \rho(t) + \rho(x \vee Ct) - \rho(x \vee t)$ (since $t \vee x \vee Ct = x \vee t$). If $t \lessdot t'$, the increase in rank from $\phi(t)$ to $\phi(t')$ is at most 2 because upon substituting t' for t, each rank on the right of the rank equation can go up by 1. The only way to achieve an increase of 2 is if $t \lessdot t'$ and $x \vee Ct \lessdot x \vee Ct'$ but $x \vee t = x \vee t'$. However, applying C_x to both sides of $x \vee t = x \vee t'$ yields $x \vee Ct = x \vee Ct'$, so this cannot occur. Thus, $\phi(t)$ is a growth.

Now consider $\phi(t) = t \wedge (A\hat{0} \vee Ct)$ applied to the length 2 interval $[a, \hat{1}]$ in the following lattice. We have $\operatorname{type}[a, \hat{1}] = \square\!\square$ and $\operatorname{type}[\phi(a), \phi(\hat{1})] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, so while the interval length is preserved, the intervals are not isomorphic. A similar construction can be done in any semi-primary lattice of type $(3, 1)$ with at least three elements of rank two by choosing a with $\operatorname{type} a = \operatorname{cotype} a = \square\!\square$. Any semi-primary lattice whose type contains $(3, 1)$ has intervals of type $(3, 1)$ in which such a construction can be done as well, provided there are at least three elements on the middle level.



We now provide a strengthening of Theorem 3.3(2) for semi-primary lattices.

Definition 4.21. For a partition λ, let $\mathcal{C}^k\lambda = (\lambda'_{k+1}, \lambda'_{k+2}, \ldots)'$ be obtained from λ by deleting the first k columns.

Theorem 4.22. *In a semi-primary lattice, $A^kC^kx = x \vee A^k\hat{0}$ and $C^kA^kx = x \wedge C^k\hat{1}$.*

Proof. We prove the first statement, and the second is proven dually. Since $A^kC^kx \geq x$ and $A^kC^kx \geq A^k\hat{0}$, we have $A^kC^kx \geq x \vee A^k\hat{0}$. Let $\lambda = \operatorname{type}\mathcal{L}$ and $\mu = \operatorname{type} x$. Then $\operatorname{type} C^kx = \mathcal{C}^k\mu$. Since $C^kx \in [\hat{0}, C^k\hat{1}]$, we have $\operatorname{type}[A^k\hat{0}, A^kC^kx] = \mathcal{C}^k\mu$ as well, by Theorem 4.17. Thus,

$$\begin{aligned}\operatorname{type}[\hat{0}, A^k C^k x] = (\operatorname{Atype}[\hat{0}, A^k C^k x])' &= (\operatorname{Atype}[\hat{0}, A^k \hat{0}])' + (\operatorname{Atype}[A^k\hat{0}, A^k C^k x])' \\ &= \lambda[k] + \mathcal{C}^k \mu = \mu \vee \lambda[k] = \operatorname{type}(x \vee A^k \hat{0}).\end{aligned}$$

The only tricky equality is the last, which follows from Corollary 3.10. Since $A^k C^k x$ is bounded below by $x \vee A^k\hat{0}$, and they both have the same type, they are equal. $\square$

4.3. Classification and representation of semi-primary lattices. We summarize what is known about representing semi-primary lattices. See Jónsson and Monk [**13**] for a detailed history of the development of this representation.

Definition 4.23 (see [13, 6.1,6.2**]).** A lattice is **primary** if it is semi-primary and no interval of length 2 has precisely 2 elements on the middle level, i.e., all intervals of type $\boxminus$ have at least 3 atoms. Clearly the dual of a primary lattice is also primary.

Definition 4.24. A **q-regular lattice** is a lattice with the property that all intervals of length 2 are either chains or have precisely $q+1$ atoms. A **q-regular semi-primary lattice** is, therefore, a semi-primary lattice in which every interval of type $\boxminus$ has $q+1$ atoms. A **regular lattice** is a lattice with the property that all intervals of length 2 that aren't chains have the same cardinal number of atoms, whether a finite $q+1$, or an infinite cardinal. All other lattices are **irregular**.

Example 4.25. We provide examples of different lattices and their classifications.

(1) If G is a finite abelian p-group, then $L(G)$ is p-regular and primary. Every interval of type $\boxminus$ is a quotient isomorphic to $L((\mathbb{Z}/p\mathbb{Z})^2)$. The subgroups of $(\mathbb{Z}/p\mathbb{Z})^2$ are itself; 0; and the groups with generators $(1, i)$ (where $i \in \mathbb{Z}/p\mathbb{Z}$) and the group with generator $(0, 1)$. So there are $p + 1$ atoms in the interval. Because $p + 1 \geq 3$ for all primes, it is primary.
(2) The lattice $L(V, N)$ over a field K is $\#K$-regular and primary. Let V be a vector space over a field K and N be a nilpotent transformation. Every interval of $L(V, N)$ of type $\boxminus$ is isomorphic to the lattice of subspaces of a two dimensional vector space over K. There are $\#K + 1$ atomic subspaces (with essentially the same construction as in the subgroup lattice), so when $\#K$ is finite, $L(V, N)$ is $\#K$-regular, and when it is infinite, it is regular. In either case, all fields have at least two elements, so $\#K + 1 \geq 3$, so $L(V, N)$ is primary.
(3) The 1-regular semi-primary lattices are products of chains. Nontrivial products of chains (at least two nonempty factors) are not primary.
(4) A direct product of semi-primary lattices is a semi-primary lattice, but in nontrivial products, is neither regular (unless all factors are chains) nor primary. Intervals of type $\boxminus$ are either wholly contained in one factor, or are the product of a chain of length 1 from one factor and a chain of length 1 from another factor. Intervals of the latter sort have exactly 2 atoms, so the product isn't primary. If all intervals of the former sort have exactly 2 atoms, this is a product of chains.
(5) If λ is a partition with two rows, we may construct an infinite number of primary irregular lattices of type λ. Take any primary lattice $\mathcal{L}$ of type λ. We require an element x which is both a cycle and a cocyle. One way to obtain such

an x is to decompose $\hat{1} = x \vee y$ where x and y are independent join-irreducibles. Then x is both a cycle and a cocycle (as $[x, \hat{1}] = [x, x \vee y] \cong [x \wedge y, y] = [\hat{0}, y]$).

The cover Ax of x is not join-irreducible, because if it is, then Ax and x are both complements to y, but have two different ranks. Thus, Ax has at least one lower cover besides x, and the meet of any such lower cover with x must be Cx by modularity. So $[Cx, Ax]$ is an interval of type $\boxminus$, and it contains an element x which has no cover relation with any other element in the lattice. Form a lattice $\mathcal{L}'$ from $\mathcal{L}$ by inserting additional atoms into $[Cx, Ax]$; the additional atoms cover Cx and are covered by Ax, and also have no other cover relations with any elements of $\mathcal{L}'$. By the local characterizations of modularity and semi-primariness, this lattice is semi-primary, for if we consider the local part of the lattice with the additional atoms, they behave exactly as x did in the original lattice $\mathcal{L}$. Each interval has at least 3 atoms, so $\mathcal{L}'$ is primary and irregular.

While two-rowed lattices usually contain a plethora of elements that are both cycles and cocycles, lattices whose type has more than two rows do not have such elements. Suppose $x > \hat{0}$ is a cycle. Then $Ax \geq x \vee A\hat{0}$ so $\operatorname{type} Ax \geq (\operatorname{type} x) \vee (1^{\lambda'_1})$, whence $\rho(Ax) \geq \rho(x) + \lambda'_1 - 1$. When $\lambda'_1 > 2$, this implies $\rho(Ax) - \rho(x) > 1$, so x isn't a cocycle. So this construction doesn't work when λ has at least 3 rows.

(6) The parameters q and λ are insufficient to classify all the q-regular semi-primary lattices. Let $G = (\mathbb{Z}/p^2\mathbb{Z})^3$. Let $\mathbb{F}_p$ be the finite field of order p; V be a 6-dimensional vector space over $\mathbb{F}_p$; and N be a nilpotent transformation on V of type $(2,2,2)$. Both $L(G)$ and $L(V, N)$ are p-regular semi-primary lattices of type $(2,2,2)$, and Butler and Hales [**1**, p. 9] prove that they are not isomorphic. Thus, for any partition $\lambda \geq (2,2,2)$, the p-subgroup lattice of type λ and the invariant subspace lattice of type λ over the field $\mathbb{F}_p$ are not isomorphic, because their subintervals of type $(2,2,2)$ are not isomorphic.

Theorem 4.26 (see [13, 6.3]**).** *In a primary lattice with at least three independent atoms, any two intervals of type* $\boxminus$ *are projective.*

Corollary 4.27. *Every primary lattice of type* λ *with* $\lambda'_1 \geq 3$ *is regular.*

Definition 4.28 (see [13, 6.6]**).** A ring R with unit is **completely primary and uniserial** if there is a two-sided ideal P of R such that every left or right ideal of R is of the form P^k, where $P^0 = R$, and $P^k = 0$ for some finite k.

Definition 4.29 (see [13, 2.1]**).** A lattice is **Arguesian** if it satisfies the following lattice-theoretic analogue of Desargues' Law: for any elements $a_0, a_1, a_2, b_0, b_1, b_2$, if

$$y = (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge \Big[\Big((a_0 \vee a_2) \wedge (b_0 \vee b_2)\Big) \ \vee \ \Big((a_1 \vee a_2) \wedge (b_1 \vee b_2)\Big)\Big],$$

then

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq (a_0 \wedge (a_1 \vee y)) \ \vee \ (b_0 \wedge (b_1 \vee y)).$$

Jónsson and Monk [**13**] show that every Arguesian lattice is modular, and that every primary lattice of type λ where the first part has multiplicity at least 4 is Arguesian. Not all primary lattices with $m_{\lambda_1} < 4$ are Arguesian, because there are non-Desarguean projective planes. Any projective plane of order q gives rise to a

q-regular primary lattice of type (1^3) by taking the points p to be the atoms and the lines l to be the coatoms, with $p < l$ in the lattice iff p is on the line l in the plane. We also attach a minimum element $\hat{0}$ and a maximum element $\hat{1}$. For examples of non-Desarguean projective planes, see, for example, [**11**, Chapter IX], or [**30**, Chapter 2.4 and Part 3].

There is a representation theorem for certain primary lattices, which generalizes the coordinatization of projective geometries.

Theorem 4.30 (see [13, 6.7,12.3]**).** *The lattice of all submodules of a finitely generated module over a completely primary and uniserial ring is primary. Conversely, any primary Arguesian lattice with* $m_{\lambda_1} \geq 3$ *(which includes all primary lattices with* $m_{\lambda_1} \geq 4$*), is isomorphic to the lattice of submodules of a finitely generated module over a completely primary and uniserial ring.*

These lattices include $L(G)$, $L(V, N)$, and the lattice of submodules of a module of finite length over a discrete valuation ring; see Macdonald [**16**, p. 85]. The residue field of the ring plays a role similar to the field in the lattice $L(V, N)$; in particular, if q is the order of the residue field, the lattice of the theorem is q-regular.

Regonatti has a representation theorem for modular lattices.

Theorem 4.31 (see [20, pp. 45–47]**).** *Let* $\mathcal{L}$ *be a modular lattice. If the number of atoms equals the number of coatoms in every interval of length 3, then* $\mathcal{L}$ *is the direct product of primary* q*-lattices (with the different factors possibly having different values of* q*).*

There is also a converse of sorts to this theorem.

Theorem 4.32 (see [13, 6.2]**).** *Every primary lattice that is not a chain is* ***simple****, that is, has no nontrivial homomorphic images.*

4.4. Enumeration in q-regular semi-primary lattices. In a q-regular semi-primary lattice $\mathcal{L}$, we often can count how many solutions $(x_1, x_2, \ldots)$ there are to a system of equations $\rho(p_i(x_1, x_2, \ldots)) = r_i$, where the p_i are lattice polynomials in the operators A and C; meets and joins; constants from the lattice; and the variables x_i, while the r_i are integers. These counts often turn out to be polynomials in q depending on the parameters r_i and the type of the lattice, but not on any further structure of the lattice. The polynomials are the same for abelian subgroup lattices, invariant subspace lattices, and any other q-regular semi-primary lattice.

The equations we consider will be derived from type specifications. For example, in this section, we will count how many atoms of cotype μ are in a q-regular semi-primary lattice of type λ. We have type$[a, \hat{1}] = \mu$ iff $\rho(a, A^k a) = \mu'_1 + \cdots + \mu'_k$, so such a are the solutions of $\rho(A^k a) = 1 + \mu'_1 + \cdots + \mu'_k$ for all k.

We will also count how many flags $f = (f_0 \leq f_1 \leq \cdots)$ there are in which type $f_i = \lambda^{(i)}$ is specified for each i. In this case, the type equations are $\rho(A^k \hat{0} \wedge f_i) = \lambda_1^{(i)\prime} + \cdots + \lambda_k^{(i)\prime}$ for all i and k. The fact that f is a flag can also be expressed as a rank equation: since $f_i \leq f_{i+1}$, we have $f_i \wedge f_{i+1} = f_i$, so $\rho(f_i \wedge f_{i+1}) = |\lambda^{(i)}|$. Or, we could use $\rho(f_i \vee f_{i+1}) = |\lambda^{(i+1)}|$.

The counts we obtain for q-regular semi-primary lattices also provide information about regular lattices with a variety structure. The proofs of the counts generally entail selecting an element within a certain interval but excluded from a lower dimension subinterval. In the variety case, we would instead select an element of the interval generic with respect to a certain property that excludes the subinterval. The closure of the solution space would have the same dimension as the degree of the polynomial specifying the number of choices in the interval, which equals the degree of the polynomial specifying the number of choices in the interval less the smaller dimension subinterval. We obtain the following principle.

Principle 4.33. *Consider a system of rank equations which has $f(q)$ solutions in all q-regular semi-primary lattices of specified type for sufficiently large q. In a semi-primary lattice of that type with a variety structure, the dimension of the closure of the set of solutions to the rank equations equals the degree of the polynomial $f(q)$.*

Definition 4.34. For a nonnegative integer n, let

$$\begin{aligned} \langle n\rangle_q &\stackrel{\text{def}}{=} (q^n - q^1)/(q-1) = 1 + q + \cdots + q^{n-1} \\ \langle n\rangle_q! &\stackrel{\text{def}}{=} \langle n\rangle_q \langle n-1\rangle_q \cdots \langle 1\rangle_q \\ \langle 0\rangle_q! &\stackrel{\text{def}}{=} 1. \end{aligned}$$

When the subscript is omitted, implicitly assume it is q, so $\langle n\rangle = \langle n\rangle_q$. We will commonly use $\langle n\rangle_{q^{-1}}$, and it should be noted that $\langle n\rangle_{q^{-1}} = q^{1-n}\langle n\rangle$ and $\langle n\rangle_{q^{-1}}! = q^{-\binom{n}{2}}\langle n\rangle!$. The common notation $(\boldsymbol{n})$ for $\langle n\rangle$ is not used here because often in this manuscript, n will be a long expression with parentheses, and the former notation is difficult to read.

Theorem 4.35. *The number of atoms in a q-regular semi-primary lattice $\mathcal{L}$ of type λ is $\langle \lambda_1'\rangle$. Dually, the number of coatoms of such a lattice is also $\langle \lambda_1'\rangle$.*

Proof. The atoms of $\mathcal{L}$ and the atoms of $[\hat{0}, A\hat{0}]$ are identical, and the type of the latter is $(1^{\lambda_1'})$, so it suffices to consider lattices of type $\lambda = (1^n)$. When $n = 0$, there are no atoms; when $n = 1$, there is one atom, $\hat{1}$, and when $n = 2$, there are $q+1$ atoms since the lattice is q-regular. So the theorem holds for $n = 0, 1, 2$. Now assume it holds for integers smaller than n, and count the number a of atoms of a q-regular semi-primary lattice of type (1^n). In two different ways, we will count triples (u, v, w) of elements of the lattice, where u, v are distinct atoms and $w = u \vee v$. First, pick u in a ways, then an atom w of $[u, \hat{1}]$ in $\langle n-1\rangle$ ways (as the interval $[u, \hat{1}]$ has type (1^{n-1})), and then let v be any of the q atoms of $[\hat{0}, w]$ other than u; altogether, there are $a\langle n-1\rangle q$ such triples. Second, pick u in a ways and v in $a-1$ ways, and then $w = u \vee v$ in one way; so there are $a(a-1)$ triples. Thus, $a(a-1) = a\langle n-1\rangle q$, so $a = \langle n-1\rangle q + 1 = \langle n\rangle$. ∎

Theorem 4.36. *In a q-regular semi-primary lattice of type λ, the number of coatoms of type $\mu \lessdot \lambda$ (dually, atoms of cotype $\mu \lessdot \lambda$) is $\langle \lambda_c'\rangle - \left\langle \lambda_{c+1}'\right\rangle$, where $c = \operatorname{col}(\lambda/\mu)$.*

Proof. The coatoms x of type μ satisfy $x \geq \hat{1}[c-1]$ but not $x \geq \hat{1}[c]$. Thus, we want the coatoms of the interval $\left[\hat{1}[c-1], \hat{1}\right]$ not in the subinterval $\left[\hat{1}[c], \hat{1}\right]$. The types of these intervals are $\mathcal{C}^{c-1}\lambda$ and $\mathcal{C}^c\lambda$, respectively, and so they contain $\langle \lambda_c'\rangle$ and $\left\langle \lambda_{c+1}'\right\rangle$ coatoms, respectively, by the preceding theorem. ∎

Definition 4.37. By Theorem 4.11, the types of elements in a flag increase in Young's lattice, so we may define for each $x \le f_l$ the **flag type** $\operatorname{ftype}_x f$ of a flag $f = (f_l, \ldots, f_h)$ as the chain of partitions

$$\begin{aligned} \operatorname{ftype}_x f &\stackrel{\text{def}}{=} (\operatorname{type}[x, f_l], \ldots, \operatorname{type}[x, f_h]) \\ \operatorname{ftype} f &\stackrel{\text{def}}{=} (\operatorname{type}[f_l, f_l], \ldots, \operatorname{type}[f_l, f_h]). \end{aligned}$$

This is a skew tableau of shape $\operatorname{type}[x, f_h]/\operatorname{type}[x, f_l]$ in which for each $k = l+1, \ldots, h$, the squares of $\operatorname{type}[x, f_k]/\operatorname{type}[x, f_{k-1}]$ are filled with k.

Theorem 4.38. *Let $\mathcal{L}$ be a q-regular semi-primary lattice of type λ, and $\boldsymbol{P}$ be a standard skew tableau of outer shape λ with distinct entries. The number of flags with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ is*

$$F^{\boldsymbol{P}}(q) \stackrel{def}{=} \prod_{i \in \boldsymbol{P}} \Big(\langle \operatorname{row}(\boldsymbol{P}, i) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, i) \rangle \Big).$$

This is a monic polynomial of degree $n(\operatorname{sh} \boldsymbol{P})$.

Proof. We have $f_h = \hat{1}$. Given $f_h \ge \cdots \ge f_l$ with types given by $\boldsymbol{P}$, we want to choose $f_{i-1} \le f_i$. If $\lambda^{(i-1)} = \lambda^{(i)}$ (equivalently, i doesn't appear in $\boldsymbol{P}$) then $f_{i-1} = f_i$. So consider $\lambda^{(i-1)} \lessdot \lambda^{(i)}$, with these differing in the cell at the bottom of column c. Then the number of coatoms f_{i-1} of $[\hat{0}, f_i]$ of type $\lambda^{(i-1)}$ is

$$\left\langle \lambda^{(i)\prime}_c \right\rangle - \left\langle \lambda^{(i)\prime}_{c+1} \right\rangle = \langle \operatorname{row}(\boldsymbol{P}, i) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, i) \rangle.$$

This is a monic polynomial of degree $\operatorname{row}(\boldsymbol{P}, i) - 1$, The number of flags of type $\boldsymbol{P}$ is the product of these polynomials over all entries i of $\boldsymbol{P}$, and is hence monic of degree $n(\operatorname{sh} \boldsymbol{P})$. $\square$

Note. Many theorems concerning flag types will be stated for saturated flags with $f = (f_0 \lessdot \cdots \lessdot f_n)$, so that $\operatorname{ftype}_x f$ contains $1, \ldots, n$; however, they can always be applied to multisaturated flags with different indices $f = (f_l \lesssim f_{l+1} \lesssim \cdots \lesssim f_h)$, where the tableau $\operatorname{ftype}_x f$ has entries $\{ k : f_{k-1} \lessdot f_k \}$. Simply change the indices used, and when some k doesn't appear in the flag type, take f_k to be the same as f_{k-1} or conversely, depending on which is constructed first in the proof. Also, when a theorem is stated concerning $\operatorname{ftype}_{\hat{0}} f$, it can be applied to $\operatorname{ftype}_x f$ with any $x \le f_l$ by applying the theorem to the semi-primary lattice $[x, \hat{1}]$ instead of $\mathcal{L} = [\hat{0}, \hat{1}]$.

Definition 4.39. The **superstandard skew tableau of shape** λ/μ is the tableau of shape λ/μ in which the entries are filled in consecutively along the first row, then the second, and so on. If the entries aren't specified, count beginning with 1. So the superstandard tableau of shape 4332/221 is

$$\begin{array}{cccc} & & 1 & 2 \\ & & 3 & \\ & 4 & 5 & \\ 6 & 7 & & \end{array}$$

Definition 4.40. The following quantities frequently appear in enumerating various objects in semi-primary lattices. Let λ be a partition.

$$M_\lambda(q) \stackrel{\text{def}}{=} \prod_{j=1}^{\lambda_1} \left\langle \lambda'_j - \lambda'_{j+1} \right\rangle! = \prod_{j=1}^{\lambda_1} \langle m_j(\lambda) \rangle!$$

$$n_0(\lambda) \stackrel{\text{def}}{=} \sum_{j=1}^{\lambda_1} \binom{m_j(\lambda)}{2}$$

We will sometimes use $M_\lambda(q^{-1}) = q^{-n_0(\lambda)} M_\lambda(q)$, which has only non-positive powers of q, including a constant term 1. For any polynomial $p(q)$ divisible by $q^{n_0(\lambda)}$, the product $p(q)M_\lambda(q^{-1})$ is a polynomial of the same degree as $p(q)$. A fact we will not use that may be of interest is that $M_\lambda(q)$ is a polynomial whose coefficients form a symmetric, unimodal sequence, i.e., if we write $M_\lambda(q) = \sum_{i=0}^{d} a_i q^i$ where d is the degree of $M_\lambda(q)$, then $a_i = a_{d-i}$ and $0 < a_0 \le \cdots \le a_{\lfloor d/2 \rfloor}$. This is because $M_\lambda(q)$ is the product of terms $\langle n \rangle$, each of which is unimodal and symmetric, and the product of unimodal symmetric polynomials is again unimodal symmetric.

Corollary 4.41. *Let $\boldsymbol{P}$ be the superstandard tableau of shape λ/μ in a q-regular semi-primary lattice of type λ. Let ν be the partition given by $m_j(\nu) = \#\{\, i \,:\, \lambda_i = \mu_i = j \,\}$. The number of flags of type $\boldsymbol{P}$ is*

$$q^{n(\lambda/\mu)} M_\lambda(q^{-1})/M_\nu(q^{-1}).$$

Proof. We evaluate the product over $i \in \boldsymbol{P}$ of

$$\langle \mathrm{row}(\boldsymbol{P}, i) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, i) \rangle = q^{\mathrm{row}_1(\boldsymbol{P}, i)} \langle \mathrm{row}(\boldsymbol{P}, i) - \mathrm{row}_1(\boldsymbol{P}, i) \rangle.$$

First consider the differences on the right side. Since $\boldsymbol{P}$ is superstandard, the differences $\mathrm{row}(\boldsymbol{P}, i) - \mathrm{row}_1(\boldsymbol{P}, i)$ equal 1 when i is not at the end of a row of $\boldsymbol{P}$, and since $\langle 1 \rangle = 1$, we can ignore these terms. The $m_j(\lambda) - m_j(\nu)$ rows of length j with at least one entry contribute consecutive factors $\langle m_j(\nu)+1 \rangle, \langle m_j(\nu)+2 \rangle, \ldots, \langle m_j(\lambda) \rangle$ to the product, which we collect into $\langle m_j(\lambda) \rangle! / \langle m_j(\nu) \rangle!$. The product of this over all row lengths j is $M_\lambda(q)/M_\nu(q)$.

Next consider the product over $i \in \boldsymbol{P}$ of the powers of q. The exponent of q is $d = \sum_i \mathrm{row}_1(\boldsymbol{P}, i)$. Consider the contribution of the cell (r, j) of λ/μ to this sum. Each cell of λ/μ not at the end of a row contributes $r - 1$ to the sum. An entry is at the end of a row when $\lambda'_{j+1} + \nu'_j < r \le \lambda'_j$, and contributes λ'_{j+1} to the sum. The sum of $r - 1$ over all squares of λ/μ is $n(\lambda/\mu)$, while the sum of $r - 1 - \lambda'_{j+1}$ over all squares at the end of rows that contain entries is $n_0(\lambda) - n_0(\nu)$. The difference is d. The total number of flags is then

$$q^{n(\lambda/\mu) - (n_0(\lambda) - n_0(\nu))} M_\lambda(q)/M_\nu(q) = q^{n(\lambda/\mu)} M_\lambda(q^{-1})/M_\nu(q^{-1}). \qquad \square$$

4.5. Lattice automorphisms. We examine lattice automorphisms of $\mathcal{L} = L(V, N)$. As usual, let V be a vector space, and choose a basis of V with elements

$$e_{ij} \text{ with } (i, j) \in \lambda, \text{ such that } Ne_{i1} = 0 \text{ and } Ne_{ij} = e_{i,j-1} \text{ when } j > 1.$$

Let $\vec{x} = (x_1, \ldots, x_{\lambda'_1})$ be given componentwise by $x_i = \mathrm{span}\{\, e_{ij} \,:\, 1 \le j \le \lambda_i \,\}$; these are N-invariant subspaces of V.

Let $A = \operatorname{Aut}(\mathcal{L})$. Let E be the group of endomorphisms (invertible linear transformations) on V that commute with N. Any endormorphism in E induces a lattice automorphism of $\mathcal{L}$, and since nonzero scalar multiples of the same endomorphism induce the same lattice automorphism, we may projectively identify E with a subgroup E' of A. However, E' may not be all of A. For example, suppose the field is $\mathbb{C}$, and for any $W \in L(V, N)$, let $\overline{W} = \left\{ \sum_{(i,j)\in\lambda} \overline{a_{ij}} e_{ij} : \sum_{(i,j)\in\lambda} a_{ij} e_{ij} \in W \right\}$; this fixes $\vec{x}$. More generally, for any field, the Galois group induces distinct lattice automorphisms fixing $\vec{x}$, and these are distinct from the automorphisms in E', so that A contains at least the product of E' with the automorphisms induced by the Galois group acting on coordinates with respect to a fixed basis.

We will not determine A, but we will show that the cardinality of the orbit under A of any saturated flag from $\hat{0}$ to $\hat{1}$ is divisible by $M_\lambda(q)$.

Theorem 4.42. *In $\mathcal{L} = L(V, N)$, the orbit of each maximal length saturated flag under the automorphism group has cardinality divisible by $M_\lambda(q)$. The minimal orbit has cardinality exactly $M_\lambda(q)$.*

Note. We may replace $\mathcal{L}$ by any q-regular Arguesian lattices of the form given by Jónsson and Monk (summarized in Theorem 4.30), but this result may not apply to non-Desarguean q-regular semi-primary lattices.

Proof. Let $V_k = \operatorname{span}\{ e_{ik} : \lambda_i = k \}$. Let G_k be the set of endomorphisms on V_k. With $m = m_k(\lambda)$,

$$\#G_k = (q^m - q^0)(q^m - q^1) \cdots (q^m - q^{m-1}) = (q-1)^m q^{\binom{m}{2}} \langle m \rangle!.$$

To any tuple $(g_1, \ldots, g_{\lambda_1}) \in G_1 \times \cdots \times G_{\lambda_1}$, associate an endomorphism g on V, commuting with N, by defining $g(e_{ij}) = N^{\lambda_i - j}(g_{\lambda_i}(e_{ik}))$ and extending linearly to all of V. Let G be the group of all such endomorphisms; then $\#G = (q-1)^l q^{n_0(\lambda)} M_\lambda(q)$, where $l = \lambda_1'$.

Define projections by linearly extending their action on a basis of V:

$$\begin{aligned} \pi_k : V &\longrightarrow V_k \\ e_{ij} &\longmapsto \begin{cases} e_{ij} & \text{if } \lambda_i = j = k; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $x \in \mathcal{L}$, define $\pi(x) = (\pi_1(x), \ldots, \pi_{\lambda_1}(x))$.

For any saturated flag $0 = f_0 \lessdot f_1 \lessdot \cdots \lessdot f_n = V$, let $G_{\pi(f)}$ be the subgroup of G fixing $\pi(f) = (\pi(f_0), \ldots, \pi(f_n))$. We can explicitly determine $G_{\pi(f)}$ as follows. For each k with $m_k(\lambda) > 0$, consider the sequence $0 = \pi_k(f_0) \le \cdots \le \pi_k(f_n)$. Among these are $m = m_k(\lambda)$ distinct spaces forming a saturated chain, $0 = W_0 \lessdot \cdots \lessdot W_m = V_k$. Initially, the action of g_k on $W_0 = 0$ is void. Assume g_k fixes each of $W_0, \ldots, W_{d-1}$. Arbitrarily choose a vector in W_d not in W_{d-1}, and extend g_k to fix W_d as well by mapping this vector to any of the $q^d - q^{d-1}$ vectors in W_d not also in W_{d-1}. Altogether there are $\prod_{d=1}^{m} (q^d - q^{d-1}) = (q-1)^m q^{\binom{m}{2}}$ possible g_k's fixing $W_0, \ldots, W_d$. Making independent choices of $g_1, \ldots, g_{\lambda_1}$ to form all g fixing $\pi(f)$, there are $(q-1)^l q^{n_0(\lambda)}$ endomorphisms in $G_{\pi(f)}$.

Thus, $\#G/\#G_{\pi(f)} = M_\lambda(q)$ is the size of the orbit of $\pi(f)$ under the action of G.

Let G_f be the subgroup of G fixing f. If f is fixed, so is $\pi(f)$, so G_f is a subgroup of $G_{\pi(f)}$. Thus, $\#G_f$ divides $\#G_{\pi(f)}$, so the size of the orbit of f under G is a multiple of $M_\lambda(q)$, specifically $\#G/\#G_f = (\#G/\#G_{\pi(f)})(\#G_{\pi(f)}/\#G_f) = M_\lambda(q) \cdot (\#G_{\pi(f)}/\#G_f)$.

Although G is not a subgroup of A, it may be projectively identified with some subgroup H of A: each element of G induces an automorphism in A, and only nonzero scalar multiples yield the same automorphism. Consider right cosets Ha with $a \in A$. Apply two cosets Ha_1 and Ha_2 to f. Either they yield the exact same orbit, or they yield disjoint orbits: if $h_1a_1(f) = h_2a_2(f)$ with $h_1, h_2 \in H$, the orbits are the same because for any $h \in H$ we have $ha_1(f) = (hh_1^{-1}h_2)a_2(f)$ and $ha_2(f) = (hh_2^{-1}h_1)a_1(f)$. Each coset yields an orbit $(Ha)(f) = H(a(f))$ with cardinality divisible by $M_\lambda(q)$, and unequal orbits are disjoint, so the orbit of f under A also has cardinality divisible by $M_\lambda(q)$.

The minimum cardinality of an orbit is in fact exactly $M_\lambda(q)$. Let $\boldsymbol{P}$ be the superstandard tableau of shape λ. Since the flag type is an automorphism invariant, the set of flags of type $\boldsymbol{P}$ is the union of orbits of A, and since there are $F^{\boldsymbol{P}}(q) = M_\lambda(q)$ such flags, they form one orbit of size $M_\lambda(q)$. $\square$

4.6. Complemented elements.

Theorem 4.43. *Let $\mathcal{L}$ be a semi-primary lattice of type λ. An element of $\mathcal{L}$ is complemented iff it has type μ and cotype ν with $\mu \cup \nu = \lambda$ (i.e., μ and ν are complementary subsets of the rows of λ).*

Proof. If x and y are complements, a join decomposition of $\hat{1}$ may be formed from any join decompositions of x and y, so we manifestly have $\operatorname{type} \hat{1} = \operatorname{type} x \cup \operatorname{type} y$. But also $[x, \hat{1}] = [x, x \vee y] \cong [x \wedge y, y] = [\hat{0}, y]$, so $\operatorname{type} \hat{1} = \operatorname{type} x \cup \operatorname{cotype} x$.

We prove the converse by induction on the length of $\mathcal{L}$. Suppose x has type μ and cotype ν where $\mu \cup \nu = \lambda$. Either $\mu_1 = \lambda_1$ or $\nu_1 = \lambda_1$ (or both). In the former case, pick a μ_1-cycle c in $[\hat{0}, x]$, and a complement c' to c in $\mathcal{L}$. Then $x \wedge c'$ and c are complementary in $[\hat{0}, x]$ because $(x \wedge c') \vee c = x \wedge (c' \vee c) = x \wedge \hat{1} = x$ and $(x \wedge c') \wedge c = x \wedge (c' \wedge c) = x \wedge \hat{0} = \hat{0}$. Thus, $\operatorname{type}(x \wedge c') \cup \operatorname{type} c = \operatorname{type} x$, so $\operatorname{type}(x \wedge c') = (\mu_2, \mu_3, \ldots)$. Also, $\operatorname{type}[x \wedge c', c'] = \operatorname{type}[x, x \vee c'] = \operatorname{type}[x, \hat{1}] = \nu$, and $\operatorname{type}[\hat{0}, c'] = (\lambda_2, \lambda_3, \ldots)$. Inductively apply the theorem to find a complement x' of $x \wedge c'$ in the subinterval $[\hat{0}, c']$ of $\mathcal{L}$, i.e., $\hat{0} = (x \wedge c') \wedge x'$ and $c' = (x \wedge c') \vee x'$. In fact, x' is a complement to x in $\mathcal{L}$: since $x' \leq c'$, we have $x \wedge x' = x \wedge (x' \wedge c') = \hat{0}$, and since $x = c \vee (x \wedge c')$, we have

$$x \vee x' = (c \vee (x \wedge c')) \vee x' = c \vee ((x \wedge c') \vee x') = c \vee c' = \hat{1}.$$

In case $\nu_1 = \lambda_1$, we dualize the above argument, choosing a cocycle c of maximum rank in $[x, \hat{1}]$, a complement c' to c in $\mathcal{L}$, and then a complement x' to $x \vee c'$ in $[c', \hat{1}]$. $\square$

Theorem 4.44. *The number of complemented cycles of length $r > 0$ in a q-regular semi-primary lattice $\mathcal{L}$ of type λ is*

$$q^{(\lambda'_1 + \cdots + \lambda'_r) - r} \langle m_r(\lambda) \rangle_{q^{-1}}.$$

Hence, the number of maximum length cycles is

$$q^{|\lambda|-\lambda_1}\langle m_{\lambda_1}(\lambda)\rangle_{q^{-1}} = q^{\lambda_2+\lambda_3+\cdots}\langle m_{\lambda_1}(\lambda)\rangle_{q^{-1}}.$$

In the trivial case $r = 0$, the number of complemented cycles is 1.

Proof. We induct on r and λ_1. The only complemented cycle of length $r = 0$ is $\hat{0}$.

When $r = 1$, the complemented atoms are the atoms whose cotype is $(\lambda'_1 - 1, \lambda'_2, \lambda'_3, \ldots)'$. Equivalently, by evaluating the cotype as $(\mathrm{Ctype}[v, \hat{1}])'$, an atom v is complemented iff $v \not\leq C\hat{1}$. The number of atoms in $\mathcal{L}$, less the number in $[\hat{0}, C\hat{1}]$, is

$$\langle \lambda'_1 \rangle - \langle \lambda'_2 \rangle = q^{\lambda'_1 - 1}\langle \lambda'_1 - \lambda'_2 \rangle_{q^{-1}} = q^{\lambda'_1-1}\langle m_1(\lambda)\rangle_{q^{-1}},$$

as the theorem claims for $r = 1$.

When $\lambda_1 = 0$, there are no cycles of positive length, and indeed we have $m_r(\lambda) = 0$ when $r > 0$.

Now take $\lambda_1 > 0$ and $r > 1$ and assume the theorem is true for smaller values. In two ways, we count pairs (c, d) where c is a complemented r-cycle in the lattice; d is a complemented $(r-1)$-cycle in $[A\hat{0}, \hat{1}]$; and $c \leq d$.

Let c be a complemented r-cycle in $\mathcal{L}$, with a complement c'. We require $d \geq c$ and $d \geq A\hat{0}$, so $d \geq A\hat{0} \vee c$. Now $\mathrm{type}[A\hat{0}, A\hat{0} \vee c] = \mathrm{type}[A\hat{0} \wedge c, c] = (r-1)$, so only $d = A\hat{0} \vee c$ is possible (as a larger d would not be an $(r-1)$-cycle). We will show that this d is complemented in $[A\hat{0}, \hat{1}]$ by showing that $d' = c' \vee A\hat{0}$ is a complement. By Corollary 3.10, $\mathrm{type}[\hat{0}, d']$ is the type of c' with the first column lengthened by 1. We have $d \vee d' = c \vee c' \vee A\hat{0} = \hat{1} \vee A\hat{0} = \hat{1}$. Also, $d \wedge d' \geq A\hat{0}$ and

$$\begin{aligned}
\rho(d \wedge d') &= \rho(d) + \rho(d') - \rho(\hat{1}) \\
&= (\rho(c) + \rho(A\hat{0}) - 1) + (\rho(c') + 1) - \rho(\hat{1}) \\
&= (\rho(c) + \rho(c') - \rho(\hat{1})) + \rho(A\hat{0}) \\
&= \rho(A\hat{0})
\end{aligned}$$

so in fact, $d \wedge d' = A\hat{0}$. So the number of pairs (c, d) is the same as the number of complemented r-cycles c of $\mathcal{L}$.

On the other hand, let d be a complemented $(r-1)$-cycle in $[A\hat{0}, \hat{1}]$, in one of $q^{(\lambda'_2+\cdots+\lambda'_r)-(r-1)}$ ways. This has type $(\lambda'_1, 1^{r-1})'$ in $\mathcal{L}$. Consider the atom $v = C^{r-1}d$ of $\mathcal{L}$. Any r-cycle c of $\mathcal{L}$ with $c \leq d$ must have $c \geq v$, for we must have $d = c \vee A\hat{0}$, and then $C^{r-1}d = C^{r-1}c \vee C^{r-1}A\hat{0} = C^{r-1}c$. Also, $\mathrm{type}[v, d] = (\lambda'_1, 1^{r-2})'$. So c is an $(r-1)$-cycle in $[v, d]$, hence a maximum length cycle, hence complemented. The number of choices of c is, by induction, $q^{(\lambda'_1+(r-2)\cdot 1)-(r-1)} = q^{\lambda'_1+1}$. So the number of pairs (c, d) is $q^{(\lambda'_1+\cdots+\lambda'_r)-r}$. By the first count, this number is also the number of complemented r-cycles. □

Definition 4.45 (see [20, p. 38**]).** In a semi-primary lattice $\mathcal{L}$, the set of complements of $x \in \mathcal{L}$ is $\Gamma_{\mathcal{L}}(x)$.

An enumerative counterpart to Theorem 4.5 is the following.

Theorem 4.46 (see [20, pp. 38–39**]).** *If c is a cycle of rank highest rank* $\mathrm{arank}\,\mathcal{L}$ *in a q-regular semi-primary lattice $\mathcal{L}$, then the number of complements to c in $\mathcal{L}$ is* $q^{\rho(\hat{1})-\rho(c)}$.

We now generalize this to enumerate the complements of all complemented elements.

Theorem 4.47. *Let x be a complemented element of type μ and cotype ν in a q-regular semi-primary lattice of type $\lambda = \mu \cup \nu$. The number of complements to x is*

$$\#\Gamma_{\mathcal{L}}(x) = q^{\sum_j \mu'_j \nu'_j}.$$

We present two proofs. The first is analytic, and the second is synthetic, so it applies to any q-regular semi-primary lattice.

Analytic proof. Let V be a vector space with a nilpotent action of type λ. Choose a basis of V with two kinds of elements,

$$\begin{array}{lll} e_{ij} \text{ with } 1 \le j \le \mu_i & Ne_{i1} = 0 & Ne_{ij} = e_{i,j-1} \text{ for } j > 1, \text{ and} \\ e'_{ij} \text{ with } 1 \le j \le \nu_i & Ne'_{i1} = 0 & Ne'_{ij} = e'_{i,j-1} \text{ for } j > 1, \end{array}$$

such that the e_{ij} span the subspace x of V. A complement to x is some y which can be expressed canonically as follows. It is spanned by the vectors $N^k z_m$ where $(m, k+1) \in \nu$ and z_m is of the form

$$z_m = e'_{m,\nu_m} + \sum_{(i,j)\in\mu:\, j\le\nu_m} a^{(m)}_{ij} e_{ij}.$$

The restriction $j \le \nu_m$ is because the space spanned by $\{z_m, Nz_m, \ldots, N^{\nu_m - 1} z_m\}$ must be annihilated by N^{ν_m}.

We can set all the parameters $a^{(m)}_{ij}$ independently to any of the q elements of the field. For given m and $j \le \nu_m$, the number of i with $(i,j) \in \mu$ is μ'_j. Thus, the total number of parameters $a^{(m)}_{ij}$ is

$$\sum_m \sum_{j\le\nu_m} \mu'_j = \sum_j \sum_{m:\,\nu_m\ge j} \mu'_j = \sum_j \nu'_j \mu'_j. \quad \square$$

Synthetic proof. We induct on λ_1. When $\lambda_1 = 0$, then $x = \hat{0}$, its only complement is $\hat{0}$, and $\mu = \nu = \lambda = \emptyset$, so $\sum_j \mu'_j \nu'_j = 0$. Now take $\lambda_1 > 0$ and assume lattices with smaller values of λ_1 obey the theorem.

Let $k = \mu'_1$. In two ways, we count flags $(c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k)$ where c_0 is a complement to x in $\mathcal{L}$ and c_k is a complement to $x \vee A\hat{0}$ in $[A\hat{0}, \hat{1}]$. Note that $\operatorname{type} x \vee A\hat{0} = (\lambda'_1, \mu'_2, \mu'_3, \ldots)'$ and $\operatorname{type} c_k = (\lambda'_1, \nu'_2, \nu'_3, \ldots)'$. So in any such sequence, we have $c_k = c_0 \vee A\hat{0}$, and $\operatorname{type} c_i = (\nu'_1 + i, \nu'_2, \nu'_3, \ldots)'$.

Let c_0 be a complement of x in $\mathcal{L}$. Successively choose $c_1, c_2, \ldots, c_k$ so that c_i is an atom of $[c_{i-1}, c_0 \vee A\hat{0}]$. This interval is elementary of rank $k - i + 1$, because it is contained in the elementary interval $[c_0, c_0 \vee A\hat{0}]$, so it has type (1^{k-i+1}), and the number of choices of c_i is $\langle k - i + 1\rangle$. The final term is $c_k = c_0 \vee A\hat{0}$. So there are

$$\#\Gamma_{\mathcal{L}}(x) \cdot \prod_{i=1}^{k} \langle k - i + 1\rangle = \#\Gamma_{\mathcal{L}}(x) \cdot \langle k\rangle!$$

sequences.

On the other hand, choose a complement c_k to $x \vee A\hat{0}$ in $[A\hat{0}, \hat{1}]$. First we show that $\operatorname{type} c_k \wedge x = (1^k)$. We have $c_k \wedge (x \vee A\hat{0}) = A\hat{0}$ because the elements on the left are complementary. By modularity, $(c_k \wedge x) \vee A\hat{0} = A\hat{0}$, so $c_k \wedge x \leq A\hat{0}$, so $c_k \wedge x = c_k \wedge A\hat{0} \wedge x = A\hat{0} \wedge x$, which has type (1^k).

Next choose c_{i-1} to be a coatom in $[\hat{0}, c_i]$ with $c_{i-1} \vee x = c_i \vee x = \hat{1}$ and $c_{i-1} \wedge x \lessdot c_i \wedge x$; thus, c_{i-1} is any coatom of $[\hat{0}, c_i]$ not also in $[c_i \wedge x, c_i]$. The type of $[\hat{0}, c_i]$ has first column $\nu'_1 + i$, so this interval has $\langle \nu'_1 + i \rangle$ coatoms. The interval $[c_i \wedge x, c_i] \cong [x, x \vee c_i] = [x, \hat{1}]$ has type ν and hence $\langle \nu'_1 \rangle$ coatoms. The type of $[\hat{0}, x \wedge c_i]$ is (1^i), so c_0 is a complement to x.

The number of sequences counted in this second fashion is

$$\begin{aligned} \#\Gamma_{[A\hat{0},\hat{1}]}(x \vee A\hat{0}) \cdot \prod_{i=1}^{k} (\langle \nu'_1 + i \rangle - \langle \nu'_1 \rangle) &= q^{\sum_{j\geq 2} \mu'_j \nu'_j} \cdot \prod_{i=1}^{k} q^{\nu'_1} \langle \nu'_1 + i - \nu'_1 \rangle \\ &= q^{\sum_{j\geq 2} \mu'_j \nu'_j} \cdot q^{\mu'_1 \nu'_1} \cdot \langle k \rangle !. \end{aligned}$$

Equate this with the number of sequences counted in the first fashion to obtain the theorem. $\square$

4.7. The algebra of independent elements. For a tuple $\vec{x} = (x_1, \ldots, x_k)$ of elements of a semi-primary lattice, and a weak composition $\mu = (\mu_1, \ldots, \mu_k)$, let

$$\begin{aligned} \vec{x}[[\mu]] &\stackrel{\text{def}}{=} x_1[\mu_1] \vee \cdots \vee x_k[\mu_k] \\ \vec{x}((\mu)) &\stackrel{\text{def}}{=} x_1(\mu_1) \wedge \cdots \wedge x_k(\mu_k). \end{aligned}$$

Ordinarily, we will apply the former (latter) to a tuple $\vec{x}$ of independent join (meet) irreducibles.

A **join decomposition** of x is a tuple $\vec{x}$ of independent join-irreducibles whose join is x, such that $\rho(x_1) \geq \cdots \geq \rho(x_k)$. A **meet decomposition** is defined dually.

Theorem 4.48. *The number of join decompositions of $\hat{1}$ in a q-regular semi-primary lattice of type λ is*

$$q^{2n(\lambda)} M_\lambda(q^{-1}).$$

Proof. Let $\lambda^{(k)} = (\lambda_k, \lambda_{k+1}, \ldots)$, and t_k be the multiplicity of λ_k in $\lambda^{(k)}$. Let $n_k = \left|\lambda^{(k)}\right| = \lambda_k + \lambda_{k+1} + \cdots$.

Choose a maximum length cycle x_1 in $\hat{1}$ in one of $q^{n_1 - \lambda_1} \langle t_1 \rangle_{q^{-1}}$ ways, and then a complement y_1 to x_1 in one of $q^{n_1 - \lambda_1} = q^{n_2}$ ways.

Suppose we have alternately chosen $y_0, x_1, y_1, \ldots, x_{k-1}, y_{k-1}$ (where $y_0 = \hat{1}$) so that x_i is a maximum length cycle, and y_i is a complement to it, in y_{i-1}, for $i = 1, \ldots, k-1$. Then type $y_i = \lambda^{(i)}$. The number of ways to pick a maximum length cycle x_k in y_{k-1} is $q^{n_{k+1}} \langle t_k \rangle_{q^{-1}}$. The number of complements y_k to x_k in y_{k-1} is $q^{n_{k+1}}$.

The number of sequences $y_0, x_1, y_1, \ldots, x_k, y_k$ of the above form with $y_k = \hat{0}$, is the number of join decompositions of $\hat{1}$, because the y's are recoverable from the x's: $y_i = x_i \vee x_{i+1} \vee \cdots$. This number is

$$\prod_k q^{2n_{k+1}} \langle t_k \rangle_{q^{-1}} = q^{2n(\lambda)} M_\lambda(q^{-1}).$$

The exponent follows from $\sum_k n_{k+1} = \sum_k (\lambda_{k+1} + \lambda_{k+2} + \cdots)$, noting that each λ_i appears $i-1$ times. For each $j > 0$, there is a string of consecutive t_k's of the form $m_j, m_j - 1, \ldots, 1$, so that the product of all $\langle t_k \rangle_{q^{-1}}$ is $M_\lambda(q^{-1})$. $\square$

If x, y are independent elements in a modular lattice, the interval $[\hat{0}, x \vee y]$ is not in general isomorphic to the direct product of lattices $[\hat{0}, x]$ and $[\hat{0}, y]$. However, we have the following result.

Lemma 4.49. *Let x, y be independent elements in a modular lattice. When $x_1, x_2 \leq x$ and $y_1, y_2 \leq y$, we have*

$$\begin{aligned}(x_1 \vee y_1) \vee (x_2 \vee y_2) &= (x_1 \vee x_2) \vee (y_1 \vee y_2) \\ (x_1 \vee y_1) \wedge (x_2 \vee y_2) &= (x_1 \wedge x_2) \vee (y_1 \wedge y_2)\end{aligned}$$

Proof. The first part is clear from associativity and commutativity. For the second part, $(x_1 \vee y_1) \wedge (x_2 \vee y_2) \geq (x_1 \wedge x_2) \vee (y_1 \wedge y_2)$ because each parenthesized expression on the left is weakly greater than each parenthesized expression on the right. Because x and y are independent, any element less than or equal to x and any less than or equal to y are independent, so the rank of the left-hand side is

$$\begin{aligned}&\rho(x_1 \vee y_1) + \rho(x_2 \vee y_2) - \rho(x_1 \vee y_1 \vee x_2 \vee y_2) \\ &\qquad = \rho(x_1) + \rho(y_1) + \rho(x_2) + \rho(y_2) - \rho(x_1 \vee x_2) - \rho(y_1 \vee y_2) \\ &\qquad\qquad = \rho(x_1 \wedge x_2) + \rho(y_1 \wedge y_2),\end{aligned}$$

which is the rank of the right-hand side, so we have equality. □

Corollary 4.50. *Let $\vec{x}$ be a join decomposition of $\hat{1}$ of type λ. Then for all compositions $\mu, \nu \leq \lambda$, we have $\vec{x}[[\mu]] \vee \vec{x}[[\nu]] = \vec{x}[[\mu \vee \nu]]$ and $\vec{x}[[\mu]] \wedge \vec{x}[[\nu]] = \vec{x}[[\mu \wedge \nu]]$. Dual equalities apply to meet decompositions of $\hat{0}$.*

Theorem 4.51. *Let $\vec{x} = (x_1, \ldots, x_k)$ be a join decomposition of $\hat{1}$ and $\vec{y} = (y_1, \ldots, y_k)$ be defined by $y_i = \vee_{j \neq i} x_j$. Then $\vec{y}$ is a meet decomposition of $\hat{0}$. Further, for any weak compositions μ, ν with $\mu_i + \nu_i = \lambda_i$, we have $\vec{x}[[\mu]] = \vec{y}((\nu))$. Further, $C\vec{x}[[\mu]] = \vec{x}[[\mu^-]]$, $A\vec{y}((\nu)) = \vec{y}((\nu^-))$, $A\vec{x}[[\mu]] = \vec{y}((\mu^+))$, and $C\vec{y}((\nu)) = \vec{y}((\nu^+))$, where $\mu_i^- = \max\{\mu_i - 1, 0\}$, $\mu_i^+ = \min\{\mu_i + 1, \lambda_i\}$, and similarly for ν.*

Proof. The intervals $[\hat{0}, x_i] = [x_i \wedge y_1, x_i]$ and $[y_i, \hat{1}] = [y_i, x_i \vee y_i]$ are perspective, so the y_i are cocycles with $\bar{\rho}(y_i) = \rho(x_i) = \lambda_i$. The element μ_i from the bottom of either interval is $\lambda_i - \mu_i = \nu_i$ from the top, so $[y_i, \hat{1}](\nu_i) = y_i \vee x_i[\mu_i] = \vec{x}[[\lambda^{(i)}]]$, where $\tilde{\lambda}$ is the weak composition $\lambda_j^{(i)} = \lambda_j$ if $j \neq i$, and $\lambda_i^{(i)} = \mu_i$. Finally,

$$\vec{y}((\nu)) = \bigwedge_i [y_i, \hat{1}](\nu_i) = \bigwedge_i \vec{x}[[\lambda^{(i)}]] = \vec{x}[[\bigwedge_i \lambda^{(i)}]] = \vec{x}[[\mu]],$$

where we applied Corollary 4.50 to obtain the third equality.

Next,

$$C\vec{y}((\nu)) = C\vec{x}[[\mu]] = C\left(\bigvee_i x_i[\mu_i]\right) = \bigvee_i C\left(x_i[\mu_i]\right) = \bigvee_i x_i[\mu_i^-] = \vec{x}[[\mu^-]] = \vec{y}((\nu^+)),$$

and similarly for the other pair of equalities. □

Definition 4.52. Given weak compositions $\mu \leq \lambda$, define weak compositions $\mathcal{A}_\lambda \mu$ and $\mathcal{C}_\mu \lambda$ in the interval $[\mu, \lambda]$ by specifying their parts:

$$\begin{aligned}(\mathcal{A}_\lambda \mu)_i &= \min\{\mu_i + 1, \lambda_i\} \\ (\mathcal{C}_\mu \lambda)_i &= \max\{\lambda_i - 1, \mu_i\}.\end{aligned}$$

The weak composition $\mathcal{A}_\lambda\mu$ adds a cell to the end of each row of μ unless it would exceed the length of the corresponding row of λ, while the weak composition $\mathcal{C}_\mu\lambda$ deletes a cell from the end of each row of λ unless it would be shorter than the same row of μ.

Corollary 4.53. *Let $\vec{x}, \vec{y}, \lambda, \mu, \nu$ be as in Theorem 4.51. Then*

$$\begin{aligned}
C_{\vec{x}[[\mu]]}\Big(\vec{x}[[\lambda]]\Big) &= \vec{x}[[\mathcal{C}_\mu\lambda]] \\
A_{\vec{x}[[\mu]]}\Big(\vec{x}[[\lambda]]\Big) &= \vec{x}[[\mathcal{A}_\lambda\mu]] \\
C_{\vec{y}((\nu))}\Big(\vec{y}((\lambda))\Big) &= \vec{y}((\mathcal{A}_\nu\lambda)) \\
A_{\vec{y}((\nu))}\Big(\vec{y}((\lambda))\Big) &= \vec{y}((\mathcal{C}_\lambda\nu))
\end{aligned}$$

Definition 4.54. Given partitions μ, λ, we say that μ is a **subcolumn partition** of λ, written $\mu \sqsubseteq \lambda$, if μ can be obtained from λ by deleting columns, or equivalently, $m_i(\mu') \le m_i(\lambda')$ for all possible column lengths i.

Lemma 4.55. *If $\mu \sqsubseteq \lambda$ then $\mu \sqsubseteq \mathcal{A}_\lambda\mu \sqsubseteq \lambda$ and $\mu \sqsubseteq \mathcal{C}_\mu\lambda \sqsubseteq \lambda$.*

Proof. We obtain $\mathcal{A}_\lambda\mu$ from μ by inserting the largest column of λ omitted from μ, while we obtain $\mathcal{C}_\mu\lambda$ from λ by deleting the largest omitted column. $\square$

Lemma 4.56. *If $\mu, \nu \sqsubseteq \lambda$ then $\mu \vee \nu, \mu \wedge \nu \sqsubseteq \lambda$.*

Proof. Let $\rho = \mu \wedge \nu$, so $\rho'_i = \min\{\mu'_i, \nu'_i\}$. Each ρ'_i is the length of some column of μ or ν and hence of λ. We must make sure that the multiplicity of each length is no greater than in λ. Suppose i is minimal with $\rho'_i = k$, and that, without loss of generality, $\mu'_i = k$. The $\big(i + m_k(\mu')\big)$th column of μ, and hence of ρ, has length strictly less than k, so $m_k(\rho') \le m_k(\mu')$. Also, $m_k(\mu') \le m_k(\lambda')$, so $m_k(\rho') \le m_k(\lambda')$. This holds for all k, so $\rho \sqsubseteq \lambda$.

The argument for joins is similar, but instead of choosing the leftmost column of length k and examining the one $m_k(\mu')$ to its right, we choose the rightmost one of length k and examine the column (if any) $m_k(\mu')$ to its left. $\square$

Theorem 4.57. *Let S be the minimal set containing $\emptyset, \lambda$ and closed under $\wedge, \vee, \mathcal{A}, \mathcal{C}$. Then S is precisely the set of all subcolumn partitions of λ.*

Proof. If $\mu \sqsubseteq \nu \sqsubseteq \lambda$ then $\mathcal{A}_\nu\mu$ adds a column to μ of length equal to the largest i with $m_i(\nu') > m_i(\mu')$, while $\mathcal{C}_\nu\mu$ subtracts a column of this length from ν. So all elements of S are subcolumn partitions of λ.

Conversely, let $\mu \sqsubseteq \lambda$ be obtained from λ by deleting columns $c_1 > c_2 > \cdots > c_k > 0$. Abbreviate $\mathcal{A}x = \mathcal{A}_\lambda x$ and $\mathcal{C}x = \mathcal{C}_\emptyset x$. Then $\mu = \mathcal{A}^{c_k-1}\mathcal{C}^{c_k}\cdots\mathcal{A}^{c_1-1}\mathcal{C}^{c_1}\lambda$, because $\mathcal{A}^{c_i-1}\mathcal{C}^{c_i}$ deletes the first c_i columns, which happen to coincide with the first c_i columns of λ, and then inserts the $c_i - 1$ largest columns of λ that are missing, which in fact are the $c_i - 1$ largest columns of λ. $\square$

Theorem 4.58. *Let $\mu \sqsubseteq \lambda = \operatorname{type}\hat{1}$. Let $\vec{x}$ and $\vec{y}$ be two join decompositions of $\hat{1}$. Then $\vec{x}[[\mu]] = \vec{y}[[\mu]]$.*

Proof. By Theorem 4.57, we can express μ as a polynomial in $\wedge, \vee, \mathcal{A}, \mathcal{C}, \emptyset, \lambda$. Replace $\emptyset$ with $\hat{0}$; λ with $\hat{1}$; $\mathcal{A}$ with A; $\mathcal{C}$ with C; and interpret $\wedge$ and $\vee$ in $\mathcal{L}$ instead of Young's lattice. This gives a decomposition-independent expression of an element equal to either of $\vec{x}[[\mu]]$ and $\vec{y}[[\mu]]$, by recursive applications of Corollaries 4.50 and 4.53. $\square$

Definition 4.59. This last theorem allows us, for all $\mu \sqsubseteq \lambda$, to define a special element $\hat{1}[[\mu]]$. Its type is μ and its cotype is $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$, or alternately, the columns of λ missing from μ. Further, for any interval $[x, y]$, we may define $[x, y][[\mu]]$ as $\vec{z}[[\mu]]$, where $\mu \sqsubseteq \operatorname{type}[x, y]$ and $\vec{z}$ is any join decomposition of y in the interval $[x, y]$. We abbreviate $[\hat{0}, y][[\mu]]$ by $y[[\mu]]$.

Theorem 4.60. *Let λ, μ, ν be partitions with $\mu + \nu = \lambda$. The unique element of type μ and cotype ν in a semi-primary lattice of type λ is $\hat{1}[[\mu]]$.*

Proof. The previous theorem establishes existence. To prove uniqueness, we induct on the length of the lattice. For $\lambda = \mu = \nu = \emptyset$, there is only one element in the lattice, so take a larger case. Suppose x has type μ and cotype ν. Either $\mu'_1 = \lambda'_1$, or $\nu'_1 = \lambda'_1$ (or both). In the former case, $A_x\hat{0} = A\hat{0}$, so that x has type $\mathcal{C}\mu$ and cotype ν in the interval $[A\hat{0}, \hat{1}]$ of type $\mathcal{C}\lambda$, and since this theorem inductively applies to this smaller interval, x is unique. In the latter case, $C_x\hat{1} = C\hat{1}$, so that x has type μ and cotype $\mathcal{C}\nu$ in $[\hat{0}, C\hat{1}]$ (of type $\mathcal{C}\lambda$), and again x is unique. $\square$

4.8. Hereditary Decompositions.

Definition 4.61. A **hereditary (join) decomposition** of a (nonsaturated) flag $f = (f_0 \leq \cdots \leq f_n)$ is a join decomposition $\vec{x}$ of f_n such that $f_i = \vec{x}[[\operatorname{type} f_i]]$ for all i.

Theorem 4.62. *Let $\mathcal{L}$ be a semi-primary lattice of type λ. Let $f = (f_0, \ldots, f_n)$ have a hereditary join decomposition in $\mathcal{L}$. Let $p(v_0, \ldots, v_n)$ be a lattice polynomial constructed from the variables $v_0, \ldots, v_n$; constant $\hat{0}$; and binary operators $\wedge, \vee, C, A$. Then* $\operatorname{type} p(f_0, \ldots, f_n)$ *is computed by substituting into $p(v_0, \ldots, v_n)$ the following: v_i becomes* $\operatorname{type} f_i$*; C becomes $\mathcal{C}$; A becomes $\mathcal{A}$; $\hat{0}$ becomes $\emptyset$; and $\wedge$ and $\vee$ are interpreted in Young's lattice instead of $\mathcal{L}$.*

Proof. Consider any hereditary decomposition $\vec{x}$ of f. Then

$$p(f_0, \ldots, f_n) = p(\vec{x}[[\operatorname{type} f_0]], \ldots, \vec{x}[[\operatorname{type} f_n]]).$$

Also $\hat{0}$ can be replaced by $\vec{x}[[\emptyset]]$. Recursively apply Corollaries 4.50 and 4.53 to this form of the polynomial to obtain the result. $\square$

Now we'll examine some conditions under which flags have hereditary join decompositions.

Theorem 4.63. *Let $x \lessdot y$ in a semi-primary lattice $\mathcal{L}$. There is a hereditary decomposition of (x, y) in which a single part is reduced in length by 1. Consequently,* $\operatorname{type} x \lessdot \operatorname{type} y$ *in Young's lattice. If the types differ in the first component, then all join decompositions of y restrict to join decompositions of x.*

Proof. If $x = \hat{0}$ then y is an atom, and the decomposition is simply y.

If $\operatorname{arank} x = \operatorname{arank} y$ then pick a cycle $c \leq x$ of rank $\operatorname{arank} x$. It will be one element of the decompositions of both x and y. Take a complement c' of c in $[\hat{0}, y]$. Then $y \vee c' \geq x \vee c' \geq c \vee c' = \hat{1}$ so $y \vee c' = x \vee c' = \hat{1}$, and then by Lemma 2.3, $c' = y \wedge c' \gtrdot x \wedge c'$. Since $\rho(c') < \rho(y)$, inductively there is a hereditary decomposition of $(x \wedge c', c')$ satisfying the conditions of the theorem, and to this decomposition we prepend a cycle c to obtain a decomposition of (x, y).

If $\operatorname{arank} x < \operatorname{arank} y$ then pick any cycle $c \leq y$ of rank $\operatorname{arank} y$. Let $d = c \wedge x \lessdot c \wedge y = c$. Let c' be a complement of d in $[\hat{0}, x]$. Then $c' \wedge c \geq c' \wedge d = \hat{0}$, but the only atom in the chain $[\hat{0}, c]$ is also in $[\hat{0}, d]$, so $c' \wedge c = \hat{0}$. Thus, $c' \vee c \gtrdot c' \vee d = x$; in fact, $c' \vee c = y$, since $c' \vee c \leq y$ and $y \gtrdot x$. Thus c' is also a complement of c in y, so we add c to any decomposition of c' to obtain a hereditary decomposition of (x, y). □

As a corollary, we obtain a different proof of Theorem 4.11.

Theorem 4.11. *Let* $[a', b'] \subseteq [a, b]$. *Then* $\operatorname{type}[a', b'] \leq \operatorname{type}[a, b]$.

Proof. By repeated applications of the preceding theorem on a saturated chain from b down to b', we have $\operatorname{type}[a, b'] \leq \operatorname{type}[a, b]$. By a dual argument, we obtain $\operatorname{type}[a', b'] \leq \operatorname{type}[a, b']$. □

We obtain an enumerative version of Theorem 4.63.

Corollary 4.64. *Let x be a coatom of type μ in a q-regular semi-primary lattice of type λ. Suppose λ and μ differ in their ith part.*

(1) *The number of complemented cycles c of length $r = \lambda_i$ with $c \not\leq x$ is*
$$q^{(\lambda'_1 + \cdots + \lambda'_r) - r}.$$

(2) *The number of hereditary decompositions of $(x, \hat{1})$ is*
$$\frac{q^{2n(\lambda) - i + 1} M_\lambda(q^{-1})}{\langle m_r(\lambda) \rangle_{q^{-1}}}.$$

Proof.

(1) Let $r = \lambda_i$. The number of complemented cycles of length r in $[\hat{0}, \hat{1}]$, less the number of complemented cycles of length r in $[\hat{0}, x]$, is
$$\begin{aligned} &q^{(\lambda'_1 + \cdots + \lambda'_r) - r} \langle m_r(\lambda) \rangle_{q^{-1}} - q^{(\mu'_1 + \cdots + \mu'_r) - r} \langle m_r(\mu) \rangle_{q^{-1}} \\ &\qquad = q^{(\lambda'_1 + \cdots + \lambda'_r) - r} \langle m_r(\lambda) \rangle_{q^{-1}} - q^{(\lambda'_1 + \cdots + \lambda'_r - 1) - r} \langle m_r(\lambda) - 1 \rangle_{q^{-1}} \\ &\qquad = q^{(\lambda'_1 + \cdots + \lambda'_r) - r}. \end{aligned}$$
by Theorem 4.47. The parts of λ' and μ' are the same except $\mu'_r = \lambda'_r - 1$, so $m_r(\mu) = m_r(\lambda) - 1$. The two terms subtracted in the second expression are each sums of consecutive powers of q, differing only in the single power given in the third expression.

(2) Let $d = (\lambda'_1 + \cdots + \lambda'_r) - r$. Consider any hereditary decomposition $\vec{x}$ of $(x, \hat{1})$. We have $x_i \leq \hat{1}$; $x_i \not\leq x$; and $Cx_i \leq x$. Let $c' = \vee_{j \neq i} x_j$. Then c' is a complement to x_i in $[\hat{0}, \hat{1}]$, and to Cx_i in $[\hat{0}, x]$. Note that in fact any complement y to Cx_i in $[\hat{0}, x]$ is a complement to x in $[\hat{0}, \hat{1}]$ (though the

converse is not true): we have $x_i \vee y \geq Cx_i \vee y = x$ but $x_i \not\leq x$, so $x_i \vee y = \hat{1}$, and so $x_i \wedge y = Cx_i \wedge y = \hat{0}$.

To enumerate the hereditary decompositions, first choose a complemented cycle $x_i \leq \hat{1}$ but $x_i \not\leq x$ in one of $q^d \langle m_r(\lambda) \rangle_{q^{-1}}$ ways. This has type $(1^r)'$ and cotype

$$\nu = (\lambda'_1 - 1, \dots, \lambda'_r - 1, \lambda'_{r+1}, \lambda'_{r+2}, \dots)'.$$

Then we pick a complement c' to Cx_i in x, with type ν and cotype (1^{r-1}) in $[\hat{0}, x]$, in one of

$$q^{\lambda'_1 + \cdots + \lambda'_{r-1} - (r-1)} = q^{d - \lambda'_r + 1} = q^{d-i+1}$$

ways by Theorem 4.47. Obtain a decomposition of c' in one of

$$q^{2n(\nu)} M_\nu(q^{-1}) = q^{2(n(\lambda)-d)} \frac{M_\lambda(q^{-1})}{\langle m_r(\lambda) \rangle_{q^{-1}}}$$

ways, and insert x_i into this decomposition as the ith term to obtain a hereditary decomposition of $(x \lessdot \hat{1})$. Multiply the number of choices together to obtain the value stated in the theorem. $\square$

The next several theorems are different generalizations of Theorem 4.63 to hereditary decompositions of flags.

Definition 4.65. An interval is **elementary** if its type is of the form (1^n).

Definition 4.66. A **vertical strip** is a skew partition λ/μ with at most one cell in each row; equivalently, $\lambda_i \geq \mu_i$ for all $i > 0$. The notation $\lambda \geq_\vee \mu$ means $\lambda \geq \mu$ and λ/μ is a vertical strip.

Dually, a **horizontal strip** is a skew partition λ/μ with at most one cell in each column; equivalently, $\lambda'_i \geq \mu'_i$ for all $i > 0$.

Proposition 4.67. *Let $[x, y]$ be an elementary interval in a semi-primary lattice. Then* $\operatorname{type} y / \operatorname{type} x$ *is a vertical strip.*

Proof. Since $[x, y]$ is elementary, it is contained in $[Cy, y]$, whence $\operatorname{type} Cy \leq \operatorname{type} x \leq \operatorname{type} y$. Since $\operatorname{type} y / \operatorname{type} Cy$ has one cell on each nonempty row of $\operatorname{type} y$, and no other cells, it is a vertical strip. $\square$

Definition 4.68. A **row-end** of a tableau $\boldsymbol{P}$ of shape λ/μ is any cell (i, λ_i) with $1 \leq i \leq \lambda'_1$.

Theorem 4.69. *Let $f = (f_0, \dots, f_n)$ be a flag in which the interval $[f_0, f_n]$ is elementary.*

(1) *f has a hereditary decomposition.*
(2) *Let $\boldsymbol{P} = (\lambda^{(0)} \lessdot \cdots \lessdot \lambda^{(n)})$ be a standard skew tableau whose entries form a vertical strip, $\lambda^{(n)}/\lambda^{(0)}$. In a q-regular semi-primary lattice of shape $\lambda^{(n)}$, the number of these flags with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *is*

$$\prod_{k \,\in\, \boldsymbol{P}} \Big(\langle l_1 \rangle - \langle l_2 \rangle \Big)$$

where

$$\begin{aligned} l_1 &= \# \textit{ row-ends of } \boldsymbol{P} \textit{ weakly right of } k \textit{ containing any of } \bullet, 1, \ldots, k \\ l_2 &= \# \textit{ row-ends of } \boldsymbol{P} \textit{ strictly right of } k \textit{ containing any of } \bullet, 1, \ldots, k \end{aligned}$$

Proof.

(1) We may assume f is saturated, because if it's not, we can always remove duplicate elements and add additional intermediate elements to saturate it; doing so does not change the type of the interval from the first to the last element, and still provides decompositions of all the original elements.

Any hereditary decomposition of $f_{n-1} \lessdot f_n$ yields a cycle c and an independent element c' with $c \vee c' = f_n$ and $Cc \vee c' = f_{n-1}$.

For $0 \leq i \leq n$, the interval $[c \wedge f_i, c \wedge f_n]$ is contained in $[\hat{0}, c]$, so its type, α, is a one row partition. Also,

$$[f_i \wedge c, f_n \wedge c] \subseteq [f_i \wedge c, c] \cong [f_i, f_i \vee c] \subseteq [f_i, f_n],$$

so α is a one column partition. Thus α is $\emptyset$ or (1). Further, $c \wedge f_{n-1} \lessdot c \wedge f_n$, so $\alpha = (1)$ for $i = 0, \ldots, n-1$. Thus $c \wedge f_i = Cc$ for $i = 0, \ldots, n-1$.

Let $g = (g_0 \leq \cdots \leq g_{n-1})$ be the flag given by $g_i = f_i \wedge c'$. The elements Cc and g_i are independent, and

$$Cc \vee g_i = Cc \vee (c' \wedge f_i) = (Cc \vee c') \wedge f_i = f_{n-1} \wedge f_i = f_i.$$

The interval $[g_0, g_{n-1}]$ is elementary because

$$Ag_0 = A(f_0 \wedge c') = Af_0 \wedge Ac' \geq f_n \wedge Ac' \geq f_{n-1} \wedge c' = g_{n-1}.$$

Since the flag g is saturated and shorter than f, by induction we find a hereditary decomposition of g, and then add c to it to obtain a hereditary decomposition of f.

(2) Set $f_n := \hat{1}$. After choosing $f_n, \ldots, f_k$, choose f_{k-1} to be a coatom of $[\hat{0}, f_k]$ of type $\lambda^{(k-1)}$ so that $[f_{k-1}, \hat{1}]$ is elementary, as follows. Let $r = \mathrm{col}(\lambda^{(k)}/\lambda^{(k-1)})$. The condition that f_{k-1} is a coatom of $[\hat{0}, f_k]$ of type $\lambda^{(k-1)}$ is equivalent to

$$\begin{aligned} f_{k-1} &\lessdot f_k \\ f_{k-1} &\geq f_k[r-1] \\ f_{k-1} &\not\geq f_k[r] \end{aligned}$$

while the elementary condition is

$$f_{k-1} \leq C\hat{1}.$$

So f_{k-1} is any coatom of $I_1 = \left[C\hat{1} \vee f_k[r-1], f_k\right]$ not also in $I_2 = \left[C\hat{1} \vee f_k[r], f_k\right]$. These are both elementary intervals since they are subintervals of $[C\hat{1}, \hat{1}]$. By (1) there is a hereditary decomposition of $(f_k, \hat{1})$, so by Theorem 4.62, $\mathrm{type}(C\hat{1} \vee f_k[r-1]) = \mathcal{C}\lambda^{(n)} \vee \lambda^{(k)}[r-1]$. Compare this rowwise to $\lambda^{(k)}$: when $r - 1 < \lambda_i^{(k)} = \lambda_i^{(n)}$ the ith row is $\lambda_i^{(k)} - 1$, and otherwise, the ith row is $\lambda_i^{(k)}$. We have $\lambda_i^{(k)} \lessdot \lambda_n^{(k)}$ when the entry at the end of row i of $\boldsymbol{P}$ is larger than k. Thus, $\rho(I_1)$ is the number of entries weakly smaller than k at the end of a row in a column weakly right of k. Similarly, $\rho(I_2)$ is the number of entries weakly smaller than k at the end of a row in a column strictly right of k. $\square$

Theorem 4.70. *Let $f = (f_0 \lessdot \cdots \lessdot f_n)$ be a flag in a semi-primary lattice $\mathcal{L}$ with $\operatorname{ftype}_{\hat{0}} f$ equal to the superstandard skew tableau of shape λ/μ.*

(1) *f has a hereditary decomposition.*
(2) *If $\mathcal{L}$ is q-regular, there are*

$$q^{n(\lambda)+n(\mu)} M_\nu(q^{-1})$$

such decompositions, where ν is the partition given by

$$m_i(\nu) = \#\{\, j : \lambda_j = \mu_j = i \,\} \text{ for } i \geq 1.$$

Proof. In both parts we induct on the number of rows in λ. When there are no rows, the flag is $(\hat{0})$, which has the empty sequence as its sole decomposition.

(1) Let the first row of $\operatorname{ftype}_{\hat{0}} f$ have entries $1, \ldots, k$, with $k = \lambda_1 - \mu_1$. Choose a cycle c of rank λ_1 with $c \leq f_k$, and take a complement c' of c in f_n. Since $\operatorname{arank} f_k = \cdots = \operatorname{arank} f_n = \lambda_1$, we see that c is a maximal cycle in each of $f_k, \ldots, f_n$, and so the type of each is the type of any relative complement of c, with a part λ_1 prepended. The element $c' \wedge f_j$ is a complement to c in f_j (with $k \leq j \leq n$) because $c \vee (c' \wedge f_j) = (c \vee c') \wedge f_j = f_n \wedge f_j = f_j$ and $c \wedge (c' \wedge f_j) \leq c \wedge c' = \hat{0}$. So the flag $g = (f_k \wedge c', \ldots, f_n \wedge c')$ of complements to c has type $\operatorname{ftype}_{\hat{0}} f$ with the first row deleted; that is, it is the superstandard skew tableau of shape $\tilde{\lambda}/\tilde{\mu}$ where $\tilde{\lambda} = (\lambda_2, \lambda_3, \ldots)$ and $\tilde{\mu} = (\mu_2, \mu_3, \ldots)$ and entries $k+1, \ldots, n$. Inductively applying the theorem to g, we obtain a hereditary decomposition $\vec{x}$ of g, to which we prepend the cycle c to obtain a hereditary decomposition $\vec{y}$ of $(f_k, \ldots, f_n)$. Finally, because the types of $f_0, \ldots, f_k$ differ in only their first component, all join decompositions of f_k are hereditary decompositions of $f_0, \ldots, f_k$, so $\vec{y}$ is a hereditary decomposition of f.
(2) We count the choices made in (1). The type of f_k is $\gamma = (\lambda_1, \mu_2, \mu_3, \ldots)$, so the number of choices of c is

$$q^{\mu_2+\mu_3+\cdots} \langle m_{\lambda_1}(\gamma) \rangle_{q^{-1}} = \begin{cases} q^{|\mu|-\mu_1} & \text{if } \lambda_1 > \mu_1; \\ q^{|\mu|-\mu_1} \langle m_{\mu_1}(\mu) \rangle_{q^{-1}} & \text{if } \lambda_1 = \mu_1. \end{cases}$$

by Theorem 4.44 (note that if $\lambda_1 > \mu_1$ then $\lambda_1 > \mu_2$, so $m_{\lambda_1}(\gamma) = 1$, and $\langle 1 \rangle_{q^{-1}} = 1$). The number of choices of c' is

$$q^{|\lambda|-\lambda_1}$$

by Theorem 4.47. Let $\tilde{\nu}$ be the partition given by

$$m_i(\tilde{\nu}) = \#\left\{ j : \tilde{\lambda}_j = \tilde{\mu}_j = i \right\}.$$

Then inductively there are

$$q^{n(\tilde{\lambda})+n(\tilde{\mu})} M_{\tilde{\nu}}(q^{-1})$$

decompositions of g.

If we fill in each cell of λ with its row number minus 1 and add up these entries, we obtain $n(\lambda)$; subtract an additional 1 from each entry not on the first row, and add them up to obtain $n(\tilde{\lambda})$. A similar argument computes $n(\tilde{\mu})$. Thus,

$$\begin{aligned} n(\tilde{\lambda}) &= n(\lambda) - |\lambda| + \lambda_1 \\ n(\tilde{\mu}) &= n(\mu) - |\mu| + \mu_1 \end{aligned}$$

Now we multiply all the numbers of choices together to obtain the number of decompositions:

$$\begin{array}{ll} q^{n(\lambda)+n(\mu)} M_{\tilde{\nu}}(q^{-1}) & \text{if } \lambda_1 > \mu_1; \\ q^{n(\lambda)+n(\mu)} \langle m_{\mu_1}(\mu)\rangle_{q^{-1}} M_{\tilde{\nu}}(q^{-1}) & \text{if } \lambda_1 = \mu_1. \end{array}$$

If $\lambda_1 > \mu_1$ then $\nu = \tilde{\nu}$, so that we have the total stated in the theorem.

If $\lambda_1 = \mu_1$ then $\nu = (\mu_1, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \ldots)$. Since the parts of λ weakly decrease, but are bounded below by the parts of μ, the first $m_{\mu_1}(\mu)$ parts of λ are also equal to μ_1, so $m_{\mu_1}(\nu) = m_{\mu_1}(\mu)$ and $M_\nu(q^{-1}) = \langle m_{\nu_1}(\nu)\rangle_{q^{-1}} M_{\tilde{\nu}}(q^{-1})$. So again, the total is as stated in the theorem. $\square$

Example 4.71. Consider

$$\boldsymbol{P} = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 2 & 3 \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & 4 & 5 & & \\ \bullet & & & & \\ 6 & & & & \end{array}$$

This has shape $\lambda/\mu = (5,5,5,5,4,4,4,4,1,1)/(5,5,4,3,3,3,3,1,1)$. The rows with no entries form the partition $\nu = (5,5,3,3,3,1)$. A given flag with type $\boldsymbol{P}$ has

$$q^{n(\lambda)+n(\mu)} M_\nu(q^{-1}) = q^{173+82}\langle 2\rangle_{q^{-1}}\langle 3\rangle_{q^{-1}}\langle 1\rangle_{q^{-1}} = q^{255}\langle 2\rangle_{q^{-1}}\langle 3\rangle_{q^{-1}}\langle 1\rangle_{q^{-1}}$$

hereditary decompositions.

Corollary 4.72. *Let f be a saturated flag and* $\operatorname{low}(f) < k < \operatorname{high}(f)$*.*

1. *If* $\operatorname{ftype}_{\hat{0}} f$ *has $k+1$ on a lower row than k, then* $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$*;*
2. *if k and $k+1$ are on the same row, then* $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{array}{|c|c|}\hline & \\ \hline\end{array}$*;*
3. *if $k+1$ is on an earlier row, either type is possible (although generically we expect* $\begin{array}{|c|c|}\hline & \\ \hline\end{array}$*).*

Proof. In the first two cases, we have $\operatorname{ftype}_{\hat{0}}(f_{k-1} \lessdot f_k \lessdot f_{k+1})$ is a superstandard skew tableau, so that these three consecutive elements have a hereditary decomposition. In (1), two join-irreducibles in the decomposition of f_{k+1} are each reduced in length by 1 to obtain f_{k-1}, while in (2), one join-irreducible is reduced in length by 2. So we obtain the types indicated.

The smallest example of (3) is

$$\operatorname{ftype}_{\hat{0}} f = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}.$$

We see that $\operatorname{type} Cf_3 = (1)$. So the only value of f_1 with $\operatorname{type}[f_1, f_3] = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$ is Cf_1, and all other atoms f_1 of $[f_0, f_2]$ yield $\operatorname{type}[f_1, f_3] = \square\square$.

More generally in (3), let $\boldsymbol{P} = (\lambda^{(k-1)} \lessdot \lambda^{(k)} \lessdot \lambda^{(k+1)})$ be a skew tableau of shape λ/μ with just 2 entries, k in cell (r_0, c_0) and $k+1$ in cell (r_1, c_1), with $c_0 < c_1$ and $r_0 > r_1$. Choose any f_k, f_{k+1} with $\operatorname{ftype}_{\hat{0}}(f_k, f_{k+1}) = (\lambda^{(k)}, \lambda^{(k+1)})$. We will show that of those $f_{k-1} \lessdot f_k$ with type $\lambda^{(k-1)}$, fewer have $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$ than $\square\square$.

We have $f_{k-1} \lessdot f_k$ and $\operatorname{type} f_{k-1} = \lambda^{(k-1)}$ iff f_{k-1} is a coatom of the interval $I_1 = \Big[f_k[c_0 - 1] \vee Cf_k, f_k\Big]$ not also in $I_2 = \Big[f_k[c_0] \vee Cf_k, f_k\Big]$.

If we further want $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$, then since $f_{k-1} \geq Cf_{k+1}$, we obtain that f_{k-1} is any coatom of $I_3 = \Big[f_k[c_0 - 1] \vee Cf_{k+1}, f_k\Big]$ not also in $I_4 = \Big[f_k[c_0] \vee Cf_{k+1}, f_k\Big]$.

Let $x_1 = f_k[c_0 - 1] \vee Cf_k$, and similarly, let $x_i = \hat{0}_{I_i}$ for $i = 1, 2, 3, 4$. Since $f_k \lessdot f_{k+1}$, these have a hereditary decomposition, so the types of $x_1, \ldots, x_4$ can be computed by Theorem 4.62 by replacing f_k and f_{k+1} with their types, and C with $\mathcal{C}$.

The type of x_1 is $\lambda^{(k)}$ with each row at least c_0 long reduced in length by 1, while in $\operatorname{type} x_2$, each row of $\lambda^{(k)}$ at least $c_0 + 1$ long is reduced in length by 1. The types of x_3 and x_4 are almost the same as x_1 and x_2 except that an additional cell is added at $(r_1, c_1 - 1)$. Since $c_1 > c_0$, this cell is not present in $\operatorname{type} x_1$, and when $c_1 > c_0 + 1$, it is not present in $\operatorname{type} x_2$. Note that in any case, we do not have $x_3 = x_4$ because the cell (r_0, c_0) is in $\operatorname{type} x_4$ but not $\operatorname{type} x_3$. Thus, $I_3 - I_4$ is nonempty, and any coatom yields f_{k-1} with $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$. Any coatom of $I_1 - I_2$ not also in $I_3 - I_4$ yields $\square\square$. Since I_1 is longer than I_3 by 1, there are roughly q times as many possibilities yielding $\square\square$ as $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ in a q-regular semi-primary lattice. $\square$

Example 4.73. Suppose

$$\operatorname{ftype} f = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 8 \\ \hline 2 & 5 & 6 \\ \cline{1-3} 7 & 9 \\ \cline{1-2}\end{array}$$

Then intervals $[f_0, f_2]$, $[f_3, f_5]$, $[f_5, f_7]$, and $[f_7, f_9]$ have type $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$. Intervals $[f_2, f_4]$ and $[f_4, f_6]$ have type $\square\square$. The types of the intervals $[f_1, f_3]$ and $[f_6, f_8]$ cannot be determined from $\operatorname{ftype} f$.

Definition 4.74. Let $\boldsymbol{A}$ be a skew tableau and $\boldsymbol{B}$ be an ordinary tableau. The tableau $\boldsymbol{A} + \boldsymbol{B}$ is obtained by appending the ith row of $\boldsymbol{B}$ to the ith row of $\boldsymbol{A}$, for all i. Its shape is $\operatorname{osh} \boldsymbol{A} + \operatorname{sh} \boldsymbol{B} = \Big((\operatorname{osh} \boldsymbol{A})' \cup (\operatorname{sh} \boldsymbol{B})'\Big)'$, so $\operatorname{osh} \boldsymbol{A}$ and $\operatorname{sh} \boldsymbol{B}$ are both subcolumn partitions of $\operatorname{osh} \boldsymbol{A} + \boldsymbol{B}$. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 8 & 10 \\ \hline 9 & 11 \\ \hline 12 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 7 & 8 & 10 \\ \hline 2 & 5 & 9 & 11 \\ \cline{1-4} 4 & 12 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

This is also defined for composition tableaux, but the relation for transposed shapes doesn't apply. The definition in terms of the chains $\boldsymbol{A} = (\lambda^{(l)}, \lambda^{(l+1)}, \ldots, \lambda^{(k)})$ and $\boldsymbol{B} = (\gamma^{(k)}, \gamma^{(k+1)}, \ldots, \gamma^{(h)})$ is

$$\boldsymbol{A} + \boldsymbol{B} = (\lambda^{(l)}, \lambda^{(l+1)}, \ldots, \lambda^{(k)} = \gamma^{(k)} + \lambda^{(k)}, \gamma^{(k+1)} + \lambda^{(k)}, \ldots, \gamma^{(h)} + \lambda^{(k)}).$$

Definition 4.75. For any tableau $\boldsymbol{A} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$, let $d\boldsymbol{A} = (\lambda^{(l)}, \ldots, \lambda^{(h-1)})$, so $\operatorname{low}(d\boldsymbol{A}) = \operatorname{low}(\boldsymbol{A})$ and $\operatorname{high}(d\boldsymbol{A}) \lessdot \operatorname{high}(\boldsymbol{A})$. This can be thought of as deleting the largest entry of $\boldsymbol{A}$, though when $\operatorname{high}(\boldsymbol{A})$ doesn't occur in $\boldsymbol{A}$, this isn't strictly correct.

Theorem 4.76. *Let f be a saturated flag. Suppose $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{A} + \boldsymbol{B}$, where $\boldsymbol{A}$ is a standard skew tableau and $\boldsymbol{B}$ is an ordinary standard tableau. Let $h = \operatorname{high}(\boldsymbol{B})$ and $k = \operatorname{high}(\boldsymbol{A}) = \operatorname{low}(\boldsymbol{B})$. Let $\mu = \operatorname{sh} \boldsymbol{A}$. Then $f_k = f_m[[\mu]]$ for $m = k, \ldots, h$, and $\operatorname{ftype}(f_k, \ldots, f_h) = \boldsymbol{B}$.*

Proof. We induct on h. When $h = k$ it is true, so take $h > k$. Then $\operatorname{ftype} df = \boldsymbol{A} + d\boldsymbol{B}$, and inductively we assume the theorem applies to df. Let $\vec{x}$ be a hereditary decomposition of $(f_{h-1} \lessdot f_h)$.

Since $\mu \sqsubseteq \operatorname{sh} \boldsymbol{A} + \boldsymbol{B} = \operatorname{type} f_h$, by Theorem 4.58, the shape μ restriction of all join decompositions of f_h are the same. Thus $f_h[[\mu]] = \vec{x}[[\mu]] = f_{h-1}[[\mu]] = f_k$. Because $f_k = \vec{x}[[\mu]]$, we have $\operatorname{type}[f_k, f_h]$ is obtained from the composition $\operatorname{type} f_h - \operatorname{type} f_k = \operatorname{sh} \boldsymbol{B}$ by sorting the parts into decreasing order to form a partition, and they already are in order. By applying the theorem to df and $\operatorname{ftype} df = d(\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{A} + d\boldsymbol{B}$, we have $\operatorname{ftype}(f_k, \ldots, f_{h-1}) = d\boldsymbol{B}$, and by adding a cell with h to achieve the shape $\operatorname{sh} \boldsymbol{B}$ of $\operatorname{type}[f_k, f_h]$, we obtain $\operatorname{ftype}(f_k, \ldots, f_h) = \boldsymbol{B}$. $\square$

Example 4.77. For all flags $f = (f_0 \lessdot \cdots \lessdot f_{11})$,

$$\text{if} \quad \operatorname{ftype} f = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 9 & 11 & \\ \cline{1-4} 6 & 10 & & & \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array} + \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 9 & 11 \\ \hline 10 & \\ \cline{1-1} \end{array} \quad \text{then} \quad \operatorname{ftype} \partial^6 f = \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 9 & 11 \\ \hline 10 & \\ \cline{1-1} \end{array}.$$

See Theorem 4.93 for a generalization of this to $\boldsymbol{B}$ being a row and column strict composition tableau.

4.9. Hall Polynomials. The first two definitions are standard.

Definition 4.78. The **reverse row word** of a skew tableau $\boldsymbol{P}$ is the sequence of digits obtained by reading the entries of $\boldsymbol{P}$ from right to left on the first row, then the second row, and so on to the bottom. For example, the reverse row word of

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & 1 & 1 & 1 \\ \bullet & \bullet & 1 & 2 & 2 & \\ \bullet & 1 & 3 & & & \\ 4 & & & & & \end{array}$$

is 111221314.

Definition 4.79. A **ballot sequence** is a sequence of integers such that in any initial segment of the sequence, there are at least as many i's as $(i+1)$'s for all $i \in \mathbb{P}$. So 111221314 is a ballot sequence, but 112221314 is not because in the initial segment 11222, there are more 2's than 1's.

Definition 4.80 (see [16, p. 90]). A tableau $\boldsymbol{P}$ representing the chain of partitions $(\lambda^{(0)} \leq \lambda^{(1)} \leq \ldots \leq \lambda^{(r)})$ is called an **LR-sequence of type** $(\mu, \nu; \lambda)$ if

(1) $\lambda^{(0)} = \mu$ and $\lambda^{(r)} = \lambda$;
(2) $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of length ν_i for $1 \leq i \leq r$;

(3) the reverse row word of $\boldsymbol{P}$ is a ballot sequence.

The number of LR-sequences of type $(\mu, \nu; \lambda)$ is the **Littlewood-Richardson coefficient** $c^{\lambda}_{\mu\nu}$. Their properties include $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu} = c^{\lambda'}_{\mu',\nu'}$, proofs of which can also be found in [**16**].

Theorem 4.81. *Let μ, ν, λ be partitions. There are polynomials $g^{\lambda}_{\mu\nu}(q)$ with integer coefficients, such that in every q-regular semi-primary lattice of type λ, the number of elements of type μ and cotype ν is $g^{\lambda}_{\mu\nu}(q)$. When $|\mu| + |\nu| \neq |\lambda|$, the polynomial is 0. Otherwise, the leading coefficient is $c^{\lambda}_{\mu\nu}$ and the degree is $n(\lambda) - n(\mu) - n(\nu)$.*

These polynomials are called **Hall polynomials**.

Hall proved the existence of polynomials $g^{\lambda}_{\mu\nu}(p)$ such that in the lattice of subgroups of an abelian p-group of type λ, the number of subgroups of type μ and cotype ν is $g^{\lambda}_{\mu\nu}(p)$; a summary of his theory is in [**9**]. He determined the degree and leading coefficient of the polynomials. Klein [**14**] was the first to explicitly determine these polynomials for all parameters, and did so in the wider context of p-modules. Macdonald [**16**, §II.4] later determined a different (but of course equal) formula, in the context of modules over discrete valuation rings, and replaced the prime p by the cardinality q of the residue field of the ring.

Macdonald's construction of these polynomials goes through to the wider context of q-regular semi-primary lattices by making minor modifications to his proofs. There are syntactic changes, since we are considering elements of a lattice instead of submodules of a module: the module M is the maximal element $\hat{1}$ of $\mathcal{L}$; the sum of modules is replaced by the join, and the intersection by the meet; and quotients of modules M/N are replaced by intervals $[N, M]$. The socle of M is replaced by $M[1] = A\hat{0}$, while $\mathfrak{p}^i N$ is replaced by $C^i N$ (where $\mathfrak{p}$ is the maximal ideal of the ring). There are further semantic changes required because submodules can be viewed as collections of vectors, while elements of a semi-primary lattice cannot, because not all semi-primary lattices have a submodule lattice representation. Nonetheless, semi-primary lattices have sufficient structure to carry out the vector dependent portions of the proof. This problem arises in his Proposition (3.4) which we redo below as Theorem 4.83 in the context of semi-primary lattices, and in his Proposition (4.4), which requires selection of equal length sequences of linearly independent vectors in two elementary quotients in M. We cannot do this, because in our wider context, the elements of an arbitrary semi-primary lattice do not further decompose into collections of vectors. He counts the number of ways of choosing each sequence, and takes the ratio. We circumvent this by choosing sequences of independent atoms in these elementary intervals, which diminishes the number of choices of each term in each sequence by a factor $q - 1$. Since the number of terms in each sequence is the same, these diminishing factors cancel when the ratio is taken.

Maley [**18**] showed that $g^{\lambda}_{\mu\nu}(p)$ is a polynomial in $(p - 1)$ with nonnegative integer coefficients. His formula, though still long, is much simpler than those of Klein and Macdonald. In his proof, the number of subgroups satisfying certain conditions is determined by a canonical construction of matrices whose row spans are the subgroups in question. This dependence on the coordinatization of groups is critical to the relative simplicity of his formula and proof, but would require extensive changes to be applied directly to semi-primary lattices. However, since Maley and Macdonald's polynomials are equal, and Macdonald's proof goes through to semi-primary lattices, Maley's formula may also be used for q-regular semi-primary lattices. Refer to Klein, Macdonald, or Maley for explicit formulas for Hall polynomials.

Definition 4.82. Let y be an element of a semi-primary lattice. Define

$$\begin{aligned}\mathrm{ALR}(y) &\stackrel{\text{def}}{=} ((\operatorname{type} A^i y)')_{i\geq 0}\\ \mathrm{CLR}(y) &\stackrel{\text{def}}{=} ((\operatorname{cotype} C^i y)')_{i\geq 0}\end{aligned}$$

More generally, if $[x,z]$ is an interval and y is in this interval, define

$$\begin{aligned}\mathrm{ALR}_{[x,z]}(y) &\stackrel{\text{def}}{=} ((\operatorname{type}[x, A_z^i y])')_{i\geq 0}\\ \mathrm{CLR}_{[x,z]}(y) &\stackrel{\text{def}}{=} ((\operatorname{type}[C_x^i y, z])')_{i\geq 0}\end{aligned}$$

Green [**8**] introduced the LR-sequence of an element (our $\mathrm{CLR}(y)$) as a means of determining $g^{\lambda}_{\mu\nu}(p)$ in the context of p-modules. He shows that for any LR-sequence S of type $(\mu',\nu';\lambda')$, the number of elements y with $\mathrm{CLR}(y)=S$ is a monic polynomial in q with degree $n(\lambda)-n(\mu)-n(\nu)$. Summing over all such LR-sequences, we obtain that $g^{\lambda}_{\mu\nu}(q)$ is a polynomial of degree $n(\lambda)-n(\mu)-n(\nu)$ with leading coefficient $c^{\lambda'}_{\mu',\nu'}=c^{\lambda}_{\mu\nu}$.

The following theorem was originally done for our $\mathrm{CLR}(y)$ by Green in the context of p-modules, and later Macdonald [**16**, p. 91] in the context of modules over discrete valuation rings. It illustrates how to recast proofs in semi-primary lattice notation, and has additional minor differences that are discussed afterwards.

Theorem 4.83. *Let $\mathcal{L}$ be a semi-primary lattice of type λ, and y be an element of type μ and cotype ν. Then $\mathrm{ALR}(y)$ is an LR-sequence of type $(\mu',\nu';\lambda')$, and $\mathrm{CLR}(y)$ is an LR-sequence of type $(\nu',\mu';\lambda')$.*

Let $x\leq y\leq z$. Let $\lambda=\operatorname{type}[x,z]$, $\mu=\operatorname{type}[x,y]$, and $\nu=\operatorname{type}[y,z]$. Then $\mathrm{ALR}_{[x,z]}(y)$ is an LR-sequence of type $(\mu',\nu';\lambda')$ and $\mathrm{CLR}_{[x,z]}(y)$ is an LR-sequence of type $(\nu',\mu';\lambda')$.

Proof. It suffices to prove this for $\mathrm{ALR}(y)$; it then holds for $\mathrm{CLR}(y)$ by applying the theorem to the dual lattice, and for the interval $[x,z]$ by applying the theorem to this sublattice. We verify that the three conditions in the definition of an LR-sequence hold. Let $\lambda^{(i)}=(\operatorname{type} A^i y)'$.

(1) $(\operatorname{type} A^0 y)'=(\operatorname{type} y)'=\mu'$, and when r is sufficiently large, $A^r y=\hat{1}$ so $(\operatorname{type} A^r y)'=\lambda'$.
(2) The number of cells in $\lambda^{(i)}/\lambda^{(i-1)}$ is $\rho(A^i y)-\rho(A^{i-1}y)=\nu'_i$. The interval $[A^{i-1}y, A^i y]$ is elementary, so $\operatorname{type} A^i y/\operatorname{type} A^{i-1}y$ is a vertical strip by Proposition 4.67, whence its transpose $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip.
(3) The reverse row word of a tableau $\boldsymbol{P}$ is a ballot sequence iff the number of i's in the first k rows is at least the number of $(i+1)$'s in the first $k+1$ rows, for all $i,k\in\mathbb{P}$. This is because if there is any initial segment of the reverse row word of $\boldsymbol{P}$ that has more $(i+1)$'s than i's for some i, we can adjust the initial sequence to be cut off after all the $(i+1)$'s in the last row partially contained in the initial sequence, and since this weakly increases the number of $(i+1)$'s and weakly decreases the number of i's, we still have a violation.

For $\boldsymbol{P} = \mathrm{ALR}(y)$, the number of entries at most i in the first k rows is

$$\Big|(\operatorname{type} A^i y)[k]\Big| = \rho((A^i y)[k]) = \rho(A^i y \wedge A^k \hat{0}).$$

Thus, the number of i's in the first k rows is the length of the interval $[A^{i-1}y \wedge A^k\hat{0}, A^i y \wedge A^k\hat{0}]$. Similarly, the number of $(i+1)$'s in the first $k+1$ rows is the length of the interval $[A^i y \wedge A^{k+1}\hat{0}, A^{i+1}y \wedge A^{k+1}\hat{0}]$. The latter interval is obtained from the former one by applying A, so the latter one has smaller length by Theorem 4.15(7). $\square$

The means used in the last sentence of this proof differs from Macdonald; he gives a surjective homomorphism from the quotient module corresponding to the former interval, to the one for the latter interval, thus showing the latter interval has smaller length. We use less structure to obtain this in the wider context of semi-primary lattices. Note also that this applies to all semi-primary lattices, not just regular ones.

Macdonald's other results [**16**, §II.1] about the lattice of submodules of a module over a discrete valuation ring do not all go through to arbitrary q-regular semi-primary lattices. The construction of the dual of a module M may not apply to a semi-primary lattice not derived from a submodule lattice, and in fact, semi-primary lattices do not have to be self-dual. He computes the number of automorphisms of a module through an analytic argument, which doesn't apply if the lattice is not a submodule lattice.

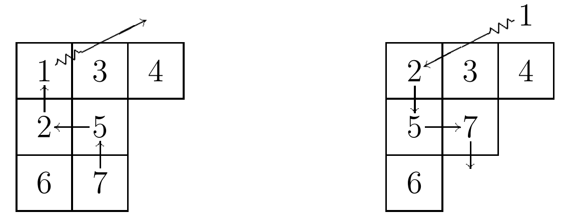
4.10. The Littlewood-Richardson sequences of adjacent elements. We will compare the tableau $\mathrm{ALR}_{[x,z]}(y)$ (or dually, $\mathrm{CLR}_{[x,z]}(y)$) to the corresponding tableau obtained by increasing one of x, y, or z by a single cover relation. The tableaux will almost be the same, except along an insertion/deletion path.

Given a cell $c \in \mathbb{P} \times \mathbb{P}$, and a skew tableau $\boldsymbol{A}$ of shape λ/μ, the entry in cell c of $\boldsymbol{A}$ is denoted $\operatorname{entry}(\boldsymbol{A}, c)$. An entry $\bullet$ has numerical value $\operatorname{low}(\boldsymbol{A})$ and denotes cells in μ, while o denotes a cell outside λ, and is considered to be a greater value than all numbers in the tableau.

Definition 4.84. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be skew tableaux, and let $c_1, \ldots, c_N$ be cells in $\mathbb{P} \times \mathbb{P}$ such that the entries in $\boldsymbol{A}$ and $\boldsymbol{B}$ agree in all cells except the c_k's. The tableau $\boldsymbol{B}$ is obtained by **inserting** e into $\boldsymbol{A}$ on the **insertion path** $c_1, \ldots, c_N$, and that $\boldsymbol{A}$ is obtained by **deleting** e from $\boldsymbol{B}$ on the **deletion path** $c_N, \ldots, c_1$, when the entries in $c_1, \ldots, c_N$ are $e_1, \ldots, e_{N-1}, o$ in $\boldsymbol{A}$ and $e, e_1, \ldots, e_{N-1}$ in $\boldsymbol{B}$, where e_k are some entries. The insertion path and deletion path have **length** N.

The tableau $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by inserting e and deleting d on the path $c_1, \ldots, c_N$ if the consecutive entries in these cells are $e_1, \ldots, e_{N-1}, d$ in $\boldsymbol{A}$, and are $e, e_1, \ldots, e_{N-1}$ in $\boldsymbol{B}$. Equivalently, $\boldsymbol{A}$ is obtained from $\boldsymbol{B}$ by inserting d and deleting e on the path $c_N, \ldots, c_1$. The insertion/deletion path has **length** $N+1$.

These paths may be represented by diagrams such as the following ones. The two tableaux shown are obtained from each other by means of the indicated paths. A squiggly arrow $\rightsquigarrow$ indicates that an entry moves into or out of the tableau, and a solid arrow indicates that an entry moves into a cell, **bumping** the entry there out to another cell or out of the tableau (according to further arrows). The number of arrows, both solid and squiggly, is the length of the path. For example, these two tableaux are obtained from each other by the indicated paths of length 4.



deletion path insertion path

Many of the insertion/deletion paths we encounter will have a similar form.

Proposition 4.85. *Let $\boldsymbol{A} = (\gamma^{(i)}, \ldots, \gamma^{(j)})$ and $\boldsymbol{B} = (\mu^{(i)}, \ldots, \mu^{(j)})$ be skew tableaux with $\gamma^{(k)} \lessdot \mu^{(k)}$ for $e \leq k < d$ (for some $d \leq j$) and $\gamma^{(k)} = \mu^{(k)}$ otherwise. Let $c_k = \mu^{(k)}/\gamma^{(k)}$ be a single cell for $k = e, \ldots, d-1$. Some consecutive c_k's may be equal; select the subsequence of $e, \ldots, j$ consisting of the first k in each run of equal c_k's, by defining the sequence $e = k_0 < k_1 < \ldots < k_N$ so that $c_{k_m} = c_{k_m+1} = \cdots = c_{k_{m+1}-1} \neq c_{k_{m+1}}$ and $c_{k_N} = c_{d-1}$. Then $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by inserting e and deleting d on the path $c_{k_0}, \ldots, c_{k_N}$.*

If we set $d = j+1$ above, so that $e \leq i \leq j$ and $k = e, \ldots, j$, then $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by inserting e along the insertion path $c_{k_0}, \ldots, c_{k_N}$, and $\boldsymbol{A}$ is obtained from $\boldsymbol{B}$ by deleting e from the deletion path $c_{k_N}, \ldots, c_{k_0}$.

In both cases, if c_{k-1} and c_k are distinct cells on the same row, then all cells between them contain k in both $\boldsymbol{A}$ and $\boldsymbol{B}$.

Proof. The entries in $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same in all cells except the c_k's, because the k's in $\boldsymbol{A}$ are in cells $\gamma^{(k)}/\gamma^{(k-1)}$, and are in $\boldsymbol{B}$ in cells $\mu^{(k)}/\mu^{(k-1)}$, and these differ only in c_k and c_{k-1}. Consecutive c_k's may be equal, but if $c_{k-1} \neq c_k$ then c_{k-1} is in both $\boldsymbol{A}$ and $\boldsymbol{B}$, so no later cell equals c_{k-1}.

The value of entry$(\boldsymbol{B}, c)$ is the smallest k for which c is in $\mu^{(k)}$, and similarly for $\boldsymbol{A}$, so entry$(\boldsymbol{B}, c_{k_m}) = k_m$ for $m = 0, \ldots, r$, and entry$(\boldsymbol{A}, c_{k_m}) = k_{m+1}$ for $m = 0, \ldots, r-1$. Since c_N isn't in $\gamma^{(d-1)}$, it follows that entry$(\boldsymbol{A}, c_N)$ is d when $d \leq j$ and o when $d = j+1$.

If c_{k-1} and c_k are distinct cells on the same row, the cells between them all appear in $\gamma^{(k)}$ and $\mu^{(k)}$ and none appear in $\mu^{(k-1)}$ or $\gamma^{(k-1)}$, so all contain k in both $\boldsymbol{A}$ and $\boldsymbol{B}$. $\square$

We will examine how $\mathrm{ALR}_{[x,z]}(y)$ changes in three situations with one of x, y, z varying to an element covering it, and the other two staying constant:

(x) $x \lessdot \tilde{x} \leq y \leq z$;
(y) $x \leq y \lessdot \tilde{y} \leq z$;
(z) $x \leq y \leq z \lessdot \tilde{z}$.

These three cases and the following definitions apply to the following lemma and theorem. Let

$$\begin{aligned} w_k = w_k(x, y, z) &= A^k y \wedge z \\ w_{kr} = w_{kr}(x, y, z) &= A^r x \wedge A^k y \wedge z \end{aligned}$$

(so $(\mathrm{type}[x, A_z^k y])'_r = \rho(w_{kr}) - \rho(w_{k,r-1})$)

$$\begin{aligned}
\boldsymbol{P} &= \mathrm{ALR}_{[x,z]}(y) \\
\mu &= \mathrm{type}[x,y] \\
\nu &= \mathrm{type}[y,z] \\
\lambda &= \mathrm{type}[x,z]
\end{aligned}$$

Define $\tilde{w}_k$, $\tilde{w}_{kr}$, $\widetilde{\boldsymbol{P}}$, $\tilde{\mu}$, $\tilde{\nu}$, and $\tilde{\lambda}$ similarly by replacing x with $\tilde{x}$ in case (x); y with $\tilde{y}$ in case (y); and z with $\tilde{z}$ in case (z).

We always have $w_{kr} \lesssim \tilde{w}_{kr}$ because A and $\wedge$ both preserve $\lesssim$, so we are interested in determining for which k, r these are equal and for which they are not. Let $S = \{ (k, r) : w_{kr} = \tilde{w}_{kr} \}$.

Lemma 4.86. *If $(k, r) \in S$, then $(k', r') \in S$, where $k' \geq 0$ and $r' \geq 0$ depend on the case:*

(x) *$r' \geq r$ and $k' - k \leq r' - r$; so $(k+1, r+1) \in S$ and $(k', r) \in S$ for $k' \leq k$.*
(y) *$k' \geq k$ and $r' - r \leq k' - k$; so $(k+1, r+1) \in S$ and $(k, r') \in S$ for $r' \leq r$.*
(z) *$k' \leq k$ and $r' \leq r$.*

Proof.

(x) Given $(k, r) \in S$, we have $w_{kr} = \tilde{w}_{kr}$, or

$$A^r x \wedge A^k y \wedge z = A^r \tilde{x} \wedge A^k y \wedge z.$$

For $r' \geq r$, applying $A_z^{r'-r}$ to both sides of the equality preserves the equality and adds $r' - r$ to both k and r, yielding $w_{k+r'-r,r'} = \tilde{w}_{k+r'-r,r'}$. Next, for $k' \leq k + r' - r$, taking the meet of each side with $A^{k'} y$ preserves the equality and changes $k + r' - r$ to k', so $w_{k',r'} = \tilde{w}_{k',r'}$.
(y) This is similar to (x), but with various variables swapped.
(z) Given $(k, r) \in S$, if $0 \leq k' \leq k$ and $0 \leq r' \leq r$ then $A^{k'} y \leq A^k y$ and $A^{r'} x \leq A^r x$, so

$$w_{kr} \wedge A^{k'} y \wedge A^{r'} x = (A^r x \wedge A^{r'} x) \wedge (A^k y \wedge A^{k'} y) \wedge z = A^{r'} x \wedge A^{k'} y \wedge z = w_{k',r'},$$

and similarly, $\tilde{w}_{kr} \wedge A^{k'} y \wedge A^{r'} x = \tilde{w}_{k',r'}$. $\square$

Theorem 4.87. *We obtain $\widetilde{\boldsymbol{P}}$ from $\boldsymbol{P}$ by insertion or deletion paths as follows; see Figure 2.*

(x) *Delete $\bullet$ from $\boldsymbol{P}$ on a path $C_N, \ldots, C_0$, where*
 (1) $\bullet = \mathrm{entry}(\boldsymbol{P}, C_0) < \cdots < \mathrm{entry}(\boldsymbol{P}, C_N)$*;*
 (2) $\mathrm{row}(C_0) \lesssim \cdots \lesssim \mathrm{row}(C_N)$*;*
 (3) *If* $\mathrm{row}(C_i) \lessdot \mathrm{row}(C_{i+1})$ *then* $\mathrm{col}(C_i) \geq \mathrm{col}(C_{i+1})$*, while if* $\mathrm{row}(C_i) = \mathrm{row}(C_{i+1})$ *then* $\mathrm{col}(C_i) < \mathrm{col}(C_{i+1})$.
(y) *Insert $\bullet$ and delete $e = \mathrm{col}(\nu/\tilde{\nu})$ from a vertical strip $C_0, C_1, \ldots, C_{e-1}$ in $\boldsymbol{P}$, where*
 (1) $\mathrm{entry}(\boldsymbol{P}, C_i) = i + 1$*;*
 (2) $\mathrm{row}(C_0) < \cdots < \mathrm{row}(C_{e-1})$*;*
 (3) $\mathrm{col}(C_0) \geq \cdots \geq \mathrm{col}(C_{e-1})$.
(z) *Insert $e = \mathrm{col}(\tilde{\nu}/\nu)$ into a "horizontal strip" $C_0, \ldots, C_N$ of $\boldsymbol{P}$, where*
 (1) $e < \mathrm{entry}(\boldsymbol{P}, C_0) < \cdots < \mathrm{entry}(\boldsymbol{P}, C_N) = o$*;*
 (2) $\mathrm{col}(C_0) < \cdots < \mathrm{col}(C_e)$.

(3) $\mathrm{row}(C_0) \geq \cdots \geq \mathrm{row}(C_e)$;

Proof.

(x) We have $\tilde{\mu} \lessdot \mu$, $\tilde{\nu} = \nu$, $\tilde{\lambda} \lessdot \lambda$, and $w_k = \tilde{w}_k$ for all k. Thus $\mathrm{type}[x, w_k] \gtrdot \mathrm{type}[\tilde{x}, \tilde{w}_k]$, so $c_k = \mathrm{type}[x, w_k]' / \mathrm{type}[\tilde{x}, w_k]'$ is a single cell of λ'. By Proposition 4.85, $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by deleting $\bullet$ from a path consisting of the distinct cells among $\ldots, c_1, c_0$.

Let $r_k = \mathrm{row}(c_k) = \min\{\, r \geq 0 \,:\, w_{kr} = \tilde{w}_{kr} \,\}$, so $(k, r) \in S$ iff $r \geq r_k$. First we show that $r_{k+1} \geq r_k$, or equivalently, $r_k + 1 \geq r_{k+1} \geq r_k$. By Lemma 4.86, $(k+1, r_k+1) \in S$ because $(k, r_k) \in S$. Thus $r_k + 1 \geq r_{k+1}$. Conversely, by Lemma 4.86, $(k, r_{k+1}) \in S$ because $(k+1, r_{k+1}) \in S$. Thus $r_{k+1} \geq r_k$.

If c_k and c_{k+1} are on the same row, c_{k+1} is weakly right of c_k. Suppose they are on consecutive rows $r, r+1$. Apply A_z to the interval $[w_{r-1,k}, w_{r,k}]$ to obtain the interval $[w_{r,k+1}, w_{r+1,k+1}]$ of weakly smaller length. The lengths are $\mathrm{col}(c_k) \geq \mathrm{col}(c_{k+1})$.

(y) We have $\tilde{\mu} \gtrdot \mu$, $\tilde{\nu} \lessdot \tilde{\nu}$, and $\tilde{\lambda} = \lambda$. For each $0 \leq k < e$, we have $w_k \lessdot \tilde{w}_k$, while for $k \geq e$, we have $w_k = \tilde{w}_k$. So $c_k = \mathrm{type}[x, \tilde{w}_k]' / \mathrm{type}[x, w_k]'$ is a single cell for $k = 0, \ldots, e-1$. Since $w_k \lessdot \tilde{w}_k$ for $k < e$, we have $w_{k+1} = A_z w_k \geq \tilde{w}_k$, so the cells c_k are distinct. By Proposition 4.85, $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by inserting $\bullet$ and deleting e on the path $c_0, \ldots, c_{e-1}$. Cell c_k contains $k+1$ in $\boldsymbol{P}$ and k in $\widetilde{\boldsymbol{P}}$.

Let $r_k = \mathrm{row}(c_k) = \min\{\, r \geq 0 \,:\, w_{kr} \lessdot \tilde{w}_{kr} \,\} > 0$. Since $(k, r_k - 1) \in S$, we have $(k+1, r_k) \in S$ by Lemma 4.86, so that $r_{k+1} > r_k$. Thus the rows of the successive cells strictly increase. Since c_k is an outer corner of $\mathrm{type}[x, \tilde{w}_k]'$, and $\mathrm{type}[x, \tilde{w}_{k+1}]' / \mathrm{type}[x, \tilde{w}_k]'$ is a horizontal strip, all cells on higher numbered rows of this strip are in columns weakly smaller than the column of c_k.

(z) We have $\tilde{\mu} = \mu$, $\tilde{\nu} \lessdot \tilde{\nu}$, and $\tilde{\lambda} \lessdot \lambda$. For each $k < e$, we have $w_k = \tilde{w}_k$, while for $k \geq e$, we have $w_k \lessdot \tilde{w}_k$. Let $c_k = \mathrm{type}[x, \tilde{w}_k]' / \mathrm{type}[x, w_k]'$ for $k \geq e$. The cells c_k need not be distinct. By Proposition 4.85, $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by inserting e along a path consisting of the distinct cells among the c_k's.

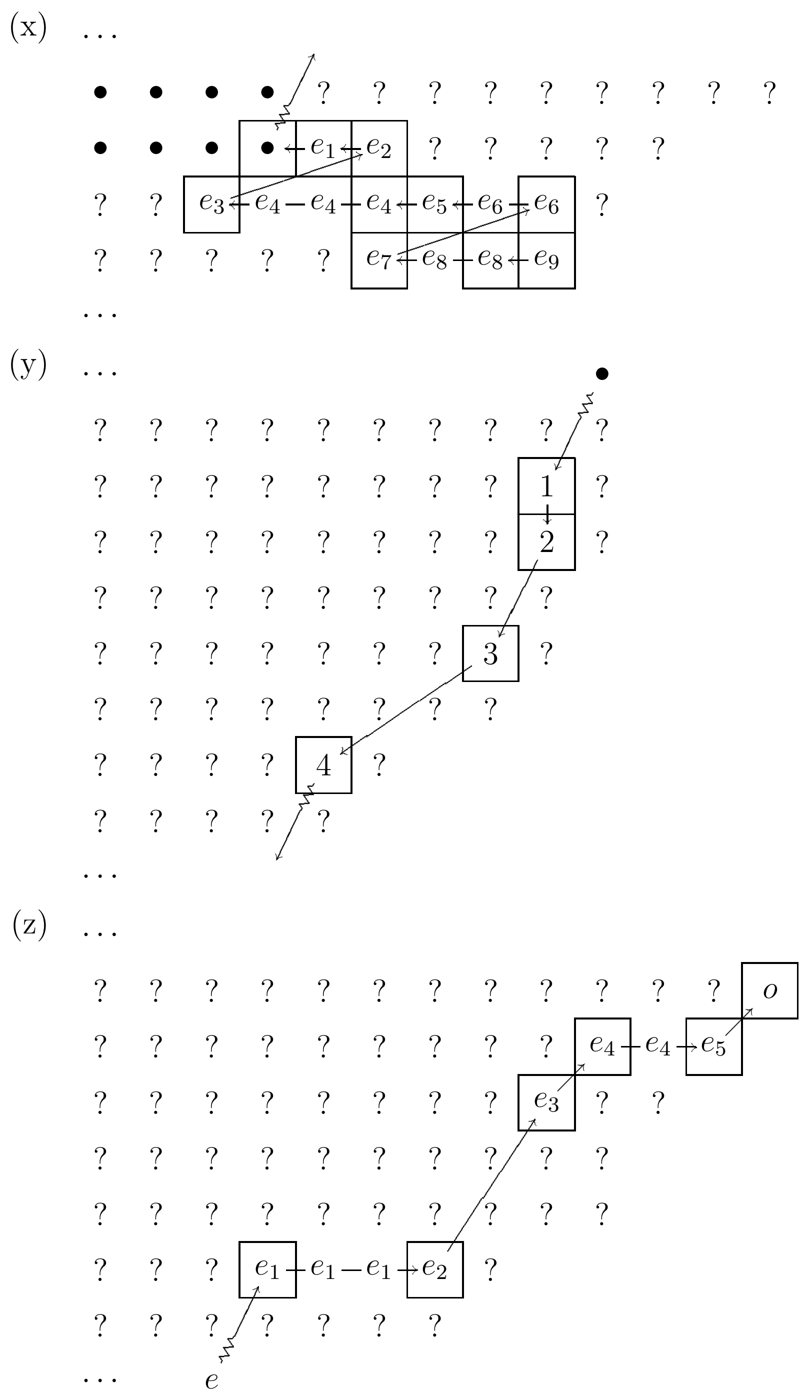
Let $r_k = \mathrm{row}(c_k) = \min\{\, r \geq 0 \,:\, w_{kr} \lessdot \tilde{w}_{kr} \,\} > 0$. So $(k+1, r_{k+1} - 1) \in S$, whence $(k, r_{k+1} - 1) \in S$ by Lemma 4.86. Thus $r_k > r_{k+1} - 1$, or equivalently, $r_k \geq r_{k+1}$, so the row numbers weakly decrease. $\square$

4.11. Applications of Littlewood-Richardson sequences.

Definition 4.88. The **straightening** of a weak composition λ is the partition $S(\lambda)$ obtained from it by sorting the parts into decreasing order. The **straightening** of a tableau of weak composition shape whose rows and columns have weakly increasing entries (ignoring the holes) is the Young tableau whose columns have the same entries in the same order as in the corresponding columns of $\boldsymbol{P}$, by pushing the entries up into the holes left by short rows. In terms of the chain of weak compositions $\boldsymbol{P} = (\lambda^{(l)} \leq \lambda^{(l+1)} \leq \cdots \leq \lambda^{(h)})$, this is

$$S(\boldsymbol{P}) \stackrel{\text{def}}{=} \Big(S(\lambda^{(l)}), S(\lambda^{(l+1)}), \ldots, S(\lambda^{(h)}) \Big)\,.$$

Definition 4.89. Let $\mu \leq \lambda$ be partitions. We define two LR-sequences in tableaux of shape λ' / μ':



Apply the insertion/deletion path to $\boldsymbol{P}$ (shown) to obtain $\widetilde{\boldsymbol{P}}$. The cells marked ? have entries that do not change. The boxed cells C_k have entries e_k in (x) and (z), and k in (y). When an entry moves horizontally, all the cells it passes across must contain that entry, since the entry weakly bounds them above and below.

FIGURE 2. Insertion/deletion paths for LR-sequences of adjacent elements.

(1)
$$\mathrm{LR}_*(\mu';\lambda') \overset{\text{def}}{=} (\mu', (\mathcal{A}_\lambda \mu)', (\mathcal{A}_\lambda^2 \mu)', \ldots)$$
is the LR-sequence of type $(\mu', S(\lambda-\mu)'; \lambda')$ obtained by filling column j of λ'/μ' with entries $1, 2, \ldots, \lambda_j - \mu_j$ from top to bottom, for each j.

(2) $\mathrm{LR}^*(\mu';\lambda')$ is the LR-sequence of type $(\mu', S(\lambda'-\mu'); \lambda')$ obtained as follows. Start with the shape λ'/μ'. Place 1 in the upper rightmost empty cell, 2 in the upper rightmost empty cell strictly below that, 3 in the upper rightmost empty cell strictly below that, and so on, until there are no lower empty cells. This fills the strip $\lambda'/\mathcal{C}_{\mu'}\lambda'$ with entries $1, 2, \ldots$ from top to bottom. In the cells remaining, do this again: place 1 in the upper rightmost empty cell, 2 in the upper rightmost empty cell strictly below that, and so on, thus filling the strip $\mathcal{C}_{\mu'}\lambda'/\mathcal{C}_{\mu'}^2\lambda'$ with entries $1, 2, \ldots$ from top to bottom. Continue in this fashion until the whole tableau is filled.

These are both LR-sequences because they are semistandard tableau with an injection from entries $k > 1$ to entries $k-1$ on higher rows.

Example 4.90. Let $\mu = (5, 4, 4, 2)$ and $\lambda = (8, 8, 6, 5, 3)$. Then

$$\mathrm{LR}_*(\mu';\lambda') = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & \bullet & 1 & 3 \\ \bullet & \bullet & \bullet & 2 & \\ \bullet & 1 & 1 & 3 & \\ 1 & 2 & 2 & & \\ 2 & 3 & & & \\ 3 & 4 & & & \end{array} \qquad \mathrm{LR}^*(\mu';\lambda') = \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \bullet & 2 \\ \bullet & \bullet & \bullet & 1 & 3 \\ \bullet & \bullet & \bullet & 4 & \\ \bullet & 1 & 2 & 5 & \\ 2 & 3 & 6 & & \\ 4 & 7 & & & \\ 5 & 8 & & & \end{array}$$

have respective contents $S(\lambda-\mu)' = S(3,4,2,3,3)' = (4,3,3,3,2)' = (5,5,4,1)$ and $S(\lambda'-\mu') = S(1,1,2,1,1,3,3,2,2) = (3,3,2,2,2,1,1,1,1)$.

Definition 4.91. The **content** of a tableau is the weak composition ν such that there are ν_i i's in the tableau for each i.

Theorem 4.92. *Let $\mu \le \lambda$ be partitions. In q-regular semi-primary lattices, for sufficiently large q, the least frequently occuring value of $\mathrm{ALR}_{[\hat{0},y]}(x)$ for pairs in the set*
$$\{\, (x,y) : x \le y \text{ and } \operatorname{type} x = \mu \text{ and } \operatorname{type} y = \lambda \,\}$$
is $\mathrm{LR}_*(\mu';\lambda')$, and the most frequently occurring value is $\mathrm{LR}^*(\mu';\lambda')$.

Proof. Fix y of type λ. The number of $x \le y$ of type μ and cotype ν is $g^\lambda_{\mu\nu}(q)$, a polynomial in q of degree $n(\lambda)-n(\mu)-n(\nu)$ with leading coefficient $c^\lambda_{\mu\nu}$. To determine the least frequently occurring LR-sequence, we seek ν with $c^\lambda_{\mu\nu} \ne 0$ that minimizes $n(\lambda)-n(\mu)-n(\nu)$, or equivalently, maximizes $n(\nu)$; the most frequently occurring LR-sequence is similarly obtained by minimizing $n(\nu)$.

Let $\boldsymbol{Q}$ be an LR-sequence of shape λ'/μ' and content ν'. Clearly, the entry in each cell is weakly bounded below by that in $\mathrm{LR}_*(\mu';\lambda')$ and above by that in $\mathrm{LR}^*(\mu';\lambda')$. Since $\mathrm{LR}_*(\mu';\lambda')$ is obtained from $\boldsymbol{Q}$ by weakly lowering the entry in each cell, the content is obtained from the partition ν' by moving squares up to higher rows, which increases $n(\nu)$. Similarly, $\mathrm{LR}^*(\mu';\lambda')$ is obtained from $\boldsymbol{Q}$ by weakly raising the entry in each cell, thus altering the content ν' by moving squares down to lower rows, decreasing $n(\nu)$. The extremes are uniquely obtained by $\mathrm{LR}_*(\mu';\lambda')$ and $\mathrm{LR}^*(\mu';\lambda')$. $\square$

A useful application of one of these LR-sequences is the following generalization of part of Theorem 4.76.

Theorem 4.93. *Let $f = (f_l, \dots, f_h)$ be a saturated flag. Suppose* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{A} + \boldsymbol{B}$, *where $\boldsymbol{A}$ is a standard tableau on entries $l+1, \dots k$ and $\boldsymbol{B}$ is a row and column strict composition tableau on distinct entries $k+1, \dots h$. Then* $\mathrm{ALR}_{[\hat{0}, f_m]}(f_k) = \mathrm{LR}_*((\operatorname{type} f_k)'; (\operatorname{type} f_m)')$ *for each $m = k, \dots, h$. Thus,* $\operatorname{ftype} \partial^{k-l} f = S(\boldsymbol{B})$.

Proof. The ALR sequence is as specified for $m = k$.

Let $\boldsymbol{P} = \mathrm{ALR}_{[\hat{0}, f_{m-1}]}(f_k)$, and assume this equals $\mathrm{LR}_*(\mu';\lambda')$ where $(\mu';\lambda') = ((\operatorname{type} f_k)'; (\operatorname{type} f_{m-1})')$. We want to find $\widetilde{\boldsymbol{P}} = \mathrm{ALR}_{[\hat{0}, f_m]}(f_k)$, of inner shape μ' and outer shape $\tilde{\lambda}' = (\operatorname{type} f_m)'$. The outer shapes of $\widetilde{\boldsymbol{P}}$ and $\boldsymbol{P}$ differ in some cell c in column k, where $k = \operatorname{row}(\boldsymbol{A} + \boldsymbol{B}, m) = \operatorname{row}(\boldsymbol{B}, m)$. By Theorem 4.87(z), we obtain $\widetilde{\boldsymbol{P}}$ from $\boldsymbol{P}$ by inserting a number on a horizontal strip terminating in c. If there are cells besides c in the insertion path, their entries in $\boldsymbol{P}$ are replaced by smaller entries in $\widetilde{\boldsymbol{P}}$. Since the columns of $\boldsymbol{P}$ consist of consecutive numbers, an entry cannot be replaced by a smaller one without violating column strictness. Thus, only c is on the insertion path, so $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by placing an entry at c and leaving all other cells intact.

Now m occurs in $\boldsymbol{B}$ in cell (k, e) where $e = \lambda_k - \mu_k + 1$. All rows of $\boldsymbol{B}$ below this cell have fewer than e entries less than m, because $\boldsymbol{B}$ is column strict. The entry in the cell above c in $\boldsymbol{P}$ (and thus $\widetilde{\boldsymbol{P}}$) is $e - 1$ (or c is at the top of the column if $\lambda_k = \mu_k = 0$), so the minimum entry possible in c in $\widetilde{\boldsymbol{P}}$ is e, and any occurrences of e in $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$ on rows above c must be in columns right of c. The number of occurrences of e in $\boldsymbol{P}$ in columns right of c is $\#\{\, r > k : \lambda_r - \mu_r \geq e \,\}$. This equals the number of rows of $\boldsymbol{B}$ below the cell with m with at least e entries less than m; it has already been shown that there are no such rows. Thus, we cannot place a number larger than e at c in $\widetilde{\boldsymbol{P}}$ because the reverse row word of $\widetilde{\boldsymbol{P}}$ would not be a ballot sequence. Since e is both a lower and upper bound for $\operatorname{entry}(\widetilde{\boldsymbol{P}}, c)$, this entry equals e.

Since $\widetilde{\boldsymbol{P}}$ is obtained from $\boldsymbol{P}$ by placing e in cell c, it is $\mathrm{LR}_*(\mu';\tilde{\lambda}')$.

We have $\operatorname{type}[f_k, f_m] = S(\operatorname{type} f_m - \operatorname{type} f_k)$ for $m = k, \dots, h$, because $\mathrm{ALR}_{[\hat{0}, f_m]}(f_k)$ is an LR-sequence of type $((\operatorname{type} f_k)', S(\operatorname{type} f_m - \operatorname{type} f_k)'; (\operatorname{type} f_m)')$. The partition $S(\operatorname{type} f_m - \operatorname{type} f_k)$ is computed by taking the shape of the subtableau of $\boldsymbol{B}$ of entries at most m, and sorting the rows into decreasing order. Thus,

$$\begin{aligned} \operatorname{ftype} \partial^{k-l} f &= \operatorname{ftype}(f_k, f_{k+1}, \dots, f_h) = \Big(\operatorname{type}[f_k, f_m]\Big)_{m=k}^{h} \\ &= \Big(S(\operatorname{type} f_m - \operatorname{type} f_k)\Big)_{m=k}^{h} = S(\boldsymbol{B}). \quad \square \end{aligned}$$

Example 4.94. Suppose

$$\operatorname{ftype}_{\hat{0}} f = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} \\ \cline{2-5} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{10} \\ \cline{1-5} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{8} & & \\ \cline{1-3} \multicolumn{1}{|c|}{11} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & & \\ \cline{1-3} \end{array} = \begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{3} \\ \cline{2-3} & \multicolumn{1}{|c|}{1} & \\ \cline{1-2} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \\ \cline{1-2} \end{array} + \begin{array}{ccc} \cline{1-2} \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} & \\ \cline{1-3} \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{10} \\ \cline{1-3} \multicolumn{1}{|c|}{8} & & \\ \cline{1-3} \multicolumn{1}{|c|}{11} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} \\ \cline{1-3} \end{array} = \boldsymbol{A} + \boldsymbol{B}.$$

For each entry of $\operatorname{ftype}_{\hat{0}} f$ that appears in $\boldsymbol{B}$, put a subscript on it indicating its column number in $\boldsymbol{B}$, and then transpose the tableau.

$$\begin{array}{cccc} \cline{3-4} & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{11_1} \\ \cline{2-4} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{12_2} \\ \cline{1-4} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{6_1} & \multicolumn{1}{c|}{8_1} & \multicolumn{1}{c|}{13_3} \\ \cline{1-4} \multicolumn{1}{|c|}{5_1} & \multicolumn{1}{c|}{9_2} & & \\ \cline{1-2} \multicolumn{1}{|c|}{7_2} & \multicolumn{1}{c|}{10_3} & & \\ \cline{1-2} \end{array}$$

To find $\mathrm{ALR}_{[f_0,f_m]}(f_4)$ for $4 \le m \le 13$, take the subtableau of entries at most m, and replace each entry by its subscript if it has one, or $\bullet$ if it doesn't. The successive values of $\mathrm{ALR}_{[f_0,f_m]}(f_4)$ from $m = 4$ through 13 are

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \\ 1 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 1 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 1 & & \\ 2 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & 1 \\ 1 & & \\ 2 & & \end{array}$$

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & 1 \\ 1 & 2 & \\ 2 & & \end{array}, \quad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 1 & 1 \\ 1 & 2 & \\ 2 & 3 & \end{array}, \quad \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \\ \bullet & 1 & 1 & \\ 1 & 2 & & \\ 2 & 3 & & \end{array}, \quad \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 2 \\ \bullet & 1 & 1 & \\ 1 & 2 & & \\ 2 & 3 & & \end{array}, \quad \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 2 \\ \bullet & 1 & 1 & 3 \\ 1 & 2 & & \\ 2 & 3 & & \end{array}$$

and $\operatorname{ftype} \partial^4 f = \operatorname{ftype}(f_4, \ldots, f_{13}) = S(B) = \begin{array}{ccc} \cline{1-3} \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{10} \\ \cline{1-3} \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{13} \\ \cline{1-3} \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{12} & \\ \cline{1-2} \multicolumn{1}{|c|}{11} & & \\ \cline{1-1} \end{array}$. Similarly,

$$\begin{aligned} \operatorname{ftype} \partial^2 d^8 f &= \operatorname{ftype}(f_2, \ldots, f_5) &= \begin{array}{cc} \cline{1-2} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \cline{1-2} \multicolumn{1}{|c|}{4} & \\ \cline{1-1} \end{array} \\ \operatorname{ftype} \partial^6 d^5 f &= \operatorname{ftype}(f_6, f_7, f_8) &= \begin{array}{|c|} \hline 7 \\ \hline 8 \\ \hline \end{array} \\ \operatorname{ftype} \partial^9 f &= \operatorname{ftype}(f_9, \ldots, f_{13}) &= \begin{array}{ccc} \cline{1-3} \multicolumn{1}{|c|}{9} & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{13} \\ \cline{1-3} \multicolumn{1}{|c|}{11} & \multicolumn{1}{c|}{12} & \\ \cline{1-2} \end{array} \\ \operatorname{ftype} \partial^{10} f &= \operatorname{ftype}(f_{10}, \ldots, f_{13}) &= \begin{array}{ccc} \cline{1-3} \multicolumn{1}{|c|}{10} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} \\ \cline{1-3} \multicolumn{1}{|c|}{11} & & \\ \cline{1-1} \end{array} \\ \operatorname{ftype} \partial^{11} f &= \operatorname{ftype}(f_{11}, f_{12}, f_{13}) &= \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline \end{array} \end{aligned}$$

Part III. Tableau games and flag configurations

5. Interval type tables in semi-primary lattices

5.1. Schützenberger's evacuation algorithm. Schützenberger [**25**] developed an involution ev on standard tableaux of any given shape, which Fomin [**4**, §8] has described in terms of a triangular array of partitions. We present both versions below. Also see Sagan [**22**, §3.11] for an exposition. Hesselink [**10**, §5] and van Leeuwen [**31**, Theorem 2.3.2] showed that in an invariant subspace lattice over an infinite field, in the Zariski topology, the cotype of a generic flag of type $\boldsymbol{P}$ is $\operatorname{ev}\boldsymbol{P}$. Hesselink also examines degenerate behavior. We will show an analogous genericity result for q-regular semi-primary lattices, and will also analyze the degenerate behavior.

Definition 5.1. Let $\boldsymbol{P}$ be a skew tableau whose entries are distinct numbers, and symbols $\star$, representing holes. We **slide** an entry e into a hole by placing e at that hole, and replacing the former cell of e with $\star$.

Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda \vdash n$. Define an operator $\Delta\boldsymbol{P}$ as follows.

(1) Replace the top left corner of $\boldsymbol{P}$ (which has the minimum element of $\boldsymbol{P}$) with $\star$.
(2) If $\star$ is not at an inner corner, slide the smaller or only of its right and lower neighbors into $\star$, and then repeat this step.
(3) Finally, $\star$ is at an inner corner. Delete the square with $\star$, obtaining a smaller tableau, $\Delta\boldsymbol{P}$.

We illustrate this with

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

First replace 1 with $\star$:

$$\begin{array}{|c|c|c|}\hline \star & \boxed{3} & 4 \\ \hline \boxed{2} & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

The right and lower neighbors of $\star$ (shown with bold frames) contain 2 and 3, the smaller of which is 2, so we next obtain

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline \star & \boxed{5} \\ \cline{1-2} \boxed{6} & 7 \\ \cline{1-2}\end{array}$$

The right and lower neighbors of $\star$ contain 5 and 6, the smaller of which is 5, so we next obtain

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & \star \\ \cline{1-2} 6 & \boxed{7} \\ \cline{1-2}\end{array}$$

Now there is no right neighbor, but there is a lower neighbor, 7.

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & 7 \\ \cline{1-2} 6 & \star \\ \cline{1-2}\end{array}$$

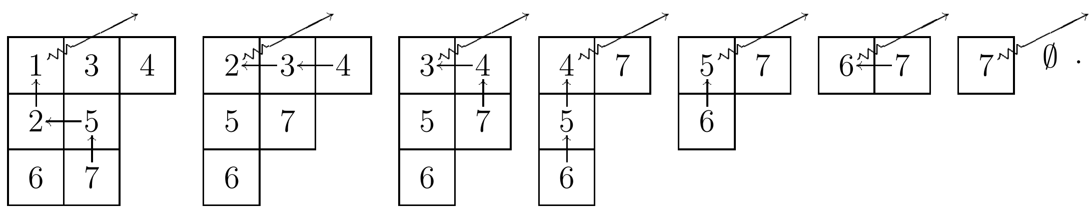
Finally there are no right or lower neighbors, so delete $\star$ to obtain

$$\Delta \boldsymbol{P} = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Apply Δ to $\Delta \boldsymbol{P}$ to obtain

$$\Delta^2 \boldsymbol{P} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 & 7 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

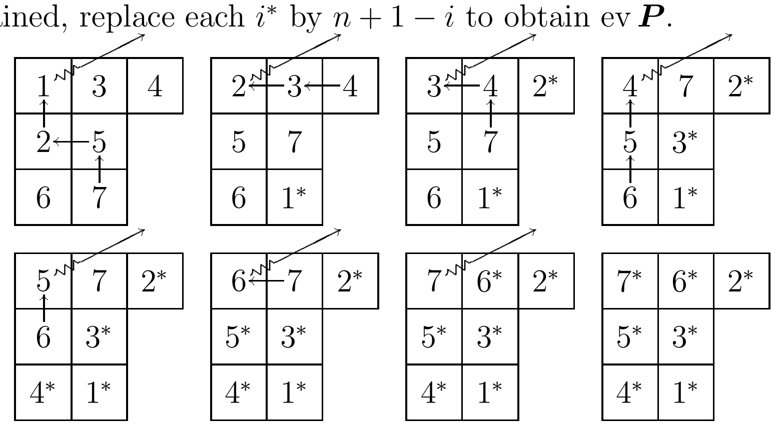
and compute $\Delta^0 \boldsymbol{P}, \Delta^1 \boldsymbol{P}, \ldots, \Delta^7 \boldsymbol{P}$:



The chain $\operatorname{sh} \Delta^n \boldsymbol{P} \lessdot \operatorname{sh} \Delta^{n-1} \boldsymbol{P} \lessdot \cdots \lessdot \operatorname{sh} \Delta^0 \boldsymbol{P}$ in Young's lattice yields a tableau $\operatorname{ev} \boldsymbol{P}$, which in this example is

$$\operatorname{ev} \boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} .$$

A convenient way to compute $\operatorname{ev} \boldsymbol{P}$ is to embed all $\Delta^i \boldsymbol{P}$ in a tableau of shape λ, by placing an entry i^* in the cell that's deleted when Δ deletes i from the tableau. The starred entries are not part of the tableau $\Delta^i \boldsymbol{P}$, but are simply a bookkeeping device to record the history of the successive shapes. When a tableau with only starred entries is obtained, replace each i^* by $n + 1 - i$ to obtain $\operatorname{ev} \boldsymbol{P}$.



Theorem 5.2. *The operator* ev *is an involution, that is,* $\operatorname{ev} \operatorname{ev} \boldsymbol{P} = \boldsymbol{P}$ *for any standard tableau* $\boldsymbol{P}$*.*

An alternate method of computing $\operatorname{ev} \boldsymbol{P}$ due to Fomin [**4**, §8] yields an easier proof of this theorem than the classical proof. We illustrate his method with the previous example. Form an upper triangular grid Λ of partitions $\lambda^{(ij)}$, where for $0 \le i \le j \le n$, the entry $\lambda^{(ij)}$ in row i, column j is the subshape of $\Delta^i \boldsymbol{P}$ consisting of entries $i + 1, \ldots, j$; see Figure 3.

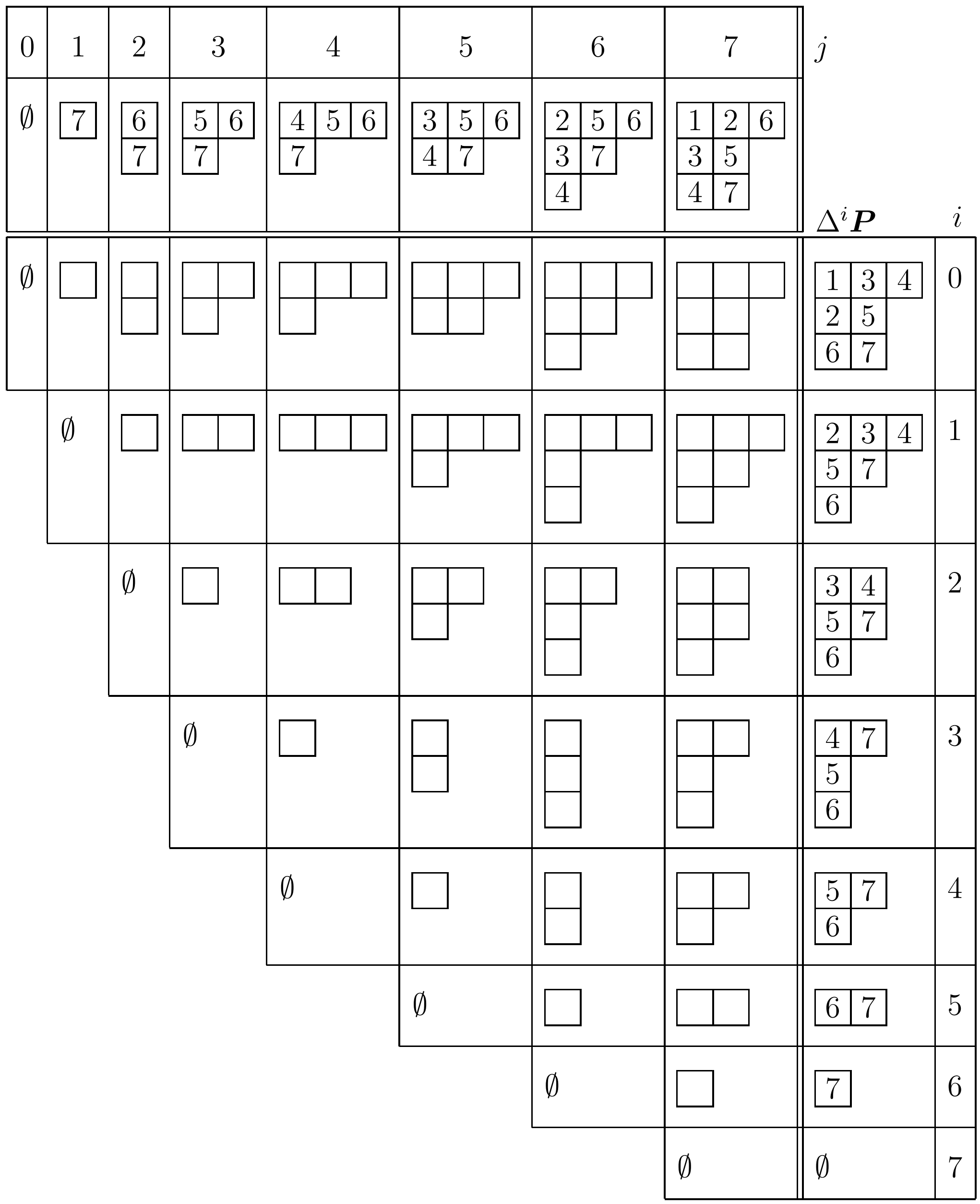


FIGURE 3. Fomin's computation of evacuation.

Not only do the rows of Λ form saturated chains in Young's lattice, but the columns do as well; comparing the contents of the cells of $\Delta^i \boldsymbol{P}$ and $\Delta^{i+1}\boldsymbol{P}$, we see that each cell of the former stays the same, increases, or is emptied in the latter. If the contents had been at most j and are emptied or increased above j, the cell is in $\lambda^{(ij)}$ but not $\lambda^{(i+1,j)}$, and otherwise is in both partitions. Thus, $\lambda^{(ij)} \geq \lambda^{(i+1,j)}$ in Young's lattice. Further, the number of cells in $\lambda^{(ij)}$ is $j-i$, so the columns of Λ also form saturated chains. Notice in the figure that the chain represented by the last column is $\operatorname{ev} \boldsymbol{P}$.

Now we show how to compute this table without first computing $\Delta^i \boldsymbol{P}$ and expanding it into a chain. We are given $\boldsymbol{P}$ of shape $\lambda \vdash n$. Form an upper triangular grid Λ with entries $\lambda^{(ij)}$, where $0 \leq i \leq j \leq n$. Set $(\lambda^{(00)}, \ldots, \lambda^{(0n)})$ to $\boldsymbol{P}$ and $\lambda^{(jj)} = \emptyset$ for $0 \leq j \leq n$. We use repeated applications of a local rule to compute the values of the remaining entries of Λ. Consider a 2 by 2 section of Λ:

$\lambda^{(ab)}$	$b=j-1$	$b=j$
$a=i$	γ	δ
$a=i+1$	α	β

If we are given α, γ, δ, we want to compute β. Since the rows of the grid increase left to right, and the columns increase top to bottom, β is between α and δ in Young's lattice. These differ by just two squares, in positions denoted x and y, where $x, y \in \mathbb{P} \times \mathbb{P}$.

Definition 5.3. A **brick** is a skew partition consisting of two adjacent squares. A **horizontal brick** is a brick with two horizontally adjacent squares, that is, λ/μ where for some j, we have $\lambda_j = \mu_j + 2$ and $\lambda_i = \mu_i$ for $i \neq j$. A **vertical brick** is a brick with two vertically adjacent squares.

If δ/α is a brick, there is only one partition between α and δ, and both γ and β are equal to it. If δ/α is not a brick, there are two intermediate partitions, one of which is γ; for evacuation, we choose β to be the other one. These two situations appear as follows; the notation $\alpha \lessdot_x \beta$ means $\alpha \lessdot \beta$ and $\beta \ominus \alpha = x$.

(a) x, y adjacent

$$\begin{array}{ccc} \gamma & \lessdot_y & \delta \\ \rotatebox{90}{$\lessdot$}_x & & \rotatebox{90}{$\lessdot$}_y \\ \alpha & \lessdot_x & \beta \end{array}$$

(b) x, y not adjacent

$$\begin{array}{ccc} \gamma & \lessdot_y & \delta \\ \rotatebox{90}{$\lessdot$}_x & & \rotatebox{90}{$\lessdot$}_x \\ \alpha & \lessdot_y & \beta \end{array}$$

Now we compare Fomin's algorithm with the classical evacuation algorithm. Given a tableau $\boldsymbol{P}$ as the top row, produce the table Λ by the local rules just given. Define the tableau $\boldsymbol{P}^{(ij)}$ (with $0 \leq i \leq j \leq n$) as the following chain of partitions, with the entry e to fill the square added between two partitions shown between them as $\boxed{\text{e}}$:

$$\boldsymbol{P}^{(ij)} = \begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\quad \lambda^{(i-1,j)} \lessdot_{\boxed{j+1}} \cdots \lessdot_{\boxed{n}} \lambda^{(i-1,n)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \rotatebox{90}{$\lessdot$}_{\boxed{\star}} \\ \lambda^{(ii)} \lessdot_{\boxed{i+1}} \lambda^{(i,i+1)} \lessdot_{\boxed{i+2}} \cdots \lessdot_{\boxed{j}} \lambda^{(ij)} \end{array}$$

Note that $\boldsymbol{P} = \Delta^0 \boldsymbol{P}$ is the top row of Λ. Given that $\Delta^{i-1}\boldsymbol{P}$ is the $(i-1)$th row of Λ, we want to show that $\Delta^i \boldsymbol{P}$ is the ith row of Λ. The tableau $\boldsymbol{P}^{(ii)}$ is $\Delta^{i-1}\boldsymbol{P}$ with the minimal entry replaced by $\star$, just as in the first step of the classical algorithm for computing $\Delta(\Delta^{i-1}\boldsymbol{P})$. The only place the chains $\boldsymbol{P}^{(i,j-1)}$ and $\boldsymbol{P}^{(ij)}$ may differ is in the 2 by 2 section of Λ,

$$\begin{matrix} \lambda^{(i-1,j-1)} & \lambda^{(i-1,j)} \\ \lambda^{(i,j-1)} & \lambda^{(ij)} \end{matrix}$$

So all entries except possibly for j and $\star$ are in the same position in both $\boldsymbol{P}^{(i,j-1)}$ and $\boldsymbol{P}^{(ij)}$. If this section is in configuration (a) then $\star$ and j are swapped, while if it is in configuration (b), they are in the same position in both tableaux. We are in configuration (a) iff the cell y of $\boldsymbol{P}^{(i,j-1)}$ containing j is adjacent, right or below, the cell x with $\star$, and is smaller than the other cell adjacent to x if the other cell is nonempty. So the rule for swapping $\star$ with an adjacent entry is the same as in the classical computation. Finally, $\star$ is at an inner corner of $\boldsymbol{P}^{(in)}$, and removing it yields the ith row of Λ as $\Delta^i\boldsymbol{P}$.

Since $\lambda^{(in)} = \operatorname{sh}\Delta^i P$, the chain represented by the last column of Λ is $\operatorname{ev}\boldsymbol{P}$.

Now every 2 by 2 segment of this table obeys the rule, given partitions α and δ, either a unique partition is between them, and both β and γ equal it, or there are two partitions between them, and β is one and γ the other. So given initial values of the final column instead of the first row, we could build the exact same table. Since the rule is symmetric, $\operatorname{ev}\operatorname{ev}\boldsymbol{P} = \boldsymbol{P}$, so ev is an involution.

5.2. The interval type table of a flag. The interval type table of a saturated flag f in a semi-primary lattice is the triangular array $\operatorname{itype} f = (\operatorname{type}[f_i, f_j])_{0\le i\le j\le n}$.

We first develop necessary (but not sufficient) conditions for a triangular array $(\lambda^{(ij)})_{0\le i\le j\le n}$ of partitions to be realizable as the interval type table of some flag. Then we show that if $\mathcal{L}$ is q-regular of type λ, and $\operatorname{sh}\boldsymbol{P} = \lambda$, most flags whose type is $\boldsymbol{P}$ have the same interval type table as as the evacuation table of $\boldsymbol{P}$.

For any saturated flag f, $\operatorname{type}[f_i, f_j]$ is increasing in Young's lattice as i decreases and j increases, and has rank $j-i$. Thus, the triangular array $\operatorname{itype} f$ is quite similar to the array described above for Fomin's version of evacuation.

$$\begin{array}{c|ccc} \operatorname{type}[f_a, f_b] & b = j-1 & & b = j \\ \hline a = i & \gamma & \lessdot & \delta \\ & \veedot & & \veedot \\ a = i+1 & \alpha & \lessdot & \beta \end{array}$$

Given $\alpha < \delta$ of ranks differing by 2, there are still either one or two elements between them in Young's lattice, and if there is only one, then both γ and β must equal it. If there are two, while the evacuation algorithm specifies that β must be one and γ the other, this needn't be the case for the interval types of flags. We will show that in a quantifiable sense, they usually are different, but also determine occasions when they may be equal.

Let $\boldsymbol{P}_i = \operatorname{ftype}\partial^i f$ denote the tableau associated to the partitions on the ith row of $\operatorname{itype} f$. We generalize the tableau operator Δ to a nondeterministic game $\widetilde{\Delta}$, such that $\boldsymbol{P}_{i+1}$ has form $\widetilde{\Delta}\boldsymbol{P}_i$ for $i = 0, \dots, n-1$. The notation $\boldsymbol{P}_{i+1} = \widetilde{\Delta}\boldsymbol{P}_i$ means that $\boldsymbol{P}_{i+1}$ is one of the possible values of $\widetilde{\Delta}\boldsymbol{P}_i$.

Let $\boldsymbol{P}$ be a standard tableau.

(1) Replace the top left corner of $\boldsymbol{P}$ with $\star$.

Consider the cell just below $\star$ and the cells in the column right of $\star$. If a cell amongst these has a larger entry than was initially at $\star$, and this entry can be slid into $\star$ to yield a tableau that is strictly increasing in rows and columns (ignoring $\star$), the cell is **eligible**. Unless $\star$ is an inner corner, there is at least one eligible cell because the smaller or only of the right and lower neighbors of $\star$ is eligible.

(2) If there are any eligible cells, choose one, slide it into $\star$, and then repeat this step.
(3) Finally, $\star$ is at an inner corner. Delete the square with $\star$, obtaining a smaller tableau, $\widetilde{\Delta}\boldsymbol{P}$.

We illustrate this with

$$\boldsymbol{P} = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

First replace 1 with $\star$; the eligible cells after the replacement are shown with bold frames:

$$\begin{array}{|c|c|c|} \hline \star & 3 & 4 \\ \hline \boxed{2} & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

Only 2 is eligible, because if any number from the second column is slid into $\star$, the first column will no longer be strictly increasing, since these numbers are larger than 2. So slide 2 into $\star$:

$$\begin{array}{|c|c|c|} \hline 2 & \boxed{3} & 4 \\ \hline \star & \boxed{5} \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

The evacuation algorithm would specify that we slide 5 into $\star$, but in this algorithm, we may choose to slide 3 into $\star$.

$$\begin{array}{|c|c|c|} \hline 2 & \star & \boxed{4} \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$$

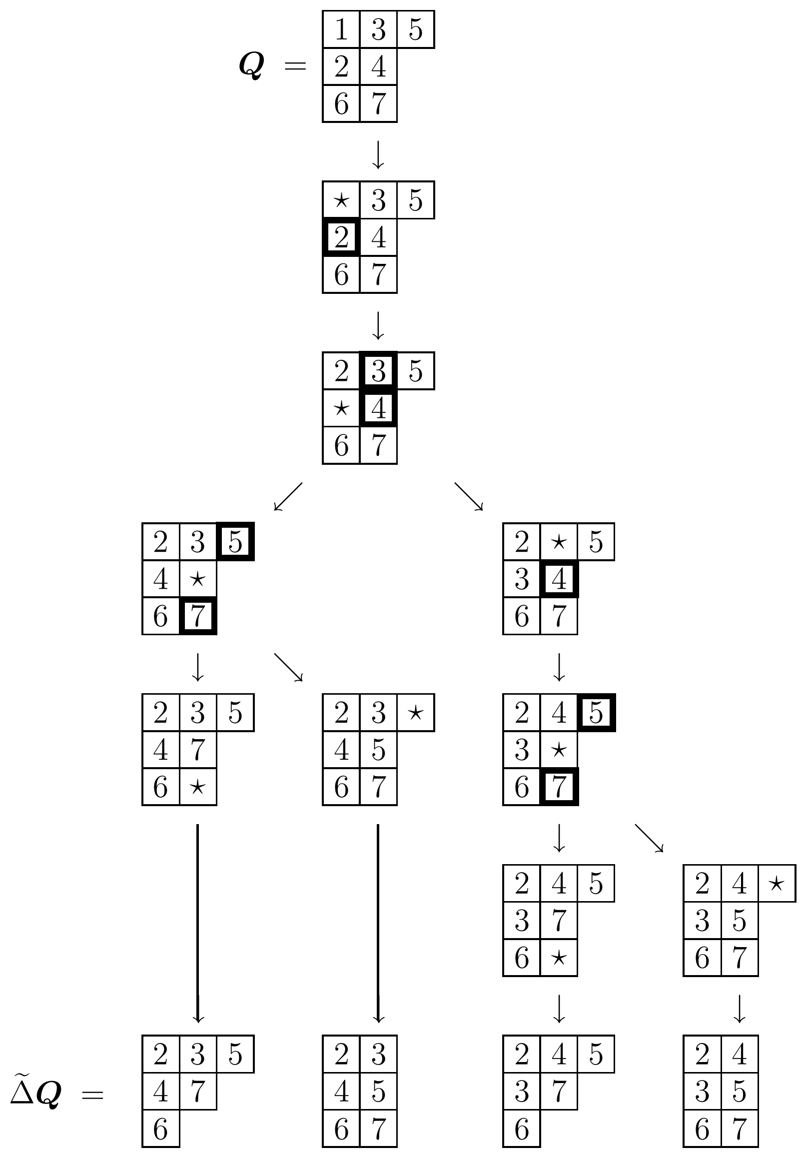
Only 4 is eligible, so slide it into $\star$. Then $\star$ is an inner corner with no further eligible cells, so $\star$ is removed to obtain as one possible value of $\widetilde{\Delta}\boldsymbol{P}$,

$$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 3 & 5 \\ \hline 6 & 7 \\ \hline \end{array}$$

This and $\Delta\boldsymbol{P}$ are the only possible values of $\widetilde{\Delta}\boldsymbol{P}$. Swapping the 4 and 5 in $\boldsymbol{P}$ leads to more possibilities, which are shown in Figure 4.

We will show that all flags f of type $\boldsymbol{P}$ have $\operatorname{ftype} \partial f = \widetilde{\Delta}\boldsymbol{P}$, and show that in a q-regular semi-primary lattice, the number of ways this happens for each possible $\widetilde{\Delta}\boldsymbol{P}$ is a nonzero monic polynomial in q.

Notation. For partitions $\alpha \lessdot \beta$, write $\alpha \lessdot_{|r|} \beta$ to mean that these partitions differ in a square in column r, that is, $\alpha'_i = \beta'_i$ for $i \neq r$, and $\alpha'_r \lessdot \beta'_r$. Write $\alpha \lessdot_x \beta$ to mean that $x \in \mathbb{P} \times \mathbb{P}$ is the cell in which the diagrams of α and β differ.



Begin with $\boldsymbol{Q}$. Replace its top left corner with $\star$. At each step, the cells marked $\boxed{k}$ are eligible to be slid to $\star$. When $\star$ reaches an inner corner, delete that cell to obtain a possible value of $\widetilde{\Delta}\boldsymbol{Q}$. The ordinary $\Delta\boldsymbol{Q}$ is the left-hand path.

FIGURE 4. All possible degenerate elementary evacuation steps.

Theorem 5.4. *Let $\mathcal{L}$ be a semi-primary lattice. Let $j - i > 1$, and consider a flag $(f_i \lessdot f_{i+1} \leq f_{j-1} \lessdot f_j)$ in $\mathcal{L}$. The interval types between them have one of the following forms, where the positions x or y of the square two partitions differ by is noted between them.*

$$\begin{array}{c|ccc} \text{(a) type}[f_a, f_b] & b = j-1 & & b = j \\ \hline a = i & \gamma & \lessdot_y & \delta \\ & \gtrdot_x & & \gtrdot_y \\ a = i+1 & \alpha & \lessdot_x & \beta \end{array} \qquad \begin{array}{c|ccc} \text{(b)} & j-1 & & j \\ \hline i & \gamma & \lessdot_y & \delta \\ & \gtrdot_x & & \gtrdot_x \\ i+1 & \alpha & \lessdot_y & \beta \end{array}$$

If δ/α is a brick, we have form (a); if $\mathrm{col}(y) \gtrdot \mathrm{col}(x)$ *but* $\mathrm{row}(y) \neq \mathrm{row}(x)$*, we may have either form; and otherwise,* $|\mathrm{col}(y) - \mathrm{col}(x)| > 1$ *and we have form (b).*

Proof. The type of an interval increases as the lower boundary is lowered or the upper boundary is raised, so we have one of forms (a) and (b). Now suppose we are given the partial flag $(f_i \lessdot f_{i+1} < f_j)$ with type$[f_i, f_j] = \delta$ and type$[f_{i+1}, f_j] = \beta$, and we want to choose a coatom f_{j-1} of $[f_{i+1}, f_j]$ with type$[f_i, f_{j-1}] = \gamma$ and type$[f_{i+1}, f_{j-1}] = \alpha$.

Define $z^k_{ab} = A^k_{f_b} f_a = A^k f_a \wedge f_b$.

Let the columns the partitions differ in be given by $\gamma \lessdot_{|s|} \delta$ and $\alpha \lessdot_{|r|} \beta$. The condition type$[f_i, f_{j-1}] = \gamma$ is satisfied precisely when f_{j-1} is a coatom of $I_1 = [z^{s-1}_{ij}, f_j]$ not also in $I_2 = [z^s_{ij}, f_j]$, while the condition type$[f_{i+1}, f_{j-1}] = \alpha$ is satisfied precisely when f_{j-1} is a coatom of $I_3 = [z^{r-1}_{i+1,j}, f_j]$ not also in $I_4 = [z^r_{i+1,j}, f_j]$. So we count the coatoms of $I_1 \cap I_3$, minus those of $I_1 \cap I_4$, minus those of $I_2 \cap I_3$, plus the doubly subtracted ones in $I_2 \cap I_4$. We tabulate all these intersections below in all cases. Each intersection has the form $[t, f_j] \cap [u, f_j] = [t \vee u, f_j]$. Note for all $k \geq 0$ that $z^{k+1}_{ij} \geq z^k_{i+1,j} \geq z^k_{ij}$: the second inequality expands to $A^k f_{i+1} \wedge f_j \geq A^k f_i \wedge f_j$, which is true because A and $\wedge$ both preserve weak cover relations, while $z^{k+1}_{ij} = A_{f_j} z^k_{ij}$ is the join of all weak upper covers of z^k_{ij} below f_j, and $z^k_{i+1,j}$ is such a weak upper cover, so the first inequality holds.

			x in each case			
sign	$[x, f_j]$	x equation	$r < s-1$	$r = s-1$	$r = s$	$r > s$
$+$	$I_1 \cap I_3$	$z^{s-1}_{ij} \vee z^{r-1}_{i+1,j}$	z^{s-1}_{ij}	z^{s-1}_{ij}	$z^{r-1}_{i+1,j}$	$z^{r-1}_{i+1,j}$
$-$	$I_1 \cap I_4$	$z^{s-1}_{ij} \vee z^r_{i+1,j}$	z^{s-1}_{ij}	$z^r_{i+1,j}$	$z^r_{i+1,j}$	$z^r_{i+1,j}$
$-$	$I_2 \cap I_3$	$z^s_{ij} \vee z^{r-1}_{i+1,j}$	z^s_{ij}	z^s_{ij}	z^s_{ij}	$z^{r-1}_{i+1,j}$
$+$	$I_2 \cap I_4$	$z^s_{ij} \vee z^r_{i+1,j}$	z^s_{ij}	z^s_{ij}	$z^r_{i+1,j}$	$z^r_{i+1,j}$
		net interval	$\emptyset$	$I_1 - I_4$	$I_3 - I_2$	$\emptyset$
		# coatoms if q-regular		$\langle \delta'_s \rangle - \langle \beta'_{r+1} \rangle$	$\langle \beta'_r \rangle - \langle \delta'_{s+1} \rangle$	

There is no way to choose f_{j-1} unless $r = s-1$ or s. If δ/α is a brick, we are in form (a). If it is not a brick, we are in form (b), except that if $\mathrm{col}(x) \lessdot \mathrm{col}(y)$, we may be in either (a) or (b). $\square$

Corollary 5.5. *Let $(f_i \lessdot f_{i+1} < f_j)$ or $(f_i < f_{j-1} \lessdot f_j)$ be given. This may be completed to $(f_i \lessdot f_{i+1} \leq f_{j-1} \lessdot f_j)$ with specified interval types, by choosing f_{j-1} (respectively, f_{i+1}) as follows, and if the lattice is q-regular, the number of such choices is as indicated.*

(1)
$$
\begin{array}{c|ccc}
\text{type}[f_a, f_b] & b = j-1 & & b = j \\
\hline
a = i & \gamma & \lessdot_{|r+1|} & \delta \\
 & \rotatebox[origin=c]{90}{$\lessdot$}_{|r|} & & \rotatebox[origin=c]{90}{$\lessdot$}_{|r+1|} \\
a = i+1 & \alpha & \lessdot_{|r|} & \beta
\end{array}
$$

f_{j-1} is a coatom of $[A^r f_i \wedge f_j, f_j]$ not in $[A^r f_{i+1} \wedge f_j, f_j]$;
$\left\langle \delta'_{r+1} \right\rangle - \left\langle \beta'_{r+1} \right\rangle$ choices.

f_{i+1} is an atom of $[f_i, C^r f_j \vee f_i]$ not in $[f_i, C^r f_{j-1} \vee f_i]$;
$\left\langle \delta'_{r+1} \right\rangle - \left\langle \gamma'_{r+1} \right\rangle$ choices.

(2)
$$
\begin{array}{c|ccc}
\text{type}[f_a, f_b] & b = j-1 & & b = j \\
\hline
a = i & \gamma & \lessdot_{|r|} & \delta \\
 & \rotatebox[origin=c]{90}{$\lessdot$}_{|t|} & & \rotatebox[origin=c]{90}{$\lessdot$}_{|t|} \\
a = i+1 & \alpha & \lessdot_{|r|} & \beta
\end{array}
$$

f_{j-1} is a coatom of $[A^{r-1} f_{i+1} \wedge f_j, f_j]$ not in $[A^r f_i \wedge f_j, f_j]$;
$\langle \beta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle$ choices.

f_{i+1} is an atom of $[f_i, C^{t-1} f_{j-1} \vee f_i]$ not in $[f_i, C^t f_{j-1} \vee f_i]$;
$\langle \gamma'_t \rangle - \left\langle \delta'_{t+1} \right\rangle$ choices.

Note. Configuration (1) of the corollary always has form (a) of the theorem; configuration (2) has form (a) for $r = t$ and (b) for $r \neq t$.

Proposition 5.6. *In a semi-primary lattice of type λ, if f is a flag of type $\boldsymbol{P}$, then* $\operatorname{ftype} d^k f = d^k \boldsymbol{P}$.

Proposition 5.7. *If f is a flag with* $\operatorname{ftype} f = \boldsymbol{P}$ *in a semi-primary lattice, then* $\operatorname{ftype} \partial f = \widetilde{\Delta} \boldsymbol{P}$. *Conversely, consider a q-regular semi-primary lattice of type $\lambda \vdash n$. The number of atoms f_1 of cotype $\mu \lessdot_{|r|} \lambda$ is $\left\langle \lambda'_{r-1} \right\rangle - \langle \lambda'_r \rangle$. Pick any such atom. Let $\boldsymbol{P}$ be a standard tableau of shape λ and $\boldsymbol{Q} = \widetilde{\Delta} \boldsymbol{P}_i$ have shape μ. The number of ways to complete $(\hat{0}, f_1, \hat{1})$ to a flag $f = (\hat{0}, f_1, f_2, \ldots, f_{n-1}, \hat{1})$ with* $\operatorname{ftype} f = \boldsymbol{P}$ *and* $\operatorname{ftype} \partial f = \boldsymbol{Q}$ *is a monic polynomial in q, given as the product over each entry j of $\boldsymbol{Q}$ of terms depending on how j moves between $\boldsymbol{P}$ and $\boldsymbol{Q}$:*

$$
\prod_{j \,\in\, \boldsymbol{Q}} \begin{cases} \langle \operatorname{row}(\boldsymbol{Q}, j) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, j) \rangle & \textit{if } \operatorname{col}(\boldsymbol{P}, j) = \operatorname{col}(\boldsymbol{Q}, j); \\ \langle \operatorname{row}(\boldsymbol{P}, j) \rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, j) \rangle & \textit{otherwise.} \end{cases}
$$

This polynomial has degree

$$
n(\lambda) - \#\ \textit{of entries that move up one cell from } \boldsymbol{P} \textit{ to } \boldsymbol{Q}.
$$

As $q \to \infty$, all but a fraction $O(q^{-1})$ of the flags f with $\operatorname{ftype} f = \boldsymbol{P}$ *have* $\operatorname{ftype} \partial f = \Delta \boldsymbol{P}$, *and all but a fraction $O(q^{-1})$ of the flags $(f_1, \ldots, f_n)$ with f_1 an atom and* $\operatorname{ftype}(f_1, \ldots, f_n) = \Delta \boldsymbol{P}$ *have* $\operatorname{ftype}(\hat{0}, f_1, \ldots, f_n) = \boldsymbol{P}$.

Proof. Let $f = (f_0, \ldots, f_n)$ be a flag of type $\boldsymbol{P}$. The first part of the theorem is obtained from Theorem 5.4 by expressing it in terms of tableaux. We begin with the tableau $\boldsymbol{P}$, and then essentially follow the proof of Section 5.1 that Fomin's computation of $\Delta \boldsymbol{P}$ agrees with the classical one. Place $\star$ in the upper left corner of $\boldsymbol{P}$, and then for each $j = 1, \ldots, n$ successively, if we are in form (a), j is in the column just right of $\star$, and we swap $\star$ and j; while if we are in form (b), we leave the tableau intact. At the end, remove $\star$. This is exactly the algorithm for $\widetilde{\Delta}\boldsymbol{P}$.

For the converse, in a q-regular semi-primary lattice of type λ, let $f_0 = \hat{0}$, $f_n = \hat{1}$, and choose an atom f_1 of cotype μ in the indicated number of ways. If $f_n, f_{n-1}, \ldots, f_j$ have been chosen so that $\operatorname{ftype}_{f_0}(f_j, \ldots, f_n)$ and $\operatorname{ftype}_{f_1}(f_j, \ldots, f_n)$ respectively agree with $\boldsymbol{P}$ and $\boldsymbol{Q}$ in entries $j+1, \ldots, n$, then the number of ways to choose f_{j-1} to extend this to entry j is as stated in the theorem, by plugging into Corollary 5.5: when j is in the same column of $\boldsymbol{P}$ and $\boldsymbol{Q}$, use form (2), and when it is in different columns, use form (1). Denote the product in the theorem by $a(q)$.

If $\operatorname{col}(\boldsymbol{P}, j) = \operatorname{col}(\boldsymbol{Q}, j)$, the degree of the jth factor is $\operatorname{row}(\boldsymbol{Q}, j) - 1$, which equals $\operatorname{row}(\boldsymbol{P}, j) - 1$ if j is in the same cell of both $\boldsymbol{P}$ and $\boldsymbol{Q}$, and equals $\operatorname{row}(\boldsymbol{P}, j) - 2$ if it moves up one cell. Let u be the number of j's that move up one cell. The degree of $a(q)$ is the sum of $\operatorname{row}(\boldsymbol{P}, j) - 1$ over all $j \in \boldsymbol{Q}$, minus u. Since 1 is the only entry in $\boldsymbol{P}$ not in $\boldsymbol{Q}$, and $\operatorname{row}(\boldsymbol{P}, 1) - 1 = 0$, this sum is the sum over all cells of $\boldsymbol{P}$ of their row number minus 1, totaling $n(\lambda)$.

Multiply $a(q)$ by the initial number of choices of f_1, to obtain the total number $b(q)$ of flags f with $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} \partial f = \boldsymbol{Q}$. It is a monic polynomial of degree $n(\lambda) + (\operatorname{row}(\lambda/\mu) - 1) - u$. In the deletion path for the game Δ, we have $u = \operatorname{row}(\lambda/\mu) - 1$ because the deletion path from λ/μ to the upper left corner only moves left one cell or up one cell at each step. In all other games $\widetilde{\Delta}$, there are entries that move to lower rows, so that $u > \operatorname{row}(\lambda/\mu) - 1$. Thus, the maximum degree $n(\lambda)$ of $b(q)$ is uniquely obtained by the game $\boldsymbol{Q} = \Delta\boldsymbol{P}$. The total number of flags f with $\operatorname{ftype} f = \boldsymbol{P}$ is $F^{\boldsymbol{P}}(q)$, which also is a monic polynomial of degree $n(\lambda)$, so the quotient $b(q)/F^{\boldsymbol{P}}(q)$ is $1 - O(q^{-1})$ as $q \to \infty$, so nearly all flags with $\operatorname{ftype} f = \boldsymbol{P}$ satisfy $\operatorname{ftype} \partial f = \boldsymbol{Q}$. The degree of $a(q)$ in the game $\Delta\boldsymbol{P}$ is $n(\lambda) - u = n(\mu)$, so also $a(q)/F^{\Delta\boldsymbol{P}}(q) = 1 - O(q^{-1})$. Thus, nearly all flags $g = (f_1, \ldots, f_n)$, where f_1 is an atom and $\operatorname{ftype} g = \Delta\boldsymbol{Q}$, satisfy $\operatorname{ftype}(\hat{0}, f_1, \ldots, f_n) = \boldsymbol{P}$. $\square$

We will extend this to show that $k = 0, \ldots, n$, all but a fraction $O(q^{-1})$ of the flags f with $\operatorname{ftype} f = \boldsymbol{P}$ satisfy $\operatorname{ftype} \partial^k f = \Delta^k \boldsymbol{P}$. First we develop notions of asymptotic enumeration and composition of operators on flags.

5.3. Generic behavior in regular semi-primary lattices. For some problems, we can do exact enumerations, and for others, we can do asymptotic enumerations showing what "usually" happens. Given $\operatorname{ftype} f$ but not f, we will compute the "usual" flag type ftype or $\operatorname{ftype}_{\hat{0}}$ of ∂f, df, $(\phi(f_l), \ldots, \phi(f_h))$ for certain latticial growths ϕ; various other flag operators; the flag cotype of f; and compositions of these.

For logical statement $S(s)$ with parameters $s = (x_1, \ldots, x_k)$, we write $s \in S$ to mean that $S(s)$ is true.

Definition 5.8 (topological). Let $S(s)$ and $T(s)$ be logical statements about indeterminates $s = (x_1, \ldots, x_k)$ in an invariant subspace lattice over an infinite field. Then $T(s)$ **is generically true of** $S(s)$ if

$$\overline{\{\, s : S(s) \text{ and } T(s) \text{ are true} \,\}} \supseteq \{\, s : S(s) \text{ is true} \,\},$$

where the closure is taken in the Zariski topology.

We obtain an analogue of this notion for q-regular semi-primary lattices by replacing closure in a specific lattice with a limit over all finite lattices. For a logical statement $S(s)$ and a finite semi-primary lattice $\mathcal{L}$, let $N_S = N_S(\mathcal{L})$ denote the number of s for which $S(s)$ is true.

Definition 5.9 (q-regular). Let $S(s)$ and $T(s)$ be logical statements about indeterminates s in semi-primary lattices. Then T **is generically true of** S if

$$\lim_{q\to\infty} \frac{\#\{\, s \in S \,:\, T(s) \,\}}{N_S} = 1$$

where the limit is taken over all q-regular semi-primary lattices as $q \to \infty$. Note that the counts in the numerator and denominator may be different for different q-regular semi-primary lattices that have the same value of q.

Let s and t be indeterminates; $S(s)$ be a logical statement about indeterminates s; and $T(t)$ be a logical statement about indeterminates t. Let ϕ be a map from the space in which s lies to the space in which t lies.

Definition 5.10. We say that ϕ **generically maps** S **to** T when $T(\phi(s))$ is generically true of $S(s)$. Equivalently, there is a function $\gamma_\phi(q)$ (of q, not $\mathcal{L}$) taking on real values from 0 to 1, vanishing as $q \to \infty$, such that

$$\#\{\, s \in S \,:\, T(\phi(s)) \,\} \geq (1 - \gamma_\phi(q)) N_S$$

in all q-regular semi-primary lattices.

Definition 5.11. ϕ **uniformly maps** S **to** T when

(1) ϕ generically maps S to T;
(2) There is a function $\epsilon(q)$ that vanishes as $q \to \infty$, such that

$$\lim_{q\to\infty} \frac{\#\left\{\, t \in T \,:\, \left|\frac{\#\{\, s\in S\,:\,\phi(s)=t \,\}}{N_S/N_T} - 1\right| < \epsilon(q) \,\right\}}{N_T} = 1.$$

Equivalently, there are functions $\delta_\phi(q)$ and $\epsilon_\phi(q)$ taking on real values from 0 to 1 and vanishing as $q \to \infty$, such that

$$\#\left\{\, t \in T \,:\, \left|\frac{\#\{\, s \in S \,:\, \phi(s) = t \,\}}{N_S/N_T} - 1\right| < \epsilon_\phi(q) \,\right\} \geq (1 - \delta_\phi(q)) N_T$$

for all q-regular semi-primary lattices.

In other words, (1) most variables $s \in S$ also satisfy $\phi(s) \in T$, and (2) conversely, most variables $t \in T$ have approximately the same number N_S/N_T of inverses under ϕ that satisfy S.

Theorem 5.12. *If ϕ uniformly maps S to T, and θ generically maps T to U, the composition $\psi = \theta \circ \phi$ generically maps S to U. If θ is uniform then so is ψ.*

Proof. Let ϕ be generic and θ be uniform. Choose γ_ϕ, δ_ϕ, ϵ_ϕ, and γ_θ as in the definitions of genericity and uniformity above. All γ, δ, and ϵ's in this proof, with any subscripts, are functions of q, while all set cardinalities are functions of $\mathcal{L}$ (and hence not solely functions of q).

For genericity, we must prove

$$\lim_{q\to\infty} \frac{\#\{\, s\in S \,:\, U(\psi(s))\,\}}{N_S} = 1.$$

Consider any q-regular semi-primary lattice. We have

$$\begin{aligned} \#\{\, t\in T \,:\, U(\theta(t))\,\} &\geq (1-\gamma_\theta)N_T \\ \#\left\{ t\in T \,:\, \left|\frac{\#\{\, s\in S \,:\, \phi(s)=t\,\}}{N_S/N_T} - 1\right| < \epsilon_\phi \right\} &\geq (1-\delta_\phi)N_T \end{aligned}$$

so the cardinality of the intersection I of the two sets on the left is at least $(1-\gamma_\theta-\delta_\phi)N_T$. Thus,

$$\begin{aligned} N_S \geq \#\{\, s\in S \,:\, U(\psi(s))\,\} &\geq \sum_{t\in T} \#\{\, s\in S \,:\, \phi(s)=t \text{ and } U(\theta(t))\,\} \\ &\geq \sum_{t\in I} \#\{\, s\in S \,:\, \phi(s)=t \text{ and } U(\theta(t))\,\} \\ &= \sum_{t\in I} \#\{\, s\in S \,:\, \phi(s)=t\,\} \\ &\geq \#I\cdot\frac{N_S}{N_T}(1-\epsilon_\theta) \\ &\geq (1-\gamma_\theta-\delta_\phi)(1-\epsilon_\theta)N_S \end{aligned}$$

Divide through by N_S to obtain

$$1 \geq \frac{\#\{\, s\in S \,:\, U(\psi(s))\,\}}{N_S} \geq (1-\gamma_\theta-\delta_\phi)\,(1-\epsilon_\phi).$$

The left and right expressions depend only on q, not on $\mathcal{L}$, and are 1 in the limit $q\to\infty$; thus the middle expression, which depends on $\mathcal{L}$, also has limit 1 as $q\to\infty$. Thus ψ generically distributes S over U.

For uniformity, we will make a number of approximations to S, T, and U.

Let

$$T_1 = \left\{ t\in T \,:\, \left|\frac{\#\{\, s\in S \,:\, \phi(s)=t\,\}}{N_S/N_T} - 1\right| < \epsilon_\phi \right\}.$$

Note $\#T_1 \geq (1-\delta_\phi)N_T$.

For $u\in U$, let $B_u = \{\, t\in T \,:\, \theta(t)=u\,\}$. At least $(1-\delta_\theta)N_U$ different u's satisfy $\left|\frac{\#B_u}{N_T/N_U}-1\right| < \epsilon_\theta$. Let $C_u = \{\, t\in T_1 \,:\, \theta(t)=u\,\}$. Then

$$\delta_\phi N_T \geq \#T - \#T_1 = \sum_{u\in U}\left(\#B_u - \#C_u\right).$$

Each term of the sum is nonnegative since $B_u\supseteq C_u$. At most $\sqrt{\delta_\phi}N_U$ values of u have $\#B_u - \#C_u > \sqrt{\delta_\phi\frac{N_T}{N_U}}$ because if more did, the rightmost expression would exceed the leftmost. So

$$U_1 = \left\{ u\in U \,:\, \#B_u - \#C_u \leq \sqrt{\delta_\phi\frac{N_T}{N_U}} \right\}$$

has cardinality at least $(1-\sqrt{\delta_\phi})N_U$.

Now let

$$\begin{aligned}
\delta_2 &= \delta_\theta + \sqrt{\delta_\phi} \\
\epsilon_2 &= \epsilon_\theta + \sqrt{\delta_\phi} \\
U_2 &= \left\{ u \in U \,:\, \left| \frac{\#\{\, t \in T_1 \,:\, \theta(t) = u \,\}}{N_T/N_U} - 1 \right| < \epsilon_2 \right\}.
\end{aligned}$$

Then U_2 contains the intersection of $\theta(T)$ and U_1, whose respective sizes are at least $(1-\delta_\theta)N_U$ and $(1-\sqrt{\delta_\phi})N_U$. Thus their intersection has size at least $(1-\delta_\theta-\sqrt{\delta_\phi})N_U$, so $\#U_2 \geq (1-\delta_2)N_U$.

For each $u \in U$, let

$$\begin{aligned}
D_u &= \{\, s \in S \,:\, \phi(s) \in T_1 \text{ and } \psi(s) = u \,\} \\
E_u &= \{\, s \in S \,:\, \psi(s) = u \,\}.
\end{aligned}$$

For at least $(1-\delta_2)N_U$ different u's, there are at least $(1-\epsilon_2)N_T/N_U$ inverses of u in T under θ that in turn have at least $(1-\epsilon_\phi)N_S/N_T$ inverses each in S under ϕ. So at least $(1-\delta_2)N_U$ different u's have $\#D_u \geq (1-\epsilon_\phi)(1-\epsilon_2)N_S/N_U$. Thus, the union of all D_u's has cardinality at least $(1-\delta_2)(1-\epsilon_\phi)(1-\epsilon_2)N_S$. Let

$$\delta_3 = 1 - (1-\delta_2)(1-\epsilon_\phi)(1-\epsilon_2).$$

We have $D_u \subseteq E_u$ so $\#D_u \leq \#E_u$. At most $\sqrt{\delta_3}N_U$ different u's can have

$$\#E_u \geq \frac{N_S}{N_U}\left((1-\epsilon_\phi)(1-\epsilon_2) + \sqrt{\delta_3}\right)$$

because if more do, we have

$$\sum_{u \in U} \#E_u > \frac{N_S}{N_U} \cdot \left((1-\delta_2)(1-\epsilon_\phi)(1-\epsilon_2) + \sqrt{\delta_3} \cdot \sqrt{\delta_3}\right) N_U = N_S,$$

but the union of the disjoint sets E_u's is a subset of S, so the sum is at most N_S.

So at least $(1-\delta_2-\sqrt{\delta_3})N_U$ different u's have

$$(1-\epsilon_\phi)(1-\epsilon_2) + \sqrt{\delta_3} \geq \frac{\#E_u}{N_S/N_U} \geq (1-\epsilon_\phi)(1-\epsilon_2).$$

So

$$\begin{aligned}
\delta_\psi &= 1 - \delta_2 - \sqrt{\delta_3} \\
\epsilon_\psi &= \max\left\{1 - (1-\epsilon_\phi)(1-\epsilon_2), \sqrt{\delta_3} + (1-\epsilon_\phi)(1-\epsilon_2) - 1\right\}
\end{aligned}$$

satisfy the definition of ψ being uniform. $\square$

Definition 5.13. The **dual** of a flag $f = (f_0, \ldots, f_n)$ is $f^* = (g_0, \ldots, g_n)$ where $g_i = f^*_{n-i}$ combines inclusion of $\mathcal{L}$ into the dual lattice $\mathcal{L}^*$ with relabeling the indices.

We now present an asymptotic analogue for finite q-regular semi-primary lattices of the result of Hesselink [**10**, §5] and van Leeuwen [**31**, Theorem 2.3.2] that the generic cotype of a flag of type $\boldsymbol{P}$ in an invariant subspace lattice over an infinite field is $\operatorname{ev}\boldsymbol{P}$.

Theorem 5.14. *Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda \vdash n$. In a q-regular semi-primary lattice of type λ, all but a fraction $O(q^{-1})$ of the flags f with* $\operatorname{ftype} f = \boldsymbol{P}$ *satisfy* $\operatorname{ftype} \partial^k = \Delta^k \boldsymbol{P}$ *for $k = 0, \ldots, n$. Consequently, all but a fraction $O(q^{-1})$ of flags f with* $\operatorname{ftype} f = \boldsymbol{P}$ *satisfy* $\operatorname{ftype} f^* = \operatorname{ev} \boldsymbol{P}$.

Proof. By Proposition 5.7, ∂ uniformly maps flags f with $\operatorname{ftype} f = \boldsymbol{P}$ to flags g from f_1 to $\hat{1}$ with $\operatorname{ftype} g = \Delta \boldsymbol{P}$. By Theorem 5.12, the composition ∂^k uniformly maps flags f with $\operatorname{ftype} f = \boldsymbol{P}$ to flags $g = (g_k, \ldots, g_n)$ such that $\operatorname{ftype} g = \Delta^k \boldsymbol{P}$ and there is a sequence $\hat{0} \lessdot g_1 \lessdot g_2 \lessdot \cdots \lessdot g_k$ with $\operatorname{cotype} g_i = \operatorname{sh} \Delta^i \boldsymbol{P}$ for $i = 0, \ldots, k$.

Thus, all but a fraction $O(q^{-1})$ of the flags f with $\operatorname{ftype} f = \boldsymbol{P}$ have $\operatorname{ftype} \partial^k f = \Delta^k \boldsymbol{P}$ for $k = 0, \ldots, n$. Since $\operatorname{cotype} f_k = \operatorname{sh}(\operatorname{ftype} \partial^k f)$ for all k, the generic cotype of f_k is $\operatorname{sh} \Delta^k \boldsymbol{P}$. Thus, the tableau $\operatorname{ftype} f^*$ encoding the cotypes is generically $(\operatorname{sh} \Delta^{n-0} \boldsymbol{P}, \operatorname{sh} \Delta^{n-1} \boldsymbol{P}, \ldots) = \operatorname{ev} \boldsymbol{P}$.

Similarly, flags g in the dual lattice with $\operatorname{ftype} g = \operatorname{ev} \boldsymbol{P}$ generically have $\operatorname{ftype} g^* = \operatorname{ev} \operatorname{ev} \boldsymbol{P} = \boldsymbol{P}$. So the uniform cotype of flags of type $\boldsymbol{P}$ is $\operatorname{ev} \boldsymbol{P}$. $\square$

5.4. Counting the number of flags achieving a partial interval type table. Consider semi-primary lattices of type $\lambda \vdash n$. An **interval type table** is a specification of partitions $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$, with $\left|\lambda^{(ij)}\right| = j - i$, and $\lambda^{(ij)}$ increasing in Young's lattice as j increases or i decreases. A **partial interval type table** has only some of the values of $\lambda^{(ij)}$ specified, but their rank and order is still the same, and the entry $\lambda^{(0n)}$ must be specified. The **shape** of an interval type table is the partition $\lambda^{(0n)}$.

When for some i, all of $\lambda^{(ii)}, \ldots, \lambda^{(in)}$ are defined, let $\boldsymbol{P}_i$ be the tableau (with entries $i+1, \ldots, n$) corresponding to this chain. If $\lambda^{(ij)}, \lambda^{(i,j+1)}, \ldots, \lambda^{(in)}$ are defined, we let $\boldsymbol{P}_i$ be the skew tableau (with entries $j+1, \ldots, n$) corresponding to these. Let $k^* = n - k$, and if $\lambda^{(i^*,j)}, \lambda^{((i+1)^*,j)}, \ldots, \lambda^{(n^*,j)}$ are defined, let $\boldsymbol{P}_j^*$ be the (skew) tableau (with entries $i+1, \ldots, n$) corresponding to them.

A (partial) interval type table is **realizable** if there is a semi-primary lattice $\mathcal{L}$ with a flag f for which $\operatorname{type}[f_i, f_j] = \lambda^{(ij)}$ whenever $\lambda^{(ij)}$ is defined. We are interested in what interval type tables are realizable, and in q-regular semi-primary lattices, determining the number of flags achieving particular (partial) interval type tables. Let $r_{ij} = \operatorname{col}(\boldsymbol{P}_i, j)$ be the column in which $\lambda^{(i,j-1)}$ and $\lambda^{(ij)}$ differ, and $r_{ij}^* = \operatorname{col}(\boldsymbol{P}_j^*, i^*)$ be the column in which $\lambda^{(ij)}$ and $\lambda^{(i+1,j)}$ differ, provided the relevant partitions are in the partial interval type table.

The condition $\boldsymbol{P}_{i+1} = \tilde{\Delta} \boldsymbol{P}_i$ may be expressed as follows: $\boldsymbol{P}_i$ is a tableau on $i+1, \ldots, n$, while $\boldsymbol{P}_{i+1}$ is a tableau on $i+2, \ldots, n$; $\operatorname{sh} \boldsymbol{P}_{i+1} \lessdot \operatorname{sh} \boldsymbol{P}_i$; and $0 \le \operatorname{col}(P_i, j) - \operatorname{col}(P_{i+1}, j) \le 1$ for all j. A transposed version of this for the $\boldsymbol{P}_j^*$'s is equivalent. These conditions generalize as follows.

Theorem 5.15. *If an interval type table is realizable, then for all $0 \le i \le j \le k \le n$, we have $0 \le r_{ik} - r_{jk} \le \lambda_1^{(ij)}$, and dually, $0 \le r_{ik}^* - r_{ij}^* \le \lambda_1^{(jk)}$.*

Proof. We show the first statement. Since the types of $[f_j, f_k]$ and $[f_j, f_{k-1}]$ differ in column r_{jk}, we have $A^r f_j \wedge f_{k-1} = A^r f_j \wedge f_k$ iff $r < r_{jk}$, and $A^r f_j \wedge f_{k-1} \lessdot A^r f_j \wedge f_k$ iff $r \ge r_{jk}$. Similar statements hold for i, j and i, k in place of j, k. So $A^{r_{jk}-1} f_j \wedge f_{k-1} = A^{r_{jk}-1} f_j \wedge f_k$. Meet both sides of this with $A^{r_{jk}-1} f_i$ (which is weakly smaller than

$A^{r_{jk}-1}f_j$) to obtain $A^{r_{jk}-1}f_i \wedge f_{k-1} = A^{r_{jk}-1}f_i \wedge f_k$. Thus $r_{jk}-1 < r_{ik}$, or equivalently, $r_{ik} - r_{jk} \geq 0$.

Next, let $t = \lambda_1^{(ij)}$. If $r_{ik} - t - 1 < 0$ then $r_{ik} - t - 1 < r_{jk}$ so $r_{ik} - r_{jk} \leq t$. If $r_{ik} - t - 1 \geq 0$, consider the equality $A^{r_{ik}-1}f_i \wedge f_{k-1} = A^{r_{ik}-1}f_i \wedge f_k$. We have $A^t f_i \geq f_j$ so we obtain an inequality $A^{r_{ik}-1}f_i \geq A^{r_{ik}-t-1}f_j$. Meet both sides of the equality with $A^{r_{ik}-t-1}f_j$ to obtain $A^{r_{ik}-t-1}f_j \wedge f_{k-1} = A^{r_{ik}-t-1}f_j \wedge f_k$, whence $r_{ik} - t - 1 < r_{jk}$, and finally, $r_{ik} - r_{jk} \leq t$. $\square$

Example 5.16. No flag can have interval type table

$$\begin{array}{|c|c|c|c|}\hline 1&4&5&6\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 3&5&6\\\hline 4\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 4&6\\\hline 5\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}\quad \begin{array}{|c|}\hline 6\\\hline\end{array}\quad \emptyset$$

because 5 moves left $\mathrm{col}(P_1, 5) - \mathrm{col}(P_3, 5) = 2$ columns from $\boldsymbol{P}_1$ to $\boldsymbol{P}_3$, but the top row of $\mathrm{type}[f_1, f_3] = \begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}$ has width 1, and $1 < 2$.

Theorem 5.17. *If the interval type table Λ is realizable, then for $0 \leq i \leq j \leq k \leq n$, if $d^{n-k}\boldsymbol{P}_i = \boldsymbol{A} + \boldsymbol{B}$ where $\boldsymbol{A}$ is a standard tableau on $\{i+1, \ldots, j\}$ and $\boldsymbol{B}$ is a row and column strict composition tableau on $\{j+1, \ldots, k\}$, then $d^{n-k}\boldsymbol{P}_j = S(\boldsymbol{B})$.*

Proof. Let f realize Λ. By Theorem 4.93 applied to $\partial^i d^{n-k} f = (f_i, \ldots, f_k)$, if $\mathrm{ftype}\,\partial^i d^{n-k} f = \boldsymbol{A} + \boldsymbol{B}$ then $\mathrm{ftype}\,\partial^j d^{n-k} f = S(\boldsymbol{B})$. Since $\mathrm{ftype}\,\partial^i f = \boldsymbol{P}_i$ and $\mathrm{ftype}\,\partial^j f = \boldsymbol{P}_j$, the result follows. $\square$

Example 5.18. If $\boldsymbol{P}_0 = \begin{array}{|c|c|c|c|}\hline 1&3&4&8\\\hline 2&5\\\cline{1-2} 6&7\\\cline{1-2}\end{array}$ then

$$\boldsymbol{P}_2 = \begin{array}{|c|c|c|}\hline 3&4&8\\\hline 5&7\\\cline{1-2} 6\\\cline{1-1}\end{array}\qquad d\boldsymbol{P}_3 = \begin{array}{|c|c|}\hline 4&7\\\hline 5\\\cline{1-1} 6\\\cline{1-1}\end{array}\qquad d\boldsymbol{P}_4 = \begin{array}{|c|c|}\hline 5&7\\\hline 6\\\cline{1-1}\end{array}\qquad d\boldsymbol{P}_5 = \begin{array}{|c|c|}\hline 6&7\\\hline\end{array}.$$

See Hesselink [**10**, §6] for stronger necessary conditions for an interval type table to be realizable.

Now we enumerate the number of flags realizing certain partial interval type tables. Initially, we are given $f_0 = \hat{0}$ and $f_n = \hat{1}$. Given a partial flag and certain of the interval types in it, we want to know how to select other elements satisfying other interval types. Throughout, f is a partial flag, with f_i the component of rank i, if present.

Let $f_\Lambda(q)$ be the number of flags realizing partial interval type table Λ in a q-regular semi-primary lattice, when this is well defined for sufficiently large q.

Proposition 5.19. *Let $\Lambda = (\lambda^{(ij)})_{0 \leq i \leq j \leq n}$ and $M = (\mu^{(ij)})_{0 \leq i \leq j \leq m}$ be partial interval type tables of respective shapes $\lambda = \lambda^{(0n)}$, $\mu = \mu^{(0m)}$. Let $\nu \vdash m+n$. Let N be a partial interval type table with indices 0 through $m+n$ and components $\nu^{(ij)} = \lambda^{(ij)}$ (when defined) for $0 \leq i \leq j \leq n$; $\nu^{(i+n,j+n)} = \mu^{(ij)}$ (when defined) for $0 \leq i \leq j \leq m$; and $\nu^{(0,m+n)} = \nu$. Then $f_N(q) = g^\nu_{\lambda\mu}(q) f_\Lambda(q) f_M(q)$ when $f_\Lambda(q)$ and $f_M(q)$ exist.*

Proof. Choose f_n of type λ and cotype ν in one of $g^\nu_{\lambda\mu}(q)$ ways. Then choose $(f_0, \ldots, f_n)$ in $f_\Lambda(q)$ ways and $(f_n, \ldots, f_{n+m})$ in $f_M(q)$ ways. $\square$

Definition 5.20. The **transpose** of interval type table Λ is $\Lambda' = (\lambda^{(n-j,n-i)})_{0\le i\le j\le n}$.

Proposition 5.21. *If either $f_\Lambda(q)$ or $f_{\Lambda'}(q)$ exists, both do and are equal.*

Proof. If either exists for sufficiently large q, all q-regular semi-primary lattices of shape $\lambda^{(0n)}$ have the same number of flags satisfying the partial interval type table Λ. Each such flag satisfies the partial interval type table Λ' in the dual lattice. $\square$

Suppose $f_0 \lessdot \ldots \lessdot f_k \le f_n$ satisfies a partial interval type table, and we want to extend this to f_{n-1}. For now, abbreviate $r = r_{0n}$ and $r_i = r_{in}$. By Theorem 5.15, $r_0 \ge r_1 \ge \cdots \ge r_k$. This splits $\{0,\ldots,k\}$ into intervals according to where the $\ge$ is $\gtrdot$: $r_i = r - j$ on $t_{u-1} < i \le t_j$, where $t_{-1} = -1$ and $t_m = k$ (with $m = r_k - r_0$).

For each i, the interval $[f_i, f_{n-1}]$ has type $\lambda^{(i,n-1)}$ precisely when f_{n-1} is a coatom of $I_i = [A^{r_i-1} f_i, f_n]$ not in $J_i = [A^{r_i} f_i, f_n]$. For i with $t_{j-1} < i \le t_j$, we have $r_i = r - j$, so $I_i = [A^{r-j-1} f_i, f_n]$ and $J_i = [A^{r-j} f_i, f_n]$. The lower bound of each interval weakly increases as i increases from $t_{j-1}+1$ up to t_j, so we have $I_{t_{j-1}+1} \subseteq \cdots \subseteq I_{t_j}$ and $J_{t_{j-1}+1} \subseteq \cdots \subseteq J_{t_j}$. Thus, the condition that f_{n-1} is in all the I_i's is equivalent to being in all of $I_{t_0}, I_{t_1}, \ldots, I_{t_m}$, while the condition that it is in none of the J_i's is equivalent to it being in none of $J_{t_{-1}+1}, \ldots, J_{t_{m-1}+1}$.

Thus, f_{n-1} is any lower cover of f_n in

$$\begin{aligned}(I_{t_0} \cap \cdots \cap I_{t_m}) &\setminus (J_{t_{-1}+1} \cup \cdots \cup J_{t_{m-1}+1}) \\ &= [A^{r-0-1} f_{t_0} \vee \cdots \vee A^{r-m-1} f_{t_m}, f_n] \setminus \left(\bigcup_{j=-1}^{m-1} [A^{r-j-1} f_{t_j+1}, f_n] \right).\end{aligned}$$

Note that in all lattices, $[a,b] \cap [c,d] = [a \vee c, b \wedge d]$, but in general, $[a,b] \cup [c,d]$ is not an interval.

Now we apply inclusion-exclusion. The number of choices of f_{n-1} is

$$\sum_{S \subseteq \{-1,\ldots,m-1\}} (-1)^{|S|} \cdot \# \text{ of coatoms in } \left[\left(\bigvee_{j=0}^{m} A^{r-j-1} f_{t_j} \right) \vee \left(\bigvee_{j\in S} A^{r-j-1} f_{t_j+1} \right), f_n \right].$$

For each j with $0 \le j < m$, if $j \in S$ then the term $A^{r-j-1} f_{t_j+1}$ in the right join absorbs the term $A^{r-j-1} f_{t_j}$ in the left join. Let $\alpha_j = 1$ if $j \in S$ and 0 otherwise. The sum simplifies to

$$\sum_{S \subseteq \{-1,\ldots,m-1\}} (-1)^{|S|} \cdot \# \text{ of coatoms in } \left[\bigvee_{j=-\alpha_{-1}}^{m} A^{r_0-j-1} f_{t_j+\alpha_j}, f_n \right]. \tag{1}$$

Once we have chosen f_{n-1} to satisfy the specified types of $[f_0, f_{n-1}], \ldots, [f_k, f_{n-1}]$, we can then choose f_{n-2} to satisfy specified types of $[f_0, f_{n-2}], \ldots, [f_k, f_{n-2}]$ in a similar fashion, by replacing n with $n-1$ throughout, and interpreting A as A_{f_n}, which we replace with $A_{f_{n-1}}$. We could continue, choosing all of $f_n, \ldots, f_{k+1}$ to satisfy specified types with $f_0, \ldots, f_k$ (note we haven't imposed any types on intervals within $f_{k+1}, \ldots, f_n$). However, this equation is generally not computable from the partial interval type table; the terms of this equation usually depend on the specific values of $f_0, \ldots, f_k$, not just on types in the partial interval type table. When $[f_0, f_k]$ is elementary, we can compute it.

Lemma 5.22. *Pick $0 \le k \le n$. Consider a partial interval type table with $\lambda^{(ij)} = (1^{j-i})$ for $0 \le i \le j \le k$, and also with partitions $\lambda^{(i,n-1)}$ and $\lambda^{(in)}$ specified for $0 \le i \le k$, so that these all satisfy Theorem 5.15. Consider a partial flag $f_0 \lessdot \dots \lessdot f_k \le f_n$ in $\mathcal{L}$ such that* $\operatorname{type}[f_i, f_n] = \lambda^{(in)}$ *for $i = 0, \dots, k$, and $[f_0, f_k]$ is elementary. The number of extensions of f by a coatom f_{n-1} of $[f_k, f_n]$ such that* $\operatorname{type}[f_i, f_{n-1}] = \lambda^{(i,n-1)}$ *for $i = 0, \dots, k$ is*

$$\begin{cases} \left\langle \lambda_r^{(kn)'} \right\rangle - \left\langle \lambda_{r+1}^{(0n)'} \right\rangle & \text{if } r_{kn} = r_{0n}; \\ \left\langle \lambda_r^{(t_0,n)'} \right\rangle - \left\langle \lambda_r^{(t_0+1,n)'} \right\rangle & \text{if } r_{kn} \lessdot r_{0n}, \end{cases}$$

where $r = r_{0n}$.

Proof. Since $[f_0, f_k]$ is elementary, we have $r_{0n} \ge r_{kn}$ by Theorem 5.15. First consider $r = r_{0n} = r_{kn}$. Then $m = 0$ and $t_m = k$. The number of choices of f_{n-1} is

$$\#\text{ of coatoms in } [A^{r-1} f_k, \hat{1}] - \#\text{ of coatoms in } [A^r f_0 \vee A^{r-1} f_k, \hat{1}]$$

and since $[f_0, f_k]$ is elementary, we have $A f_0 \ge f_k$ so $A^r f_0 \ge A^{r-1} f_k$, so this reduces to

$$\#\text{ of coatoms in } [A^{r-1} f_k, \hat{1}] - \#\text{ of coatoms in } [A^r f_0, \hat{1}] = \left\langle \lambda_r^{(kn)'} \right\rangle - \left\langle \lambda_{r+1}^{(0n)'} \right\rangle.$$

Next, if $r = r_{0n} \gtrdot r_{kn}$, then $m = 1$, and the number of choices of f_{n-1} is obtained from Equation (1):

$$\sum_{S \subseteq \{-1,0\}} (-1)^{|S|} \cdot \#\text{ of coatoms in } \left[\bigvee_{j=-\alpha_{-1}}^{1} A^{r-j-1} f_{t_j + \alpha_j}, f_n \right].$$

Now $[f_0, f_k]$ is elementary, so the subinterval $[f_{t_j+\alpha_j}, f_{t_{j+1}+\alpha_{j+1}}]$ is too. Thus

$$A^{r-j-1} f_{t_j+\alpha_j} \ge A^{r-j+1-1} f_{t_{j+1}+\alpha_{j+1}}$$

whence the $j = 1$ term of the join is always absorbed by the $j = 0$ term, so we may let j run up to 0 instead of 1. Also, if $-1 \in S$, the $j = -1$ term absorbs the $j = 0$ term, so the terms for $S = \{-1, 0\}$ and $S = \{-1\}$ are equal except for sign, and hence cancel. We are then left with the terms $S = \emptyset$ and $S = \{0\}$, yielding

$$\begin{aligned} &\#\text{ of coatoms in } [A^{r-1} f_{t_0}, f_n] - \#\text{ of coatoms in } [A^{r-1} f_{t_0+1}, f_n] \\ &\qquad = \left\langle \lambda_r^{(t_0,n)'} \right\rangle - \left\langle \lambda_r^{(t_0+1,n)'} \right\rangle. \qquad \square \end{aligned}$$

We now generalize Proposition 5.7.

Proposition 5.23. *Given partial interval type table $\boldsymbol{P}_0, \dots, \boldsymbol{P}_k$ where $\boldsymbol{P}_{i+1} = \widetilde{\Delta} \boldsymbol{P}_i$ and $1, \dots, k$ are all in the first column of $\boldsymbol{P}_0$, the number of flags f in a q-regular semi-primary lattice with* $\operatorname{ftype} \partial^i f = \boldsymbol{P}_i$ *(for $i = 0, \dots, k$) is*

$$\prod_{j=0}^{k-1} \bigl(\langle l_1 \rangle - \langle l_2 \rangle \bigr) \cdot \prod_{j=k+1}^{n} \begin{cases} \langle \operatorname{row}(\boldsymbol{P}_k, j) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}_0, j) \rangle & \text{if } \operatorname{col}(\boldsymbol{P}_k, j) = \operatorname{col}(\boldsymbol{P}_0, j); \\ \langle \operatorname{row}(\boldsymbol{P}_t, j) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}_{t+1}, j) \rangle & \text{if } \operatorname{col}(\boldsymbol{P}_k, j) \lessdot \operatorname{col}(\boldsymbol{P}_0, j), \end{cases}$$

where for each j,

$$
\begin{array}{lcl}
t & & \textit{is the unique value for which } \mathrm{col}(\boldsymbol{P}_0, j) = \mathrm{col}(\boldsymbol{P}_t, j) \gtrdot \mathrm{col}(\boldsymbol{P}_{t+1}, j); \\
l_1 & = & \textit{\# row-ends of } \boldsymbol{P}_n^* \textit{ weakly right of } k \textit{ containing any of } j^*, \ldots, (n-1)^*, \bullet; \\
l_2 & = & \textit{\# row-ends of } \boldsymbol{P}_n^* \textit{ strictly right of } k \textit{ containing any of } j^*, \ldots, (n-1)^*, \bullet.
\end{array}
$$

Proof. Because $1, \ldots, k$ are in the first column of $\boldsymbol{P}_0$, the interval $[f_0, f_k]$ is elementary. To obtain the first product, choose $f_1, \ldots, f_k$ in the number of ways given by the dual to Theorem 4.69. To obtain the second, choose $f_{n-1}, f_{n-2}, \ldots, f_k$ in the number of ways given by Lemma 5.22. $\square$

5.5. Counting the number of flags achieving a full interval type table. The previous section provides necessary but insufficient conditions for partial interval type tables to be realizable, and provides means of counting the number of flags achieving certain partial interval type tables. A conjectured extension to full interval type tables is as follows.

Definition 5.24. Consider an interval type table $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$ of shape $\lambda \vdash n$. If there is q_0 depending on Λ such that for all $q \ge q_0$, the number of flags realizing Λ in q-regular semi-primary lattices is a only function of q and not of the particular lattice, let $f_\Lambda(q)$ be this number of flags. If there is no such q_0, let $q_0 = \infty$.

Conjecture 5.25. *The functions $f_\Lambda(q)$ have the following properties.*

1. *$f_\Lambda(q) \in \mathbb{Z}[q]$.*
2. *As a polynomial, $f_\Lambda(q)$ is divisible by $M_\lambda(q)$, and when the polynomial is nonzero, the degree is in the range $n(\lambda) - n_0(\lambda) \le \deg(f_\Lambda) \le n(\lambda)$.*
3. *The maximum degree $n(\lambda)$ is attained exactly in the circumstance that Λ is the evacuation table of a standard tableau, that is, for some standard tableau $\boldsymbol{P}$, we have $\boldsymbol{P}_i = \Delta^i \boldsymbol{P}$ for $i = 0, \ldots, n$.*
4. *When $f_\Lambda(t)$ is identically 0, no semi-primary lattice with at least $q_0 + 1$ atoms in each interval of type $\substack{\square\\\square}$ realizes Λ, including irregular semi-primary lattices.*

Partial proof. When (1) holds for some Λ, (2) and (3) do as well:

(2) By Theorem 4.42, the orbit of a flag under the automorphism group of certain q-regular semi-primary lattices has cardinality divisible by $M_\lambda(q)$. The interval type table of a flag is invariant under lattice automorphisms, so the set of flags with itype $f = \Lambda$ is a union of orbits of flags. Thus, $f_\Lambda(q)$ is divisible by $M_\lambda(q)$. The degree of $M_\lambda(q)$ is $n(\lambda) - n_0(\lambda)$, and the maximum degree of $f_\Lambda(q)$ is the degree of the number of flags of type $\boldsymbol{P}_0$, namely $n(\lambda)$.

(3) By Theorem 5.14, all but a fraction $O(q^{-1})$ of flags with ftype $f = \boldsymbol{P}$ have interval type table $\boldsymbol{P}_i = \Delta^i \boldsymbol{P}$, so given $\boldsymbol{P}_0 = \boldsymbol{P}$, only this interval type table achieves the same degree $n(\lambda)$ as $F^{\boldsymbol{P}}(q)$. $\square$

The polynomials for all interval type tables with $n \le 8$, and for some larger cases, were computed using Mathematica by the method to be described in this section. In the computed cases, all parts of the conjecture hold, and $q_0 = 1$. A summary of the results is in Appendix A. In most of the computed cases, the polynomial is monic and is of the form, $M_\lambda(q)$ times a power of q times products of falling factorials

$(q-1)\cdots(q-m)$. However, for lattices of type $(3,2,2,1)$, there is one polynomial with leading coefficient 2, and for lattices of certain other types, there are sometimes a few polynomials that do not factor in this fashion. There is a refinement of the decomposition of flags into interval type tables that decomposes these classes even further, and it is possible that these finer classes have polynomials that are of this form; see Section 8.5 for a description.

The method is as follows. Fix a partition $\lambda \vdash n$. We will consider all the possible interval type tables and the number of flags realizing them in semi-primary lattices of type λ. Let L be a superset of the realizable type tables of shape λ. For the computations summarized in the appendix, L was obtained by taking all interval type tables of shape λ that satisfy Theorems 5.15 and 5.17, and whose transpose Λ' also satisfies these. These conditions are necessary but not sufficient for a type table to be realizable; thus, all realizable type tables are in L, but there may be nonrealizable ones as well. A smaller collection of possibly realizable interval type tables could be obtained by finding stronger necessary conditions for a type table to be realizable, but in the cases computed, the further steps in this method weeded out all the nonrealizable ones.

For each interval type table Λ in L, introduce a variable N_Λ, which will represent the number of flags achieving Λ in a semi-primary lattice of type λ. We develop equations in the N_Λ's from the assumption that we are working in a q-regular semi-primary lattice. Some of the equations will fail to hold for small q, but they will all hold for all q-regular semi-primary lattices with q sufficiently large. Hence, if it happens that there are enough equations to solve for some of the unknowns N_Λ, we will have a count of how many flags achieve those Λ depending only on q and Λ, and not on any further structure of the semi-primary lattice. This N_Λ is the $f_\Lambda(q)$ described in the conjecture. If Λ is not realizable, then $N_\Lambda = 0$. Beyond the possible constraint on small q, the equations in N_Λ's to be developed are true for all q-regular semi-primary lattices, even if two different semi-primary lattices with the same q have different values of a particular N_Λ. It is tempting to introduce equations $N_\Lambda = N_{\Lambda'}$, but we can not, because a q-regular semi-primary lattice needn't be self-dual, and the proof in Proposition 5.21 that $f_\Lambda(q) = f_{\Lambda'}(q)$ has the caveat that this is true provided $f_\Lambda(q)$ exists, meaning it depends only on q, not the particular q-regular lattice. Since we do not know whether the polynomials exist, we cannot apply this.

Given a partial interval type table Γ of shape λ, if we can count how many saturated flags achieve Γ (for example, by applying the theorems of the previous section), we can obtain an equation in the N_Λ's by setting this equal to the sum of N_Λ over all extensions Λ of Γ in L. For example, given a tableau $\boldsymbol{P}$, the number of flags of type $\boldsymbol{P}$ is $F^{\boldsymbol{P}}(q)$, so

$$F^{\boldsymbol{P}}(q) = \sum_{\{\,\Lambda \in L\ :\ \boldsymbol{P}_0 = \boldsymbol{P}\,\}} N_\Lambda.$$

We now develop equations relating flags that differ in only one element. Choose k with $0 < k < n$. Let Γ be a partial interval type table in which $\lambda^{(ij)}$ is defined for all $0 \le i \le j \le n$ unless $i = k$ or $j = k$, and in which $\lambda^{(k-1,k+1)} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. It is sometimes possible to generate equations involving all extensions of Γ to full interval type tables.

Definition 5.26. Two saturated flags f and f' are **k-adjacent** if $f_i = f'_i$ whenever $i \neq k$. The relation of being k-adjacent is an equivalence relation on the set of all flags in a lattice. Each k-adjacency class of flags has a partial interval type table Γ associated to it, namely $\gamma^{(ij)} = \operatorname{type}[f_i, f_j]$ for $i \neq k$ and $j \neq k$. Two interval type tables $(\lambda^{(ij)})$ and $(\gamma^{(ij)})$ in L are **k-adjacent** if $\lambda^{(ij)} = \gamma^{(ij)}$ whenever $i \neq k$ and $j \neq k$, and each k-adjacency class of interval type tables has a partial interval type table Γ associated to it.

A **k-partial flag** is a flag $(f_0 \lessdot \cdots \lessdot f_{k-1} < f_{k+1} \lessdot \cdots \lessdot f_n)$ where $\rho(f_i) = i$ for $i \neq k$. A **k-partial interval type table** is a partial interval type table $\Gamma = (\gamma^{(ij)})$ defined for all $0 \leq i \leq j \leq n$ with $i \neq k$ and $j \neq k$. Let f be a k-partial flag with $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Let Γ be the k-partial interval type table of f. Let $\Lambda_1, \ldots, \Lambda_m$ be the k-adjacency class of Γ in L. For some (but not all) Γ, the number of extensions of f to a saturated flag achieving each of $\Lambda_1, \ldots, \Lambda_m$ is the same for all f realizing Γ. If for all f realizing Γ, exactly n_t extensions of f have interval type table Λ_t for $t = 1, \ldots, m$, we have the equation $N_{\Lambda_1}/n_1 = \cdots = N_{\Lambda_m}/n_m$.

Lemma 5.27. *Fix k. Let f be a k-partial flag with* $\operatorname{type}[f_{k-1}, f_{k+1}] = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. *Let $0 \leq i < k$ and set $\alpha = \operatorname{type}[f_i, f_{k-1}]$ and $\gamma = \operatorname{type}[f_i, f_{k+1}]$.*

1. *γ/α is not a horizontal brick. It consists of two squares, x and y, with $\operatorname{row}(x) < \operatorname{row}(y)$ and $\operatorname{col}(x) \geq \operatorname{col}(y)$.*
2. *Choose $f_k \in (f_{k-1}, f_{k+1})$ and let $\beta = \operatorname{type}[f_i, f_k]$. If x and y are vertically adjacent then $\beta \gtrdot_x \alpha$. Otherwise, all but one $f_k \in (f_{k-1}, f_{k+1})$ yield $\beta \gtrdot_x \alpha$, and the unique f_k with $\beta \gtrdot_y \alpha$ is $f_k = f_{k-1} \vee A^{\operatorname{col}(y)}_{f_{k+1}} f_i$.*

Now let $k < i \leq n$. Similar statements hold with $\alpha = \operatorname{type}[f_{k+1}, f_i]$, $\beta = \operatorname{type}[f_k, f_i]$ and $\gamma = \operatorname{type}[f_{k-1}, f_i]$, and the unique f_k with $\beta \gtrdot_y \alpha$ is $f_k = f_{k+1} \wedge C^{\operatorname{col}(y)}_{f_{k-1}} f_i$.

Proof. First we consider $0 \leq i < k$. If we extend f to a saturated flag by choosing some $f_k \in (f_{k-1}, f_{k+1})$, what can $\beta = \operatorname{type}[f_i, f_k]$ be? Let the two squares of γ/α be denoted x and y. Either $\beta \gtrdot_x \alpha$, or $\beta \gtrdot_y \alpha$. By Corollary 4.72, if γ/α were a horizontal brick, we would have $\operatorname{type}[f_{k-1}, f_{k+1}] = \square\square$, so it is not a horizontal brick. So assume $\operatorname{row}(x) < \operatorname{row}(y)$ (and hence $\operatorname{col}(x) \geq \operatorname{col}(y)$). If γ/α is a vertical brick, then all saturated extensions of f by a choice of f_k have $\beta \gtrdot_x \alpha$. If γ/α is not a brick, exactly one choice of f_k has $\beta \gtrdot_y \alpha$, namely $f_k = f_{k-1} \vee A^{\operatorname{col}(y)}_{f_{k+1}} f_i$. For suppose $\operatorname{type} f_k \gtrdot_y \alpha$. Let $z = A^{\operatorname{col}(y)}_{f_{k+1}} f_i$. Since $\operatorname{type} f_k$ and $\operatorname{type} f_{k+1}$ agree in the first $\operatorname{col}(y)$ columns, $f_k \geq z$; and since $\operatorname{type}[f_i, f_{k-1}]$ and $\operatorname{type}[f_i, f_{k+1}]$ do not agree in these columns, $f_{k-1} \not\geq z$. Since $f_k \geq f_{k-1}$, we get $f_k \geq f_{k-1} \vee z$. Since $[f_{k-1}, f_{k+1}]$ is elementary, we may, by Theorem 4.62 or Corollary 3.10, compute

$$\operatorname{type}[f_i, f_{k-1} \vee A^{\operatorname{col}(y)}_{f_{k+1}} f_i] = \operatorname{type}[f_i, f_{k-1}] \vee \mathcal{A}^{\operatorname{col}(y)}_{\gamma} \emptyset = \alpha \oplus y.$$

Next, let $k < i \leq n$ and $\alpha = \operatorname{type}[f_{k+1}, f_i]$ and $\gamma = \operatorname{type}[f_{k-1}, f_i]$. If we extend f to a saturated flag by choosing some $f_k \in (f_{k-1}, f_{k-1})$, what can $\beta = \operatorname{type}[f_i, f_k]$ be? Again, γ/α consists of two squares x and y with $\operatorname{row}(x) < \operatorname{row}(y)$. If $\operatorname{col}(x) = \operatorname{col}(y)$ then all such extensions have $\beta \gtrdot_x \alpha$, while if $\operatorname{col}(y) < \operatorname{col}(x)$, there is exactly one extension with $\beta \gtrdot_y \alpha$, namely $f_k = f_{k+1} \wedge C^{\operatorname{col}(y)}_{f_{k-1}} f_i$, and all other extensions have $\beta \gtrdot_x \alpha$. $\square$

Definition 5.28. Let Λ be a interval type table, and fix $k = 1, \ldots, n-1$ and $i \neq k$. When $0 \leq i < k$, say i is **specializable** when k and $k+1$ are not in adjacent cells in $\boldsymbol{P}_i$; that Λ is i**-special** when $\operatorname{col}(\boldsymbol{P}_i, k) < \operatorname{col}(\boldsymbol{P}_i, k+1)$; and that Λ is i**-generic** when $\operatorname{col}(\boldsymbol{P}_i, k) \geq \operatorname{col}(\boldsymbol{P}_i, k+1)$. Dually, when $n \geq i > k$, say i is specializable when $n-k$ and $n-k+1$ are not in adjacent cells in $\boldsymbol{P}_i^*$; that Λ is i-special when $\operatorname{col}(\boldsymbol{P}_i^*, n-k) < \operatorname{col}(\boldsymbol{P}_i^*, n-k+1)$; and that Λ is i-generic when $\operatorname{col}(\boldsymbol{P}_i^*, n-k) \geq \operatorname{col}(\boldsymbol{P}_i^*, n-k+1)$. In the notation of the preceding lemma, i is specializable when γ/α is not a brick, while Λ is i-special when $\beta \gtrdot_y \alpha$ and is i-generic when $\beta \gtrdot_x \alpha$.

A k-partial interval type table has at most one extension that is i-generic for all i; such an extension is called **generic** and is denoted Λ_0.

See Figure 5 for an example.

We consider several possibilities about the distribution of i-special and i-generic extensions of Γ. Throughout, Γ and k are fixed, and $\gamma^{(k-1,k+1)} = \boxminus$. In terms of the tableau representation of an interval type table, $\boldsymbol{P}_{k-1}$ has $\begin{array}{|c|c|}\hline k & k+1 \\ \hline\end{array}$ in its upper left corner iff $\gamma^{(k-1,k+1)} = \boxbar$, while it has $\begin{array}{|c|}\hline k \\ \hline k+1 \\ \hline\end{array}$ in its upper left corner iff $\gamma^{(k-1,k+1)} = \boxminus$.

Proposition 5.29. *Let $i \neq k$, and L be a list of interval type tables of shape λ that includes all realizable ones, and possibly some unrealizable ones. Consider the k-adjacency class of Γ in L. The following hold for all semi-primary lattices (including irregular ones).*

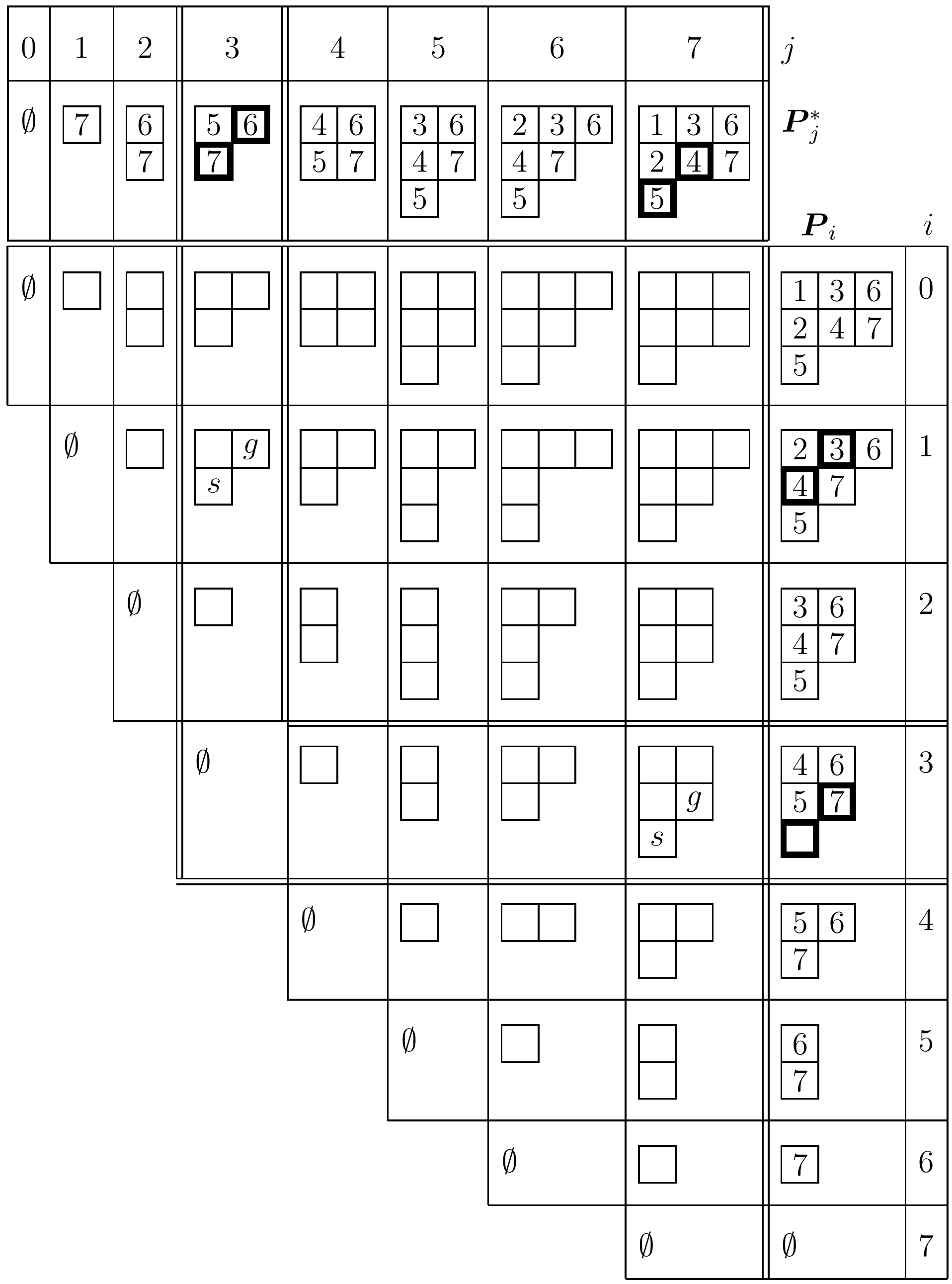
1. *If i is specializable but there are no i-special extensions of Γ in L, then all extensions of Γ in L are nonrealizable, and so should be removed from L.*
2. *If no extensions of Γ are i-generic, then again, no extensions of Γ in L are realizable, and so all such extensions should be removed.*

Proof.

1. If Γ is realized by flag f, then by Lemma 5.27, f_k can be replaced so as to achieve an i-special extension of Γ. Since L is a superset of all realizable interval type tables, $\operatorname{itype} f$ is in L. Thus, if L has no i-special extensions of Γ, no flag realizes Γ, so no flag realizes any extension of Γ, and all extensions of Γ can be removed from L.
2. If Γ is realized by a flag f, the interval $[f_{k-1}, f_{k+1}]$ has type $\boxminus$, so it has more than one element on the middle level, but at most one element yields an i-special interval type table, so all the others must yield an i-generic interval type table. $\square$

Let Λ_0 be the unique extension of Γ that is i-generic for all $i \neq k$. Given any k-partial flag f realizing Γ, at most $n-2$ choices of $f_k \in (f_{k-1}, f_{k+1})$ are i-special for some i's, and the rest are i-generic for all i; thus, all but at most $n-2$ elements in (f_{k-1}, f_{k+1}) achieve Λ_0. The bound $n-2$ can be lowered on a case by case basis, because sometimes, the special elements for different i coincide. We now show that if any extensions of Γ are realizable, then Λ_0 is realizable, except in pathological cases.

Pathological case. Suppose the k-adjacency class of Γ does not contain Λ_0, but does contain an i-special extension of Γ for each specializable i, and an i-generic extension of Γ for every $i \neq k$. Then it is conceivable that some q-regular lattice



Both 1 and 7 are specializable. Choose the square marked g for a generic partition, or the one marked s for a special partition. For the tableaux, the generic value is shown, and the special value is obtained by swapping the contents of the highlighted cells. When $i < k$ is specializable, there are two possibilities for $\boldsymbol{P}_i$; if there are c specializable values of $i < k$, there are 2^c possibilities for $\boldsymbol{P}_k$; and in all other cases, there is only one value of $\boldsymbol{P}_i$. Dual statements hold for $\boldsymbol{P}^*_j$. It turns out that all four combinations of special and generic choices in this example are realizable.

FIGURE 5. An interval type table adjacency class with $k = 3$.

with $q \leq n-2$ exists in which Γ is realizable. It is also conceivable that an irregular lattice exists in which some interval $[f_{k-1}, f_{k+1}]$ of type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ happens to have fewer than n atoms, and that Γ is realized with this f_{k-1} and f_{k+1}.

Usual case. For all $q > n-2$, if Γ is realizable, it is necessary that Λ_0 be realizable, so if $\Lambda_0 \notin L$, the class is not realizable. So assume Λ_0 is realizable. Then for each specializable i, there is a unique i-special extension of each k-partial flag realizing Γ. However, the i-special extensions of two different flags realizing Γ do not have to have the same interval type tables, as in Example 5.36 at the end of this section. If the i-special extensions of all flags realizing Γ happen to have the same interval type table, we can generate an equation in the N_Λ's.

Theorem 5.30. *Suppose the k-adjacency class of Γ in L has Λ_0, and has exactly one i-special member for each $i \neq k$ (these can coincide for different i). Let the class be $\Lambda_0, \ldots, \Lambda_m$. If $\mathcal{L}$ is q-regular with $q \geq m$, then*

$$\frac{N_{\Lambda_0}}{q+1-m} = N_{\Lambda_1} = \cdots = N_{\Lambda_m};$$

if $q = m-1$ then $N_{\Lambda_0} = 0$ while $N_{\Lambda_1} = \cdots = N_{\Lambda_m}$, and if $q < m-1$, then $N_{\Lambda_0} = \cdots = N_{\Lambda_m} = 0$.

Proof. Each of $\Lambda_1, \ldots, \Lambda_m$ may be i-special for multiple values of i, but by assumption, no two of them are special for the same value of i. Then every k-partial flag f realizing Γ has one extension realizing each of $\Lambda_1, \ldots, \Lambda_m$, and the remaining extensions realize Λ_0. If the lattice is q-regular, then $q+1-m$ extensions realize Λ_0. The equation for $q \geq m$ and the $q = m-1$ case both follow.

If $q+1 < m$, we see there are not enough elements in (f_{k-1}, f_{k+1}) to realize each of $\Lambda_1, \ldots, \Lambda_m$, but if any extension of Γ can be realized, all of these can. Thus, none of them are realizable, so $N_{\Lambda_0} = \cdots = N_{\Lambda_m} = 0$. $\square$

Note.

(1) This theorem is true even when none of $\Lambda_0, \ldots, \Lambda_m$ are realizable; if they are not realizable, $N_{\Lambda_0} = \cdots = N_{\Lambda_m} = 0$, which is consistent with all the cases in the theorem.

(2) It is possible that we may usually not need to consider $q < m$ as a special case; in the known cases, summarized in Appendix A, the N_Λ's are polynomials in q, and the polynomials in this theorem have N_{Λ_0} divisible by $(q-1)\cdots(q-m+1)$ and $N_{\Lambda_1}, \ldots, N_{\Lambda_m}$ divisible by $(q-1)\cdots(q-m+2)$, so they vanish when required without introducing $q < m$ as a special case in the equations.

We will work out some examples, according to the following procedure.

Procedure 5.31. Fix a partition λ.

(1) Create a superset L of all realizable interval type tables of shape λ. This can be done by taking all interval type tables of shape λ which satisfy Theorems 5.15 and 5.17, and whose transpose also satisfies these.

(2) For each $k = 1, \ldots, n-1$, consider the interval type tables from step (1) that satisfy $\lambda^{(k-1,k+1)} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$ (or equivalently, the interval type tables for which $\boldsymbol{P}_{k-1}$ has $\begin{array}{|c|}\hline k\\\hline k+1\\\hline\end{array}$ in the upper left corner). Split these into k-adjacency

classes: two of these interval type tables are k-adjacent provided their respective $\boldsymbol{P}_0, \dots, \boldsymbol{P}_{k-1}$ agree in all cells except possibly have some k and $(k+1)$'s reversed, and their respective $\boldsymbol{P}_{k+1}, \dots, \boldsymbol{P}_n$ are equal. They may differ on $\boldsymbol{P}_k$.

(3) For each k-adjacency class formed in step (2), consider
   (a) Does the class contain an i-generic and at least one i-special interval type table for each specializable i? If not, then by Proposition 5.29, none of the interval type tables in this class ever occur, so they should be removed from L.
   (b) Does the class contain a generic interval type table? If not, the interval type tables in it never occur for sufficiently large q, but may occur for small q in certain pathological cases.
   (c) Does the class contain exactly one i-special interval type table for each specializable i? If so, denote the generic interval type table by Λ_0 and the others by $\Lambda_1, \dots, \Lambda_r$. We then obtain $N_0/(q+1-r) = N_1 = \dots = N_r$ by Theorem 5.30.
   (d) When there is a generic interval type table; at least one i-special table for each specializable i; and more than one for some i, we do nothing with the class.

(4) Combine the equations in step (3c) from different adjacency classes to express as many different N_Λ's as possible as a rational function of q times a particular N_Λ.

(5) When $\boldsymbol{P}$ is the superstandard tableau of shape λ, all flags of type $\boldsymbol{P}$ have a hereditary decomposition by Theorem 4.70. Thus the interval type tables of all such flags are the same, say $\tilde{\Lambda}$. We thus have $N_{\tilde{\Lambda}} = F^{\boldsymbol{P}}(q)$, and now all other N_Λ's that can be expressed as multiples of $N_{\tilde{\Lambda}}$ can be computed as well.

(6) For most λ listed in Appendix A, all N_Λ are computed by the previous step. In most of the computed cases, even when a Λ is in a class of form (3d) for some k, it usually is also in a class of form (3c) for other k's, so there are equations in it. However, there are cases when Λ only occurs in adjacency classes of form (3d), and cases when the equations generated by (3c) do not express all N_Λ as multiples of $N_{\tilde{\Lambda}}$. Additional equations can sometimes be generated to deal with this; see Example 5.36.

Theorem 5.32. *In a semi-primary lattice of type* (n)*, there is only one flag, and it has interval type table* $\lambda^{(ij)} = (j-i)$*. In a semi-primary lattice of type* (1^n)*, there are* $\langle n\rangle!$ *flags, all with interval type table* $\lambda^{(ij)} = (j-i)$*.*

Proof. For either $\lambda = (n)$ or $\lambda = (1^n)$, there is only one subpartition of λ of each rank, so the type tables are as indicated. For $\lambda = (n)$, the only flag is $f_i = \hat{1}[i]$ (for $0 \le i \le n$). For $\lambda = (1^n)$, we set $f_n = \hat{1}$, and choose f_{i-1} to be a coatom of $[\hat{0}, f_i]$ in one of $\langle i\rangle$ ways, for $i = n, n-1, \dots, 1$. $\square$

Example 5.33. Let $\lambda = (2,1)$. The interval type tables satisfying $\boldsymbol{P}_{i+1} = \widetilde{\Delta}\boldsymbol{P}_i$ are as follows.

$\Lambda_1 = \Lambda_2'$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 3 \\ \hline\end{array}$	$\emptyset$
$\Lambda_2 = \Lambda_1'$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 3 \\ \hline\end{array}$	$\emptyset$
$\Lambda_3 = \Lambda_3'$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 3 \\ \hline\end{array}$	$\emptyset$

Let $N_i = N_{\Lambda_i}$ denote the number of flags with interval type table Λ_i. For each $k = 1, \ldots, n-1$, we partition the interval type tables with $\lambda^{(k-1,k+1)} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ into k-adjacency classes, and apply Theorem 5.30 when possible to compute the ratios of the N_i's within each class.

First consider $k = 1$. The interval type tables with $\boldsymbol{P}_0$ having $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ in the upper left corner are Λ_2 and Λ_3. These are k-adjacent to each other because they only differ in $\boldsymbol{P}_k$. The partition $\lambda^{(13)}$ is $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ in Λ_2 and $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$ in Λ_3, so Λ_2 is generic while Λ_3 is 3-special. Thus the ratio $N_2 : N_3$ is $(q+1) - 1 : 1 = q : 1$.

Next consider $k = 2$. The interval type tables with $\boldsymbol{P}_1$ having $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$ in the upper left corner are Λ_1 and Λ_3. These differ only in the positions of 2's and 3's, so they are 2-adjacent. Note that Λ_1 is generic while Λ_3 is 0-special, so the ratio $N_1 : N_3$ is $(q+1) - 1 : 1 = q : 1$.

Thus, the ratio $N_1 : N_2 : N_3$ is $q : q : 1$. All flags of type $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ have type table Λ_1, and by Theorem 4.38, the number of flags of this type is

$$\Big(\langle 2\rangle - \langle 1\rangle\Big) \cdot \Big(\langle 1\rangle - \langle 0\rangle\Big) \cdot \Big(\langle 1\rangle - \langle 0\rangle\Big) = q.$$

Thus $N_1 = N_2 = q$ and $N_3 = 1$.

Example 5.34. Let $\lambda = (2,2)$. The interval type tables satisfying $\boldsymbol{P}_{i+1} = \widetilde{\Delta}\boldsymbol{P}_i$ are as follows. Again, let N_i be the number of flags with type table Λ_i.

$\Lambda_1 = \Lambda_1'$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 4 \\ \hline\end{array}$	$\emptyset$
$\Lambda_2 = \Lambda_4'$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 3 \\ \hline 4 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 4 \\ \hline\end{array}$	$\emptyset$
$\Lambda_3 = \Lambda_3'$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 3 \\ \hline 4 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 4 \\ \hline\end{array}$	$\emptyset$
$\Lambda_4 = \Lambda_2'$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 4 \\ \hline\end{array}$	$\emptyset$
$\Lambda_5 = \Lambda_5'$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 3 \\ \hline 4 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 4 \\ \hline\end{array}$	$\emptyset$

If

$$\boldsymbol{P}_0 = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 3&4\\\hline\end{array}$$

then $\boldsymbol{P}_2 = \begin{array}{|c|c|}\hline 3&4\\\hline\end{array}$ by Theorem 4.93, so Λ_2 is not realizable. Also, if

$$\boldsymbol{P}_0 = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$$

then $\boldsymbol{P}_2 = \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$, so Λ_4 is not realizable. Thus $N_2 = N_4 = 0$.

Next, we partition the remaining interval type tables into k-adjacency classes, and apply Theorem 5.30. For $k = 1$ we have $N_3 : N_5 = q : 1$; for $k = 2$ we have $N_1 : N_5 = q : 1$; for $k = 3$ we have $N_3 : N_5 = q : 1$. So $N_1 : N_3 : N_5 = q : q : 1$. Also, N_1 is the total number of flags of type $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$, which is

$$\Big(\langle 2\rangle - \langle 0\rangle\Big) \cdot \Big(\langle 2\rangle - \langle 1\rangle\Big) \cdot \Big(\langle 1\rangle - \langle 0\rangle\Big) \cdot \Big(\langle 1\rangle - \langle 0\rangle\Big) = (q+1)q,$$

so $N_1 = N_3 = q(q+1)$ and $N_5 = q+1$.

Example 5.35. Let $\lambda = (2,2,1)$. We list all interval type tables satisfying the various theorems listed in step (1) of our procedure, but omit the constant values $\boldsymbol{P}_4 = \begin{array}{|c|}\hline 5\\\hline\end{array}$ and $\boldsymbol{P}_5 = \emptyset$ from each.

$\Lambda_1{=}\Lambda'_6$	$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_2{=}\Lambda'_2$	$\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 4&5\\\hline\end{array}$
$\Lambda_3{=}\Lambda'_8$	$\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_4{=}\Lambda'_{11}$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_5{=}\Lambda'_{12}$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_6{=}\Lambda'_1$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 4&5\\\hline\end{array}$
$\Lambda_7{=}\Lambda'_{13}$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_8{=}\Lambda'_3$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 4&5\\\hline\end{array}$
$\Lambda_9{=}\Lambda'_9$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{10}{=}\Lambda'_{15}$	$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{11}{=}\Lambda'_4$	$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array}$	$\begin{array}{|c|c|}\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{12}{=}\Lambda'_5$	$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{13}{=}\Lambda'_7$	$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&4\\\hline 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{14}{=}\Lambda'_{14}$	$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{15}{=}\Lambda'_{10}$	$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$
$\Lambda_{16}{=}\Lambda'_{16}$	$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$	$\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}$

Next, for each $k = 1, 2, 3, 4$, we form the k-adjacency classes. In the table below, Λ_i is abbreviated i. The column $\square\!\square$ lists those Λ_i for which $\lambda^{(k-1,k+1)} = \square\!\square$ (so these Λ_i are in singleton k-adjacency classes). The generic member of each class is listed first. The classes listed in "Bad classes" either don't have a generic member, or don't have precisely one i-special member for each specializable i. The classes listed in "Good classes" have a generic member, and for each specializable i, precisely one member is i-special. In general, for $i_1 \neq i_2$, we could have the i_1-special and i_2-special members being different, but this example is sufficiently small that the same nongeneric member is i-special for all specializable i.

k	$\square\!\square$	Bad classes	Good classes
1	$1, 2, 3$	(9)	$(4,5), (6,8), (7,10), (11,13), (12,15), (14,16)$
2	$4, 6, 7$	$(3,9,10)$	$(1,5), (2,8), (11,12), (13,14), (15,16)$
3	$1, 11, 13$	$(8,9,15)$	$(2,3), (4,5), (6,12), (7,14), (10,16)$
4	$2, 6, 8$	(9)	$(1,3), (4,7), (5,10), (11,12), (13,15), (14,16)$

First we explain the bad classes. For $k = 2$, both Λ_9 and Λ_{10} are 0-specializations of Λ_3. Since there is more than one 0-specialization, we cannot compute anything from this class. For $k = 3$, both Λ_9 and Λ_{15} are 5-specializations of Λ_8, so again, we cannot compute anything from this class.

For $k = 4$, the generic interval type table that corresponds to Λ_9 is obtained from Λ_9 by examining each $\boldsymbol{P}_i$ with $i < 4$ and swapping 4 and 5 if 4 is strictly below and left of 5. In general, we might also have to modify $\boldsymbol{P}_4$, but $\boldsymbol{P}_4 = \boxed{5}$ is forced in this case. So the generic interval type table corresponding to Λ_9 is

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\quad\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}\quad\begin{array}{|c|}\hline 5\\\hline\end{array}\quad\emptyset$$

which is not in the list of interval type tables that passed all the necessity tests of step (1) of the procedure; specifically, since $\boldsymbol{P}_0 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$, we must have $\boldsymbol{P}_2 = \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$ by Theorem 5.17. So the class with 9 does not have a generic interval type table, and hence we cannot use it to compute any ratios.

The transpose of Λ_9 is itself Λ_9, and the trouble with $k = 1$ is the transpose of the $k = 4$ problem.

We can actually show that Λ_9 is never realized in any semi-primary lattice, including irregular ones. The 4-partial interval type table corresponding to Λ_9 is

$$\Gamma \;=\; \begin{array}{|c|c|}\hline 1&3\\\hline 2&?\\\hline ?\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&?\\\hline 3\\\cline{1-1} ?\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 3&?\\\hline ?\\\cline{1-1}\end{array}\quad\begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array}\quad\begin{array}{|c|}\hline 5\\\hline\end{array}\quad\emptyset$$

The only interval type table in the list that extends Γ is Λ_9. Thus, there is no 0, 1, or 2-generic extension of Γ, so by Proposition 5.29, no extension of Γ is realizable in any semi-primary lattice. Thus, $N_9 = 0$, and Λ_9 should be removed from the list. Once it is removed, $(3, 10)$ becomes a good class for $k = 2$ and $(8, 15)$ becomes a good class for $k = 3$, but we will see that we have enough equations to solve for all the N_i without these two good classes.

The good classes all have the form (x, y) where x is a generic interval type table and y is a special one. Since there is just one special interval type table in each class, we obtain $N_x = q \cdot N_y$. By looking at the last class for each k, we see that 16 happens always to be special, so we can relate the ratios with N_{16} for the different k's, obtaining $N_{10} = N_{14} = N_{15} = q \cdot N_{16}$. We next see that 10, 14, 15 occur as special elements in conjunction with generic elements 5, 7, 12, 13 for various k, so we get $N_5 = N_7 = N_{12} = N_{13} = q \cdot N_{10} = q^2 \cdot N_{16}$. Next, 5, 7, 12, 13, occur as special elements in conjunction with generic elements 1, 4, 6, 11, so $N_1 = N_4 = N_6 = N_{11} = q \cdot N_5 = q^3 \cdot N_{16}$. From $(1, 3)$, we obtain $N_3 = N_1/q = q^2 N_{16}$, and from $(6, 8)$ we obtain $N_8 = N_6/q = q^2 N_{16}$. From $(2, 3)$ we obtain $N_2 = q \cdot N_3 = q^3 N_{16}$, and now we have expressed all but N_9 as multiples of N_{16}.

Next we note that all flags of type $\boldsymbol{P} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ have interval type table Λ_1, since this tableau is superstandard. Thus, $N_1 = F^{\boldsymbol{P}}(q) = q^3(q+1)$, so $N_1 = N_2 = N_4 = N_6 = N_{11} = q^3(q+1)$; $N_3 = N_5 = N_7 = N_8 = N_{12} = N_{13} = q^2(q+1)$; $N_{10} = N_{14} = N_{15} = q(q+1)$; and $N_{16} = q+1$.

With larger λ, we will generally find that the good classes have different numbers of elements in them, so that the ratios are not always of the form $N_0 = qN_1$, but more generally of the form $N_0/(q+1-r) = N_1 = \cdots = N_r$. Also, in this example we were able to express all the N_i as multiples of a single N_i, because the graph determined by $\{x, y\}$ is an edge iff x and y are in a good class together, has a single component. In general, the graph may have multiple components, and within each component, we can express the ratios of the N_i's by Theorem 5.30, but then we need additional equations to completely solve the problem. For the lattice types computed in Appendix A, the graph had multiple components for types $(3, 3, 1)$, $(3, 3, 2)$, $(3, 3, 3)$, $(4, 2, 2)$, $(4, 3, 1)$, $(3, 2, 2, 1)$, $(3, 3, 1, 1)$, and $(4, 2, 1, 1)$; there was one large component with almost all of the type tables, and several small components with just a handful of type tables. The large component contained the type table for the superstandard tableau of shape λ, so most values of N_Λ could be computed as described above. We illustrate how to choose additional equations by considering $(3, 3, 1)$.

Example 5.36. Let $\lambda = (3, 3, 1)$ and perform Procedure 5.31 to determine all the interval type tables and the values of N_Λ. At the end, it turns out there is a single type table, Λ_0 below, which has neither been shown to be realizable or not realizable, and which always occurs in adjacency classes of the form (3d) of the procedure. The

$k = 3$ adjacency class of Λ_0 is as follows, and is also shown in Figure 5.

Λ_0	$\begin{array}{ccc}1&3&6\\2&4&7\\5\end{array}$	$\begin{array}{ccc}2&3&6\\4&7\\5\end{array}$	$\begin{array}{cc}3&6\\4&7\\5\end{array}$	$\begin{array}{cc}4&6\\5&7\end{array}$	$\begin{array}{cc}5&6\\7\end{array}$	$\begin{array}{c}6\\7\end{array}$	$\begin{array}{c}7\end{array}$	$\emptyset$	generic
Λ_1	$\begin{array}{ccc}1&3&6\\2&4&7\\5\end{array}$	$\begin{array}{ccc}2&4&6\\3&7\\5\end{array}$	$\begin{array}{cc}3&6\\4&7\\5\end{array}$	$\begin{array}{cc}4&6\\5&7\end{array}$	$\begin{array}{cc}5&6\\7\end{array}$	$\begin{array}{c}6\\7\end{array}$	$\begin{array}{c}7\end{array}$	$\emptyset$	1-special
Λ_2	$\begin{array}{ccc}1&3&6\\2&4&7\\5\end{array}$	$\begin{array}{ccc}2&3&6\\4&7\\5\end{array}$	$\begin{array}{cc}3&6\\4&7\\5\end{array}$	$\begin{array}{cc}4&6\\5\\7\end{array}$	$\begin{array}{cc}5&6\\7\end{array}$	$\begin{array}{c}6\\7\end{array}$	$\begin{array}{c}7\end{array}$	$\emptyset$	7-special
Λ_3	$\begin{array}{ccc}1&3&6\\2&4&7\\5\end{array}$	$\begin{array}{ccc}2&4&6\\3&7\\5\end{array}$	$\begin{array}{cc}3&6\\4&7\\5\end{array}$	$\begin{array}{cc}4&6\\5\\7\end{array}$	$\begin{array}{cc}5&6\\7\end{array}$	$\begin{array}{c}6\\7\end{array}$	$\begin{array}{c}7\end{array}$	$\emptyset$	1,7-special

The values of N_Λ are known from the procedure for all Λ of shape $(3, 3, 1)$ except Λ_0. Here are two different ways to find N_{Λ_0}.

(1) The number of flags of type $\boldsymbol{P} = \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4&7\\\hline 5\\\cline{1-1}\end{array}$ is $F^{\boldsymbol{P}}(q)$. In the equation

$$F^{\boldsymbol{P}}(q) = \sum_{\{\,\Lambda\in L\;:\;\boldsymbol{P}_0=\boldsymbol{P}\,\}} N_\Lambda,$$

the only unknown is N_{Λ_0}, so

$$N_{\Lambda_0} = F^{\boldsymbol{P}}(q) - \sum_{\{\,\Lambda\in L\;:\;\boldsymbol{P}_0=\boldsymbol{P}\text{ and }\Lambda\neq\Lambda_0\,\}} N_\Lambda.$$

This is similar to step (5) of the procedure.

(2) A 3-partial flag either has one extension to Λ_3 and q to Λ_0, or one to Λ_1, one to Λ_2, and $q - 1$ to Λ_0. Thus $N_{\Lambda_1} = N_{\Lambda_2}$ and

$$N_{\Lambda_0} = qN_{\Lambda_3} + (q - 1)N_{\Lambda_1}.$$

This is similar to step (3c) of the procedure.

For the other exceptional partitions listed before the example, there were sometimes several interval type tables not in the large component of the graph. In all such cases, the first method above was used to find some N_Λ within each component, and then the ratio of this to other N_Λ within the component was used to find the remaining ones.

Note. Hesselink [**10**] analyzes necessary and sufficient conditions for the realizability of "typricies," which are equivalent to interval type tables, but with a different formalism. He has several interesting pathological examples, on which some light may be shed by our enumerative approach.

(1) In [**10**, §5.7,§9.3], he gives typricies that are realized in all invariant subspace lattices over fields of characteristic greater than 2, but not in fields of characteristic 2. At least one of these is not a pathological case in our formalism, but rather, an interval type table whose polynomial $f_\Lambda(q)$ is divisible by $q-2$.

The polynomial for the example in [**10**, §5.7], where $\lambda = (4, 2)$, turns out to be $(q-1)(q-2)$. The example in [**10**, §9.3] has $n = 21$, which is much larger than the polynomials that were computed with Mathematica.

(2) Also in [**10**, §9.3], there is a typrix that is only realized in fields of characteristic 2. In our treatment, it is a pathological case, and may correspond to a polynomial $f_\Lambda(q) = 0$ when $q \geq q_0 = 3$, provided other q-regular semi-primary lattices with $q \geq 3$ also do not realize it. Again, $n = 21$, so it was too large to compute.

(3) Finally, in [**10**, §9.4], there is a typrix that is realizable in an invariant subspace lattice over the field $\mathbb{C}$ but not $\mathbb{R}$, and a similar construction for certain other pairs of fields in [**10**, §9.5–9.6]. These correspond to the "escape clause" $q_0 = \infty$ of Definition 5.24. In the graph relating the ratios of N_Λ's, described above Example 5.36, such an interval type table is not in the main component, or in any other component where the polynomials are known to exist.

5.6. Semi-primary flags indexed by set partitions.

Definition 5.37. A **partition** π of a set S is a set $\{B_1, \ldots, B_k\}$ of nonempty subsets of S, with $B_1 \cup \cdots \cup B_k = S$ and $B_i \cap B_j = \emptyset$ when $i \neq j$. An **ordered partition** π of S is a tuple $(B_1, \ldots, B_k)$ of subsets (which may be empty) of S whose union is S and whose pairwise intersections are $\emptyset$. In both cases, each B_i is called a **block** of π.

Partitions have been well studied; see, for example, [**27**].

Definition 5.38. The **type** of a set partition $\pi = \{B_1, \ldots, B_k\}$ is the integer partition

$$(\#B_1, \ldots, \#B_k),$$

where we arrange the blocks so these numbers are in weakly decreasing order. For an n-element set, the number of partitions of type λ is

$$c_\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots m_1!m_2!\cdots}.$$

The total number of partitions of an n-element set is the n**th Bell number** $B(n)$.

The **type** of an ordered partition $\pi = (B_1, \ldots, B_k)$ is the weak composition

$$(\#B_1, \ldots, \#B_k).$$

For a weak composition $\lambda \vdash n$, the number of ordered partitions of type λ of an n-set is the multinomial coefficient

$$\binom{n}{\lambda_1, \lambda_2, \ldots} = \frac{n!}{\lambda_1!\lambda_2!\cdots}.$$

Definition 5.39. A **standard (skew) composition tableau** is a row strict tableau $\boldsymbol{P}$ of (skew) weak composition shape, with distinct entries. Almost equivalently, it is a multichain $(\lambda^{(l)} \leq \cdots \leq \lambda^{(h)})$ of weak compositions. When entry j is in row i of $\boldsymbol{P}$, the composition $\lambda^{(j)}$ is obtained from $\lambda^{(j-1)}$ by adding a square to row i, and when j does not appear, $\lambda^{(j)} = \lambda^{(j-1)}$. The shape of $\boldsymbol{P}$ is $\lambda^{(h)}/\lambda^{(l)}$, which is skew when $\lambda^{(l)} \neq \emptyset$. The parameters $l = \text{low}(\boldsymbol{P})$ and $h = \text{high}(\boldsymbol{P})$ are a minor additional structure we require that is not ordinarily present in a tableau.

We encode the ordered partition $(B_1, \ldots, B_k)$ by a standard composition tableau $\boldsymbol{P}$ of shape λ whose ith row has the entries of B_i written in increasing order from left to right.

Notation. For a composition λ, the partition obtained by sorting the parts into decreasing order is called the **straightening** of the composition, and is denoted $S(\lambda)$. For compositions λ and μ, write $\lambda \equiv \mu$ to mean $S(\lambda) = S(\mu)$. The straighening of a skew composition tableau $\boldsymbol{P} = (\lambda^{(l)} \lessdot \cdots \lessdot \lambda^{(h)})$ is the skew Young tableau $S(\boldsymbol{P}) = (S(\lambda^{(l)}), \ldots, S(\lambda^{(h)}))$. For composition tableaux, write $\boldsymbol{P} \equiv \boldsymbol{Q}$ to mean $S(\boldsymbol{P}) = S(\boldsymbol{Q})$.

Definition 5.40. Let $\boldsymbol{P} = (\lambda^{(l)} \lessdot \cdots \lessdot \lambda^{(h)})$ be any standard composition tableau, with $l = \operatorname{low}(\boldsymbol{P})$ and $h = \operatorname{high}(\boldsymbol{P})$. The **interval type table** of $\boldsymbol{P}$ is

$$\operatorname{itype} \boldsymbol{P} = \left(S(\lambda^{(j)} - \lambda^{(i)})\right)_{l \leq i \leq j \leq h}.$$

Two standard composition tableau have the same interval type table iff they encode the same unordered set partition, that is, if we permute the order of the rows of one to obtain the other. Thus, the numbers of these interval type tables are enumerated by c_λ and $B(n)$.

These interval type tables were introduced by Hesselink [**10**, p. 103], under the name "elementary typrices," and with a different formalism. He showed that all such interval type tables are realized in all invariant subspace lattices of shape $S(\operatorname{sh} \boldsymbol{P})$. We will enumerate how many flags realize each one in q-regular semi-primary lattices.

Definition 5.41. Let $\mathcal{L}$ be a semi-primary lattice, $\vec{x}$ be a join decomposition of $\hat{1}$, and $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ be a skew composition tableau, with $\operatorname{osh} \boldsymbol{P} = (\rho(x_1), \rho(x_2), \ldots) \equiv \operatorname{type} \mathcal{L}$. The flag $f(\boldsymbol{P}, \vec{x})$ with components $f_i = \vec{x}[[\lambda^{(i)}]]$ for $i = l, \ldots, h$ is a **hereditary flag**.

These flags are interesting because all (multisaturated) flags in a lattice that is a product of chains are hereditary flags; all flags in a semi-primary lattice whose type has two columns are hereditary flags; and the computed data in Appendix A shows empirically that at least for small n, the interval type tables of hereditary flags form a large class of the realized interval type tables.

Proposition 5.42. *Let $\vec{x}$ be a decomposition of $\hat{1}$ into independent join-irreducibles and $\boldsymbol{P}$ be a standard composition tableau, with $\operatorname{sh} \boldsymbol{P} = (\rho(x_1), \rho(x_2), \ldots)$. Then $\operatorname{itype} \boldsymbol{P} = \operatorname{itype} f(\boldsymbol{P}, \vec{x})$.*

Proof. This follows from Corollary 4.53. $\square$

Definition 5.43. A **special composition tableau** is a standard skew composition tableau of shape λ/μ with $\mu = \lambda[k]$ for some k; in other words, no entries occur in columns $1, \ldots, k$, and no $\bullet$'s appear in columns $k+1, k+2, \ldots$.

Definition 5.44. Let $f = (f_l, \ldots, f_h)$ be a flag. Recall that $\partial f = (f_{l+1}, \ldots, f_h)$ and $df = (f_l, \ldots, f_{h-1})$. Define $Af = (A_{f_h} f_l, \ldots, A_{f_h} f_h)$. For an integer $k = l, \ldots, h$, define $A_k f = (A_{f_k} f_l, \ldots, A_{f_k} f_k) = Ad^{h-k} f$.

Definition 5.45. For a skew composition tableau $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$, define $d\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h-1)})$, $\mathcal{C}\boldsymbol{P} = (\mathcal{C}\lambda^{(l)}, \ldots, \mathcal{C}\lambda^{(h)})$ and $\mathcal{A}\boldsymbol{P} = (\mathcal{A}_\lambda\lambda^{(l)}, \ldots, \mathcal{A}_\lambda\lambda^{(h)})$, where $\lambda = \lambda^{(h)}$. For $k = l, \ldots, h$, define $\mathcal{A}_k\boldsymbol{P} = \mathcal{A}d^{h-k}\boldsymbol{P} = (\mathcal{A}_\lambda\lambda^{(l)}, \ldots, \mathcal{A}_\lambda\lambda^{(k)})$ where $\lambda = \lambda^{(k)}$. The tableau $d\boldsymbol{P}$ deletes all h's from $\boldsymbol{P}$. The tableau $\mathcal{C}\boldsymbol{P}$ is $\boldsymbol{P}$ with the first column deleted. The tableau $\mathcal{A}\boldsymbol{P}$ is obtained from $\boldsymbol{P}$ by performing the following operation on each nonempty row: if there are no numeric entries, the row stays intact, and if there are, insert $\bullet$ at the left, slide the row one cell right, and delete the rightmost entry. For $\mathcal{A}_k\boldsymbol{P}$, first delete all cells whose entries are larger than k, and evaluate $\mathcal{A}$ on the tableau so obtained. Note that if $\boldsymbol{P}$ is special, then so are $\mathcal{C}\boldsymbol{P}$, $\mathcal{A}\boldsymbol{P}$, and $\mathcal{A}_k\boldsymbol{P}$.

Example 5.46.

$$\boldsymbol{P} = \begin{array}{llll} 2 & 5 & 6 & \\ 1 & 4 & 7 & 8 \\ 3 & & & \end{array} \qquad \mathcal{C}\boldsymbol{P} = \begin{array}{lll} 5 & 6 & \\ 4 & 7 & 8 \end{array} \qquad \mathcal{A}\boldsymbol{P} = \begin{array}{llll} \bullet & 2 & 5 & \\ \bullet & 1 & 4 & 7 \\ \bullet & & & \end{array} \qquad \mathcal{A}_5\boldsymbol{P} = \begin{array}{ll} \bullet & 2 \\ \bullet & 1 \\ \bullet & \end{array}$$

$$d\boldsymbol{P} = \begin{array}{lll} 2 & 5 & 6 \\ 1 & 4 & 7 \\ 3 & & \end{array} \quad \text{if } h = 8, \qquad \text{or} \qquad d\boldsymbol{P} = \begin{array}{llll} 2 & 5 & 6 & \\ 1 & 4 & 7 & 8 \\ 3 & & & \end{array} \quad \text{if } h > 8.$$

Since our tableaux are actually chains of compositions rather than fillings of subsets of $\mathbb{P} \times \mathbb{P}$ with entries, there is an ambiguity in evaluating $d\boldsymbol{P}$ and some other operators. There is no ambiguity in the chain form, but in the pictorial form, the hidden values of $\operatorname{low}(\boldsymbol{P})$ and $\operatorname{high}(\boldsymbol{P})$ must be tracked through careful bookkeeping. Note $\mathcal{C}$ and $\mathcal{A}$ affect neither the low nor high value, and d lowers the high value by 1. Be very careful in evaluating compositions of these operators pictorially: since $\operatorname{high}(\boldsymbol{P}) = \operatorname{high}(\mathcal{A}\boldsymbol{P})$, but $\operatorname{high}(\boldsymbol{P}) \notin \mathcal{A}\boldsymbol{P}$, the composition $d\mathcal{A}\boldsymbol{P}$ has the same pictorial appearance as $\mathcal{A}\boldsymbol{P}$, rather than removing the highest occurring entry in it. The chain form of these two are different, because $\mathcal{A}\boldsymbol{P}$ has one more term than $d\mathcal{A}\boldsymbol{P}$.

Proposition 5.47. *If $\boldsymbol{P}$ is a special composition tableau, f is a flag, and* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$, *then* $\operatorname{itype} Af = \operatorname{itype} \mathcal{A}\boldsymbol{P}$.

Proof. We may assume that $\boldsymbol{P}$ is a tableau of ordinary composition shape: if $\operatorname{sh} \boldsymbol{P} = \lambda/\mu$ with $\mu = \lambda[k]$, then apply this proposition to the flag f and tableau $\mathcal{C}^k\boldsymbol{P}$, and note that $\operatorname{itype} \boldsymbol{P} = \operatorname{itype} \mathcal{C}^k\boldsymbol{P}$ and $\operatorname{itype} \mathcal{A}\boldsymbol{P} = \operatorname{itype} \mathcal{A}\mathcal{C}^k\boldsymbol{P}$.

Without loss of generality, the flag is indexed $f = (f_0, \ldots, f_n)$, and since A will only be applied to flags whose top element is f_n, yielding the operator A_{f_n}, we assume $f_n = \hat{1}$. We induct on the length of the flag. For flags of length 0 or 1 it is trivially true. Now consider any flag f with $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$ for some $\boldsymbol{P} = (\lambda^{(0)}, \ldots, \lambda^{(n)})$, and assume the theorem holds for all flags of smaller length. Let $\lambda = \lambda^{(n)}$. Let $\boldsymbol{Q} = (\lambda^{(j)} - \lambda^{(1)})_{1 \le j \le n}$ and $\nu = \lambda^{(n)} - \lambda^{(1)}$. The smaller flag ∂f has $\operatorname{itype} \partial f = (\lambda^{(j)} - \lambda^{(i)})_{1 \le i \le j \le n} = \operatorname{itype} \boldsymbol{Q}$, so inductively applying the proposition to ∂f,

$$\begin{aligned} \operatorname{itype} A\partial f &= \operatorname{itype} \mathcal{A}\boldsymbol{Q} \\ &\equiv \Big(\mathcal{A}_\nu(\lambda^{(j)} - \lambda^{(1)}) - \mathcal{A}_\nu(\lambda^{(i)} - \lambda^{(1)})\Big)_{1 \le i \le j \le n} \\ &= \Big(\mathcal{A}_\lambda\lambda^{(j)} - \mathcal{A}_\lambda\lambda^{(i)}\Big)_{1 \le i \le j \le n}. \end{aligned}$$

It remains to show that $\operatorname{ftype} Af \equiv (\mathcal{A}_\lambda \lambda^{(j)} - \mathcal{A}_\lambda \lambda^{(0)})_{0 \le j \le n}$, which is equivalent to showing $\operatorname{ftype} Af \equiv \mathcal{CA}\boldsymbol{P}$ because $\lambda^{(0)} = \emptyset$ so that $\mathcal{A}_\lambda \lambda^{(0)}$ is the first column of $\mathcal{A}_\lambda \lambda^{(j)}$ for all j.

If $Af_0 = Af_1$ then $\operatorname{ftype} Af$ and $\operatorname{ftype} A\partial f$ agree (prepend $\emptyset$ to the latter as the 0th term to obtain the former). When this happens, $\operatorname{type}[f_0, f_n] \equiv \lambda^{(n)} - \lambda^{(0)}$ and $\operatorname{type}[f_1, f_n] \equiv \lambda^{(n)} - \lambda^{(1)}$ are either equal, so $1 \notin \boldsymbol{P}$, or differ in their first column, so 1 is the only entry in its row of $\boldsymbol{P}$.

So assume 1 is in $\boldsymbol{P}$ on a row of length at least two; without loss of generality, it is the first row.

By restricting the flag, $\operatorname{itype} \partial f = \operatorname{itype} \boldsymbol{Q}$, and inductively, $\operatorname{itype} A\partial f = \operatorname{itype} \mathcal{A}\boldsymbol{Q}$. Consider the following ordinary Young tableaux:

$$\begin{array}{rclcl} \boldsymbol{P}^{(1)} & = & \operatorname{ftype} f & \equiv & \boldsymbol{P} \\ \boldsymbol{P}^{(2)} & = & \operatorname{ftype} \partial f & \equiv & \boldsymbol{Q} \\ \boldsymbol{P}^{(3)} & = & \operatorname{ftype} A\partial f & \equiv & \mathcal{CA}\boldsymbol{Q} \\ \boldsymbol{P}^{(4)} & = & \operatorname{ftype} Af & \stackrel{?}{\equiv} & \mathcal{CA}\boldsymbol{P} \end{array}$$

A standard Young tableau is uniquely determined by specifying its entries and their columns (though not all such specifications yield standard Young tableau). The entries in $\boldsymbol{P}^{(4)}$ are the entries of $\boldsymbol{P}^{(3)}$ and a 1, because in the two flags $A\partial f = (Af_1 \lesssim f_2 \lesssim \ldots \lesssim Af_n)$ and $Af = (Af_0 \lessdot Af_1 \lesssim Af_2 \lesssim \ldots \lesssim Af_n)$, the cover relations and equalities are the same, except for the addition of $Af_0 \lessdot Af_1$; an entry e occurs in the tableau when $Af_{e-1} \lessdot Af_e$. The proposed value of $\boldsymbol{P}^{(4)}$ has the correct entries, and has the correct shape because $\operatorname{type}[Af_0, Af_n] = \mathcal{C}(\operatorname{type} \mathcal{L}) \equiv \mathcal{C}\lambda$.

Let $e^{(m)} = \operatorname{col}(\boldsymbol{P}^{(m)}, e)$, and $a_e = 1$ if $\operatorname{row}(\boldsymbol{P}, e) = 1$ and $a_e = 0$ otherwise. The entry 1 does not occur in $\boldsymbol{P}^{(2)}$, and for other entries e of $\boldsymbol{P}^{(1)}$, we have $e^{(2)} = e^{(1)} - a_e$.

By inductively applying the proposition to ∂f, we have $\operatorname{ftype} A\partial f = \mathcal{CA}\boldsymbol{P}$, so $e^{(3)} = e^{(2)}$ when $1 \ne e \in \mathcal{A}\boldsymbol{P}$, and no other values of e occur in $\boldsymbol{P}^{(3)}$.

Since $\partial A f = A\partial f$ holds for all flags of positive length, $\boldsymbol{P}^{(3)} = \operatorname{ftype} \partial Af = \widetilde{\Delta}\boldsymbol{P}^{(4)}$. See Section 5.2 for a description of $\widetilde{\Delta}$. One way to describe the tableau game $\boldsymbol{P}^{(3)} = \widetilde{\Delta}\boldsymbol{P}^{(4)}$ is that the columns of all entries stay the same, except for some sequence of entries $e_1 < e_2 < \ldots < e_k$, where e_i is in column i of $\boldsymbol{P}^{(4)}$ for $1 \le i \le k$; in column $i-1$ of $\boldsymbol{P}^{(3)}$ for $2 \le i \le k$; and $e_1 = 1$ isn't in $\boldsymbol{P}^{(3)}$. The column in which $\operatorname{sh} \boldsymbol{P}^{(4)} = \operatorname{cotype} Af_0$ and $\operatorname{sh} \boldsymbol{P}^{(3)} = \operatorname{cotype} Af_1$ differ is $\lambda_1 - 1$, so $k = \lambda_1 - 1$. Let $b_e = 1$ if $e \in \{e_2, \ldots, e_k\}$ and 0 otherwise. Then $e^{(4)} = e^{(3)} + b_e$ when $1 < e \in \mathcal{A}\boldsymbol{P}$, and $1^{(4)} = 1$. So $e^{(4)} = e^{(1)} - a_e + b_e$ when $1 < e \in \mathcal{A}\boldsymbol{P}$.

The tableaux $\boldsymbol{P}^{(4)}$ and $\mathcal{A}\boldsymbol{P}$ have the same entries. Let $e \in \mathcal{A}\boldsymbol{P}$. Then

$$\begin{array}{rcl} e^{(4)} & = & \min\{\, r \,:\, A^r(Af_0) \wedge Af_e \gtrdot A^r(Af_0) \wedge Af_{e-1} \,\} \\ & = & \min\left\{ r \,:\, A(A^r\hat{0} \wedge f_e) \gtrdot A(A^r\hat{0} \wedge f_{e-1}) \right\} \\ & \ge & \min\left\{ r \,:\, A^r\hat{0} \wedge f_e \gtrdot A^r\hat{0} \wedge f_{e-1} \right\} \\ & = & e^{(1)} \end{array}$$

Thus, for $e > 1$, we must have $b_e = 1$ whenever $a_e = 1$. Since exactly k values of $e > 1$ have $a_e = 1$ and the same number have $b_e = 1$, this determines the values of all a_e and b_e, so $e^{(4)} = e^{(1)}$ for all $e \in \mathcal{A}\boldsymbol{P}$, whence $\boldsymbol{P}^{(4)} \equiv \mathcal{CA}\boldsymbol{P}$. $\square$

Proposition 5.48. *Suppose $(f, \boldsymbol{P})$ satisfies f is a multisaturated flag; $\boldsymbol{P}$ is a special composition tableau;* $\operatorname{ftype}_{\hat{0}} f \equiv \boldsymbol{P}$*; and* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$*. Then $(df, d\boldsymbol{P})$ and $(Af, \mathcal{A}\boldsymbol{P})$ do too.*

Proof. Let $l = \operatorname{low}(f) = \operatorname{low}(\boldsymbol{P})$ and $h = \operatorname{high}(f) = \operatorname{high}(\boldsymbol{P})$. If f is a multisaturated flag, so are df and Af. We always have $\operatorname{ftype}_{\hat{0}} df = d(\operatorname{ftype}_{\hat{0}} f)$, and we delete column h of the arrays $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$ to obtain $\operatorname{itype} df = \operatorname{itype} d\boldsymbol{P}$.

By the preceding proposition, $\operatorname{itype} Af = \operatorname{itype} \mathcal{A}\boldsymbol{P}$. Let $A = A_{f_h}$. Let k be the maximum component of $\lambda^{(l)}$. Since $\operatorname{ftype}_{\hat{0}} f \equiv \boldsymbol{P}$ where $\boldsymbol{P}$ is a special composition tableau, we have $f_l = f_h[k]$, so $Af_l = f_h[k+1]$. Also, $\operatorname{type}[\hat{0}, f_i] = \operatorname{type} f_l + \operatorname{type}[f_l, f_i]$ and $\operatorname{type}[\hat{0}, Af_i] = \operatorname{type} Af_l + \operatorname{type}[Af_l, Af_i]$ for each $i = l, \ldots, h$. So we form $\operatorname{ftype}_{\hat{0}} Af$ by prepending the partition $\operatorname{type} Af_l$ to the left side of the tableau $\operatorname{ftype} Af$. This partition is formed from $\boldsymbol{P}$ by taking its outer shape, restricting each row to length $k+1$, and sorting the lengths into decreasing order. The result agrees with $\mathcal{A}\boldsymbol{P}$. $\square$

Definition 5.49. Let $A = \{a_1 < \cdots < a_r\}$ and $B = \{b_1 < \cdots < b_s\}$ be sets of integers, with either $A = B$ or with A and B disjoint. When $A = B$, let $\operatorname{dom}(A, A) = 0$. When $A \cap B = \emptyset$, let $\operatorname{dom}(A, B)$ be the largest number $t \leq \min\{r, s\}$ for which $a_{r-h} > b_{t-h}$ for $h = 0, \ldots, t-1$, i.e., the last t entries of A are consecutively greater than the first t entries of B. Clearly all numbers less than t have this property and no number larger than t does, for if t' possesses this property and we replace t' by $t' - 1$, the right side of each inequality is lowered (and the last inequality is dropped). Finally, for a standard composition tableau $\boldsymbol{P}$ whose set of entries on the ith row is B_i, let $\operatorname{dom}(\boldsymbol{P}) = \sum_{i,j} \operatorname{dom}(B_i, B_j)$.

Example 5.50.

$$\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline 1 & 5 & 7 & 10 \\ \hline 2 & 8 & 9 \\ \cline{1-3} 3 \\ \cline{1-1} 4 & 6 & 11 \\ \cline{1-3} \end{array} \qquad \begin{array}{c|cccc} \multicolumn{5}{c}{\operatorname{dom}(B_i, B_j)} \\ i \diagdown j & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 2 & 1 & 2 \\ 2 & 3 & 0 & 1 & 2 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array}$$

The sum of all $\operatorname{dom}(B_i, B_j)$ is $\operatorname{dom}(\boldsymbol{P}) = 19$. For instance, the entry $\operatorname{dom}(B_1, B_4) = 2$ is computed as follows.

$$\begin{array}{cccc} \multicolumn{4}{c}{\operatorname{dom} = 3 \text{ fails}} \\ 1 & 5 & 7 & 10 \\ & \vee & \vee & \not\vee \\ & 4 & 6 & 11 \end{array} \qquad \qquad \begin{array}{ccccc} \multicolumn{5}{c}{\operatorname{dom} = 2 \text{ works}} \\ 1 & 5 & 7 & 10 & \\ & & \vee & \vee & \\ & & 4 & 6 & 11 \end{array}$$

Lemma 5.51. *Let $\boldsymbol{P}$ be a standard composition tableau, and the set of entries on the jth row be B_j. Fix an integer k, and denote the entries on the kth row $B_k = \{a_1 < \cdots < a_r\}$. Let μ be the composition $\mu_j = \operatorname{dom}(B_k, B_j)$ when $j \neq k$, and $\mu_k = r$. In a semi-primary lattice, for any flag f with* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$*, we have*

$$\operatorname{type} f_{a_r} \wedge Af_{a_{r-1}} \wedge \cdots \wedge A^{r-1} f_{a_1} \wedge A^r f_0 \equiv \mu.$$

If we replace one or more of the a_i's by $a_i - 1$ on the left side, we replace μ_k with $r - 1$ on the right side.

Proof. Let $z_m = f_{a_r} \wedge Af_{a_{r-1}} \wedge \cdots \wedge A^{r-m} f_{a_m}$. Using the relations $A(x \wedge y) = Ax \wedge Ay$ and $A_y x = Ax \wedge y$ for elements x, y of a modular lattice, it is easy to verify that $A_{a_m} A_{a_{m+1}} \cdots A_{a_r} f = (A^{r-m+1} f_0 \wedge z_m, \ldots, A^{r-m+1} f_{a_m} \wedge z_m)$, and that the a_mth component of this is z_m and the a_{m-1}th component is z_{m-1}. We seek $\nu = \operatorname{type} z_0$, which by Proposition 5.48, is the straightening of the inner shape of $\operatorname{ftype}_{\hat{0}} g = \mathcal{A}_{a_1} \mathcal{A}_{a_2} \cdots \mathcal{A}_{a_r} \boldsymbol{P}$.

Let $t = \operatorname{dom}(B_k, B_j)$. In the expression for $\operatorname{ftype}_{\hat{0}} g$, the effect of $\mathcal{A}_{a_i}$ on a row is to delete all entries larger than a_i, and if any numerical entries remain, insert a $\bullet$ on the left, slide all entries right one cell, and delete the highest remaining entry. Write $B_j = \{b_1 < \cdots < b_s\}$. Since $a_{r-h} \geq b_{t-h}$ for $0 \leq h < t$, when we apply $\mathcal{A}_{a_{r-h}}$, row j has $h+1$ $\bullet$'s followed by at least $b_1, \ldots, b_{t-h}$. Thus, row j of $\operatorname{ftype}_{\hat{0}} g$ has at least t $\bullet$'s, so $\nu_j \geq t$.

If $t = r$ or $t = s$, this row is exactly t $\bullet$'s, so $\nu_j = t$. We now consider $t < \min\{r, s\}$. For some $h = 0, \ldots, t$ we have $a_{r-h} < b_{(t+1)-h}$. When we apply $\mathcal{A}_{r-h}$, we first delete all entries larger than a_{r-h}, leaving at most $b_1, \ldots, b_{t-h}$, then delete one more entry, and insert $\bullet$. So after applying $\mathcal{A}_{r-h}$, the entries on row j are $h+1$ $\bullet$'s followed by at most $b_1, \ldots, b_{t-1-h}$, for a maximum length of t. The maximum length of the row after applying the remaining operators $\mathcal{A}_{a_{r-h-1}}, \ldots, \mathcal{A}_{a_1}$ is t, so $\nu_j \leq t$. As the reverse inequality was already shown to hold, $\nu_j = t$ in all cases when $j \neq k$.

When $j = k$, applying $\mathcal{A}_{a_{r-h}}$ yields $h+1$ $\bullet$'s followed by $a_1, \ldots, a_{r-h-1}$ when $h < r$, and yields r $\bullet$'s when $h \geq r$. So $\nu_k = r$.

Now replace one or more a_i's by $a_i - 1$ in the expression $z' = f_{a_r} \wedge Af_{a_{r-1}} \wedge \cdots \wedge A^{r-1} f_{a_1} \wedge A^r f_0$, and evaluate $\operatorname{type} z'$ similarly to the evaluation of $\operatorname{type} z_1$ above. When $j \neq k$, we have $b_m \leq a_i$ iff $b_m \leq a_i - 1$ for all m and i because $a_i \neq b_m$ (they're on different rows). Since all comparisons of entries on row j with either a_i or $a_i - 1$ are the same, the rows besides the kth in $\operatorname{ftype}_{\hat{0}} (\mathcal{A}_{a_1} \ldots \mathcal{A}_{a_r} f)$ are the same upon any substitutions $a_i \to a_i - 1$ in the subscripts of $\mathcal{A}$'s. But row k shrinks by one $\bullet$ when one or more such substitutions are made. $\square$

We now classify and enumerate how many flags realize each type table of the form $\operatorname{itype} \boldsymbol{P}$ for each standard composition tableau $\boldsymbol{P}$.

Theorem 5.52. *If* $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$ *for some standard composition tableau* $\boldsymbol{P}$*, the flag is hereditary. The number of hereditary decompositions of a given hereditary flag is* $q^{\operatorname{dom}(\boldsymbol{P})}$*, and the number of flags with interval type table* $\operatorname{itype} \boldsymbol{P}$ *is*

$$f_{\operatorname{itype} \boldsymbol{P}}(q) = q^{2n(\lambda) - \operatorname{dom}(\boldsymbol{P})} M_\lambda(q^{-1}).$$

Synthetic proof. Let f be a flag and $\boldsymbol{P}$ be a standard composition tableau of shape λ such that $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$. Let B_i be the set of entries on the ith row of $\boldsymbol{P}$.

We seek a join decomposition $\vec{y}$ of f. Fix i and let the entries on the ith row of $\boldsymbol{P}$ be $a_1 < a_2 < \cdots < a_r$ where $r = \lambda_i$. If such a decomposition exists, the cycle y_i is complemented, as the join of the other cycles is its complement, and it satisfies $C^{r-m} y_i \leq f_{a_m}$ but $C^{r-m} y_i \not\leq f_{a_m - 1}$ for $m = 1, \ldots, r$. Conversely, a simple induction on the successive elements $f_0, \ldots, f_n$ shows that all independent selections of $y_1, \ldots, y_k$ satisfying these conditions are hereditary decompositions of f.

We have $C^{r-m} y_i \leq f_a$ iff $y_i \leq A^{r-m} f_a$ by Theorem 3.3(6), so we need $y_i \leq A^{r-m} f_a$ and $y_i \not\leq A^{r-m} f_{a-1}$ for i, m, a as given above. Also, y_i is a cycle of rank r, so $y_i \leq A^r f_0$.

Let

$$\begin{aligned} z &= A^0 f_{a_m} \wedge \cdots \wedge A^{r-1} f_{a_1} \wedge A^r f_0 \\ z' &= A^0 f_{a_m-1} \wedge \cdots \wedge A^r f_{a_1-1} \wedge A^r f_0 \end{aligned}$$

It is necessary that $y_i \leq z$, and it is sufficient, though not *a priori* necessary, that $y_i \not\leq z'$. Let $\mu_j = \nu_j = \operatorname{dom}(B_i, B_j)$ for $1 \leq j \leq k$, except $\mu_i = \lambda_i$ and $\nu_i = \lambda_i - 1$. By the preceding lemma, $\operatorname{type} z \equiv \mu$ and $\operatorname{type} z' \equiv \nu$, so that $z' \lessdot z$, and also, $\operatorname{type}(z \wedge A^{r-m} f_{b_m-1}) \equiv \nu$ for $m = 0, \ldots, r$, so that $z \wedge A^{r-m} f_{b_m-1} = z'$. Thus in all cases when $y_i \leq z$ and $y_i \leq A^{r-m} f_{a-1}$, we do in fact have $y_i \leq z'$ as well.

Thus, we require y_i to be a complemented cycle in $[\hat{0}, z]$ not in $[\hat{0}, z']$. By Corollary 4.64, there are $q^{|\mu|-\lambda_i}$ ways to choose such a cycle when the lattice is q-regular; when plugging into the corollary, note that μ has exactly λ_i columns, so that the cells in the first λ_i columns are all of the squares.

The exponent is $|\mu| - \lambda_i = \sum_j \operatorname{dom}(B_i, B_j)$. Choose all of $y_1, \ldots, y_k$ in this fashion to yield

$$q^{\sum_{i,j} \operatorname{dom}(B_i, B_j)} = q^{\operatorname{dom}(\boldsymbol{P})}$$

different hereditary decompositions of f. Since there are $q^{2n(\lambda)} M_\lambda(q^{-1})$ join decompositions $\vec{y}$ of $\hat{1}$, which split into classes according to the value of $f(\boldsymbol{P}, \vec{y})$, the quotient of these is $f_{\operatorname{itype} \boldsymbol{P}} = q^{2n(\lambda) - \operatorname{dom}(\boldsymbol{P})} M_\lambda(q^{-1})$. $\square$

Analytic proof of the enumeration only. Let f be a saturated hereditary flag and $\boldsymbol{P} = (\lambda^{(0)}, \ldots, \lambda^{(n)})$ a standard composition tableau with $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$. Let V be a vector space with a nilpotent action N of type $\lambda = S(\lambda^{(n)})$. Any hereditary decomposition of f gives rise to bases of V with elements and action

$$e_{ij} \text{ with } 1 \leq j \leq \lambda_i \qquad N e_{i1} = 0 \quad N e_{ij} = e_{i,j-1} \text{ for } j > 1.$$

For convenience, we also set $e_{ij} = 0$ when $j < 1$, but these of course do not contribute to a basis.

By the synthetic proof above, any flag f with interval type table $\operatorname{itype} \boldsymbol{P}$ can be represented by some such basis, with f_m the span of all e_{ij} with $(i,j) \in \operatorname{sh} \boldsymbol{P}$. For a given f, how many such canonical bases are there? If m is in cell (i,j), then the coset $e_{ij} + f_{m-1}$ of V spans f_m modulo f_{m-1}. So we can find the most general form of another such basis of f as follows. The ith row of $\boldsymbol{P}$ has entries $B_i = \{b_{i1} < b_{i2} < \ldots < b_{i,\lambda_i}\}$. For $i = 1, \ldots, k$, let

$$\begin{aligned} e'_{ij} &= N^{\lambda_i - j} \left(e_{i,\lambda_i} + \sum_{(i',j') \in I_i} c^{(i)}_{i',j'}\, e_{i',j'} \right) \\ &= e_{ij} + \sum_{(i',j') \in I_i} c^{(i)}_{i',j'}\, e_{i',j'+j-\lambda_i} \\ y_i &= \operatorname{span} e'_{i1}, \ldots, e'_{i,\lambda_i} \end{aligned}$$

where the I_i and $c^{(i)}{}_{(i',j')}$ are to be determined so that $f_m = \operatorname{span} \left\{ e'_{ij} \,:\, (i,j) \in \lambda^{(m)} \right\}$ in some canonical fashion. In general, the e_{ij} term could have any nonzero coefficient besides 1, but we will normalize so that it is 1. The following conditions ensure we obtain distinct bases in a canonical fashion.

(1) $I_i \subset \lambda[\lambda_i]$ because e'_{ij} is annihilated by N^j. The coefficient 1 on e_{ij} both ensures it is not annihilated by N^{j-1}, and provides a normalization.
(2) I_i has no points (i', j') with $i' = i$, for by subtracting a linear combination of $Ne'_{ij}, N^2 e'_{ij}, \ldots$ from e'_{ij}, we can erase any apparent coefficient of $e_{i',j'}$ with $i' = i$ and $j' < j$ from e'_{ij}.
(3) $(i', j') \in I_i$ requires $b_{i',j'+j-\lambda_i} < b_{ij}$ whenever $\lambda_i \geq j > \lambda_i - j'$. For pick i, and consider $m = b_{ij} \in B_i$. Assume that $f_{m-1} = f(\boldsymbol{P}, \vec{y})_{m-1}$ holds. The span of f_{m-1} and e'_{ij} is f_m provided that all the nonzero $e_{i',j'}$ occurring with nonzero coefficients in e'_{ij} have $b_{i',j'} < m = b_{ij}$, i.e, when $(i', j') \in I_i$ and $j' + j - \lambda_i > 0$, we have $b_{i',j'+j-\lambda_i} < b_{ij}$.

Conditions (1)–(3) are succinctly summarized by

$$I_i = \{\, (i', j') \,:\, j' \leq \operatorname{dom}(B_i, B_{i'}) \,\},$$

because (1) is equivalent to $\operatorname{dom}(B_i, B_{i'}) \leq \min\{\lambda_i, \lambda_{i'}\}$, which always is true; (2) is equivalent to $\operatorname{dom}(B_i, B_{i'}) = 0$ when $i' = i$; and these conditions with (3) essentially define $\operatorname{dom}(B_i, B_{i'})$.

We can independently choose $c^{(i)}_{i',j'} \in K$ for $(i', j') \in I_i$. Thus, there are $q^{\operatorname{dom}(\boldsymbol{P})}$ canonical bases yielding f.

There are $q^{2n(\lambda)} M_\lambda(q^{-1})$ join decompositions of V, each of which yields a canonical basis of the above form, and so we obtain the result in the theorem. □

Theorem 5.53. *In a semi-primary lattice whose type is a two column partition, all flags are hereditary.*

Proof. Let f be a saturated flag in the lattice. By Theorem 5.52, it suffices to show that $\operatorname{itype} f = \operatorname{itype} \boldsymbol{P}$ for some standard composition tableau $\boldsymbol{P}$. Let $\boldsymbol{P}_i = \operatorname{ftype} \partial^i f$ for $0 \leq i \leq n$. Let e be an entry in the second column of $\boldsymbol{P}_0$. Let i be minimal with e in the first column of $\boldsymbol{P}_i$; at most one entry from $\boldsymbol{P}_{i-1}$ moves into column 1 of $\boldsymbol{P}_i$, so each i has at most one e associated to it. We will show that $\boldsymbol{P}_e$ is obtained from $\boldsymbol{P}_{e-1}$ by deleting e and sliding the first column up to fill the hole; no entry moves from the second column to the first. Because this is true for all e in the second column of $\boldsymbol{P}_0$, we thus have that $\boldsymbol{P}$ has rows $\boxed{i \mid e}$ for each e in the second column of $\boldsymbol{P}_0$, and rows $\boxed{i}$ for each remaining i in the first column of $\boldsymbol{P}_0$.

The tableaux $\boldsymbol{P}_{e-1}$ and $\boldsymbol{P}_e$ have the same second column iff $Af_{e-1} \wedge f_j = Af_e \wedge f_j$ for all $j \geq e$ iff this holds for maximum j (when $f_j = \hat{1}$) iff $Af_{e-1} = Af_e$. Since e is in the second column of $\boldsymbol{P}_0$, we have $A\hat{0} \wedge f_{e-1} = A\hat{0} \wedge f_e$. Apply A to this to obtain $A^2\hat{0} \wedge Af_{e-1} = A^2\hat{0} \wedge Af_e$. Since $\operatorname{type} \boldsymbol{\mathcal{L}}$ has two columns, substitute $A^2\hat{0} = \hat{1}$ to obtain $Af_{e-1} = Af_e$. □

Definition 5.54. Let σ be a permutation and $\boldsymbol{P}$ a composition tableau. Then $\sigma\boldsymbol{P}$ is the composition tableau of the same shape as $\boldsymbol{P}$ whose entries on a row are $\{\sigma(a_1), \ldots, \sigma(a_k)\}$ arranged in order, where $a_1, \ldots, a_k$ are the entries on that row of $\boldsymbol{P}$. Below, $(i, i+1)$ denotes the permutation exchanging i and $i+1$.

Theorem 5.55. *Let λ be an integer partition. Let $\boldsymbol{P}$ be a standard composition tableau of type λ.*

(1) $n(\lambda) \leq \operatorname{dom}(\boldsymbol{P}) \leq 2n(\lambda) - n_0(\lambda)$.
(2) $\Big|\operatorname{dom}(\boldsymbol{P}) - \operatorname{dom}\big((i, i+1)\boldsymbol{P}\big)\Big| \leq 1$.

(3) *As* $\boldsymbol{P}$ *ranges over all standard composition tableaux of shape* λ, $\operatorname{dom}(\boldsymbol{P})$ *takes on every value in the interval given in (1).*

Proof.

(1) Let $A = \{a_1 < \cdots < a_r\}$ and $B = \{b_1 < \cdots < b_s\}$ be disjoint sets of integers with $r \geq s$. We will show that

$$s \leq \operatorname{dom}(A, B) + \operatorname{dom}(B, A) \leq 2s - \delta_{rs}.$$

First we establish the lower bound. Let $t = \operatorname{dom}(A, B)$ and $u = \operatorname{dom}(B, A)$. We want to show that $t+u \geq s$. If $t = s$, we are done. Otherwise, $t < s \leq r$, so we have $b_{t+1-h} > a_{r-h}$ for some $h \in \{0, \ldots, t\}$, so that $b_{t+1} \geq b_{t+1-h} > a_{r-h} \geq a_{s-t}$ (as $r \geq s$ and $h \leq t$, so $r - h \geq s - t$). Then for $k = 0, \ldots, s - t - 1$, we have $b_{s-k} \geq b_{s-(s-t-1)} = b_{t+1} > a_{s-t} \geq a_{s-t-k}$ so $\operatorname{dom}(B, A) \geq s - t$. Thus $u \geq s - t$ so $u + t \geq s$.

Now we establish the upper bound. Each of $\operatorname{dom}(A, B)$ and $\operatorname{dom}(B, A)$ is less than both r and s, the smaller of which is s, so $\operatorname{dom}(A, B) + \operatorname{dom}(B, A) \leq 2s$. Now suppose $r = s = t$. Then $a_i > b_i$ for all $i = 1, \ldots, r$, so we do not have $b_s > a_r$; thus, $t' < t$, so $\operatorname{dom}(A, B) + \operatorname{dom}(B, A) \leq 2s - 1$.

Now let λ be a partition and $\boldsymbol{P}$ be a standard composition tableau of shape λ whose set of entries on the ith row is B_i. The lower bound on $\operatorname{dom}(\boldsymbol{P})$ is obtained by summing the lower bound for each pair of rows.

$$\begin{aligned} \operatorname{dom}(\boldsymbol{P}) &= \sum_{(i,j):\, 0<i<j\leq\lambda'_1} \operatorname{dom}(B_i, B_j) + \operatorname{dom}(B_j, B_i) \\ &\geq \sum_{(i,j):\, 0<i<j\leq\lambda'_1} \lambda_j \;=\; \sum_j (j-1)\lambda_j \;=\; n(\lambda) \end{aligned}$$

The upper bound is similarly obtained by summing the upper bound for pairs of rows.

$$\begin{aligned} \operatorname{dom}(\boldsymbol{P}) &= \sum_{(i,j):\, 0<i<j\leq\lambda'_1} \operatorname{dom}(B_i, B_j) + \operatorname{dom}(B_j, B_i) \\ &\leq \sum_{(i,j):\, 0<i<j\leq\lambda'_1} 2\lambda_j - \delta_{\lambda_i,\lambda_j} \\ &= 2n(\lambda) - \sum_{r>0} \binom{m_r}{2} \;=\; 2n(\lambda) - n_0(\lambda) \end{aligned}$$

The sum of $2\lambda_j$ is double the sum computed for the lower bound. The sum of the δ terms is the number of pairs of parts of λ of equal length, which is as indicated.

(2) Let $\boldsymbol{P}$ have rows $B_1, \ldots, B_k$, and $\boldsymbol{Q} = (i, i+1)\boldsymbol{P}$ have rows $B'_1, \ldots, B'_k$. If neither or only one of i and $i+1$ occurs in $\boldsymbol{P}$, then $\operatorname{dom}(\boldsymbol{P}) = \operatorname{dom}(\boldsymbol{Q})$ because all comparisons with i and with $i+1$ are equivalent. If both are on the same row then $\boldsymbol{Q} = \boldsymbol{P}$, so $\operatorname{dom}(\boldsymbol{P}) = \operatorname{dom}(\boldsymbol{Q})$.

We are left with i and $i+1$ occurring in $\boldsymbol{P}$ on different rows. Assume without loss of generality that i is in B_1 and $i+1$ is in B_2. Then

$$\operatorname{dom}(\boldsymbol{P}) - \operatorname{dom}(\boldsymbol{Q}) = \operatorname{dom}(B_1, B_2) + \operatorname{dom}(B_2, B_1) - \operatorname{dom}(B'_1, B'_2) - \operatorname{dom}(B'_2, B'_1)$$

because all comparisons of an element from the other rows with i or with $i+1$ are the same; only when i and $i+1$ are compared can the sense of the comparison be reversed.

Let $B_1 = \{a_1 < a_2 < \cdots < a_r\}$ and $B_2 = \{b_1 < b_2 < \cdots < b_s\}$, and $t = \operatorname{dom}(B_1, B_2)$. Let $B_1' = \{a_1' < a_2' < \cdots < a_r'\}$ and $B_2 = \{b_1' < b_2' < \cdots < b_s'\}$, and $t' = \operatorname{dom}(B_1', B_2')$. Let $a_j = i$ and $b_k = i+1$, so that $a_k' = i+1$ and $b_j' = i$ and in all other cases, $a_h = a_h'$ and $b_h = b_h'$. We have $a_{r-h} > b_{t-h}$ for $h = 0, \ldots, t-1$, but this fails if t is replaced by $t+1$. If the only failure in the latter case is of the form $i > i+1$, then when we swap i and $i+1$ to obtain B_1' and B_2', this failure vanishes, so that $t' \geq t+1$. The failure $i > i+1$ is replaced by success $i+1 > i$, or rather $a_k' > b_j'$. We cannot have $t' > t+1$, because we would either exceed the length of B_2', or would obtain a comparison $i+1 = a_k' > b'_{j+t'-(t+1)} > i$ (for if $t' > t+1$ then $j + t' - (t+1) > j$ so $b'_{j+t'-(t+1)} > b_j' = i$); but that requires there to be an integer strictly between i and $i+1$, and there isn't one. Thus, $t' = t$ or $t+1$, i.e., $\operatorname{dom}(B_1, B_2) - \operatorname{dom}(B_1', B_2') = 0$ or -1.

Similarly, $\operatorname{dom}(B_2, B_1) - \operatorname{dom}(B_2', B_1') = 0$ or $+1$; the opposite sign arises from the fact that the rows i and $i+1$ are reversed.

Thus, $\operatorname{dom}(\boldsymbol{P}) - \operatorname{dom}(\boldsymbol{Q}) = -1$, 0, or 1.

(3) Let $\boldsymbol{P}_1$ be the superstandard tableau of shape λ. Then $\operatorname{dom}(B_i, B_j) = 0$ when $i < j$, and λ_i when $i > j$. There are $i-1$ values of j less than each i, so $\operatorname{dom}(\boldsymbol{P}) = \sum_i (i-1)\lambda_i = n(\lambda)$.

Now let $\boldsymbol{P}_2$ be transpose of the superstandard tableau of shape λ'. Then for $i \neq j$,

$$\operatorname{dom}(B_i, B_j) = \begin{cases} \lambda_j & \text{if } \lambda_i > \lambda_j \text{ (so } i < j); \\ \lambda_j - 1 & \text{if } \lambda_i = \lambda_j \text{ and } i < j; \\ \lambda_i & \text{if } i > j. \end{cases}$$

The sum of these over all i and j is the same as the upper bound given in part (1) of this theorem.

Any permutation σ for which $\boldsymbol{P}_1 = \sigma \boldsymbol{P}_2$ can be decomposed as a product of elementary transpositions $(i, i+1)$, and so we can obtain a sequence of composition tableaux from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$ differing by transpositions; by (2), the successive tableaux in this sequence have values of dom that either are equal or go up or down by 1, so all the values from the lower bound to the upper bound are obtained. □

Recall the polynomial $f_\Lambda(q)$ and parameter q_0 of Conjecture 5.25; the polynomial is the number of flags with type table Λ in a q-regular semi-primary lattice when $q \geq q_0$.

Corollary 5.56. *Let λ be an integer partition. Type tables Λ with $f_\Lambda(q) = q^d M_\lambda(q)$ exist for each $0 \leq d \leq n(\lambda) - n_0(\lambda)$.*

Proof. This follows from Theorems 5.52 and 5.55, and the fact that $M_\lambda(q^{-1}) = q^{-n_0(\lambda)} M_\lambda(q)$. □

Theorem 5.57. *Let Λ be an interval type table of shape λ for which the polynomial $f_\Lambda(q)$ and parameter q_0 of Conjecture* 5.25 *exist. If there is no standard composition* *tableau* $\boldsymbol{P}$ *for which* Λ *has the form* $\operatorname{itype}\boldsymbol{P}$*, then either* $q_0 > 1$*, or* $f_\Lambda(q)$ *does not have the form* $q^d M_\lambda(q)$*.*

Proof. Consider the 1-regular semi-primary lattice which is the direct product of chains of lengths $\lambda_1, \lambda_2, \ldots$. All its flags are hereditary flags, and so if Λ is not of the form $\operatorname{itype}\boldsymbol{P}$ for some $\boldsymbol{P}$, there are no flags with interval type table Λ. Thus, either $q_0 > 1$, or $q-1$ is a divisor of $f_\Lambda(q)$. $\square$

Note. For all the polynomials listed in Appendix A, $q_0 = 1$ and $q-1$ is a divisor of all polynomials except those of the form $\Lambda = \operatorname{itype}\boldsymbol{P}$.

6. Tableaux games describing operations on flags

6.1. Operations on flags. We now consider a number of operations on multisaturated flags. We will work in the generality of multisaturated flags rather than saturated flags because certain operations, such as A and C, turn saturated flags into multisaturated flags, and we want to look at compositions of operations. The indexing of the elements of flags and tableaux is important as well. For a flag $f = (f_0, \ldots, f_n)$, we have $\partial f = (f_1, \ldots, f_n)$, whose components are indexed $1, \ldots, n$, not $0, \ldots, n-1$; the tableau game $\operatorname{ftype}\partial f = \widetilde{\Delta}(\operatorname{ftype} f)$ is nicer this way, as it can be expressed by moving numbers along a single path and leaving the rest of the tableau intact, rather than also decrementing every entry of the tableau, and a similar phenomenon occurs with other games. The price to pay for these saturation and indexing issues is bookkeeping to track the indices used in flags and tableau, as already seen in Example 5.46. For multisaturated flags, there are only two numbers that must be tracked: the low index and the high index of a flag or tableau. The indices between them do not change. For nonsaturated flags (Section 6.8), the intermediate indices will be changed as well.

Here are all the operations on multisaturated flags.

flag g	$\operatorname{low}(g)$	$\operatorname{high}(g)$
$f=(f_l, f_{l+1}, \ldots, f_h)$	l	h
$\partial f=(f_{l+1}, \ldots, f_h)$	$l+1$	h
$df=(f_l, \ldots, f_{h-1})$	l	$h-1$
$Cf=(C_{f_l} f_l, \ldots, C_{f_l} f_h)$	l	h
$C_x f=(C_x f_l, \ldots, C_x f_h)$	l	h
$Af=(A_{f_h} f_l, \ldots, A_{f_h} f_h)$	l	h
$A_x f=(A_x f_l, \ldots, A_x f_h)$	l	h
$f^*=(f_h^*, \ldots, f_l^*)$	$n-h$	$n-l$
$f \vee x=(f_l \vee x, \ldots, f_h \vee x)$	l	h
$f \wedge x=(f_l \wedge x, \ldots, f_h \wedge x)$	l	h

Certain pairs of operators commute: ∂, d; ∂, C_x; ∂, A; ∂, A_x; d, C; d, C_x; d, A_x. The pairs ∂, d; C, A; C_x, A_{x^*}; $\vee x, \wedge x^*$ are dual, in the sense that $(\partial(f^*))^* = df$ and $(d(f^*))^* = \partial f$, and similarly for the others.

The value of n must be specified, but generally, we start with a flag $f = (f_0, \ldots, f_n)$ and apply some sequence of operators to it; the high index may no longer be n when dualizing, but we still use the initial n rather than the new high index.

For each operator, we would like to determine the following, to the extent possible. We may ask the same questions with ftype replaced by ftype_y.

(1) What are the possible values of $\mathrm{ftype}\, g$ given $\mathrm{ftype}\, f$, and which value, if any, is generic or uniform?
(2) How many flags f (or pairs f and x) achieve specified values of $\mathrm{ftype}\, f$ and $\mathrm{ftype}\, g$?
(3) Inversely, given g and a tableau $\boldsymbol{P}$, how many flags f are there with $\mathrm{ftype}\, f = \boldsymbol{P}$ such that applying the operator yields g?

For the various operations, we will develop the **tableau game** $\mathcal{G}$ transforming $\boldsymbol{P} = \mathrm{ftype}\, f = (\lambda^{(l)} \lessdot \cdots \lessdot \lambda^{(h)})$ to $\mathcal{G}\boldsymbol{P} = \mathrm{ftype}\, g = (\mu^{(l')} \lessdot \cdots \lessdot \mu^{(h')})$. Either or both ftype's may be replaced by ftype_y, as indicated in the particular game. There are two equivalent formulations of each game: there are rules describing how to manipulate the numbers and cells of the tableau $\boldsymbol{P}$ to form $\mathcal{G}\boldsymbol{P}$, and there are rules describing how to compute the partitions $\mu^{(k)}$ from the partitions $\lambda^{(k)}$. In the latter formulation, we write the partitions $\lambda^{(l)}, \ldots, \lambda^{(h)}$ in columns l through h on one line, and then apply rules that ultimately yield partitions $\mu^{(l')}, \ldots, \mu^{(h')}$ in columns l' through h' on the line below it. We can compose operations by using that line as the input to yet another operator, whose result is written beneath it; the evacuation of Figure 3 shows the computation of successive powers of Δ in this format.

The tableau and partition formulations of each game are equivalent, provided we keep track of $\mathrm{low}(\boldsymbol{P})$, $\mathrm{high}(\boldsymbol{P})$, $\mathrm{low}(\mathcal{G}\boldsymbol{P})$, and $\mathrm{high}(\mathcal{G}\boldsymbol{P})$ through careful bookkeeping in the tableau formulation of the game; in the partition formulation, we keep track of these quantities automatically.

Depending on the operation, the value of $\mathrm{ftype}\, g$ may not be uniquely determined by the value of $\mathrm{ftype}\, f$. If multiple values are possible, we define two similar games: $\widetilde{\mathcal{G}}\boldsymbol{P}$ is a nondeterministic game yielding all possible values of $\mathrm{ftype}\, g$ given $\boldsymbol{P} = \mathrm{ftype}\, f$, and $\mathcal{G}\boldsymbol{P}$ is a deterministic game, usually yielding the generic value. For most games (the Robinson-Schensted games being the important exception), the generic value is uniform, so that the uniform value of a composition of operators on a flag is described by composing the generic games.

6.2. The backwards jeu de taquin slide, $j^c(\boldsymbol{P})$. Let $\mathrm{sh}\, \boldsymbol{P} = \lambda/\mu$; the parameter c is an inner corner of μ.

Let $\hat{0} \lessdot x \leq f_l$. Uniformly,

$$j^c, \tilde{\jmath}^c : \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_x\, f \qquad c = \mathrm{type}[\hat{0}, f_l]/\,\mathrm{type}[x, f_l].$$

(That is, the game j^c uniformly maps $\{\, f \,:\, \mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P} \,\}$ to $\{\, f \,:\, \mathrm{ftype}_x\, f = j^c(\boldsymbol{P}) \,\}$, and in all degenerate cases, $\mathrm{ftype}_x\, f = \tilde{\jmath}^c(\mathrm{ftype}_{\hat{0}}\, f)$.)

Tableau Rules. The items tagged with j or $\tilde{\jmath}$ are performed only in one or the other game, and the untagged items are done in both games.

(1) Place $\star$ at c and let $e = \bullet$ be the value that was there.

($2j$) If $\star$ is at an inner corner, delete it, and we are done. Otherwise, choose the smaller or only of the cells just below and just right of $\star$, and slide that entry into $\star$. Repeat step (2).

($2\tilde{\jmath}$) Choose one of these alternatives, when the provision allows.

(a) Slide the entry e' below $\star$ into $\star$, provided the numeric entries on the rows continue to increase left to right.
(b) Slide any entry $e' > e$ from the column left of $\star$ into $\star$, provided the rows continue to increase left to right and the columns top to bottom.
(c) Delete $\star$ and we are done, provided $\star$ is at an inner corner.

Then let $e := e'$ and repeat step $(2\tilde{\jmath})$ until alternative (c) is chosen.

See Figure 6.

Local Rules. Place $\lambda^{(l)} \ominus c$ in column l. Then for $k = l+1, l+2, \dots, h$, propagate as follows.

$$\begin{array}{r|c|c|c|c|l} \text{column} & l & & k-1 & k & \\ \hline \boldsymbol{P} & \lambda^{(l)} & & \gamma & \delta & \\ \tilde{\jmath}^c\boldsymbol{P} & (\lambda^{(l)} \ominus c) & & \alpha & (\beta) & \longrightarrow \end{array}$$

The nonparenthesized values are given, and the parenthesized values are computed, according to which of several cases occurs.

$\boldsymbol{\alpha = \gamma}$: Let $\beta := \delta$.
$\boldsymbol{\delta = \gamma}$: Let $\beta := \alpha$.
$\boldsymbol{\delta/\alpha}$ is a brick: Let β be the unique partition between α and δ.
$\boldsymbol{\delta/\alpha}$ is two nonadjacent squares ($\boldsymbol{j^c}$): There are two partitions between α and δ, and β is the one that's not γ.
$\boldsymbol{\delta/\alpha}$ is two nonadjacent squares ($\boldsymbol{\tilde{\jmath}^c}$): If $\mathrm{col}(\delta/\gamma) \gtrdot \mathrm{col}(\gamma/\alpha)$, choose either partition between α and δ to be β, and otherwise, choose the partition unequal to γ to be β.

Theorem 6.1. *Let $\boldsymbol{P}$ be a tableau of shape λ/μ. Let $\nu \lessdot \mu$ and $c = \mu/\nu$. For all flags f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$, and all atoms x of $[\hat{0}, f_l]$ with $\mathrm{type}[x, f_l] = \nu$, we have $\mathrm{ftype}_x\, f = \tilde{\jmath}^c(\mathrm{ftype}_{\hat{0}}\, f)$.*

Fix $\boldsymbol{Q} = \tilde{\jmath}^c\boldsymbol{P}$. Let $r = \mathrm{col}(\lambda/\mu^{(h)})$. In a q-regular semi-primary lattice of type λ, the number of pairs (x, f) where f is a flag with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and x is an atom of $[\hat{0}, f_l]$ with $\mathrm{type}[x, f_l] = \nu$, and $\mathrm{ftype}_x\, f = \boldsymbol{Q}$, is

$$\Big(\langle\lambda'_r\rangle - \big\langle\lambda'_{r+1}\big\rangle\Big) \cdot \prod_{k \,\in\, \boldsymbol{P}} \begin{cases} \langle\mathrm{row}(\boldsymbol{Q}, k)\rangle - \langle\mathrm{row}_1(\boldsymbol{P}, k)\rangle & \text{if } \mathrm{col}(\boldsymbol{P}, k) = \mathrm{col}(\boldsymbol{Q}, k); \\ \langle\mathrm{row}(\boldsymbol{P}, k)\rangle - \langle\mathrm{row}_1(\boldsymbol{Q}, k)\rangle & \text{if } \mathrm{col}(\boldsymbol{P}, k) \gtrdot \mathrm{col}(\boldsymbol{Q}, k). \end{cases}$$

This is a polynomial of degree

$$n(\mathrm{sh}\,\boldsymbol{P}) + n(\mathrm{osh}\,\boldsymbol{P}/\,\mathrm{osh}\,\boldsymbol{Q}) - \#\text{ of entries that move up one cell from } \boldsymbol{P} \text{ to } \boldsymbol{Q},$$

which is uniquely maximized in the uniform game $\boldsymbol{P} = j^c(\boldsymbol{P})$.

Proof. See Proposition 5.7, which counts the number of flags f with specified values of $\mathrm{ftype}\, f$ and $\mathrm{ftype}\,\partial f$. The proof is essentially the same, except that here we allow the flag to be multisaturated by setting $f_{k-1} := f_k$ when $k \notin \boldsymbol{Q}$ (which simply introduces factors of 1 that can be ignored), and we included the number $\langle\lambda'_r\rangle - \big\langle\lambda'_{r+1}\big\rangle$ of choices of x. Also, the final cell c of the deletion path was fixed at $(1,1)$ in the proposition, but is given here as a parameter. $\square$

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 3 \\ \hline {}^{c}\bullet & 2 & 5 \\ \hline 1 & 4 & 8 \\ \hline 6 & 7 & \\ \cline{1-2} \end{array} \qquad\qquad c = (2,1)$$

The possible computations of $\tilde{\jmath}^c(\boldsymbol{P})$ are as follows.

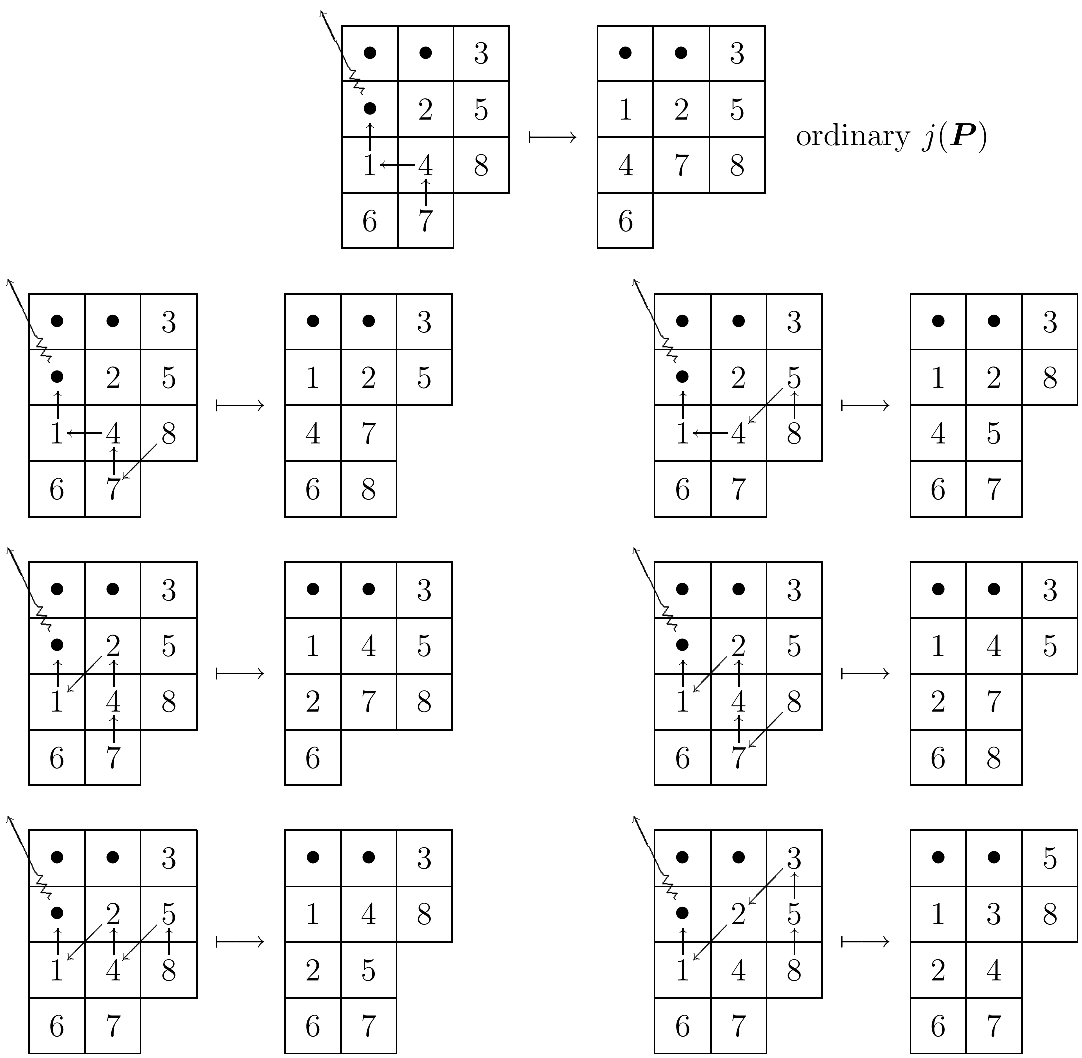


FIGURE 6. All possible degenerate *jeu de taquin* slides.

Note. The following conventions explicitly described in this game will be adopted for all games without further explicit descriptions.

(1) The notation
$$\mathcal{G}, \widetilde{\mathcal{G}} : \operatorname{ftype}_x f \longmapsto \operatorname{ftype}_y \phi(f)$$
means $\widetilde{\mathcal{G}}$ is a nondeterministic game with $\operatorname{ftype}_y \phi(f) = \widetilde{\mathcal{G}}(\operatorname{ftype}_x f)$ for all flags f, and that the game $\mathcal{G}$ is a deterministic version of it, usually giving a uniform or generic value, as noted.

(2) Differences in the rules for $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are noted by tagging the rules that differ with $\mathcal{G}$ or $\widetilde{\mathcal{G}}$; untagged rules are the same in both.

(3) In the local section shown of the grid of partitions for a game, the unparenthesized partitions are given, and the parenthesized ones are computed as a (possibly nondeterministic) function of them. If there is a simple formula for a parenthesized one, it is given, and otherwise, it is given a name, such as "β," and then rules for computing β are presented.

6.3. The *jeu de taquin*, $j(\boldsymbol{P})$. Uniformly,
$$j, \tilde{\jmath} : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype} f.$$

Tableau Rules. Let $\boldsymbol{P}$ be a skew tableau. Choose any inner corner c of the inner shape of $\boldsymbol{P}$ and apply the backwards slide j^c (for j), or the degenerate backwards slide $\tilde{\jmath}^c$ (for $\tilde{\jmath}$), and repeat again with another inner corner of the resulting tableau, until a tableau of ordinary shape is achieved.

The tableau $j(\boldsymbol{P})$ is independent of the sequence of inner corners chosen! For a detailed analysis of this game and a combinatorial proof of the uniqueness of $j(\boldsymbol{P})$, see the original source [**26**] or an exposition [**22**, §3.9]. We provide the following algebraic proof.

Definition 6.2. For a standard tableau $\boldsymbol{R}$ on $1, \ldots, m$ with i in cell c_i, let $j^{\boldsymbol{R}} = j^{c_1} \cdots j^{c_m}$.

Theorem 6.3.

(1) *Let μ be a partition. The tableau operator $j^{\boldsymbol{R}}$ is the same for all standard tableau $\boldsymbol{R}$ of shape μ. Simply denote this operator by j.*

(2) *Generically,* $\operatorname{ftype} f = j(\operatorname{ftype}_{\hat{0}} f)$ *in q-regular semi-primary lattices.*

Proof. Let $\boldsymbol{P}$ be a standard skew tableau of shape λ/μ and $\boldsymbol{R}$ be a standard tableau of shape $\mu \vdash m$. Let A be the set of flags f with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, and let X_f be the set of flags x with $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_m = f_l$ and $\operatorname{ftype} x^* = \boldsymbol{R}$. For all flags $f \in A$, there are $F^{\boldsymbol{R}}(q)$ flags in X_f. By genericity of the composition $j^{\boldsymbol{R}}$ of uniform games, all but a fraction $O(q^{-1})$ of pairs $\{\,(x, f) : f \in A \text{ and } x \in X_f\,\}$ satisfy $\operatorname{ftype} f = \operatorname{ftype}_{x_m} f = j^{\boldsymbol{R}}(\boldsymbol{P})$. This expression does not depend on which $x \in X_f$ was chosen, and the size of X_f is independent of $f \in A$, so in fact, all but a fraction $O(q^{-1})$ of flags $f \in A$ satisfy $\operatorname{ftype} f = j^{\boldsymbol{R}}(\boldsymbol{P})$. Since this holds for any standard tableau $\boldsymbol{R}$ of shape μ, in fact, $j^{\boldsymbol{R}}(\boldsymbol{P})$ is independent of $\boldsymbol{R}$. □

Corollary 6.4. *Let μ be a partition and $\boldsymbol{P}$ be a standard tableau. Then $j(\mu + \boldsymbol{P}) = S(\boldsymbol{P})$.*

Proof. All flags f with $\operatorname{ftype}_{\hat{0}} f = \mu + \boldsymbol{P}$ have $\operatorname{ftype} f = S(\boldsymbol{P})$ by Theorem 4.93, and generically have $\operatorname{ftype} f = j(\boldsymbol{P})$ by the preceding theorem, so $j(\boldsymbol{P}) = S(\boldsymbol{P})$. $\square$

This can also be proved without the machinery of flags by showing $j^c(\mu + \boldsymbol{P}) = (\mu \ominus c) + \boldsymbol{P}$, i.e., that the slide j^c simply slides c's row one cell left, deleting $\bullet$ from c, and does nothing to any other row.

6.4. The elementary evacuation step, $\Delta\boldsymbol{P}$. Uniformly,

$$\Delta, \widetilde{\Delta} : \operatorname{ftype} f \longmapsto \operatorname{ftype} \partial f.$$

Tableau Rules. If the top left corner of $\boldsymbol{P}$ is larger than $l+1$ then $\widetilde{\Delta}\boldsymbol{P} := \boldsymbol{P}$, and we are done. Otherwise, let $c = (1,1)$ be the upper left corner of $\boldsymbol{P}$, place $\bullet$ there, and perform the backwards slide j^c (for Δ) or $\tilde{\jmath}^c$ (for $\widetilde{\Delta}$) into it.

Local Rules. Place $\emptyset$ in column $l+1$. Then for $k = l+2, \ldots, h$, propagate by the same rule as in j^c (for Δ) or $\tilde{\jmath}^c$ (for $\widetilde{\Delta}$).

$$\begin{array}{r|c|c|c|c|c|l}
\text{column} & \multicolumn{1}{c}{l} & \multicolumn{1}{c}{l+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1} & \multicolumn{1}{c}{k} & \\ \cline{2-3}\cline{5-6}
\boldsymbol{P} & \emptyset & \lambda^{(l+1)} & & \gamma & \delta & \\ \cline{2-3}\cline{5-6}
\widetilde{\Delta}\boldsymbol{P} & & (\emptyset) & & \alpha & (\beta) & \longrightarrow \\ \cline{2-3}\cline{5-6}
\end{array}$$

The game Δ is uniform; see Proposition 5.7 for further details, including an enumeration of the number of flags achieving $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} \partial f = \boldsymbol{Q}$. The proof there is for saturated flags, but is trivially extended to multisaturated flags, with the same enumerative formulas.

6.5. Remove the low entry, $\partial\boldsymbol{P} = (\lambda^{(l+1)}, \ldots, \lambda^{(h)})$. Uniformly (in fact, identically),

$$\begin{aligned}
\partial &: \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} \partial f \\
\partial &: \quad \operatorname{ftype} f \longmapsto \operatorname{ftype}_{f_l} \partial f.
\end{aligned}$$

Tableau Rules. Replace any occurrence of $l+1$ in $\boldsymbol{P}$ by $\bullet$. See Figure 7.

Local Rules. Place partitions $\lambda^{(l+1)}, \ldots, \lambda^{(h)}$ in columns $l+1$ through h.

$$\begin{array}{r|c|c|c|l}
\text{column} & \multicolumn{1}{c}{l} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k} & \\ \cline{2-2}\cline{4-4}
\boldsymbol{P} & \lambda^{(l)} & & \lambda^{(k)} & \qquad \text{for } l+1 \le k \le h \\ \cline{2-2}\cline{4-4}
\partial\boldsymbol{P} & & & (\lambda^{(k)}) & \\ \cline{2-2}\cline{4-4}
\end{array}$$

All flags f with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ have $\operatorname{ftype}_{\hat{0}} \partial f = \partial\boldsymbol{P}$. In a regular semi-primary lattice whose type is $\operatorname{osh} \boldsymbol{P}$, all flags g with $\operatorname{ftype}_{\hat{0}} g = \partial\boldsymbol{P}$ have the same number of extensions to a flag $f = (f_l, g_{l+1}, \ldots, g_h)$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$; if $\lambda^{(l)} = \lambda^{(l+1)}$, then $f_l := g_{l+1}$ in one way, and if $\lambda^{(l)} \lessdot \lambda^{(l+1)}$ then f_l is any coatom of $[\hat{0}, f_{l+1}]$ of type $\lambda^{(l)}$.

All flags f with $\operatorname{ftype} f = \boldsymbol{P}$ have $\operatorname{ftype}_{f_l} \partial f = \partial\boldsymbol{P}$. In a lattice whose type is $\operatorname{sh} \boldsymbol{P}$, all flags g up to $\hat{1}$ with with $\operatorname{ftype}_{\hat{0}} g = \partial\boldsymbol{P}$ have one extension, $f = (\hat{0}, g_{l+1}, \ldots, g_h)$, with $\operatorname{ftype} f = \boldsymbol{P}$.

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2}\end{array}$$

$$\partial\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline \bullet & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2}\end{array} \quad \text{if } l = 1, \qquad \partial\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2}\end{array} \quad \text{if } l < 1.$$

$$d\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{if } h = 9, \qquad d\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline 2 & 5 & 6 \\ \hline 3 & 9 \\ \cline{1-2}\end{array} \quad \text{if } h > 9.$$

$$\mathcal{C}\boldsymbol{P} = \begin{array}{|c|c|}\hline \bullet & 4 \\ \hline 5 & 6 \\ \hline 9 \\ \cline{1-1}\end{array}$$

FIGURE 7. Deterministic games ∂, d, $\mathcal{C}$.

6.6. Remove the high entry, $d\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h-1)})$. Uniformly (in fact, identically),

$$\begin{aligned} d :\ \ \operatorname{ftype} f &\longmapsto \operatorname{ftype} df \\ d : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} df. \end{aligned}$$

Tableau Rules. Delete any cell with h from $\boldsymbol{P}$. See Figure 7.

Local Rules. Place partitions $\lambda^{(l)}, \ldots, \lambda^{(h-1)}$ in columns l through $h-1$.

$$\begin{array}{r} \text{column} \\ \boldsymbol{P} \\ d\boldsymbol{P} \end{array} \begin{array}{c} k \\ \hline \multicolumn{1}{|c|}{\lambda^{(k)}} \\ \hline \multicolumn{1}{|c|}{(\lambda^{(k)})} \\ \hline \end{array} \quad \begin{array}{c} h \\ \hline \multicolumn{1}{|c|}{\lambda^{(h)}} \\ \hline \multicolumn{1}{|c|}{} \\ \hline \end{array} \qquad \text{for } l \le k \le h-1$$

All flags f with $\operatorname{ftype} f = \boldsymbol{P}$ have $\operatorname{ftype} df = d\boldsymbol{P}$, and all with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ have $\operatorname{ftype}_{\hat{0}} df = d\boldsymbol{P}$.

In a semi-primary lattice whose type is osh $\boldsymbol{P}$, all flags g with $\operatorname{ftype}_{\hat{0}} g = d\boldsymbol{P}$ have exactly one extension $f = (g_l, \ldots, g_{h-1}, \hat{1})$ to a flag with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, and all flags g from $\hat{0}$ with $\operatorname{ftype} g = d\boldsymbol{P}$ have exactly one extension $f = (g_l, \ldots, g_{h-1}, \hat{1})$ to a flag with $\operatorname{ftype} f = \boldsymbol{P}$.

6.7. Delete the left column, $\mathcal{C}\boldsymbol{P} = (\mathcal{C}\lambda^{(l)}, \ldots, \mathcal{C}\lambda^{(h)})$. Uniformly,

$$\begin{aligned} \mathcal{C} :\ \ \operatorname{ftype} f &\longmapsto \operatorname{ftype} C f \\ \mathcal{C} : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} C_{\hat{0}} f. \end{aligned}$$

Tableau Rules. Delete the first column of $\boldsymbol{P}$. See Figure 7.

Local Rules. Place partitions $\mathcal{C}\lambda^{(l)}, \ldots, \mathcal{C}\lambda^{(h)}$ in columns l through h.

$$\begin{array}{r} \text{column} \\ \boldsymbol{P} \\ \mathcal{C}\boldsymbol{P} \end{array} \begin{array}{c} k \\ \hline \multicolumn{1}{|c|}{\lambda^{(k)}} \\ \hline \multicolumn{1}{|c|}{(\mathcal{C}\lambda^{(k)})} \\ \hline \end{array} \qquad \text{for } l \le k \le h$$

We will examine $\mathrm{ftype}_{\hat{0}}$, and ftype is treated by setting $\hat{0}$ to f_l.

All flags f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ yield $\mathrm{ftype}_{\hat{0}}\, f = \mathcal{C}\boldsymbol{P}$.

For the inverse problem, we show that all flags g to $C\hat{1}$ with type $\mathcal{C}\boldsymbol{P}$ can be expressed $g = Cf$ where f has type $\boldsymbol{P}$. Since $f_n = \hat{1}$, it is mandatory that $g_n = C\hat{1}$. So select any flag g to $C\hat{1}$ with $\mathrm{ftype}_{\hat{0}}\, g = \mathcal{C}\boldsymbol{P}$. Given

$$\longleftarrow \text{type} \boxed{\begin{matrix} (f_{k-1}) & \lessdot & f_k \\ Cf_{k-1} & \lessdot & Cf_k \end{matrix}} = \boxed{\begin{matrix} \gamma & \lessdot & \delta \\ \alpha & \lessdot & \beta \end{matrix}}$$

(that is, the partitions $\alpha, \beta, \gamma, \delta$ are given; f_k, Cf_k, and the desired value of Cf_{k-1} are given; but f_{k-1} is not given), we want to find all possible f_{k-1}. Let w be the given desired value of Cf_{k-1}.

If $\gamma = \delta$ then $f_{k-1} := f_k$ so $g_{k-1} := g_k$ and $\alpha := \beta$.

$$\boxed{\begin{matrix} \gamma & = & \delta \\ \gtrdot\!\!\mid & & \gtrdot\!\!\mid \\ \mathcal{C}\gamma & = & \mathcal{C}\delta \end{matrix}} \qquad \text{1 choice}$$

So let $\gamma \lessdot_{|r|} \delta$. The intervals $[\hat{0}, Cf_k]$ and $[A\hat{0}, f_k]$ are isomorphic by Theorem 4.17. Since $w \le Cf_k$, the type of $A_{f_k}w$ is $(\delta_1', \alpha_1', \alpha_2', \alpha_3', \ldots)'$, which agrees with γ in all columns except possibly the first. Any f_{k-1} with $Cf_{k-1} = w$ must have $f_{k-1} \le A_{f_k}w$ because the interval $[w, f_{k-1}]$ is elementary. If $r = 1$ then $\alpha = \beta$ and $w = Cf_k$; set f_{k-1} to be any lower cover of f_k whose type is γ.

$$\boxed{\begin{matrix} \gamma & \lessdot_{|1|} & \delta \\ \gtrdot\!\!\mid & & \gtrdot\!\!\mid \\ \mathcal{C}\gamma & = & \mathcal{C}\delta \end{matrix}} \qquad \begin{matrix} \langle\delta_1'\rangle - \langle\delta_2'\rangle = \\ \quad \langle \mathrm{row}(\boldsymbol{P}, k)\rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k)\rangle \\ \text{choices} \end{matrix}$$

If $r > 1$ then type $A_{f_k}w = \gamma$ so $f_{k-1} = A_{f_k}w$.

$$(r > 1) \quad \boxed{\begin{matrix} \gamma & \lessdot_{|r|} & \delta \\ \gtrdot\!\!\mid & & \gtrdot\!\!\mid \\ \mathcal{C}\gamma & \lessdot_{|r-1|} & \mathcal{C}\delta \end{matrix}} \qquad \text{1 choice}$$

Written in terms of tableaux, given a flag g up to $C\hat{1}$ with $\mathrm{ftype}_{\hat{0}}\, g = \mathcal{C}\boldsymbol{P}$, the number of flags f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $Cf = g$ is

$$\frac{F^{\boldsymbol{P}}(q)}{F^{\mathcal{C}\boldsymbol{P}}(q)} = \prod_{k\,:\ \mathrm{col}(\boldsymbol{P},k) = 1} \Big(\langle \mathrm{row}(\boldsymbol{P}, k)\rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k)\rangle\Big).$$

6.8. Saturating a flag, $\mathcal{S}\boldsymbol{P}$. Let $f = (f_l \le \cdots \le f_h)$, but no longer assume each $\le$ is $\lessdot$. The rules given previously for tableau games $\partial, d, \mathcal{C}$ do not change, and give the same transformations of flag types: for $\partial\boldsymbol{P}$, drop the low partition from $\boldsymbol{P}$, or equivalently, replace $\mathrm{low}(\boldsymbol{P}) + 1$ by $\bullet$; for $d\boldsymbol{P}$, drop the high partition, or equivalently, delete all cells whose entry is $\mathrm{high}(\boldsymbol{P})$; and for $\mathcal{C}\boldsymbol{P}$, apply $\mathcal{C}$ to each partition in $\boldsymbol{P}$, or equivalently, delete the first column of $\boldsymbol{P}$. The situation for most other games is more complicated, but can be handled in a systematic fashion.

Saturate the flag f by adding additional elements, as follows. For some i, let $r = \rho(f_{i-1}, f_i)$, and introduce additional indicies ordered $i - 1 = i_0 < i_1 < i_2 < \cdots < i_r = i$. Choose an extension of $(f_{i-1} < f_i)$ to a saturated flag $(f_{i-1} = f_{i_0} \lessdot f_{i_1} \lessdot f_{i_2} \lessdot \cdots \lessdot f_{i_r} = f_i)$; there is always some way to do this, and usually there are many. After doing this for all $l < i \le h$, let f^s be the flag so obtained. We introduce a game that uniformly determines the type of the saturated flag when $\boldsymbol{P}$ is semistandard. When $\boldsymbol{P}$ is row and column weak but not necessarily semistandard, we shall see that something close to uniformity is obtained.

$$\begin{aligned} \mathcal{S}, \widetilde{\mathcal{S}}:&\ \operatorname{ftype} f \longmapsto \operatorname{ftype} f^s \\ \mathcal{S}, \widetilde{\mathcal{S}}:&\ \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} f^s \end{aligned}$$

Tableau Rules. For each symbol i in $\boldsymbol{P}$, do the following. Suppose $\boldsymbol{P}$ has a total of r entries equal to i.

- $\boldsymbol{\mathcal{S}}$**:** Traverse $\boldsymbol{P}$ by going down the first column, then down the second column, and so on, and replace the occurrences of i by $i_1, \ldots, i_r$ in the order they are encountered.
- $\tilde{\boldsymbol{\mathcal{S}}}$**:** Replace the symbols i by $i_1, \ldots, i_r$ in any fashion so that the subtableau formed by $i_1, \ldots, i_r$ is a standard skew tableau.

Example 6.5.

$$\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \qquad \mathcal{S}\boldsymbol{P} = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1_3 & 1_5 \\ \hline 1_1 & 1_2 & 1_4 & 2_3 \\ \hline 2_1 & 2_2 \\ \cline{1-2} \end{array}$$

There are 5 ways to place subscripts on the 1's so that the subscripts are row and column strict, and 3 ways to similarly place them on the 2's, so there are 15 possible values of $\widetilde{\mathcal{S}}\boldsymbol{P}$.

In $\operatorname{ftype}_x f^s$, the cell of $\operatorname{type}[x, f_{i_k}]/\operatorname{type}[x, f_{i_{k-1}}]$ is filled with the symbol i_k, and the cells of $\operatorname{type}[x, f_i]/\operatorname{type}[x, f_{i-1}]$ are a standard skew tableau on entries $i_1, \ldots, i_r$. Not all f_{i-1} and f_i allow saturations permitting any given skew tableau filling, but given a filling, we can begin with arbitrary f_i and form a descending chain of elements terminating in some f_{i-1} so that the filling is achieved.

Theorem 6.6.

(1) *Given tableaux* $\boldsymbol{P}$ *and* $\boldsymbol{Q} = \widetilde{\mathcal{S}}\boldsymbol{P}$, *the number of pairs of flags* (f, f^s) *with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, $\operatorname{ftype}_{\hat{0}} f^s = \boldsymbol{Q}$, *and* f^s *a saturation of* f *is* $F^{\boldsymbol{Q}}(q)$.

(2) *Let* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *and* n_{ij} *be the number of entries in column* j *of* $\boldsymbol{P}$ *equal to* i. *There are*

$$\prod_{l<j\leq h} \langle n_{ij} \rangle !$$

saturations f^s *of* f *with* $\operatorname{ftype}_{\hat{0}} f = \mathcal{S}\boldsymbol{P}$.

(3) *When* $\boldsymbol{P}$ *is semistandard,* $\mathcal{S}$ *uniformly maps flags with* $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ *to saturations* f^s *with* $\operatorname{ftype}_{\hat{0}} f^s = \mathcal{S}\boldsymbol{P}$.

Proof.

(1) Choose any of $F^{\boldsymbol{Q}}(q)$ flags f^s with $\operatorname{ftype}_{\hat{0}} f^s = \boldsymbol{Q}$. The only possible choice of f is the subflag $f = (f^s_l, f^s_{l+1}, \ldots, f^s_h)$.

(2) Choose i with $l < i \leq h$. Let $N_j = n_{i1} + \cdots + n_{ij}$ and $I_j = i_{N_j}$. Let $\mu = \lambda^{(i-1)}$ and $\lambda = \lambda^{(i)}$. Since the desired type of f_{I_j} is $\mu \vee \lambda[j]$ and $f_{i-1} \leq f_{I_j} \leq f_i$, we have $f_{I_j} = f_{i-1} \vee f_i[j]$ for all j. Choose a saturation of the type $(1^{n_{ij}})$ interval $[f_{I_{j-1}}, f_{I_j}]$ to a flag $(f_{I_{j-1}} \lessdot f_{I_{j-1}+1} \lessdot \cdots \lessdot f_{I_j})$ in any of $\langle n_{ij} \rangle !$ ways. Multiply the number of choices over all i and j to obtain the total in the theorem.

(3) If $\boldsymbol{P}$ is semistandard, each $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. We show that generically, $[f_{i-1}, f_i]$ is a chain, so that there is only one saturation of it. Let y have type $\lambda^{(i)}$ and let $\mu = \lambda^{(i-1)}$. The number of $x \leq y$ with type μ and cotype ν is the Hall polynomial $g^{\lambda}_{\mu\nu}(q)$, of degree $n(\lambda) - n(\mu) - n(\nu)$; we seek to maximize this degree, or equivalenty, minimize $n(\nu)$. Since λ/μ is a horizontal strip of some length r, the value $\nu = (r)$ with $n(\nu) = 0$ uniquely minimizes $n(\nu)$.

So generically, $[f_{i-1}, f_i]$ is a chain for each i, and f has only one saturation, whose type is therefore $\mathcal{S}\boldsymbol{P}$. Every saturated flag of type $\mathcal{S}\boldsymbol{P}$ has exactly one inverse of type $\boldsymbol{P}$, so the map is uniform. $\square$

When $\boldsymbol{P}$ is not semistandard, there need not be a generic value of $\operatorname{ftype}_{\hat{0}} f^s$. For example, in q-regular semi-primary lattices of type $\lambda \vdash r$, if $f_0 = \hat{0}$ and $f_1 = \hat{1}$, there are $F^{\boldsymbol{Q}}(q)$ saturations of type $\boldsymbol{Q}$ for each standard tableau $\boldsymbol{Q}$ of shape λ on $1_1, \ldots, 1_r$. These polynomials all have degree $n(\lambda)$, so asymptotically, each tableau has probability $1/f^{\lambda}$ of occurring as the type of a saturation.

More generally, by Theorem 4.92, if $\operatorname{type} f_i / \operatorname{type} f_{i-1} = \lambda/\mu$, then generically, $\operatorname{type}[f_{i-1}, f_i] = \nu$, where $\nu = (S(\lambda' - \mu'))'$. Let $r = |\nu|$. Let

$$T = \{\, \boldsymbol{Q} \,:\, \boldsymbol{Q} \text{ is standard on } i_1, \ldots, i_r, \text{ with } \operatorname{sh} \boldsymbol{Q} = \lambda/\mu \text{ and } \operatorname{sh} j(\boldsymbol{Q}) = \nu \,\}\,.$$

As $q \to \infty$, over all saturations f^s of flags f of type $\boldsymbol{P}$, each tableau in T has probability $(\#T)^{-1} + O(q^{-1})$ of being the subtableau replacing the i's, while any tableau not in T has probability $o(q^{-1})$. Every flag f^s of type $\boldsymbol{Q}$ arises from a single flag f of type $\boldsymbol{P}$, so we have a condition similar to uniformity.

Note that $\mathcal{S}\boldsymbol{P} \in T$ because applying j to the subtableau of $i_1, \ldots, i_r$ simply deletes the $\bullet$'s at the top of each column, pushes the columns up to be aligned at the top, and then pushes the entries on each row horizontally left to fill any holes, thus obtaining a tableau of shape ν (this is transpose to Corollary 6.4).

In the following, the type of a flag may be interpreted as $\operatorname{ftype} f$ or $\operatorname{ftype}_x f$, as appropriate.

Theorem 6.7. *Let $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ be a growth on semi-primary lattices, and extend it to an operator on flags by $\phi(f_l, \ldots, f_h) = (\phi(f_l), \ldots, \phi(f_h))$. Suppose ϕ generically (respectively, uniformly) transforms multisaturated flags of type $\boldsymbol{P}$ to flags of type $\mathcal{G}\boldsymbol{P}$. If $\boldsymbol{P}$ is row and column weak but not necessarily standard, ϕ generically (uniformly) maps flags of type $\boldsymbol{P}$ to flags of type $\mathcal{S}^{-1}\mathcal{G}\mathcal{S}(\boldsymbol{P})$, where $\mathcal{S}^{-1}$ drops the subscripts from the entries. The degenerate possibilities are all of the form $\mathcal{S}^{-1}\widetilde{\mathcal{G}}\mathcal{S}(\boldsymbol{P})$.*

A similar statement holds for order-reversing growths, that is, $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that if $x \leq y$ then $\phi(y) \leq \phi(x)$. Set $\phi(f_l, \ldots, f_h) = (\phi(f_h), \ldots, \phi(f_l))$ in the above.

Proof. Since there are always saturations of type $\mathcal{S}\boldsymbol{P}$, the possible outcomes of a game on a nonsaturated flag may be determined by saturating it with type $\mathcal{S}\boldsymbol{P}$, applying further operators, and later "unsaturating" by dropping the subscripts from the entries; the value of $\phi(f)$ does not depend on the saturation, so the added elements can be removed.

Every flag of type $\boldsymbol{P}$ has the same number of saturations of type $\mathcal{S}\boldsymbol{P}$, and every saturated flag of type $\mathcal{S}\boldsymbol{P}$ has a single unsaturated inverse flag f of type $\boldsymbol{P}$. This is quite similar to uniformity, but with a multivalued saturation function instead of a singly valued function, so saturation composes with other generic or uniform operators in a fashion similar to Theorem 5.12 by essentially the same proof. □

All operators we describe on semi-primary lattices have the form described in the theorem, except for $d\boldsymbol{P}$ and $\partial\boldsymbol{P}$, which already have been described for nonsaturated flags. The theorem does not hold for $d\boldsymbol{P}$, for suppose $\rho(f_{h-1}, f_h) = r > 1$. We saturate f to obtain $f^s = (\ldots, f_{h_{r-1}}, f_{h_r})$. Then $df^s = (\ldots, f_{h_{r-1}})$, and so the type of this still has subscripted h's, which turn to ordinary h's when we apply $\mathcal{S}^{-1}$. However, $df = (f_l, \ldots, f_{h-1})$, so there are no h's in ftype df. A similar problem occurs for ∂f.

6.9. The dual of a flag, $\operatorname{ev}\boldsymbol{P}$**.** Uniformly,

$$\operatorname{ev}, \widetilde{\operatorname{ev}} : \operatorname{ftype} f \longmapsto \operatorname{ftype} f^*.$$

By Theorem 5.14, the uniform cotype of saturated flags of type $\boldsymbol{P}$ is $\operatorname{ev}\boldsymbol{P}$ when $\boldsymbol{P}$ is a standard tableau on $1, \ldots, n$. This is trivially extended to multisaturated flags via $\operatorname{ev}\boldsymbol{P} = (\operatorname{sh}\Delta^{n-k}\boldsymbol{P})_{k=n-h}^{l}$, with degenerate possibilities given by using $\widetilde{\Delta}$ instead of Δ.

Tableau Rules. Let $\boldsymbol{P}$ be a standard tableau on distinct entries. Repeatedly apply the operator Δ (for ev) or $\widetilde{\Delta}$ (for $\widetilde{\operatorname{ev}}$) to the subtableau of unstarred entries, and when an entry k is deleted, place an entry k^* in the cell that is concurrently deleted. Finally, a tableau with only starred entries is obtained, and each k^* is replaced by $n+1-k$.

Local Rules. This does not have a representation in the form of two lines of partitions, but rather, has the form of a triangular array of partitions; see Section 5.2.

This can also be extended to flags that aren't from $\hat{0}$ to $\hat{1}$. The following map is uniform on flags in semi-primary lattices of type λ.

$$\operatorname{ev}^\lambda, \widetilde{\operatorname{ev}}^\lambda : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}^*} f^*$$

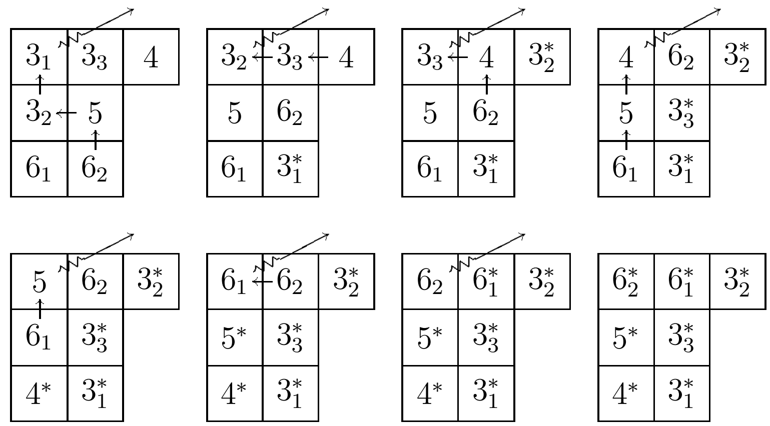
Tableau Rules. Let $\boldsymbol{P}$ be a standard skew tableau of shape μ/ν with $\mu \le \lambda$. Let $r = |\nu|$ and $s = |\lambda| - |\mu|$. Fill ν with entries $l_1, l_2, \ldots, l_r$ by traversing its cells down the first column, then down the second column, and so on. Fill λ/μ with entries $(h+1)_1, (h+1)_2, \ldots, (h+1)_s$ in a similar fashion. Apply the ordinary ev or $\widetilde{\operatorname{ev}}$ as described above, but replace all $(h+1)_i^*$'s by $\bullet$'s and remove all cells with l_i^*'s.

This map is uniform: extend the flag $f = (f_l, \ldots, f_h)$ to the flag $(\hat{0}, f_l, \ldots, f_h, \hat{1})$ (filling ν with l's and λ/μ with $(h+1)$'s), saturate it with a canonical saturation ($\mathcal{S}$), dualize it (apply ev), unsaturate it ($\mathcal{S}^{-1}$), and remove the leading $\hat{1}^*$ (replacing the $(h+1)_i$'s by $\bullet$'s) and the trailing $\hat{0}^*$ (removing the cells with l_i's).

Example 6.8. Let $\lambda = (3,3,2)$ and $n = 7$. In semi-primary lattices of type λ, consider flags $f = (f_3, f_4, f_5)$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$, whose saturation to shape λ is $\boldsymbol{P}^s$:

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 4 \\ \hline \bullet & 5 \\ \cline{1-2} \end{array} \qquad \boldsymbol{P}^s = \begin{array}{|c|c|c|}\hline 3_1 & 3_3 & 4 \\ \hline 3_2 & 5 \\ \cline{1-2} 6_1 & 6_2 \\ \cline{1-2} \end{array}.$$

This is a relabeling of the tableau used throughout Section 5.1. Do repeated applications of Δ, keeping track of the vacated cells by replacing them with starred deleted entries:



Replace 6_i's by $\bullet$'s and remove 3_i's to obtain that uniformly, we expect

$$\mathrm{ftype}_{\hat{0}^*} f^* = \mathrm{ev}^\lambda \boldsymbol{P} = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline 5^* \\ \cline{1-1} 4^* \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}.$$

7. The Robinson-Schensted correspondence

7.1. The Robinson-Schensted correspondence. Let S_n be the group of permutations on $1, \ldots, n$. Robinson [**3**] and Schensted [**24**] created an algorithmic bijection between ordered pairs of standard tableaux of the same shape with entries $1, \ldots, n$, and S_n. See Sagan [**22**] for a comprehensive account. There have been many generalizations of it, including for repeated entries (Knuth [**15**]), skew tableaux (Sagan and Stanley [**23**]), bijections of a similar nature with other structures (Fomin [**4**]), and oscillating chains in Young's lattice (Roby [**21**]). The basic algorithm is as follows.

Robinson-Schensted Row Insertion. Let $\boldsymbol{P}$ be a standard tableau on distinct entries, and k be a number not in $\boldsymbol{P}$. Insert k in the first row by either replacing the smallest number on the row larger than k with k, or by appending k to the end of the row if it is larger than all entries on it. If a number was replaced, insert it into the second row in the same fashion. Continue this on successive rows until the process terminates by appending a number to a row. The final tableau is the result of row inserting k into $\boldsymbol{P}$.

Robinson-Schensted Algorithm. Let $\sigma \in S_n$. Let $\boldsymbol{P}_0 = \emptyset$, and for $k = 1, \ldots, n$, form $\boldsymbol{P}_k$ by row inserting $\sigma(k)$ into $\boldsymbol{P}_{k-1}$. Define $\boldsymbol{P}(\sigma) = \boldsymbol{P}_n$, called the **insertion tableau**. Form a tableau $Q(\sigma)$ on $1, \ldots, n$, called the **recording tableau**, by placing k at the cell where inserting $\sigma(k)$ into $\boldsymbol{P}_{k-1}$ terminates, so that $\boldsymbol{Q}(\sigma) = (\mathrm{sh}\, \boldsymbol{P}_0, \ldots, \mathrm{sh}\, \boldsymbol{P}_n)$.

The inverse map is given by a deletion process.

Robinson-Schensted Deletion. Let $\boldsymbol{P}$ be a standard tableaux on distinct entries, and c be an inner corner of $\boldsymbol{P}$. The deletion step beginning at c is as follows. Let e be the entry at c. Remove cell c. If e is on the top row, we are done. Otherwise, go up one row. Replace the largest entry smaller than e with e and let e be the number that was replaced. If we're on the top row, we're done, and otherwise, go up another row, and replace the largest entry smaller than e with e. Continue in this fashion until a number is removed from the top row. That number is said to be **deleted** from the tableau.

Inverse Robinson-Schensted Algorithm. Let $(\boldsymbol{P}, \boldsymbol{Q})$ be a pair of standard tableaux of equal shape with entries $1, \ldots, n$. Let $c_k \in \mathbb{P} \times \mathbb{P}$ be the cell containing k in $\boldsymbol{Q}$. Let $\boldsymbol{P}_n = \boldsymbol{P}$, and for $k = n, \ldots, 1$, form $\boldsymbol{P}_{k-1}$ from $\boldsymbol{P}_k$ by a deletion step beginning at cell c_k, resulting in some entry $\sigma(k)$ being deleted from the tableau. Denote the permutation σ obtained in this fashion by $\sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.

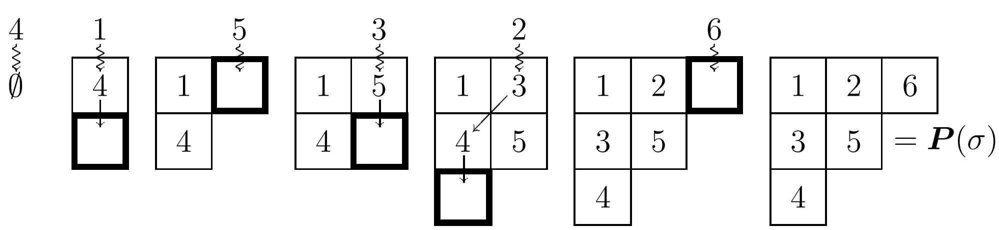
Theorem 7.1 (Robinson-Schensted Correspondence). *The maps*

$$\sigma \mapsto (\boldsymbol{P}(\sigma), \boldsymbol{Q}(\sigma)) \qquad \text{and} \qquad (\boldsymbol{P}, \boldsymbol{Q}) \mapsto \sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$$

are inverse bijections between S_n and pairs of standard tableaux of the same shape on $1, \ldots, n$.

See Robinson [**3**] and Schensted [**24**] for the original proof.

Example 7.2. The **one-line** form of a permutation σ is the sequence of numbers $\sigma(1)\,\sigma(2)\cdots$. Let $\sigma = 4\ 1\ 5\ 3\ 2\ 6$. The sequence of tableaux $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_6$ and the insertion paths are as follows. The outer corner in which the path will terminate is shown in bold.



$$\boldsymbol{Q}(\sigma) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

The inverse Robinson-Schensted algorithm is performed via deletion paths obtained by translating the arrows one tableau right and reversing their direction.

7.2. Bruhat order. Let $\sigma \in S_n$ and $A(\sigma) = (a_{ij})_{0 \le i,j \le n}$ be the matrix with entries $a_{ij} = \#\{\, j' : j' \le j \text{ and } \sigma(j') \le i' \,\}$. Note that $\sigma(j) = \min\{\, i : a_{ij} > a_{i,j-1} \,\}$ and $\sigma(j) = i$ iff $a_{ij} > a_{i,j-1}$ but $a_{i-1,j} = a_{i-1,j-1}$.

The **Bruhat order** is a graded partial order on S_n given by $\sigma \le \tau$ iff $A(\sigma) \ge A(\tau)$ entrywise. See [**29**, p. 528] and [**31**, p. 83] for discussion of the Bruhat order in this form, and [**17**, pp. 5–8] for an in-depth discussion of other descriptions and properties of the order.

7.3. Relative positions of flags in modular lattices. Throughout this section, $f = (f_0 \lessdot \cdots \lessdot f_n)$ and $f' = (f'_0 \lessdot \cdots \lessdot f'_n)$ are saturated flags from $\hat{0}$ to $\hat{1}$ in a modular lattice.

Definition 7.3. This **relative position** of flags f and f' is the permutation $\sigma(f, f')$ given by $\sigma(k) = \min\Big\{ i \,:\, f_i \wedge f'_k \gtrdot f_i \wedge f'_{k-1} \Big\}$ for $k = 1, \ldots, n$.

We can also replace $f_i \wedge f'_k \gtrdot f_i \wedge f'_{k-1}$ by $\rho(f_i \wedge f'_k) \gtrdot \rho(f_i \wedge f'_{k-1})$ in the above. The relative position is completely determined by the ranks $\rho(f_i \wedge f'_k)$ for $0 \leq i, k \leq n$.

Another construction of $\sigma(f, f')$ is as follows. For each $0 \leq k \leq n$, consider $g^{(i)} = (f_i \wedge f'_0 \leqq \cdots \leqq f_i \wedge f'_n) = f' \wedge f_i$. The bottom element of $g^{(i)}$ is $\hat{0}$ and the top element is $f'_n \wedge f_i = f_i$, so of the weak cover relations, i are strict. Clearly $g^{(i-1)} = g^{(i)} \wedge f_{i-1}$. Consider the $\leqq$ relations in $g^{(i)}$ and in $g^{(i-1)}$. One $\lessdot$ in $g^{(i)}$ becomes $=$ in $g^{(i-1)}$, and all other $\lessdot$ or $=$ remain the same. If $f_i \wedge f'_{k-1} \lessdot f_i \wedge f'_k$ but $f_{i-1} \wedge f'_{k-1} = f_{i-1} \wedge f'_k$, define $\sigma(k) = i$. This associates a unique k to each $i = n, \ldots, 1$, and since all relations are $\lessdot$ in $g^{(n)}$ and $=$ in $g^{(0)}$, there is for each k a unique i when the kth relation switches from $\lessdot$ to $=$. So $\sigma(f, f')$ is a permutation.

Theorem 7.4 (Steinberg [29]). *In the invariant subspace lattice over an infinite field, the generic relative position of saturated flags (f, f') with* ftype $f = \boldsymbol{P}$ *and* ftype $f' = \boldsymbol{Q}$ *is $\sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.*

See also van Leeuwen [**31**].

Conjecture 7.5. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be standard tableaux of shape λ. In q-regular semi-primary lattices of type λ, as $q \to \infty$, all but $O(q^{-1})$ pairs (f, f') of flags with* ftype $f = \boldsymbol{P}$ *and* ftype $f' = \boldsymbol{Q}$ *have $\sigma(f, f') = \sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.*

The tableaux $\boldsymbol{P} = (\lambda^{(0)}, \ldots, \lambda^{(n)})$ and $\boldsymbol{Q} = (\mu^{(0)}, \ldots, \mu^{(n)})$ are chains of partitions in Young's lattice, which is modular. Their relative position $\sigma(\boldsymbol{P}, \boldsymbol{Q})$ in Young's lattice is the permutation $\sigma(j) = i$ where the cell with i in $\boldsymbol{P}$ has j in $\boldsymbol{Q}$: comparing the tableaux representing the chains $\boldsymbol{P} \wedge \mu^{(j)}$ and $(\boldsymbol{P} \wedge \mu^{(j)}) \wedge \mu^{(j-1)} = \boldsymbol{P} \wedge \mu^{(j-1)}$, we see that the cell with j in $\boldsymbol{Q}$ is deleted from the former and all other cells stay intact. The deleted entry is whatever was in that cell in $\boldsymbol{P}$.

Conjecture 7.6. *Let (f, f') be a pair of saturated flags in a semi-primary lattice with* ftype $f = \boldsymbol{P}$ *and* ftype $f' = \boldsymbol{Q}$*. Then $\sigma(\boldsymbol{P}, \boldsymbol{Q}) \leq \sigma(f, f') \leq \sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$ holds in the Bruhat order.*

We will see later, in Corollary 9.20, that this holds in all semi-primary lattices whose type has two rows. For arbitrary types, The upper bound holds for invariant subspace lattices over infinite fields (see [**31**, Theorem 2.5.4]), and the lower bound is a consequence of the following.

Theorem 7.7. *Let L and M be modular lattices, and $\phi : L \mapsto M$ be a rank and order preserving map. Let f and f' be saturated flags in L from $\hat{0}$ to x. Then $\sigma\Big(\phi(f), \phi(f')\Big) \leq \sigma(f, f')$.*

Proof. The map ϕ preserves order, but not necessarily meets and joins. We have $f_i \wedge f'_j \le f_i, f'_j$ so $\phi(f_i \wedge f'_j) \le \phi(f_i), \phi(f'_j)$, whence $\phi(f_i \wedge f'_j) \le \phi(f_i) \wedge \phi(f'_j)$. Take the ranks of both sides: $\rho(f_i \wedge f'_j) \le \rho\Big(\phi(f_i) \wedge \phi(f'_j)\Big)$. Thus, $\sigma(f, f') \ge \sigma\Big(\phi(f), \phi(f')\Big)$ in the Bruhat order. $\square$

We consider an enumeration problem for pairs of flags of specified types in specified relative positions, similar to the enumeration problem for the interval type table of a flag considered in Section 5.5.

Conjecture 7.8. *Given standard tableaux* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *of equal shape* λ *on* $1, \ldots, n$, *and a permutation* $\sigma \in S_n$, *the number of pairs* (f, f') *of flags with* $\operatorname{ftype} f = \boldsymbol{P}$, $\operatorname{ftype} f' = \boldsymbol{Q}$, *and* $\sigma(f, f') = \sigma$ *is often a polynomial in* q, *depending only on* $\boldsymbol{P}$, $\boldsymbol{Q}$, *and* σ, *and on no further structure of the lattice. The polynomial is divisible by* $M_\lambda(q)$. *As a function of* σ, *its degree increases as* σ *increases in the Bruhat order from* $\sigma(\boldsymbol{P}, \boldsymbol{Q})$ *to* $\sigma_{\mathrm{RS}}(\boldsymbol{P}, \boldsymbol{Q})$.

We consider three square arrays of partitions that each refine the notion of relative position. Their components $\lambda^{(ij)}$ for $0 \le i, j \le n$ are as follows.

	$\lambda^{(ij)}$	$\rho(f_i \wedge f'_j)$
(1)	$\operatorname{type}(f_i \wedge f'_j)$	$\lvert\lambda^{(ij)}\rvert$
(2)	$\operatorname{type}[f_i, f_i \vee f'_j]$	$j - \lvert\lambda^{(ij)}\rvert$
(3)	$\operatorname{type}(f_i \vee f'_j)$	$i + j - \lvert\lambda^{(ij)}\rvert$

The relative position of (f, f') can be computed from any of these, because the values of $\rho(f_i \wedge f'_j)$ can be determined. In subsequent sections it is shown that given $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} f' = \boldsymbol{Q}$, computation of (1) is similar to Robinson-Schensted deletion; (2) is similar to a transposed deletion operating on columns instead of rows; and (3) is similar to insertion in skew tableaux.

Conjecture 7.9. *For all but* $O(q^{-1})$ *pairs* (f, f') *with* $\operatorname{ftype} f = \boldsymbol{P}$ *and* $\operatorname{ftype} f' = \boldsymbol{Q}$, *in (1) the tableaux* $\boldsymbol{P}_n, \ldots, \boldsymbol{P}_0$ *obtained by Robinson-Schensted deletion from* $(\boldsymbol{P}, \boldsymbol{Q})$ *are* $\boldsymbol{P}_k = (\lambda^{(0k)}, \ldots, \lambda^{(nk)})$, *and in (2) the same, but with the transposed deletion process.*

We will later see that (3) does not have a generic value.

In the next section, we will consider enumeration of the number of pairs (f, x) where $x \lessdot \hat{1}$ and the desired values of $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \wedge x)$ are given. Subsequent sections will consider similar problems for (2) and (3), and will encounter similar difficulties. Let $\boldsymbol{P}_{n-1}$ be the tableau obtained from a Robinson-Schensted deletion step in $\boldsymbol{P}$ beginning at cell $(\operatorname{sh} \boldsymbol{P})/\operatorname{type} x$. The most common value of $\operatorname{ftype}_{\hat{0}}(f \wedge x)$ given $\boldsymbol{P}$ and $\operatorname{type} x$ is $\boldsymbol{P}_{n-1}$. To compute the relative position, we look at flags $f \wedge f'_k$. Since $f \wedge f'_{k-1} = (f \wedge f'_k) \wedge f'_{k-1}$, it would be tempting to repeat the process to obtain a series of Robinson-Schensted deletion steps. The uniformity condition in Theorem 5.12 that we use elsewhere to compose steps is *not* met here: $f \mapsto f \wedge x$ generically, but not uniformly, maps flags with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ to flags g with $\operatorname{ftype}_{\hat{0}} g = \boldsymbol{P}_{n-1}$; topologically, $\{\, f \wedge x \,:\, \operatorname{ftype}_{\hat{0}}(f \wedge x) = \boldsymbol{P}_{n-1} \text{ and } \operatorname{ftype}_{\hat{0}} f = \boldsymbol{P} \,\}$ is not dense in $\{\, g \,:\, g_n = x \text{ and } \operatorname{ftype}_{\hat{0}} g = \boldsymbol{P}_{n-1} \,\}$, while enumeratively, the cardinality of the former set is a fraction $O(q^{-1})$ of the latter, rather than $1-O(q^{-1})$. Some other technique is required to continue. Steinberg [**29**] resolves this for invariant subspace lattices over infinite fields by using properties of the Zariski topology and a larger lattice (all pairs of saturated flags in the vector space V, without the condition of invariance under N) that concurrently embeds invariant subspace lattices of all types of order $\dim V$.

7.4. Robinson-Schensted deletion, $\mathcal{D}_c\boldsymbol{P}$. Let $\lambda/\mu = \operatorname{sh}\boldsymbol{P}$; the parameter c is either $\emptyset$ or an inner corner of λ not in μ.

Pick $x \leq f_h$. The following map is generic but not uniform.

$$\begin{array}{lll} \mathcal{D}_c, \widetilde{\mathcal{D}}_c: & \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \wedge x) & c = \lambda/(\operatorname{type}[f_l \wedge x, f_h \wedge x]) \\ \mathcal{D}_c, \widetilde{\mathcal{D}}_c: & \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \wedge x) & c = \lambda/\operatorname{type} x \end{array}$$

Tableau Rules. See Figure 8. Let $\boldsymbol{P}$ be a standard skew tableau on distinct entries, and c be an inner corner of $\operatorname{osh}\boldsymbol{P}$ or $c = \emptyset$. If $c = \emptyset$ then $\widetilde{\mathcal{D}}_c\boldsymbol{P} = \boldsymbol{P}$. Otherwise, proceed as follows.

We insert an entry e into a row by replacing its largest entry (possibly $\bullet$) smaller than e with e; the number it replaces is **bumped** from the row.

- **($\mathcal{D}_c$):** Begin by letting e be the entry at c and removing cell c from the tableau. Insert e into the row above c, bumping out a number. Insert that number one row higher, bumping out another number. Continue this until either an entry is bumped out from the top row, or $\bullet$ is bumped out.
- **($\widetilde{\mathcal{D}}_c$):** Begin by letting e be the entry at c and removing cell c from the tableau. Either choose any higher up row and insert e on it (provided the column e lands in continues to increase from top to bottom), or choose to stop. Take the bumped out number, and either choose a still higher row to insert it into, provided the column remains increasing, or choose to stop. Continue in this fashion until optionally stopping; bumping out $\bullet$; or bumping out an entry from the top row.

The final value bumped out is said to be **deleted** from the tableau.

Local Rules. If $c = \emptyset$, let $\mu^{(h)} := \lambda^{(h)}$, and if c is a single cell, let $\mu^{(h)}$ be $\lambda^{(h)}$ with c deleted. Then for $k = h, h-1, \ldots, l+1$, propagate as follows.

$$\begin{array}{c|c|c|c|l} & k-1 & k & h & \text{column} \\ \cline{2-4} & \gamma & \delta & \lambda^{(h)} & \boldsymbol{P} \\ \cline{2-4} \longleftarrow & (\alpha) & \beta & (\lambda^{(h)} \ominus c) & \widetilde{\mathcal{D}}_c\boldsymbol{P} \\ \cline{2-4} \end{array}$$

$\boldsymbol{\beta = \gamma \lessdot \delta}$ does not hold: Let $\alpha := \beta \wedge \gamma$.
$\boldsymbol{\beta = \gamma \lessdot \delta}$ holds: Let $c := \delta/\beta$.
- **$\mathbf{row}(\boldsymbol{c}) = \mathbf{1}$:** Let $\alpha := \gamma$.
- **$\mathbf{row}(\boldsymbol{c}) > \mathbf{1}$ ($\mathcal{D}_c$):** Delete the last cell of γ on the row above c to obtain α.
- **$\mathbf{row}(\boldsymbol{c}) > \mathbf{1}$ ($\widetilde{\mathcal{D}}_c$):** Choose either to let $\alpha := \gamma$, or to delete any inner corner of γ whose row is less than $\operatorname{row}(c)$ to obtain α.

The unique k such that $\lambda^{(k)} \gtrdot \lambda^{(k-1)}$ and $\mu^{(k)} = \mu^{(k-1)}$ is the entry deleted from the tableau.

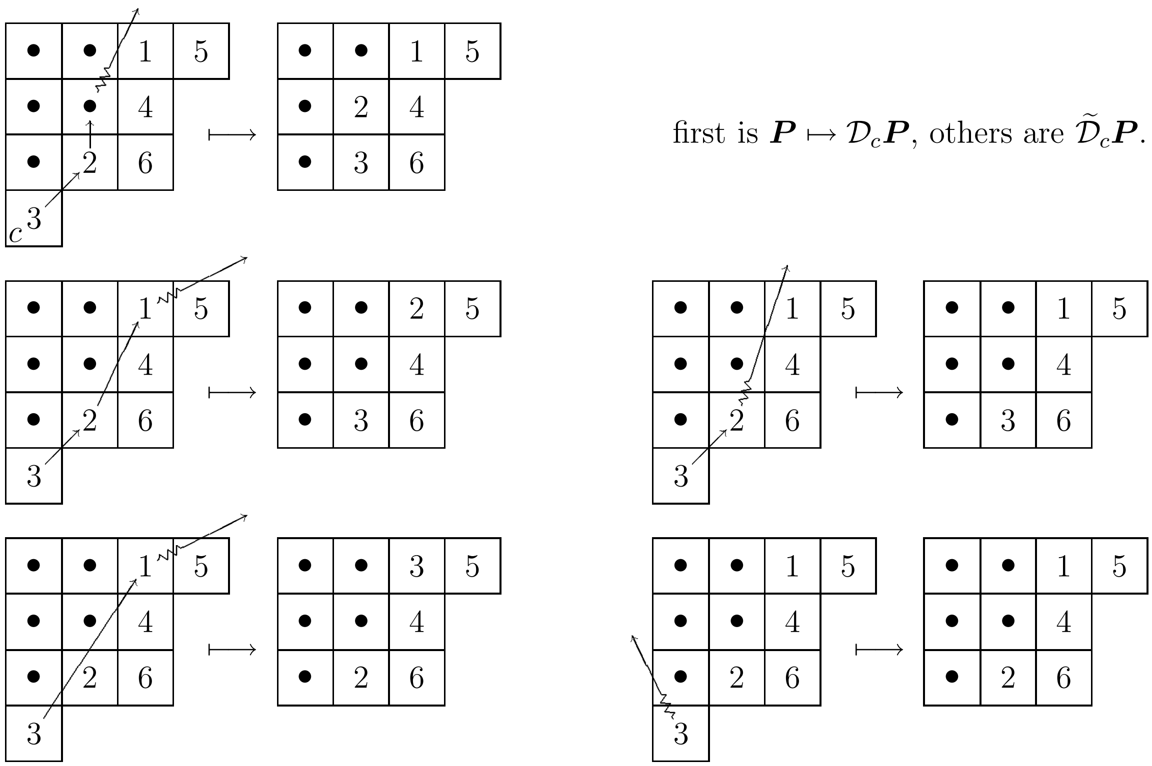


FIGURE 8. All possible degenerate skew Robinson-Schensted deletion games.

We will show that the tableau interpretation and the local rule interpretation of the games $\mathcal{D}_c$ and $\widetilde{\mathcal{D}}_c$ are the same, and that they describe $f \wedge x$.

Proposition 7.10. *The tableau rules and local rules given above for $\mathcal{D}_c$ and $\widetilde{\mathcal{D}}_c$ describe the same game.*

Proof. Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\widetilde{\mathcal{D}}_c\boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(h)})$ (as given by the tableau rules). Since each cell of $\boldsymbol{P}$ either vanishes, stays the same, or increases in $\widetilde{\mathcal{D}}_c\boldsymbol{P}$, we have $\mu^{(k)} \leq \lambda^{(k)}$ for $l \leq k \leq h$. If e is the entry bumped out, then $\mu^{(k)} = \lambda^{(k)}$ for $l \leq k < e$ because no cells with entries less than e are changed. Also, $\mu^{(k)} \lessdot \lambda^{(k)}$ for $e \leq k \leq h$ because the number of entries less than or equal to k in $\boldsymbol{P}$ is exactly one more than in $\widetilde{\mathcal{D}}_c\boldsymbol{P}$ since e was bumped out.

For $k = l, \ldots, h$, let $\boldsymbol{P}^{(k)} = (\lambda^{(l)} \wedge \mu^{(k)}, \ldots, \lambda^{(k)} \wedge \mu^{(k)}, \mu^{(k+1)}, \ldots, \mu^{(h)})$, which we picture as follows:

$$\begin{array}{lll} & \lambda^{(l)} \lessdot_{\boxed{l+1}} \lambda^{(l+1)} \lessdot_{\boxed{l+2}} \cdots \lessdot_{\boxed{k}} & \lambda^{(k)} \\ & & \gtrdot \ \boxed{} \nearrow e \ = c \\ \boldsymbol{P}^{(k)} \ = & & \mu^{(k)} \lessdot_{\boxed{k+1}} \mu^{(k+1)} \lessdot_{\boxed{k+2}} \cdots \lessdot_{\boxed{h}} \mu^{(h)} \end{array}$$

Reading left to right, the cells added in successive partitions are filled with the boxed entries, except that the entry $e \leq k$ placed in the square (if any) of $\lambda^{(k)}/\mu^{(k)}$ is removed, possibly to be filled later in the chain with an entry larger than k.

The tableau $\boldsymbol{P}^{(h)}$ is $\boldsymbol{P}$ with the initial cell c, containing an entry e, deleted. Now compare $\boldsymbol{P}^{(k)}$ and $\boldsymbol{P}^{(k-1)}$, successively for $k = h, h-1, \ldots, l+1$. The two chains differ only in the partitions

$$\begin{array}{cc} \lambda^{(k-1)} & \lambda^{(k)} \\ \mu^{(k-1)} & \mu^{(k)} \end{array}$$

Let $c_0 = \lambda^{(k)}/\mu^{(k)}$ and $c_1 = \lambda^{(k-1)}/\mu^{(k-1)}$. If these are the same, the tableaux $\boldsymbol{P}^{(k-1)}$ and $\boldsymbol{P}^{(k)}$ are equal. We will show that the local rules for $\widetilde{\mathcal{D}}_c$ mimic the changes in these tableaux when c_0 and c_1 are unequal. When c_0 and c_1 are different squares, we have

$$\boxed{\begin{array}{ccc} \lambda^{(k-1)} & \lessdot_{c_0} & \lambda^{(k)} \\ \dot\vee_{c_1} & & \dot\vee_{c_0} \\ \mu^{(k-1)} & \lessdot_{c_1} & \mu^{(k)} \end{array}}$$

Thus, the cell c_0 that was deleted in $\boldsymbol{P}^{(k)}$ to make way for a larger entry contained k. To form $\boldsymbol{P}^{(k-1)}$, bump the entry $e < k$ out of c_1 and place k in c_1.

By the local rules, c_1 is any inner corner of $\lambda^{(k-1)}$ on an earlier row than c_0, and by the tableau rules, k displaces the rightmost entry on an earlier row than c_0, provided the column remains increasing; these correspond to each other. In the generic game $\mathcal{D}_c$, this is restricted by $1 \le \operatorname{row}(c_1) \lessdot \operatorname{row}(c_0)$.

When c_0 is a square but c_1 is $\emptyset$, the value k is bumped from the tableau. In $\widetilde{\mathcal{D}}_c$, choosing $c_1 = \emptyset$ is permitted at any step, while in $\mathcal{D}_c$, it is only permitted when $\operatorname{row}(c_0) = 1$. $\square$

Lemma 7.11. *In a q-regular semi-primary lattice, the number of solutions to*

$$\operatorname{type}\boxed{\begin{array}{ccc} (f_{k-1}) & \lessdot & f_k \\ \underline{\dot\vee} & & \underline{\dot\vee} \\ (g_{k-1}) & \lessdot & g_k \end{array}} = \boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \underline{\dot\vee} & & \underline{\dot\vee} \\ \alpha & \lessdot & \beta \end{array}} \qquad f_{k-1} \neq g_k$$

(that is, $\alpha, \beta, \gamma, \delta$ are given partitions, $f_k \ge g_k$ are given elements of respective types $\delta \ge \beta$, and we seek elements f_{k-1}, g_{k-1} to satisfy the remaining relations) depends on the partitions, as follows.

(1) $(\beta \neq \gamma)$ $\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \underline{\dot\vee} & & \underline{\dot\vee} \\ \alpha & \lessdot_{|r|} & \beta \end{array}}$ $\qquad \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle$

(2) $(r \neq s)$ $\boxed{\begin{array}{ccc} \gamma & \lessdot_{|s|} & \delta \\ \dot\vee_{|r|} & & \dot\vee_{|s|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}}$ $\qquad \left(\langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle \right) (q-1)$
further, in this case, $r > s$,
even when the lattice is not q-regular.

(3) $\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \dot\vee_{|r|} & & \dot\vee_{|r|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}}$ $\qquad \left(\langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle \right) q$

Proof.

(1) Let f_{k-1} be any lower cover of f_k of type γ, in one of $\langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle$ ways. Since $g_k \le f_k$, also $f_{k-1} \wedge g_k \le g_k$. This meet is an upper bound for g_{k-1}. We have $\beta \wedge \gamma \lessdot \beta$, so $\operatorname{type}(f_{k-1} \wedge g_k) \le \operatorname{type} f_{k-1} \wedge \operatorname{type} g_k = \gamma \wedge \beta$. Thus $f_{k-1} \wedge g_k$ has rank at least one lower than g_k, but also it has rank at most one lower, so $g_{k-1} := f_{k-1} \wedge g_k$ in one way.

(2) In the remaining cases, $|\alpha| \lessdot |\beta| = |\gamma| \lessdot |\delta|$, and $f_{k-1} \wedge g_k = g_{k-1}$ and $f_{k-1} \vee g_k = f_k$, since f_{k-1}, g_k are distinct elements in the rank two interval $[g_{k-1}, f_k]$.

For any $f_{k-1} \lessdot f_k$ of type γ, we have $f_k[s-1] \leq g_k, f_{k-1}$ by Theorem 3.8 as the types of f_k, g_k, f_{k-1} agree in the first $s-1$ columns. Thus $f_k[s-1] \leq g_k \wedge f_{k-1} = g_{k-1}$, so $\alpha = \operatorname{type} g_{k-1}$ also agrees with β, γ, and δ in the first $s-1$ columns, whence $r \geq s$.

Pick any $g_{k-1} \lessdot g_k$ with $\operatorname{type} g_{k-1} = \alpha$, in one of $\langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle$ ways.

Next we determine which elements of (g_{k-1}, f_k) are suitable choices for f_{k-1}. We have $\operatorname{type}[g_{k-1}, f_k] = \boxminus$ by applying Corollary 4.72 to $\operatorname{ftype}_{\hat{0}}(g_{k-1}, g_k, f_k) = (\alpha, \beta, \delta)$ because the cell β/α is weakly right and strictly above δ/β.

If $r > s$, there are two partitions between α and δ. The element $g_{k-1} \vee f_k[s]$ has type $\alpha \vee \delta[s]$ (add the left square of δ/α to α) but the other q elements (including g_k) have type γ, so there are $q-1$ choices of f_{k-1} distinct from g_k.

(3) Continuing the analysis of the previous case, if $r = s$, then δ/α is a vertical brick, so all $q+1$ elements in (g_{k-1}, f_k) have type γ, and there are q choices of f_{k-1} (since it is distinct from g_k, which is also in this interval). $\square$

Theorem 7.12. *Consider a q-regular semi-primary lattice of type λ. Let $\mu \lessdot \lambda$ and $c = \lambda/\mu$. Let $\boldsymbol{P}$ be a skew tableau of outer shape λ, and fix $\boldsymbol{Q} = \widetilde{\mathcal{D}}_c \boldsymbol{P}$.*

Pick $x \lessdot \hat{1}$ of type μ. The number of flags f with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \wedge x) = \boldsymbol{Q}$ is a product of terms that depend on how entries move from $\boldsymbol{P}$ to $\boldsymbol{Q}$:

$$\prod_{k \,\in\, \boldsymbol{Q}} \begin{cases} \langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle & \text{if } k \text{ is in the same cell;} \\ \Big(\langle \operatorname{row}(\boldsymbol{Q}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k)\rangle\Big) \cdot q & \text{if } k \text{ moves up one cell;} \\ \Big(\langle \operatorname{row}(\boldsymbol{Q}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k)\rangle\Big) \cdot (q-1) & \text{if } k \text{ moves up and right.} \end{cases}$$

This is a polynomial of degree

$$n(\operatorname{sh} Q) + \textit{length of deletion path} - 1.$$

It is maximized by having the deletion path contain a cell on every row above c, which happens uniquely in the generic game $\boldsymbol{Q} = \mathcal{D}_c \boldsymbol{P}$. The game is not uniform, however.

Conversely, in all semi-primary lattices, for any flag $f = (f_l \leq \cdots \leq f_h)$ and element $x \leq f_h$,

$$\begin{aligned} \operatorname{ftype}(f \wedge x) &= \widetilde{\mathcal{D}}_c \operatorname{ftype} f && \textit{where } c = (\operatorname{type}[f_l, f_h])/(\operatorname{type}[f_l \wedge x, f_h \wedge x]), \\ \operatorname{ftype}_{\hat{0}}(f \wedge x) &= \widetilde{\mathcal{D}}_c \operatorname{ftype}_{\hat{0}} f && \textit{where } c = (\operatorname{type} f_h)/(\operatorname{type} x). \end{aligned}$$

The generic case in the invariant subspace lattice over an infinite field was first done by Steinberg [**29**, Lemma 1.2], and later by van Leeuwen [**31**, Lemma 2.5.1].

Proof. Since $g_k = f_k \wedge x$, we have $g_{k-1} = f_{k-1} \wedge x = f_{k-1} \wedge f_k \wedge x = f_{k-1} \wedge g_k$. Since $x \leq \hat{1}$, we have $g_k \leq f_k$, and $\mu^{(k)} \leq \lambda^{(k)}$. We will construct f and g with $g = f \wedge x$ using these local conditions.

Begin with $f_h := \hat{1}$ and $g_h := x$. For some $k = h, h-1, \ldots, l+1$, we have

$$\longleftarrow \operatorname{type} \boxed{\begin{array}{ccc} (f_{k-1}) & \lessdot & f_k \\ \rotatebox{90}{$\lessdot$} & & \rotatebox{90}{$\lessdot$} \\ (g_{k-1}) & \lessdot & g_k \end{array}} = \boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \rotatebox{90}{$\lessdot$} & & \rotatebox{90}{$\lessdot$} \\ \alpha & \lessdot & \beta \end{array}}$$

(that is, f_k and g_k and all the partitions are given, and we will choose f_{k-1} and g_{k-1} to satisfy this).

$\gamma = \delta$: Then $f_{k-1} = f_k$; meet both sides with x to obtain $g_{k-1} = g_k$, so that $\alpha = \beta$ also. Neither tableau has an entry k.

$$\boxed{\begin{array}{ccc}\gamma & = & \delta\\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$}\\ \alpha & = & \beta\end{array}} \qquad \text{1 choice}$$

$\alpha = \beta$ and $\gamma \lessdot \delta$: Then $\beta = \alpha \leq \beta \wedge \gamma \leq \gamma \lessdot \delta$. Since β is a weak lower cover of δ and $\beta \leq \gamma \lessdot \delta$, in fact, $\alpha = \beta = \gamma$. So $f_{k-1} := g_{k-1} := g_k$ in one way, and k is deleted on the deletion path from $\boldsymbol{P}$ to $\boldsymbol{Q}$.

$$\boxed{\begin{array}{ccc}\gamma & \lessdot & \delta\\ \| & & \rotatebox{90}{$\lessdot$}\\ \alpha & = & \beta\end{array}} \qquad \text{1 choice}$$

$\alpha \lessdot \beta$ and $\gamma \lessdot \delta$: If $\beta = \delta$ then $g_k = f_k$; meet both sides with f_{k-1} to obtain $g_{k-1} = f_{k-1}$ also, so $\alpha = \gamma$. In view of this, the configurations of partitions in Lemma 7.11 are all the configurations that arise when $\alpha \lessdot \beta$ and $\gamma \lessdot \delta$. The possibilities are as follows.

If $\beta \neq \gamma$, then k is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$.

$$(\beta \neq \gamma) \quad \boxed{\begin{array}{ccc}\gamma & \lessdot_{|r|} & \delta\\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$}\\ \alpha & \lessdot_{|r|} & \beta\end{array}} \qquad \begin{array}{l}\langle\delta'_r\rangle - \left\langle\delta'_{r+1}\right\rangle \; = \\ \quad \langle\mathrm{row}(\boldsymbol{P},k)\rangle - \langle\mathrm{row}_1(\boldsymbol{P},k)\rangle \\ \text{choices}\end{array}$$

If $\beta = \gamma$, then k is on the deletion path. It can move strictly up and right from $\boldsymbol{P}$ to $\boldsymbol{Q}$, bumping a smaller entry (as β/α is a cell in δ, so that cell in $\boldsymbol{P}$ has a smaller entry than k):

$$(r > s) \quad \boxed{\begin{array}{ccc}\gamma & \lessdot_{|s|} & \delta\\ \rotatebox{90}{$\lessdot$}_{|r|} & & \rotatebox{90}{$\lessdot$}_{|s|}\\ \alpha & \lessdot_{|r|} & \beta\end{array}} \qquad \begin{array}{l}\Big(\langle\beta'_r\rangle - \left\langle\beta'_{r+1}\right\rangle\Big)\,(q-1) \; = \\ \quad \Big(\langle\mathrm{row}(\boldsymbol{Q},k)\rangle - \langle\mathrm{row}_1(\boldsymbol{Q},k)\rangle\Big)(q-1) \\ \text{choices;}\end{array}$$

or it can move up one cell from $\boldsymbol{P}$ to $\boldsymbol{Q}$:

$$\boxed{\begin{array}{ccc}\gamma & \lessdot_{|r|} & \delta\\ \rotatebox{90}{$\lessdot$}_{|r|} & & \rotatebox{90}{$\lessdot$}_{|r|}\\ \alpha & \lessdot_{|r|} & \beta\end{array}} \qquad \begin{array}{l}\Big(\langle\beta'_r\rangle - \left\langle\beta'_{r+1}\right\rangle\Big)\,q \; = \\ \quad \Big(\langle\mathrm{row}(\boldsymbol{Q},k)\rangle - \langle\mathrm{row}_1(\boldsymbol{Q},k)\rangle\Big)q \\ \text{choices.}\end{array}$$

Now multiply all the numbers of choices given over all $k \in \boldsymbol{P}$, that is, all k with $\gamma \lessdot \delta$, to obtain the number of choices of f. The deleted entry k, if any, yields a factor of 1, so the product may be taken over all $k \in \boldsymbol{Q}$, to obtain the formula in the statement of the theorem. Each term of the product has degree $\mathrm{row}(\boldsymbol{Q},k) - 1$, plus 1 if there is a factor of q or $q-1$ (that is, if k is on the deletion path, but is not the entry bumped out). So the degree is the sum over all entries of $\boldsymbol{Q}$ of their row number minus 1 (totalling $n(\mathrm{sh}\, Q)$), plus the number of cells in $\boldsymbol{Q}$ on the deletion path, which is one less than the length of the deletion path since the deleted entry isn't counted in this length. The degree is uniquely maximized by the ordinary Robinson-Schensted deletion path, with each entry (but the smallest) on the path moving up one row, until it terminates at the top row or a $\bullet$. So $\mathcal{D}_c\boldsymbol{P}$ is the generic value of $\widetilde{\mathcal{D}}_c\boldsymbol{P}$.

It is not, however, uniform. Suppose β and γ are distinct lower covers of δ and β/α is strictly below and left of δ/β. In terms of tableau, k is larger than the entry deleted in the game $\mathcal{D}_c\boldsymbol{P}$, and is to the left of the cell on the deletion path on the row above k; thus, this configuration arises for some k unless the deletion path moves the first entry in each row to the first entry of the preceding row (and deletes the smallest entry of the tableau). By applying Corollary 4.72 to $\operatorname{ftype}_{\hat{0}}(g_{k-1}, g_k, f_k) = (\alpha, \beta, \delta)$, we generically expect $\operatorname{type}[g_{k-1}, f_k] = \square\!\square$, but in fact, g_{k-1} is the meet of two lower covers of f_k, so $\operatorname{type}[g_{k-1}, f_k] = \begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$. Only a fraction at most $O(q^{-1})$ of the lower covers of g_k of type α may be chosen as g_{k-1}, so most flags g from $\hat{0}$ to x with $\operatorname{ftype} g = \boldsymbol{Q}$ cannot be represented as $f \wedge x$ with $\operatorname{ftype} f = \boldsymbol{P}$.

For the converse statement in the theorem, ignoring all the enumerative parts of the above proof shows that

$$\begin{aligned}
&\mathcal{D}_c, \widetilde{\mathcal{D}}_c : \ \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \wedge x) && \text{where } f_l \leq x \lessdot f_h \text{ and } c = \lambda/(\operatorname{type}[f_l, x]) \\
&\mathcal{D}_c, \widetilde{\mathcal{D}}_c : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \wedge x) && \text{where } x \lessdot f_h \text{ and } c = \lambda/\operatorname{type} x.
\end{aligned}$$

If $x = f_h$, we have $f \wedge x = f$ and $c = \emptyset$, and indeed, $\widetilde{\mathcal{D}}_\emptyset$ does not change $\boldsymbol{P}$. If $f_l \not\leq x \lessdot f_h$, then $f_l \vee x = f_h$, because f_h is an upper bound of the two; there is nothing between x and f_h; and the join is not x since $f_l \not\leq x$. Thus,

$$[f_l \wedge x, f_h \wedge x] = [f_l \wedge x, x] \cong [f_l, f_l \vee x] = [f_l, f_h],$$

so $c = \emptyset$ and $\operatorname{ftype}(f \wedge x) = \operatorname{ftype} f = \widetilde{\mathcal{D}}_\emptyset \operatorname{ftype} f$. $\square$

7.5. Sample enumeration of the number of flags realizing a meet insersection type table.

Example 7.13. In a q-regular semi-primary lattice of type $(3, 1)$, we'll classify and enumerate all pairs of flags (f, f') with $\operatorname{ftype} f = \boldsymbol{P}$ and $\operatorname{ftype} f' = \boldsymbol{Q}$ where

$$\boldsymbol{P} = \boldsymbol{Q} = \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}.$$

They will be classified according to their **meet type table** $\Lambda = (\lambda^{(ij)})_{0 \leq i,j \leq n}$ where $\lambda^{(ij)} = \operatorname{type}(f_i \wedge f'_j)$. We encode such a table by a sequence of tableaux $\boldsymbol{P}_n, \ldots, \boldsymbol{P}_0$ where $\boldsymbol{P}_j = (\lambda^{(0j)}, \ldots, \lambda^{(nj)})$. The transpose of $\Lambda = (\lambda^{(ij)})_{0 \leq i,j \leq n}$ is $\Lambda' = (\lambda^{(ji)})_{0 \leq i,j \leq n}$. Apply the degenerate Robinson-Schensted deletion algorithm to the pair $(\boldsymbol{P}, \boldsymbol{Q})$ by performing degenerate deletion steps $\widetilde{\mathcal{D}}_c$ on $\boldsymbol{P}$ with c successively being the cell with $n, n-1, \ldots, 1$ in $\boldsymbol{Q}$, and forming a permutation σ by listing the deleted entries in reverse order. The possibilities are in Figure 9. Let N_k be the number of pairs (f, f') of flags realizing meet type table Λ_k in a q-regular semi-primary lattice.

(1) First we show that Λ_7 never occurs in any semi-primary lattice, so that $N_7 = 0$. By Theorem 4.76, $f_1 = \hat{1}[[\square]] = C\hat{1}$ because

$$\operatorname{ftype}_{\hat{0}} f = \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} = \begin{ytableau} 1 \end{ytableau} + \begin{ytableau} 2 \\ 3 \\ 4 \end{ytableau}.$$

Also, $f'_1 = \hat{1}[[\square]]$ for the same reason, so $f_1 \wedge f'_1 = \hat{1}[[\square]]$ and $\boldsymbol{P}_1 = \begin{ytableau} 1 \end{ytableau}$. Thus $N_7 = 0$.

	σ	$\boldsymbol{P}_4$	$\boldsymbol{P}_3$	$\boldsymbol{P}_2$	$\boldsymbol{P}_1$	$\boldsymbol{P}_0$
$\Lambda_1 = \Lambda_1'$	1 4 3 2	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 3 \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 4 \end{array}$	1	$\emptyset$
$\Lambda_2 = \Lambda_3'$	1 3 4 2	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 3 \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 3 \end{array}$	1	$\emptyset$
$\Lambda_3 = \Lambda_2'$	1 4 2 3	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 2 \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 4 \end{array}$	1	$\emptyset$
$\Lambda_4 = \Lambda_4'$	1 2 4 3	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 2 \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 2 \end{array}$	1	$\emptyset$
$\Lambda_5 = \Lambda_5'$	1 3 2 4	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$	$\begin{array}{ll} 1 & 3 \end{array}$	1	$\emptyset$
$\Lambda_6 = \Lambda_6'$	1 2 3 4	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$	$\begin{array}{ll} 1 & 2 \end{array}$	1	$\emptyset$
$\Lambda_7 = \Lambda_7'$	3 4 1 2	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 4 \\ 3 & \end{array}$	$\begin{array}{ll} 3 & 4 \end{array}$	3	$\emptyset$
$\Lambda_8 = \Lambda_8'$	1 4 3 2	$\begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array}$	$\begin{array}{ll} 1 & 4 \\ 3 & \end{array}$	$\begin{array}{ll} 1 & 4 \end{array}$	1	$\emptyset$

FIGURE 9. All possible degenerate Robinson-Schensted games.

(2) We evalutate N_k for other k's. Since $f_1 = f_1' = \hat{1}[[\square]]$, we have $\mathrm{type}[f_1, f_4] = \mathrm{type}[f_1', f_4'] = (1^3)$, so $\mathrm{type}[f_i, f_j] = \mathrm{type}[f_i', f_j'] = (1^2)$ for $(i, j) \in \{(1, 3), (2, 4)\}$. Thus, we could use an approach similar to that of Section 5.5: develop a system of equations in the N_k's by replacing f_2 with some other atom in the interval $[f_1, f_3]$, and similarly replacing f_3, f_2', f_3', to see how it could change the meet type table and whether a ratio of the frequencies of the possibilities can be determined. In larger examples, however, it is often not possible to determine from Λ whether $\mathrm{type}[f_i, f_j] = \square\!\square$ or $\genfrac{}{}{0pt}{}{\square}{\square}$ for $j - i = 2$.

We give here a more direct approach than a system of equations, but it, too, often fails in larger examples when types of intervals of length 2 cannot be determined.

Since $\Lambda_2 = \Lambda_3'$, there is a bijection between pairs of flags with meet type tables Λ_3 and Λ_2, namely $(f, f') \mapsto (f', f)$, so $N_2 = N_3$. We'll determine N_3. First expand Λ_3 as a square grid of partitions; see Figure 10. The tableau $\boldsymbol{P}$ is the chain going up the right-hand edge and the tableau $\boldsymbol{Q}$ is the chain going across the top edge. The games $\widetilde{\mathcal{D}}_c$ are played on columns from right to left, rather than on rows from top to bottom. This way, the orientation of the grid so obtained agrees with that in Roby [**21**].

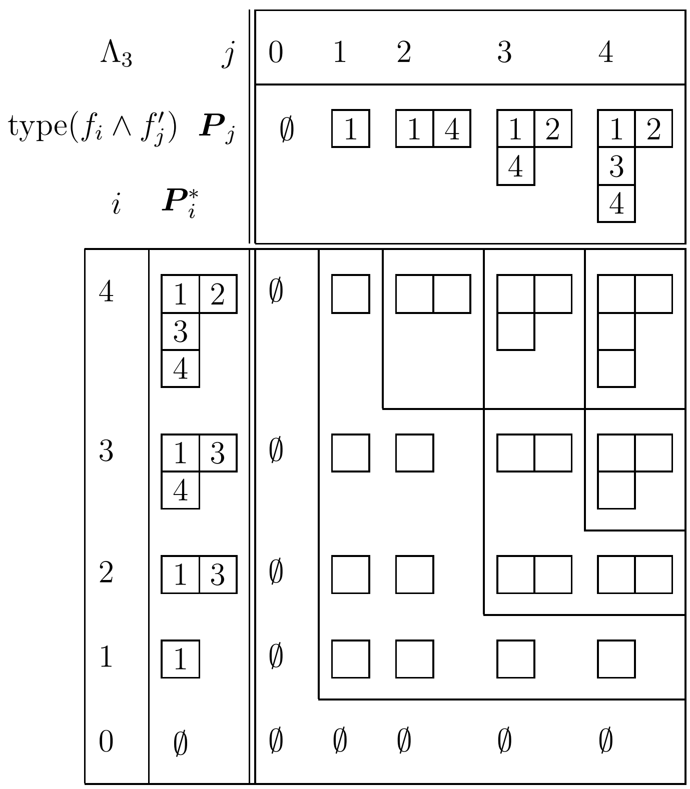


FIGURE 10. The graphical meet type table of Λ_3.

The grid is divided into regions of adjacent grid cells whose partitions are equal, because $f_i \wedge f_j'$ is constant on each such region: since $f_i \wedge f_j' \geq f_{i-1} \wedge f_j'$ always holds, if $\lambda^{(ij)} = \lambda^{(i-1,j)}$ then $f_i \wedge f_j' = f_{i-1} \wedge f_j'$, and similarly, if $\lambda^{(ij)} = \lambda^{(i,j-1)}$ then $f_i \wedge f_j' = f_i \wedge f_{j-1}'$. Form a graded graph (Figure 11) by rotating the grid an eighth of a revolution counterclockwise, collapsing each region to a node, and forming an edge between regions that share a horizontal or vertical boundary (not just a corner). (More generally, if we form such a graph for any pair of flags (f, f') in a lower semimodular lattice by coalescing regions of constant $f_i \wedge f'_j$ into single nodes, the graph is the Hasse diagram of a lower semimodular sublattice.)

$$\Lambda_3 : \; q^3(q-1)(q+1)$$

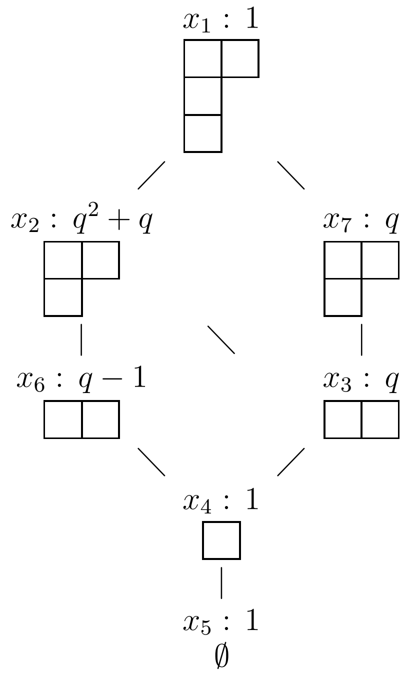


FIGURE 11. The types in the meet semilattice of Λ_3.

The values of $f_i \wedge f'_j$ on various regions are denoted $x_1, \ldots, x_7$, and we choose these values in the order given by the subscripts. The number of choices of each element is specified after the colon. First choose $x_1 = \hat{1}$ of type $(3,1)$ in one way. Choose $x_2 \lessdot x_1$ with type $x_2 = (2,1)$ in $\langle 3\rangle - \langle 1\rangle = q^2 + q$ ways, and similarly for x_3, x_4, x_5. The interval $[x_4, x_2]$ has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ as noted previously, but in larger examples, we wouldn't always be able to determine this type, and the procedure would break down now. Here we can continue: x_6 is an atom of $[x_4, x_2]$ of type $\square\square$ in the lattice, not equal to x_3. Of the $q+1$ atoms in the interval, one has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, namely $x_2[[\begin{smallmatrix}\square\\\square\end{smallmatrix}]]$, and the q others, including x_3, have type $\square\square$, so there are $q-1$ choices of x_6. Similarly, the interval $[x_3, x_1]$ has type $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, and all atoms in it have type $(2,1)$, so there are q choices of elements not equal to x_2. Thus, $N_3 = (q^2+q)q(q-1)q = q^3(q-1)(q+1)$.

(3) In general, how many solutions $x_4 \in (x_1, x_3)$ are there in a configuration

$$\begin{tikzcd}[column sep=small]
 & \operatorname{type} x_1 = \delta \arrow[dl, dash] \arrow[dr, dash] & \\
\operatorname{type} x_2 = \gamma \arrow[dr, dash] & & (\operatorname{type} x_4 = \beta) \arrow[dl, dash] \\
 & \operatorname{type} x_3 = \alpha &
\end{tikzcd}$$

where x_1, x_2, x_3 are given and $\text{type}[x_3, x_1] = \yng(1,1)$? There are $q+1$ elements in the interval (x_3, x_1). If δ/α is a brick, they all have the same type, and if it is not a brick, one has the type given by adding the left-hand square of δ/α to α and the other q have the type given by adding the right-hand square: if the columns of the squares of δ/α are $r < s$ then $z = x_1[r] \vee x_3$ is the unique solution of $z \in (x_3, x_1)$ with $\text{type}\, x_3 \lessdot_{|r|} \text{type}\, z \lessdot_{|s|} \text{type}\, x_1$. Altogether, the number of choices of x_4 in every circumstance is as follows.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|s|} & \delta \\ \lessdot_{|r|} & & \lessdot_{|r|} \\ \alpha & \lessdot_{|s|} & \beta \end{array}} \qquad \begin{cases} 1 & \text{if } r > s, \text{ namely } x_1[s] \vee x_3; \\ q & \text{if } r < s \text{ since only } x_2 \text{ has type } \gamma; \\ q & \text{if } r = s, \text{ since all } x \in (x_3, x_1) \\ & \quad \text{have equal type and } x_4 \neq x_2. \end{cases}$$

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|s|} & \delta \\ \lessdot_{|r|} & & \lessdot_{|s|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{cases} 0 & \text{if } r < s \text{ since only} \\ & \quad x_2 = x_3[r] \vee x_1 \text{ has type } \gamma; \\ q-1 & \text{if } r > s \text{ since } x_4 \neq x_1 \\ & \quad \text{but they have equal type;} \\ q & \text{if } r = s. \end{cases}$$

By applying Corollary 4.72 to $\text{ftype}_{\hat{0}}(x_3, x_2, x_1) = (\alpha \lessdot_{|r|} \gamma \lessdot_{|s|} \delta)$ in both diagrams, if $r \geq s$ then $\text{type}[x_3, x_1] = \yng(1,1)$ holds necessarily, while if $r < s$ it may or may not hold; in particular, in the first configuration with $r < s$ and δ/α not a brick, it is necessary to specify that $\text{type}[x_3, x_1] = \yng(1,1)$ because without doing so, $\text{type}[x_3, x_1] = \yng(2)$ is possible.

(4) In Figure 12, we solve for N_1, N_4, N_5, N_6, N_8 in a similar fashion. In each case we choose a flag $\hat{1} = x_1 \gtrdot \cdots \gtrdot x_5 = \hat{0}$, and then fill in intervals of type $\yng(1,1)$. All length two intervals required to have type $\yng(1,1)$ do, because $x_4 = \hat{1}[[\yng(1)]]$ and they are subintervals of $[x_4, \hat{1}]$. The highest degree polynomial is N_1, of degree 6, corresponding to the ordinary Robinson-Schensted algorithm on $(\boldsymbol{P}, \boldsymbol{Q})$.

Example 7.14. This method can break down.

(1) In a semi-primary lattice of type $\yng(2,2)$, how many flags (not of maximal length) $f = (f_2 \lessdot f_3 \lessdot f_4 = \hat{1})$, $f' = (f'_2 \lessdot f'_3 \lessdot f'_4 = \hat{1})$ have the following meet meet type table?

$$\Lambda = (\lambda^{(ij)})_{2 \leq i,j \leq 4} = \begin{array}{c|ccc} \multicolumn{4}{c}{\text{type}(f_i \wedge f'_j)} \\ i \backslash j & 2 & 3 & 4 \\ \hline 4 & \yng(2) & \yng(2,1) & \yng(2,2) \\ 3 & \yng(1) & \yng(1,1) & \yng(2,1) \\ 2 & \emptyset & \yng(1) & \yng(2) \end{array}$$

Let $z_{ij} = f_i \wedge f'_j$, and try to choose the z_{ij}'s in a similar fashion to the previous example. In what order do we choose the z_{ij}'s? First, $z_{44} = \hat{1}$. Then choose z_{43} or z_{34}; the type table is symmetric, so without loss of generality, choose z_{43} in one of $\langle 2 \rangle = q+1$ ways. The initial saturated chains

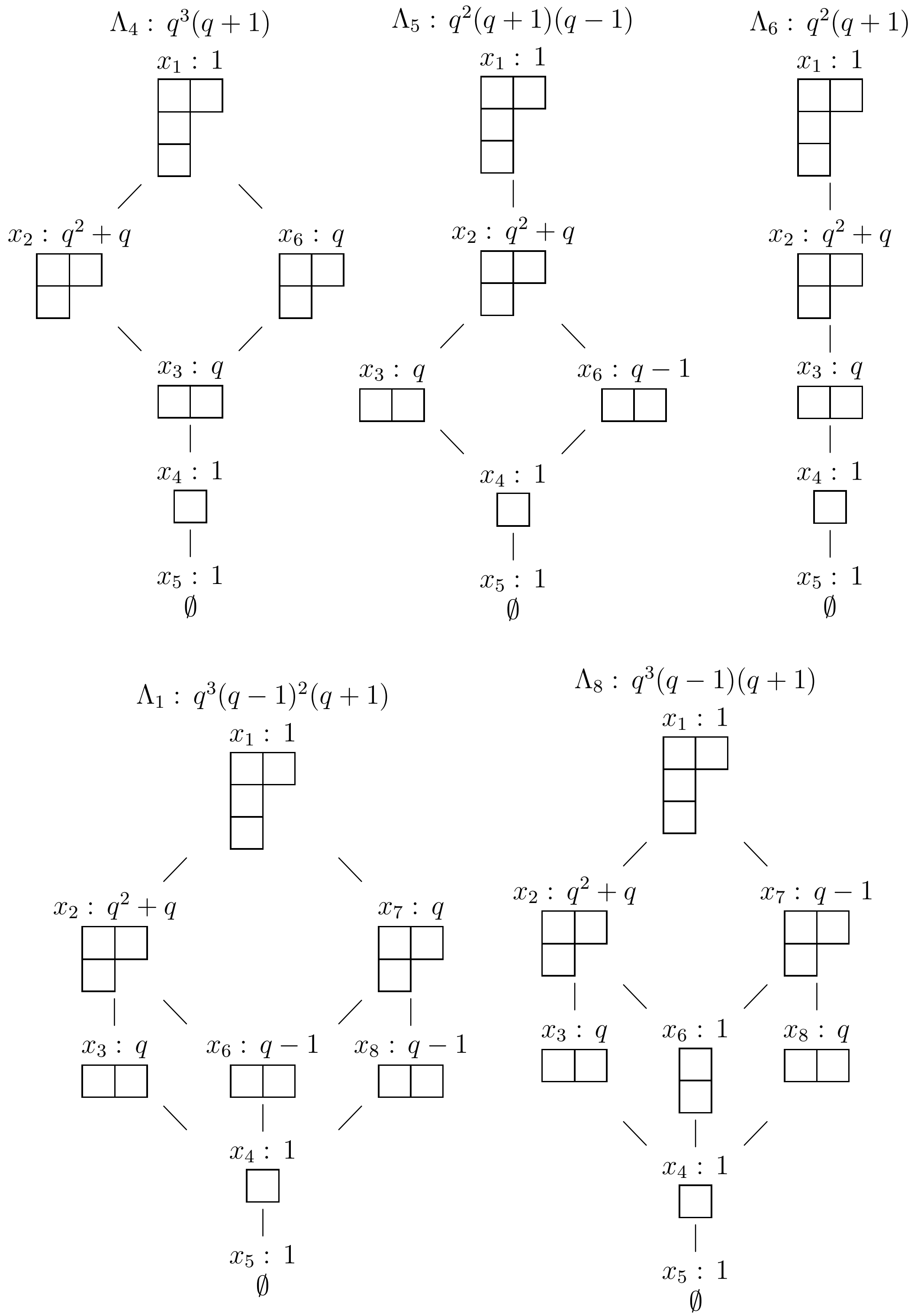


Figure 12. More meet semilattices.

we could form by continuing in this fashion are (a) $(z_{44}, z_{43}, z_{42}, z_{32}, z_{22})$; (b) $(z_{44}, z_{43}, z_{33}, z_{32}, z_{22})$; or (c) $(z_{44}, z_{43}, z_{33}, z_{23}, z_{22})$. Continuing with either (b) or (c), we will ultimately have $\operatorname{ftype}_{\hat{0}}(z_{32} \lessdot z_{33} \lessdot z_{43}) = \begin{smallmatrix}\bullet & 3\\ 2 & \end{smallmatrix}$, so that we do not know whether $\operatorname{type}[z_{32}, z_{43}] = \square\!\square$ or $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$; we need to know this when selecting z_{42}. With (a) on the other hand, we ultimately have a flag $(z_{23} \lessdot z_{33} \lessdot z_{34})$ with the same problem when choosing z_{24}.

We can enumerate the pairs of flags with meet type table Λ by another approach. The pair (x_{24}, x_{42}) is of two independent 2-cycles whose join is $\hat{1}$. There are q^2+q join decompositions of $\hat{1}$ by Theorem 4.48. The interval $[x_{42}, \hat{1}]$ has type $\square\!\square$ and the only intermediate element is $x_{43} = x_{42} \vee A\hat{0} = x_{42} \vee x_{24}[1]$, while $x_{34} = x_{24} \vee x_{42}[1]$. Since variables on the upper and right boundaries are all expressed in terms of the same join decomposition of $\hat{1}$, their meets have the types shown in Λ. Thus, all $q^2 + q$ join decompositions of $\hat{1}$ yield pairs of flags whose meet type table is Λ, so there are $q^2 + q$ such (f, f').

(2) Now take any partition λ, let $n = |\lambda|$, and form the grid $\Gamma = (\gamma^{(ij)})_{n+2 \le i,j \le n+4}$ where $\gamma^{(ij)} = \lambda \cup \lambda^{(i-n,j-n)}$. Try to find flags $f = (f_{n+2}, f_{n+3}, f_{n+4})$ and $f' = (f'_{n+2}, f'_{n+3}, f'_{n+4})$ with meet type table Γ. Let $z_{ij} = f_i \wedge f'_j$. The problems with sequences (a), (b), (c) above happen again, *mutatis mutandis.* If $\lambda'_1 > \lambda'_2$, then $\gamma^{(n+2,n+4)}/\gamma^{(n+2,n+2)}$ and $\gamma^{(n+4,n+2)}/\gamma^{(n+2,n+2)}$ are no longer horizontal bricks, so $\operatorname{type}[z_{n+2,n+2}, z_{n+2,n+4}]$ and $\operatorname{type}[z_{n+2,n+2}, z_{n+4,n+2}]$ no longer need to be $\square\!\square$; either or both could be $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$. The join decomposition approach cannot be applied. Further *ad hoc* methods could be developed to solve this problem. It would be preferable to find a systematic approach to enumerate the number of flags realizing a wide variety of meet type tables.

7.6. Transposed Robinson-Schensted deletion, $\mathcal{D}^*_c\boldsymbol{P}$. Let $\lambda/\mu = \operatorname{sh}\boldsymbol{P}$; the parameter c is either $\emptyset$ or an inner corner of λ not in μ.

Pick $x \ge f_l$. The following map is generic but not uniform.

$$\begin{array}{llll}
\mathcal{D}^*_c, \widetilde{\mathcal{D}}^*_c: & \operatorname{ftype} f \longmapsto \operatorname{ftype}(f \vee x) & & x \gtrdot f_l \text{ and } c = \lambda/(\operatorname{type}[f_l \vee x, f_h \vee x]) \\
\mathcal{D}^*_c, \widetilde{\mathcal{D}}^*_c: & \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_x(f \vee x) & & x \gtrdot \hat{0} \text{ and } c = \lambda/\operatorname{type}[x, f_h \vee x]
\end{array}$$

The games $\mathcal{D}_c$ and $\mathcal{D}^*_c$ are the same except that the roles of rows and columns are reversed. The games $\widetilde{\mathcal{D}}_c$ and $\widetilde{\mathcal{D}}^*_c$ are not transposed versions of each other, however.

Tableau Rules. Let $\boldsymbol{P}$ be a standard skew tableau on distinct entries, and c be an inner corner of $\operatorname{osh}\boldsymbol{P}$ or $c = \emptyset$. If $c = \emptyset$ then $\widetilde{\mathcal{D}}_c\boldsymbol{P} = \boldsymbol{P}$. Otherwise, proceed as follows.

($\boldsymbol{\mathcal{D}^*_c}$): We insert an entry e into a column by replacing its largest entry (possibly $\bullet$) smaller than e with e; the number it replaces is **bumped** from the column.

($\boldsymbol{\widetilde{\mathcal{D}}^*_c}$): We insert an entry e into a column by placing e where the largest entry (possibly $\bullet$) smaller than it is, then moving that entry up one cell, moving the entry there up one cell, and so on, until we either choose to stop, or we are forced to because moving up an entry would cause a row not to increase left to right. At least the cell where e is placed changes, and possibly more do.

The final entry displaced that does not displace the one above it is **bumped** from the column.

Begin by letting e be the entry at c and removing cell c from the tableau. Insert e one column left of c. Insert the bumped entry one column left of that, bumping out another number. Continue this until either an entry is bumped out from the first column, or $\bullet$ is bumped out. The final entry bumped out is said to be **deleted** from the tableau.

The local rules are a cross between transposing the rules to $\widetilde{\mathcal{D}}_c$ on the one hand, and the rules to $\tilde{\jmath}^{c'}$ played in reverse on the other hand.

Local Rules. If $c = \emptyset$, let $\mu^{(h)} := \lambda^{(h)}$, and if c is a single cell, let $\mu^{(h)}$ be $\lambda^{(h)}$ with c deleted. Then for $k = h, h-1, \ldots, l+1$, propagate as follows.

$$\begin{array}{cc|c|c|c|l} & k-1 & k & & h & \text{column} \\ \cline{2-3}\cline{5-5} \longleftarrow & \gamma & \delta & & \lambda^{(h)} & \boldsymbol{P} \\ \cline{2-3}\cline{5-5} & (\alpha) & \beta & & (\lambda^{(h)} \ominus c) & \widetilde{\mathcal{D}}^*_c\boldsymbol{P} \\ \cline{2-3}\cline{5-5} \end{array}$$

> **$\boldsymbol{\beta = \gamma \lessdot \delta}$ does not hold:** Let $\alpha := \beta \wedge \gamma$.
> **$\boldsymbol{\beta = \gamma \lessdot \delta}$ holds:** Let $c := \delta/\beta$.
> > **$\mathbf{col}(\boldsymbol{c}) = \mathbf{1}$:** Let $\alpha := \gamma$.
> > **$\mathbf{col}(\boldsymbol{c}) > \mathbf{1}$ ($\boldsymbol{\mathcal{D}_c}$):** Delete the bottom cell of γ in the column left of c to obtain α.
> > **$\mathbf{col}(\boldsymbol{c}) > \mathbf{1}$ ($\boldsymbol{\widetilde{\mathcal{D}}_c}$):** Choose one of the following as α:
> > - delete the cell above c in γ, provided there is one and doing so yields a partition;
> > - delete the bottom cell of γ in the column left of c; or
> > - γ.

The tableau rules and local rules describe the same game. The proof is similar to Proposition 7.10, but the positions of consecutive cells on the path are different.

Theorem 7.15. *Consider a q-regular semi-primary lattice of type λ. Let $\mu \lessdot \lambda$ and $c = \lambda/\mu$. Let $\boldsymbol{P}$ be a skew tableau, and fix $\boldsymbol{Q} = \widetilde{\mathcal{D}}^*_c\boldsymbol{P}$. Let e be the deleted entry, or $e = \mathrm{low}(\boldsymbol{P}) = \bullet$ if no entry is deleted.*

Pick an atom x of cotype μ. There are

$$\prod_{k \in \boldsymbol{P}} \begin{cases} \langle \mathrm{row}(\boldsymbol{Q},k)\rangle - \langle \mathrm{row}_1(\boldsymbol{P},k)\rangle & \text{if } k > e \text{ and } \mathrm{col}(\boldsymbol{P},k) = \mathrm{col}(\boldsymbol{Q},k); \\ \langle \mathrm{row}(\boldsymbol{P},k)\rangle - \langle \mathrm{row}_1(\boldsymbol{Q},k)\rangle & \text{if } k > e \text{ and } \mathrm{col}(\boldsymbol{P},k) \gtrdot \mathrm{col}(\boldsymbol{Q},k); \\ q^{\mathrm{row}(\boldsymbol{P},k)-1} & \text{if } k = e \\ \langle \mathrm{row}(\boldsymbol{P},k)\rangle - \langle \mathrm{row}_1(\boldsymbol{P},k)\rangle & \text{if } k < e, \end{cases}$$

flags f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_x(f \vee x) = \boldsymbol{Q}$. This is a polynomial of degree

$$n(\mathrm{sh}\,\boldsymbol{P}) - \#\text{ of entries on deletion path that move up one cell}$$

It is maximized by having the deletion path contain a cell in every column weakly left of c, which happens uniquely in the generic (but not uniform) game $\boldsymbol{Q} = \mathcal{D}^*_c\boldsymbol{P}$.

Conversely, in all semi-primary lattices, for any flag $f = (f_l \lessdot \cdots \lessdot f_h)$,

$$\begin{array}{rcll} \operatorname{ftype}(f \vee x) & = & \widetilde{\mathcal{D}}^*_c \operatorname{ftype} f & \textit{where } x \gtrdot f_l \textit{ and} \\ & & & c = (\operatorname{type}[f_l, f_h])/(\operatorname{type}[f_l \vee x, f_h \vee x]) \\ \operatorname{ftype}_x(f \vee x) & = & \widetilde{\mathcal{D}}^*_c \operatorname{ftype}_{\hat{0}} f & \textit{where } x \gtrdot \hat{0} \textit{ and} \\ & & & c = (\operatorname{type}[\hat{0}, f_h])/(\operatorname{type}[x, f_h \vee x]). \end{array}$$

The generic case in the invariant subspace lattice over an infinite field was done by van Leeuwen [**31**, Lemma 2.5.2].

Proof. We begin with $f_h := g_h := \hat{1}$. For some k with $l < k \leq h$, after having chosen $f_h \gtrdot f_{h-1} \gtrdot \cdots \gtrdot f_k$ we have

$$\longleftarrow \operatorname{type} \boxed{\begin{array}{ccc} [\hat{0}, (f_{k-1})] & & [\hat{0}, f_k] \\ \dot{\wedge}\!\mid & & \dot{\wedge}\!\mid \\ [x, (g_{k-1})] & & [x, g_k] \end{array}} = \boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \dot{\vee}\!\mid & & \dot{\vee}\!\mid \\ \alpha & \lessdot & \beta \end{array}}$$

We will choose f_{k-1} to satisfy these types, in one of several ways. If $\gamma = \delta$ then $f_{k-1} := f_k$ so $g_{k-1} := g_k$ and $\alpha = \delta$. There is no entry k in $\boldsymbol{P}$ or $\boldsymbol{Q}$.

$$\boxed{\begin{array}{ccc} \gamma & = & \delta \\ \dot{\vee}\!\mid & & \dot{\vee}\!\mid \\ \alpha & = & \beta \end{array}} \qquad \text{1 choice of } f_{k-1} \text{ and } g_{k-1}$$

Assume that $\gamma \lessdot \delta$.

$\boldsymbol{x \leq f_{k-1}}$**:** We want to extend the partial flag $(\hat{0} \lessdot x < f_k)$ to $(\hat{0} \lessdot x \leq f_{k-1} \lessdot f_k)$, with specified types for each interval. By Theorem 5.4 and Corollary 5.5, we have the following possible relations among the partitions, and the following number of choices of f_{k-1} in each case.

The entry k can bump out the bottommost entry smaller than k in the column just left of k:

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r+1|} & \delta \\ \dot{\vee}_{|r|} & & \dot{\vee}_{|r+1|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \left\langle \delta'_{r+1} \right\rangle - \left\langle \beta'_{r+1} \right\rangle \; = \\ \quad \langle \operatorname{row}(\boldsymbol{P}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k) \rangle \\ \text{choices} \end{array}$$

or k can stay in the same cell ($r \neq s$) or move up one cell ($r = s$):

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \dot{\vee}_{|s|} & & \dot{\vee}_{|s|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \beta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \; = \\ \quad \langle \operatorname{row}(\boldsymbol{Q}, k) \rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

$\boldsymbol{x \not\leq f_{k-1}}$ **and** $\boldsymbol{x \leq f_k}$**:** Then $g_{k-1} = f_{k-1} \vee x = f_k$ and $g_k = f_k \vee x = f_k$, so $\alpha = \beta \lessdot \delta$. Also

$$[\hat{0}, f_{k-1}] = [x \wedge f_{k-1}, f_{k-1}] \cong [x, x \vee f_{k-1}] = [x, f_k]$$

so $\alpha = \beta = \gamma \lessdot_{|r|} \delta$ for some r, and k is the entry deleted on the deletion path.

Pick f_{k-1} to be a coatom of $[\hat{0}, f_k]$ of type γ not also in $[x, f_k]$. The coatoms of type γ are the coatoms of $[A^{r-1}\hat{0} \wedge f_k, f_k]$ not also in $[A^r\hat{0} \wedge f_k, f_k]$. Since x is an atom, $A\hat{0} \geq x$, so coatoms of both intervals are weakly larger than x

unless $r = 1$. Thus we must have $r = 1$, and f_{k-1} is any coatom of $[\hat{0}, f_k]$ not also in $[x, f_k]$, in one of $\langle \delta'_1 \rangle - \langle \beta'_1 \rangle$ ways.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|1|} & \delta \\ \parallel & & \rotatebox{90}{$\lessdot$}_{|1|} \\ \alpha & = & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_1 \rangle - \langle \beta'_1 \rangle \; = \; q^{\mathrm{row}(\boldsymbol{P},k)-1} \\ \text{choices} \end{array}$$

$\boldsymbol{x \not\leq f_k}$**:** Then $[\hat{0}, f_k] \cong [x, f_k \vee x]$. We must have $\alpha = \gamma$ and $\beta = \delta$ for all $f_{k-1} \lessdot f_k$. Entry k is smaller than the deleted entry, and does not move. Choose f_{k-1} to be any coatom f_{k-1} of $[\hat{0}, f_k]$ with type γ.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \parallel & & \parallel \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \; = \\ \quad \langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

Now multiply all the numbers of choices given over all $k \in \boldsymbol{P}$, that is, all k with $\gamma \lessdot \delta$, to obtain the number of choices of f stated in the theorem. When $k > e$ and $\mathrm{col}(\boldsymbol{P}, k) = \mathrm{col}(\boldsymbol{Q}, k)$, the number of choices of f_{k-1} has degree $\mathrm{row}(\boldsymbol{Q}, k) - 1$, which equals $\mathrm{row}(\boldsymbol{P}, k) - 1$ when k is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$, and equals $\mathrm{row}(\boldsymbol{P}, k) - 2$ when k moves up one cell. For all other $k \in \boldsymbol{P}$, the number of choices of f_{k-1} has degree $\mathrm{row}(\boldsymbol{P}, k) - 1$. So the total degree of the number of choices of f is the sum of $\mathrm{row}(\boldsymbol{P}, k) - 1$ over all $k \in \boldsymbol{P}$ (totalling $n(\mathrm{sh}\,\boldsymbol{P})$), minus the number of k's which move up one cell from $\boldsymbol{P}$ to $\boldsymbol{Q}$. This degree is maximized by never moving up one cell and always choosing to move to the next column left, which happens uniquely in the generic game $\mathcal{D}_c^* \boldsymbol{P}$. See the note following this proof for further commentary on maximizing this degree.

The game $\mathcal{D}_c^*$ is not in general uniform, however, as we do not obtain most flags in $[x, \hat{1}]$ of type $\boldsymbol{Q}$ as possible images of $f \vee x$ over f's of type $\boldsymbol{P}$. Suppose there is some k with $\mathrm{row}(\boldsymbol{P}, k) < \mathrm{row}(\boldsymbol{Q}, k)$, which happens in all games except those whose deletion path simply shifts a row horizontally left one cell (possibly deleting the leftmost entry). The number of choices of f_{k-1} has degree $\mathrm{row}(\boldsymbol{P}, k) - 1$, while the number of choices of a coatom in $[x, \hat{1}]$ of type $\mu^{(k-1)}$ has degree $\mathrm{row}(\boldsymbol{Q}, k) - 1$, which is larger. Thus, at most a fraction $O(q^{-1})$ of all flags in $[x, \hat{1}]$ of type $\boldsymbol{Q}$ are possible images of $f \vee x$ over all f of type $\boldsymbol{P}$ in $\mathcal{L}$, for most games $\boldsymbol{P}$ and $\boldsymbol{Q}$.

For the converse statement in the theorem, ignoring the enumerative portions of the above proof yields that the valid configurations of partitions α, β, γ, δ are precisely those given by the local rules for the game $\widetilde{\mathcal{D}_c^*}$. $\square$

Note. The formula in Proposition 5.7 for the number of flags f with $\mathrm{ftype}\, f = \boldsymbol{P}$ and $\mathrm{ftype}\, \partial f = \boldsymbol{Q} = \widetilde{\Delta} \boldsymbol{P}$ is a special case of the formula in Theorem 7.15 for the number of flags f such that $\mathrm{ftype}\, f = \boldsymbol{P}$ and $\mathrm{ftype}(f \vee x) = \boldsymbol{Q} = \widetilde{\mathcal{D}_c^*} \boldsymbol{P}$, yet the generic games Δ and $\mathcal{D}_c^*$ are different. This is because the boundary conditions of the deletion paths are different; in $\widetilde{\Delta}$, it must terminate at the top left corner, while in $\widetilde{\mathcal{D}_c^*}$, it must begin at c. In the case of $\widetilde{\Delta}$, we minimize the number of entries bumped up one cell by only allowing an entry to switch columns by moving horizontally one cell, yielding the generic game Δ.

7.7. Robinson-Schensted internal insertion, $\mathcal{I}_c \boldsymbol{P}$. The game $\mathcal{I}_c \boldsymbol{P}$ below is the internal insertion game R_c defined by Sagan and Stanley in [**23**, p. 164], and $\widetilde{\mathcal{I}}_c \boldsymbol{P}$ is a nondeterministic variant.

When $x \gtrdot f_l$, we have

$$\mathcal{I}_c, \widetilde{\mathcal{I}}_c : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}}(f \vee x) \qquad c = \operatorname{type} x / \operatorname{type} f_l.$$

This is neither generic nor uniform. Unlike most other games considered, there usually is not a single value of $\operatorname{ftype}_{\hat{0}}(f \vee x)$ that occurs generically, that is, with asymptotic probability 1; there is some finite number N of outcomes that each asymptotically occur with probability $1/N$, and all other outcomes asymptotically occur with probability 0. The concept of uniformity does not similarly generalize to this situation, however, and the game $\mathcal{I}_c$ is usually not one of the N "generic" possibilities. See the note at the end of this section for further discussion.

Tableau Rules. Let $\boldsymbol{P}$ be a standard skew tableau of shape λ/μ on distinct entries, and c be an outer corner of μ or $\emptyset$. If $c = \emptyset$ then $\widetilde{\mathcal{I}}_c \boldsymbol{P} = \boldsymbol{P}$. Otherwise, proceed as follows.

We insert an entry e into a row by replacing its smallest entry larger than e with e, **bumping out** that entry, or if all entries are smaller than e, by appending e to the end of the row.

- **($\boldsymbol{\mathcal{I}_c}$):** Place $\bullet$ at c. If c is an outer corner of λ then stop, and otherwise, let e be the entry that was at c. Insert e one row lower; if e is placed at an outer corner, stop, and otherwise, insert the bumped out number yet another row one lower, continuing until a number is placed at an outer corner.
- **($\boldsymbol{\widetilde{\mathcal{I}}_c}$):** Place $\bullet$ at c. If c is an outer corner of λ then stop, and otherwise, let e be the entry that was at c. Choose either to stop, or choose any lower row, and insert e on it, provided that the column e lands in continues to increase from top to bottom, and if e is appended to the end of the row, it is at an outer corner. If indeed e is appended, stop. Otherwise, either choose to stop, or choose a lower row, and try to insert the bumped out number in the same fashion. Continue this until either optionally stopping, or until forced to stop by placing a number at an outer corner.

If we terminate by placing e at an outer corner, this is an insertion path with no entry deleted. If we terminate $\widetilde{\mathcal{I}}_c$ by optionally stopping, the resulting tableau has the same outer shape as $\boldsymbol{P}$, and the final entry bumped out is said to be **deleted** from the tableau.

Local Rules. Let $\lambda := \operatorname{osh} \boldsymbol{P}$. Place $\lambda^{(l)} \oplus c$ in column l. Then for $k = l+1, l+2, \ldots, h$, propagate as follows.

$$\begin{array}{r|c|c|c|c|l}
\text{column} & \multicolumn{1}{c}{l} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{k-1} & \multicolumn{1}{c}{k} & \\
\cline{2-2}\cline{4-5}
\boldsymbol{P} & \lambda^{(l)} & & \gamma & \delta & \\
\cline{2-2}\cline{4-5}
\widetilde{\mathcal{I}}_c\boldsymbol{P} & (\lambda^{(l)} \oplus c) & & \alpha & (\beta) & \longrightarrow \\
\cline{2-2}\cline{4-5}
\end{array}$$

- **$\boldsymbol{\gamma \lessdot \alpha = \delta}$ is false:** Let $\beta := \alpha \vee \delta$.
- **$\boldsymbol{\gamma \lessdot_{|r|} \alpha = \delta}$ ($\boldsymbol{\mathcal{I}_c}$):** Let $\beta \gtrdot_{|s|} \alpha$ with s maximal subject to $s \leq r$ and α has an outer corner in column s.
- **$\boldsymbol{\gamma \lessdot_{|r|} \alpha = \delta}$ ($\boldsymbol{\widetilde{\mathcal{I}}_c}$):** Choose any $\beta \gtrdot \alpha$ with $\operatorname{col}(\beta/\alpha) \leq r$, or choose to let $\beta := \alpha$.

Theorem 7.16. *For any flag f and $x \gtrdot f_l$, we have* $\operatorname{ftype}(f \vee x) = \widetilde{\mathcal{I}}_c(\operatorname{ftype} f)$, *where* $c = \operatorname{type} x / \operatorname{type} f_l$.

Fix c, $\boldsymbol{P}$, and $\boldsymbol{Q}$ with $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c \boldsymbol{P}$, and let e be the deleted entry, or $e = \infty$ if no entry is deleted. In a q-regular semi-primary lattice of type $\operatorname{osh} \boldsymbol{Q}$, *the number of pairs (x, f) where f is a flag and $x \gtrdot f_l$, with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \vee x) = \boldsymbol{Q}$, is a product of factors depending on how entries k move from $\boldsymbol{P}$ to $\boldsymbol{Q}$.*

$$\prod_{k \in \boldsymbol{P}} \begin{cases} \langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle & \text{if } k \geq e\text{; else, assume } k < e\text{:} \\ \langle \operatorname{row}(\boldsymbol{Q}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{Q}, k)\rangle & \text{if } k \text{ stays in the same cell;} \\ \Big(\langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle\Big) q & \text{if } k \text{ moves down one cell;} \\ \Big(\langle \operatorname{row}(\boldsymbol{P}, k)\rangle - \langle \operatorname{row}_1(\boldsymbol{P}, k)\rangle\Big)(q-1) & \text{if } k \text{ moves strictly left and down.} \end{cases}$$
$$\times g^{\operatorname{osh} \boldsymbol{Q}}_{\operatorname{osh} \boldsymbol{P},(1)}(q)$$

This is a polynomial of degree

$$\begin{cases} n(\operatorname{sh} \boldsymbol{P}) + \text{length of insertion/deletion path} - 2 & \text{if } e < \infty; \\ n(\operatorname{osh} \boldsymbol{Q}) - n(\operatorname{osh} \boldsymbol{P}) + n(\operatorname{sh} P) + \text{length of insertion path} - 1 & \text{if } e = \infty. \end{cases}$$

For specified values of $\boldsymbol{Q}$ and $\lambda \lessdot \operatorname{sh} \boldsymbol{Q}$, this is maximized over all $(c, \boldsymbol{P})$ where $\operatorname{osh} \boldsymbol{P} = \lambda$ *and $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c \boldsymbol{P}$, by a unique pair for which $\boldsymbol{Q} = \mathcal{I}_c \boldsymbol{P}$.*

Proof. Suppose $x \gtrdot f_l$ and $g = f \vee x$. Since $f_k \leq f_k \vee x$, we have $\lambda^{(k)} \leq \mu^{(k)}$ for all k. Whenever $f_{k-1} = f_k$, we also have $g_{k-1} = g_k$, while when $f_{k-1} \lessdot f_k$, we have $g_{k-1} \leq g_k$. At most one k satisfies $f_{k-1} \lessdot f_k$ and $g_{k-1} = g_k$: for such a k we have $f_{k'} = g_{k'}$ for all $k' \geq k$, so that $f_{k'-1} \lessdot f_{k'} \Rightarrow g_{k'-1} \lessdot g_{k'}$ for $k' \geq k$. If there is such a k, denote it by e, and then $\boldsymbol{Q}$ is obtained from $\boldsymbol{P}$ by inserting $\bullet$ and deleting e along some path; if there is no such k, let $e = \infty$, and then $\boldsymbol{Q}$ is obtained from $\boldsymbol{P}$ by inserting $\bullet$ along some path.

Now we solve for flags f and elements $x \gtrdot f_l$ with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \vee x) = \boldsymbol{Q}$ for some tableaux $\boldsymbol{P}$ and $\boldsymbol{Q}$. If f and g are flags and x is an element with $g = f \vee x$, then for each k we have $g_k = f_k \vee x$, so

$$g_k = g_{k-1} \vee g_k = g_{k-1} \vee f_k \vee x = (g_{k-1} \vee x) \vee f_k = g_{k-1} \vee f_k.$$

This relation allows us to select successive pairs (f_k, g_k) by looking at a local part of the flags. While the local rules for the game $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c \boldsymbol{P}$ were described in the order $k = l, l+1, \ldots, h$ to satisfy the initial condition given by c, the selection process proceeds in the order $k = h, h-1, \ldots, l+1$.

If $e < \infty$, let $f_h := g_h = \hat{1}$, and otherwise, let $g_h := \hat{1}$ and f_h be a coatom with $\operatorname{type} f_h = \operatorname{osh} \boldsymbol{P}$, in one of $g^{\operatorname{osh} \boldsymbol{Q}}_{\operatorname{osh} \boldsymbol{P},(1)}(q)$ ways.

For some $k = h, h-1, \ldots, l+1$ we have

$$\longleftarrow \operatorname{type} \boxed{\begin{matrix} (f_{k-1}) & \lessdot & f_k \\ \curlywedge\!\!\!| & & \curlywedge\!\!\!| \\ (g_{k-1}) & \lessdot & g_k \end{matrix}} = \boxed{\begin{matrix} \gamma & \lessdot & \delta \\ \curlywedge\!\!\!| & & \curlywedge\!\!\!| \\ \alpha & \lessdot & \beta \end{matrix}}$$

(that is, f_k and g_k and all the partitions are given, and we will choose f_{k-1} and g_{k-1} to satisfy this).

$\gamma = \delta$: Then $f_{k-1} := f_k$, and $g_k = f_k \vee g_{k-1} = f_{k-1} \vee g_{k-1} = g_{k-1}$ so $g_{k-1} := f_{k-1}$ and $\alpha = \beta$. Neither tableau has an entry k.

$$\boxed{\begin{array}{ccc} \gamma & = & \delta \\ \rotatebox{90}{$\leqslant\!\cdot$} & & \rotatebox{90}{$\leqslant\!\cdot$} \\ \alpha & = & \beta \end{array}} \qquad \text{1 choice}$$

$\alpha = \beta$ and $\gamma \lessdot \delta$: The only way to resolve the weak cover relations is $\gamma \lessdot \delta = \alpha = \beta$. So $g_{k-1} := g_k$, and f_{k-1} is any lower cover of f_k of type γ. The entry k is deleted on the path from $\boldsymbol{P}$ to $\boldsymbol{Q}$.

$$\boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \rotatebox{90}{$\lessdot$} & & \Vert \\ \alpha & = & \beta \end{array}} \qquad \begin{array}{l} \langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \; = \\ \quad \langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \\ \text{choices} \end{array}$$

$\alpha \lessdot \beta$ and $\gamma \lessdot \delta$: If $\alpha = \gamma$ then $g_{k-1} = f_{k-1}$; join both sides with f_k to obtain $g_k = f_k$ also, so $\beta = \delta$. In view of this, the configurations of partitions in Lemma 7.11 (but turned upside down, so the roles of f and g are reversed, and the roles of α and β are swapped with γ and δ) are all the configurations that arise when $\alpha \lessdot \beta$ and $\gamma \lessdot \delta$. The possibilities are as follows.

If $\alpha \neq \delta$, then k is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$.

$$(\alpha \neq \delta) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{90}{$\leqslant\!\cdot$} & & \rotatebox{90}{$\leqslant\!\cdot$} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \langle \beta'_r \rangle - \left\langle \beta'_{r+1} \right\rangle \; = \\ \quad \langle \mathrm{row}(\boldsymbol{Q}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k) \rangle \\ \text{choices} \end{array}$$

The vertical relations are equality when k is larger than the deleted entry, and in this case, the count is also $\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle$.

If $\alpha = \delta$, then k is on the deletion path. It can move down and left from $\boldsymbol{P}$ to $\boldsymbol{Q}$, bumping a larger entry (as all entries at most k in $\boldsymbol{P}$ are located in the cells of δ, and β/α is an outer corner of δ):

$$(r > s) \quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{90}{$\lessdot$}_{|r|} & & \rotatebox{90}{$\lessdot$}_{|s|} \\ \alpha & \lessdot_{|s|} & \beta \end{array}} \qquad \begin{array}{l} \Big(\langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \Big)(q-1) \; = \\ \quad \Big(\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \Big)(q-1) \\ \text{choices;} \end{array}$$

or it can move down one cell from $\boldsymbol{P}$ to $\boldsymbol{Q}$, bumping a larger entry:

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ \rotatebox{90}{$\lessdot$}_{|r|} & & \rotatebox{90}{$\lessdot$}_{|r|} \\ \alpha & \lessdot_{|r|} & \beta \end{array}} \qquad \begin{array}{l} \Big(\langle \delta'_r \rangle - \left\langle \delta'_{r+1} \right\rangle \Big) q \; = \\ \quad \Big\langle (\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle \Big) q \\ \text{choices.} \end{array}$$

Finally we set $x := g_l$, and then $g = f \vee x$.

Multiply the numbers of choices together to obtain the polynomial stated in the theorem. The factors $\langle \mathrm{row}(\boldsymbol{P}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{P}, k) \rangle$ have degree $\mathrm{row}(\boldsymbol{P}, k) - 1$, as does the factor $\langle \mathrm{row}(\boldsymbol{Q}, k) \rangle - \langle \mathrm{row}_1(\boldsymbol{Q}, k) \rangle$ when it occurs, as it occurs when k is in the same cell of $\boldsymbol{P}$ and $\boldsymbol{Q}$. The factors q and $q-1$ have degree 1, and occur when $k \in \boldsymbol{P}$ is on the insertion/deletion path, but isn't the deleted entry; the total number of such k's is two less than the length of the insertion/deletion path if an entry is deleted ($e < \infty$), or one less if none is ($e = \infty$). When no entry is deleted, there is also the Hall polynomial, of degree $\mathrm{row}(\mu^{(h)}/\lambda^{(h)}) - 1$. The total degree is as stated in the theorem.

When there is a deleted entry, the total degree equals

$$n(\operatorname{sh}\boldsymbol{Q}) + (\operatorname{row}(c) - 1) + \text{length of insertion path} - 1.$$

For fixed $\boldsymbol{Q}$ and fixed $\lambda \lessdot \operatorname{sh}\boldsymbol{Q}$, consider $(c, \boldsymbol{P})$ as stated in the theorem. Given the first cell c and the last cell $c' = (\operatorname{osh}\boldsymbol{Q})/\lambda$ on the insertion path, the length of the insertion path is at most $\operatorname{row}(c') - \operatorname{row}(c) + 1$, uniquely obtained by inserting $\bullet$ at c and then bumping entries to successive rows without skipping any rows (this might not be possible, given c and c'). So an upper bound for the degree is $n(\operatorname{sh}\boldsymbol{Q}) + \operatorname{row}(c') - 1$. This is obtained only by $\boldsymbol{P} = \mathcal{D}_{c'}\boldsymbol{Q}$, with c set to the cell of the deleted entry in this game. $\square$

Note. For all other games yet and still to be considered, we sought the value of $\boldsymbol{Q}$ that, for a specified $\boldsymbol{P}$, maximized the number of pairs of flags (f, g) satisfying the game. The maximization in this theorem is *backwards*, because that turns out to be the condition required to get the classically defined internal insertion game.

Let's consider the ordinary maximization question. Given $\boldsymbol{P}$ and c, what game $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c\boldsymbol{P}$ maximizes the degree of the number of choices of (f, x)? In a given lattice $\mathcal{L}$, the shape of $\boldsymbol{Q}$ is predetermined as $\operatorname{type}\mathcal{L}$, and the degree is maximized by maximizing the length of the insertion/deletion path. We consider the case when there is a deleted entry, and when there is no deleted entry.

Let

$$\boldsymbol{P} = \begin{array}{lll} \bullet & \bullet & 1 \\ \bullet & 3 & \\ 2 & & \end{array}$$

and $c = (1, 3)$ be the cell with 1. The possible values of $\boldsymbol{Q} = \widetilde{\mathcal{I}}_c\boldsymbol{P}$ are listed below, along with the factor for each k; the Hall polynomial; and the product of all factors, equal to the total number of pairs (f, x) with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}}(f \vee x) = \boldsymbol{Q}$ in a lattice whose type is $\operatorname{osh}\boldsymbol{Q}$.

$\widetilde{\mathcal{I}}_c\boldsymbol{P}$	$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & 3 & \\ 2 & & \end{array}$	$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 2 & & \end{array}$	$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & 3 & \\ 1 & & \end{array}$	$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 2 & 3 & \end{array}$ $\mathcal{I}_c\boldsymbol{P}$	$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & 1 & \\ 2 & & \\ 3 & & \end{array}$
$k = 1$	1	$1 \cdot (q-1)$	$1 \cdot (q-1)$	$1 \cdot (q-1)$	$1 \cdot (q-1)$
2	q^2	q^2	q^2	q^2	q^2
3	q	q	q	$q \cdot q$	$q \cdot (q-1)$
Hall	1	1	1	$q^2 + q$	$q^3 + q^2$
total	q^3	$q^3(q-1)$	$q^3(q-1)$	$q^5(q-1)(q+1)$	$q^5(q-1)^2(q+1)$

Of the three games $\widetilde{\mathcal{I}}_c\boldsymbol{P}$ with a deleted entry, corresponding to $f_h := \hat{1}$, two achieve the maximum degree 4, so there is not a unique generic value. A generic value can be thought of as a value that occurs with probability $1 - O(q^{-1})$ as $q \to \infty$. As $q \to \infty$, these two values each occur with probability $\frac{1}{2} + O(q^{-1})$, and the third with probability $o(q^{-1})$; it might be interesting to do an analysis generalizing the notion of genericity to multiple outcomes with well defined asymptotically positive probabilities.

In the games without a deleted entry, so that $f_h \lessdot \hat{1}$ and $x \not\leq f_h$, the value of $\widetilde{\mathcal{I}}_c\boldsymbol{P}$ achieving the maximum degree is *not* $\mathcal{I}_c\boldsymbol{P}$.

8. The leftward and rightward vertical strip games

We introduce new tableau games similar in spirit to evacuation and Robinson-Schensted that can be expressed in terms of tableaux, or in terms of grids of partitions. They are all uniform.

$$\begin{aligned} \mathcal{L}, \widetilde{\mathcal{L}} : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} Af \\ \mathcal{CL}, \mathcal{C}\widetilde{\mathcal{L}} : \ \operatorname{ftype} f &\longmapsto \operatorname{ftype} Af \\ \mathcal{R}, \widetilde{\mathcal{R}} : \operatorname{ftype}_{\hat{0}} f &\longmapsto \operatorname{ftype}_{\hat{0}} Cf \end{aligned}$$

Note that for any flag f, the intervals $[f_k, A_{f_h} f_k]$ and $[C_{f_l} f_k, f_k]$ are elementary, so that $\operatorname{type}(A_{f_h} f_k)/\operatorname{type} f_k$ and $\operatorname{type} f_k/\operatorname{type}(C_{f_l} f_k)$ are vertical strips by Proposition 4.67. In the games j^c and Δ, we formed a deletion path by creating a hole $\star$ into which an entry would slide. In the vertical strip games, we instead form vertical strips of holes $\star$ into which entries slide. In the leftward vertical strip game, a vertical strip is formed that moves left while entries move into it, and in the rightward vertical strip game, the directions are reversed.

8.1. The leftward vertical strip game, $\mathcal{L}\boldsymbol{P}$.

$$\mathcal{L}, \widetilde{\mathcal{L}} : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Af$$

Tableau Rules. We form a sequence of tableau $\boldsymbol{P}^{(h)}, \ldots, \boldsymbol{P}^{(l)}$. Set $\boldsymbol{P}^{(h)} := \boldsymbol{P}$. To transform $\boldsymbol{P}^{(k)}$ to $\boldsymbol{P}^{(k-1)}$ for $k = h, \ldots, l{+}1$, if there is no k in $\boldsymbol{P}^{(k)}$, let $\boldsymbol{P}^{(k-1)} := \boldsymbol{P}^{(k)}$, and otherwise do the following.

- **($\boldsymbol{\mathcal{L}}$):** Slide k to the bottom leftmost $\star$ in a column strictly right of k, provided there is such a $\star$; if there is not, replace k by $\star$.
- **($\widetilde{\boldsymbol{\mathcal{L}}}$):** If the cell right of k has $\star$, slide k into it; otherwise, choose either to slide k into the bottommost $\star$ in any column right of k, or to replace k by $\star$ without putting a new k anywhere.

Finally, replace all $\star$'s with $\bullet$'s in $\boldsymbol{P}^{(l)}$ to obtain $\widetilde{\mathcal{L}}\boldsymbol{P}$.

Local Rules. Let $\mu^{(h)} := \lambda^{(h)}$, and then for $k = h, h{-}1, \ldots, l{+}1$, propagate as follows.

$$\begin{array}{ccccl} & k-1 & k & h & \text{column} \\ \longleftarrow & \alpha & \beta & \lambda^{(h)} & \boldsymbol{P} \\ & (\gamma) & \delta & (\lambda^{(h)}) & \widetilde{\mathcal{L}}\boldsymbol{P} \end{array}$$

- **$\boldsymbol{\beta = \alpha}$:** Let $\gamma := \delta$.
- **$\boldsymbol{\beta \gtrdot_{|s|} \alpha}$ ($\boldsymbol{\mathcal{L}}$):** If δ/β has no cells in columns $s+1, s+2, \ldots$, let $\gamma := \beta$, and otherwise, choose minimum $r > s$ so that δ/β has a cell in column r, and set $\gamma \lessdot_{|r|} \delta$.
- **$\boldsymbol{\beta \gtrdot_{|s|} \alpha}$ ($\widetilde{\boldsymbol{\mathcal{L}}}$):** If δ/α has a horizontal brick, remove the right square of the brick from δ to obtain γ. Otherwise, δ/α is a vertical strip. Either choose an $r > s$ such that column r of δ/α is nonempty, and set $\gamma \lessdot_{|r|} \delta$, or choose to let $\gamma := \delta$.

Note that these rules ensure δ/β and γ/α are vertical strips, and that $\alpha \le \beta \le \gamma \le \delta$.

Theorem 8.1. *The local rules and the tableau rules for $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ describe the same game.*

Proof. Apply the local rules to $\boldsymbol{P}$ to determine a possible value of $\widetilde{\mathcal{L}}\boldsymbol{P}$. Define the tableau $\boldsymbol{P}^{(k)}$ (with $l \leq k \leq h$) as the following chain of partitions, with the entry e to fill the squares added between two partitions shown between them as $\boxed{\text{e}}$:

$$\boldsymbol{P}^{(k)} \; = \; \begin{array}{llll} \mu^{(l)} \lessdot_{\boxed{l+1}} \mu^{(l+1)} \lessdot_{\boxed{l+2}} \cdots \lessdot_{\boxed{k}} & \mu^{(k)} & & \\ & \rotatebox{90}{$\lessdot$}_{\boxed{\star}} & & \\ & \lambda^{(k)} & \lessdot_{\boxed{k+1}} \cdots \lessdot_{\boxed{h}} & \lambda^{(h)} \end{array}$$

Since $\mu^{(h)} = \lambda^{(h)}$, we have $\boldsymbol{P}^{(h)} = \boldsymbol{P}$. We will show that the tableau rules describe the transformation from $\boldsymbol{P}^{(k)}$ to $\boldsymbol{P}^{(k-1)}$.

When $\beta = \alpha$, there is no k in $\boldsymbol{P}^{(k)}$, and both sets of rules yield $\boldsymbol{P}^{(k-1)} := \boldsymbol{P}^{(k)}$.

There is a horizontal brick in δ/α exactly when $\star$ is in the cell right of k in $\boldsymbol{P}^{(k)}$; moving k right one cell is the same as removing the right square of the brick from δ to obtain γ.

When δ/α is a vertical strip and $\beta \gtrdot_{|s|} \alpha$, letting $\gamma := \delta$ is the same as replacing k with $\star$, and letting $\gamma \lessdot_{|r|} \delta$ for some $r > s$ is the same as sliding k to the bottommost $\star$ in column r. A partition γ with $\alpha \leq \gamma \lessdot_{|r|} \delta$ exists provided $r > s \geq 1$ and column r of δ/α is nonempty.

Finally, given $\boldsymbol{P}^{(l)}$, replacing the $\star$'s by $\bullet$'s yields the tableau $(\lambda^{(l)}, \ldots, \lambda^{(h)})$. $\square$

Example 8.2. We will compute all possible values of $\widetilde{\mathcal{L}}\boldsymbol{P}$ for

$$\boldsymbol{P} = \boldsymbol{P}^{(8)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 3 & 8 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

We begin with $k = 8$. There is no $\star$ in the tableau, so there is nowhere to slide 8, so 8 is replaced by $\star$.

$$\boldsymbol{P}^{(7)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

There is no $\star$ in any column right of 7, so 7 is replaced by $\star$, and similarly, the same then happens with 6.

$$\boldsymbol{P}^{(6)} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \qquad \boldsymbol{P}^{(5)} = \begin{array}{|c|c|c|} \hline 1 & 4 & \star \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Both columns right of 5 contain $\star$. For $\widetilde{\mathcal{L}}\boldsymbol{P}$, there are choices as to how to proceed, which we'll examine later. To compute $\mathcal{L}\boldsymbol{P}$, slide 5 to the lower $\star$ of column 2, obtaining

$$\boldsymbol{P}^{(4)} = \begin{array}{|c|c|c|}\hline 1 & 4 & \star \\ \hline 2 & \star \\ \cline{1-2} 3 & 5 \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

Next, the cell just right of 4 has $\star$, so slide 4 there.

$$\boldsymbol{P}^{(3)} = \begin{array}{|c|c|c|}\hline 1 & \star & 4 \\ \hline 2 & \star \\ \cline{1-2} 3 & 5 \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

The column right of 3 has two $\star$'s; slide 3 to the lower one.

$$\boldsymbol{P}^{(2)} = \begin{array}{|c|c|c|}\hline 1 & \star & 4 \\ \hline 2 & 3 \\ \cline{1-2} \star & 5 \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

The column right of 2 has a $\star$, where we slide 2.

$$\boldsymbol{P}^{(1)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \star & 3 \\ \cline{1-2} \star & 5 \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

There are no $\star$'s in columns right of 1, so 1 is replaced by $\star$.

$$\boldsymbol{P}^{(0)} = \begin{array}{|c|c|c|}\hline \star & 2 & 4 \\ \hline \star & 3 \\ \cline{1-2} \star & 5 \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

Replace the $\star$'s by $\bullet$'s to obtain

$$\mathcal{L}\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & 2 & 4 \\ \hline \bullet & 3 \\ \cline{1-2} \bullet & 5 \\ \cline{1-2} \bullet \\ \cline{1-1}\end{array}$$

There are several places we could have made degenerate choices. The first was at $k = 5$; to compute $\mathcal{L}\boldsymbol{P}$, we put 5 in the lower $\star$ of column 2, but to compute $\widetilde{\mathcal{L}}\boldsymbol{P}$, we can either do that, slide 5 to the $\star$ in column 3, or replace 5 by $\star$ without sliding 5 somewhere else. In the second case, we have

$$\boldsymbol{P}^{(4)} = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

There is no $\star$ in columns right of 4, so 4 is replaced by $\star$.

$$\boldsymbol{P}^{(3)} = \begin{array}{|c|c|c|}\hline 1 & \star & 5 \\ \hline 2 & \star \\ \cline{1-2} 3 & \star \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

Each of 3, 2, and 1 in turn have $\star$ just to their right, so we are successively forced to have

$$\boldsymbol{P}^{(2)} = \begin{array}{|c|c|c|}\hline 1 & \star & 5 \\ \hline 2 & \star \\ \cline{1-2} \star & 3 \\ \cline{1-2} \star \\ \cline{1-1}\end{array} \qquad \boldsymbol{P}^{(1)} = \begin{array}{|c|c|c|}\hline 1 & \star & 5 \\ \hline \star & 2 \\ \cline{1-2} \star & 3 \\ \cline{1-2} \star \\ \cline{1-1}\end{array} \qquad \boldsymbol{P}^{(0)} = \begin{array}{|c|c|c|}\hline \star & 1 & 5 \\ \hline \star & 2 \\ \cline{1-2} \star & 3 \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

Finally, replace the $\star$'s by $\bullet$'s to obtain another possible value of $\widetilde{\mathcal{L}}\boldsymbol{P}$.

We list all possible choices in Figure 13.

Example 8.3. Now consider the skew tableau

$$\boldsymbol{P} = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 6 \\ \hline \bullet & 7 \\ \cline{1-2} \bullet & 8 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

The movements of $k = 8, 7, 6$ are similar to the previous example, since the entries larger than 4 are in the same positions in both tableaux.

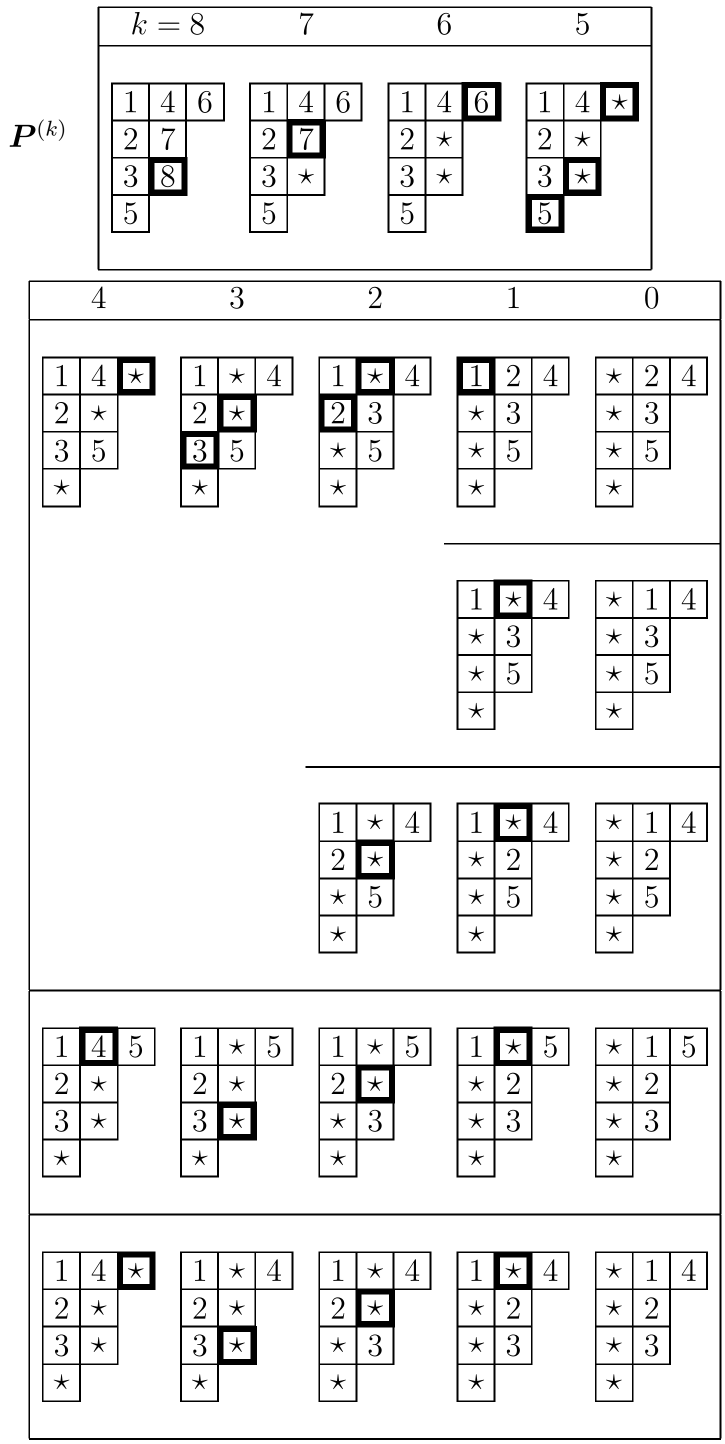
$$\begin{array}{ccc} \boldsymbol{P}^{(7)} & \boldsymbol{P}^{(6)} & \boldsymbol{P}^{(5)} \\ \begin{array}{|c|c|c|}\hline \bullet & \bullet & 6 \\ \hline \bullet & 7 \\ \cline{1-2} \bullet & \star \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline \bullet & \bullet & 6 \\ \hline \bullet & \star \\ \cline{1-2} \bullet & \star \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline \bullet & \bullet & \star \\ \hline \bullet & \star \\ \cline{1-2} \bullet & \star \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \end{array}$$

For $\boldsymbol{P}^{(4)}$, we have three choices again.

$$\begin{array}{|c|c|c|}\hline \bullet & \bullet & \star \\ \hline \bullet & \star \\ \cline{1-2} \bullet & 5 \\ \cline{1-2} \star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \bullet & \bullet & 5 \\ \hline \bullet & \star \\ \cline{1-2} \bullet & \star \\ \cline{1-2} \star \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \bullet & \bullet & \star \\ \hline \bullet & \star \\ \cline{1-2} \bullet & \star \\ \cline{1-2} \star \\ \cline{1-1}\end{array}$$

Replace all $\star$'s by $\bullet$'s to obtain the possible values of $\widetilde{\mathcal{L}}\boldsymbol{P}$, the first of which is $\mathcal{L}\boldsymbol{P}$.

$$\begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet \\ \cline{1-2} \bullet & 5 \\ \cline{1-2} \bullet \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \bullet & \bullet & 5 \\ \hline \bullet & \bullet \\ \cline{1-2} \bullet & \bullet \\ \cline{1-2} \bullet \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet \\ \cline{1-2} \bullet & \bullet \\ \cline{1-2} \bullet \\ \cline{1-1}\end{array}$$



A missing tableau abbreviates the tableau above it. The possible transformations of $\boldsymbol{P}^{(k)}$ to $\boldsymbol{P}^{(k-1)}$ are indicated: either slide k to a $\boxed{\star}$ or change $\boxed{k}$ to $\boxed{\star}$. Finally, replace $\star$'s by $\bullet$'s in $\boldsymbol{P}^{(0)}$ to obtain $\widetilde{\mathcal{L}}\boldsymbol{P}$. The first game is $\mathcal{L}\boldsymbol{P}$.

FIGURE 13. All possible degenerate leftward vertical strip games.

8.2. The rightward vertical strip game, $\mathcal{R}\boldsymbol{P}$.

$$\mathcal{R}, \widetilde{\mathcal{R}} : \operatorname{ftype}_{\hat{0}} f \longmapsto \operatorname{ftype}_{\hat{0}} Cf$$

Tableau Rules. Let $\boldsymbol{P}$ be a standard skew tableau on distinct entries. We form a sequence of tableau $\boldsymbol{P}^{(l)}, \dots, \boldsymbol{P}^{(h)}$. Set $\boldsymbol{P}^{(l)} := \boldsymbol{P}$. To transform $\boldsymbol{P}^{(k-1)}$ to $\boldsymbol{P}^{(k)}$ for $k = l+1, \dots, h$, if there is no k in $\boldsymbol{P}^{(k-1)}$, let $\boldsymbol{P}^{(k)} := \boldsymbol{P}^{(k-1)}$, and otherwise do the following.

- **($\boldsymbol{\mathcal{R}}$):** Slide k to the $\star$ that's upper rightmost in the columns strictly left of k, provided there is such a $\star$; if there is not, replace k by $\star$.
- **($\widetilde{\boldsymbol{\mathcal{R}}}$):** If the cell left of k has $\star$, slide k into it; otherwise, choose either to slide k into the topmost $\star$ in any column left of k, or to replace k by $\star$ without putting a new k anywhere.

Finally, delete all the cells with $\star$ from $\boldsymbol{P}^{(h)}$ to obtain $\widetilde{\mathcal{R}}\boldsymbol{P}$.

See Figure 14 for an example.

Local Rules. Let $\mu^{(l)} := \lambda^{(l)}$, and then for $k = l+1, l+2, \dots, h$, propagate as follows.

$$\begin{array}{r|c|c|c|c|l}
\text{column} & l & & k-1 & k & \\
\cline{2-2}\cline{4-5}
\boldsymbol{P} & \lambda^{(l)} & & \gamma & \delta & \\
\cline{2-2}\cline{4-5}
\widetilde{\mathcal{R}}\boldsymbol{P} & (\lambda^{(l)}) & & \alpha & (\beta) & \longrightarrow \\
\cline{2-2}\cline{4-5}
\end{array}$$

- **$\boldsymbol{\delta = \gamma}$ or $\boldsymbol{\delta \gtrdot_{|1|} \gamma}$:** Let $\beta := \alpha$.
- **$\boldsymbol{\delta \gtrdot_{|r|} \gamma}$ for some $\boldsymbol{r > 1}$ ($\boldsymbol{\mathcal{R}}$):** If δ/α has no cells in columns $1, \dots, r-1$, let $\beta := \alpha$, and otherwise, choose maximum $s < r$ such that δ/α has a cell in column s, and set $\beta \gtrdot_{|s|} \alpha$.
- **$\boldsymbol{\delta \gtrdot_{|r|} \gamma}$ for some $\boldsymbol{r > 1}$ ($\widetilde{\boldsymbol{\mathcal{R}}}$):** If δ/α has a horizontal brick, add the left square of the brick to α to obtain β. Otherwise, δ/α is a vertical strip. Either choose an $s < r$ such that column s of δ/α is nonempty and set $\beta \gtrdot_{|s|} \alpha$, or choose to let $\beta := \alpha$.

Note that these rules ensure δ/β and γ/α are vertical strips, and that $\alpha \mathrel{\underline{\lessdot}} \beta \leq \gamma \mathrel{\underline{\lessdot}} \delta$.

Theorem 8.4. *The local rules and the tableau rules for $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ describe the same game.*

Proof. Apply the local rules to $\boldsymbol{P}$ to determine a possible value of $\widetilde{\mathcal{R}}\boldsymbol{P}$. Define the tableau $\boldsymbol{P}^{(k)}$ (with $l \leq k \leq h$) as the following chain of partitions, with the entry e to fill the squares added between two partitions shown between them as $\boxed{\text{e}}$:

$$\boldsymbol{P}^{(k)} \;=\; \begin{array}{llllllll}
 & & & & & \lambda^{(k)} & \lessdot_{\boxed{k+1}} \cdots \lessdot_{\boxed{h}} & \lambda^{(h)} \\
 & & & & & \rotatebox{90}{$\lessdot$}_{\boxed{\star}} & & \\
\mu^{(l)} & \lessdot_{\boxed{l+1}} & \mu^{(l+1)} & \lessdot_{\boxed{l+2}} & \cdots \lessdot_{\boxed{k}} & \mu^{(k)} & &
\end{array}$$

Since $\mu^{(l)} = \lambda^{(l)}$, we have $\boldsymbol{P}^{(l)} = \boldsymbol{P}$. We will show that the tableau rules describe the transformation from $\boldsymbol{P}^{(k-1)}$ to $\boldsymbol{P}^{(k)}$.

When $\delta = \gamma$, there is no k in $\boldsymbol{P}^{(k-1)}$, and both sets of rules yield $\boldsymbol{P}^{(k)} := \boldsymbol{P}^{(k-1)}$.

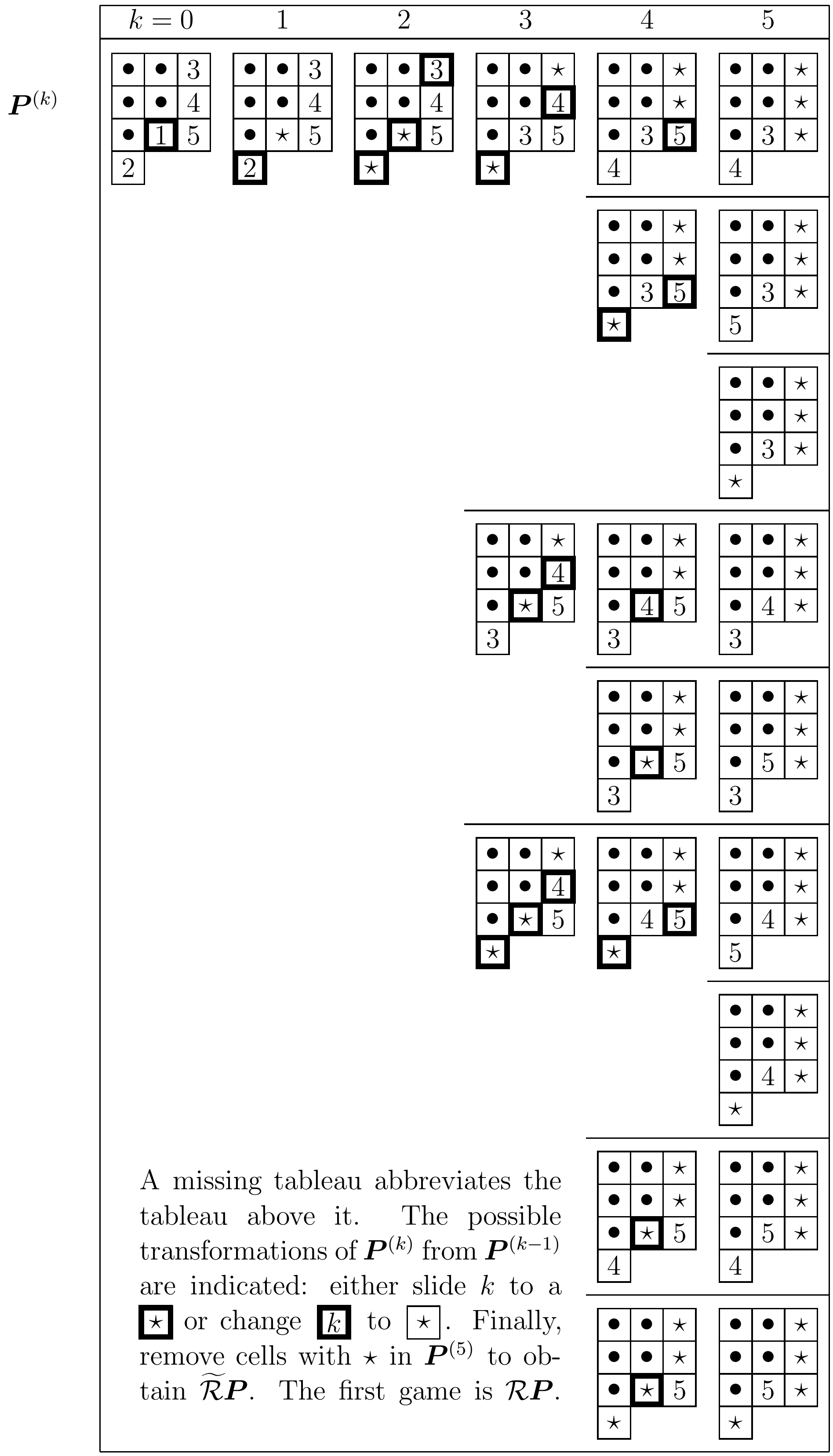


FIGURE 14. All possible degenerate rightward vertical strip games.

There is a horizontal brick in δ/α exactly when $\star$ is in the cell left of k in $\boldsymbol{P}^{(k-1)}$; moving k left one cell is the same as adding the left square of the brick to α to obtain β.

When δ/α is a vertical strip and $\delta \gtrdot_{|r|} \gamma$, letting $\beta := \alpha$ is the same as replacing k with $\star$, and letting $\beta \gtrdot_{|s|} \alpha$ for some $s < r$ is the same as sliding k to the topmost $\star$ in column s. A partition β with $\gamma \geq \beta \gtrdot_{|s|} \alpha$ exists provided $r > s \geq 1$ and column s of γ/α is nonempty.

Finally, given $\boldsymbol{P}^{(h)}$, deleting the cells with $\star$ yields the tableau $(\mu^{(l)}, \ldots, \mu^{(h)})$. $\square$

Next we prove that $\operatorname{ftype}_{\hat{0}} Af = \widetilde{\mathcal{L}}(\operatorname{ftype} f)$ and $\operatorname{ftype}_{\hat{0}} Cf = \widetilde{\mathcal{R}}(\operatorname{ftype} f)$, and show that $\mathcal{L}$ and $\mathcal{R}$ are the uniform values.

Definition 8.5. The notation $x \leq_{\mathrm{e}} y$ means $x \leq y$ and $[x, y]$ is an elementary interval.

Definition 8.6. For a vertical strip λ/μ and integer $t \geq 0$, define

$$\begin{aligned}
\operatorname{vlen}(\lambda, \mu, t) &\stackrel{\text{def}}{=} |\lambda| - |\lambda[t] \vee \mu| \;=\; \#\{\, i : \lambda_i \gtrdot \mu_i \geq t \,\} \\
&= \#\text{ of cells in } \lambda/\mu \text{ in columns } t+1, t+2, \ldots \\
\operatorname{vlen}'(\lambda, \mu, t) &\stackrel{\text{def}}{=} |\mu| - |\mathcal{C}\lambda \vee \mu[t]| \;=\; \#\{\, i : \lambda_i = \mu_i > t \,\} \\
&= \#\text{ of rows of } \lambda \text{ of length greater than } t \text{ without cells of } \lambda/\mu
\end{aligned}$$

Evaluate the right-hand expressions for vlen and vlen$'$ by summing the contribution of each row in the middle expressions. In vlen, the contribution of row i is $\lambda_i - \max\{\min\{\lambda_i, t\}, \mu_i\}$. Since $\lambda_i \geq \mu_i$ for all i, this contribution is 1 when both $\lambda_i > t$ and $\lambda_i \gtrdot \mu_i$, and is 0 otherwise. Similarly, the contribution of row i in vlen$'$ is $\mu_i - \min\{\max\{\lambda_i - 1, 0\}, \mu_i, t\}$, which is 0 when either $\lambda_i \gtrdot \mu_i$ or $\mu_i \leq t$, and is 1 otherwise. Also note that $\operatorname{vlen}(\lambda, \mu, t) + \operatorname{vlen}'(\lambda, \mu, t) = \lambda'_{t+1}$ because the sum is the number of parts of λ of length at least $t+1$ that equal or cover the corresponding part of μ, and all parts do because λ/μ is a vertical strip.

Let $\boldsymbol{P}$ be a skew tableau whose entries are distinct integers, and a vertical strip of $\star$'s. Let k be an integer in $\boldsymbol{P}$ and $t = 0$ or 1. Let $r = \operatorname{col}(\boldsymbol{P}, k)$. Define

$$\begin{aligned}
\operatorname{vlen}_t(\boldsymbol{P}, k) &\stackrel{\text{def}}{=} \#\text{ of } \star\text{'s in } \boldsymbol{P} \text{ at least } t \text{ columns right of } k \\
\operatorname{vlen}'_t(\boldsymbol{P}, k) &\stackrel{\text{def}}{=} \operatorname{row}_t(\boldsymbol{P}, k) - \operatorname{vlen}_t(\boldsymbol{P}, k)
\end{aligned}$$

When t is omitted, $t = 0$. If μ is the shape of the subtableau of $\boldsymbol{P}$ whose entries are $\bullet, l+1, \ldots, k$, and λ is the shape of the subtableau whose entries are $\bullet, l+1, \ldots, k, \star$, and $r = \operatorname{col}(\boldsymbol{P}, k)$, then $\operatorname{vlen}_t(\boldsymbol{P}, k) = \operatorname{vlen}(\lambda, \mu, r+t-1)$ and $\operatorname{vlen}'_t(\boldsymbol{P}, k) = \operatorname{vlen}'(\lambda, \mu, r+t-1)$.

Lemma 8.7. *In a q-regular semi-primary lattice, the number $T(\alpha, \beta, \gamma, \delta)$ of solutions (f_{k-1}, g_{k-1}) to*

$$\operatorname{type}\begin{array}{|ccc|}\hline (f_{k-1}) & \lessdot & f_k \\ & \diagdown & \\ \vee\!|_{\mathrm{e}} & \vee\!|_{\mathrm{e}} & \vee\!|_{\mathrm{e}} \\ & \diagdown & \\ (g_{k-1}) & \lessdot & g_k \\ \hline \end{array} = \begin{array}{|ccc|}\hline \gamma & \lessdot & \delta \\ & \diagdown & \\ \vee\!|_{\mathrm{v}} & \vee\!|_{\mathrm{v}} & \vee\!|_{\mathrm{v}} \\ & \diagdown & \\ \alpha & \lessdot & \beta \\ \hline \end{array}$$

in various situations is as follows.

$$(1)\quad \boxed{\begin{array}{ccc} \gamma & = & \delta \\ & \diagdown & \\ \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} \\ & & \diagdown \\ \alpha & = & \beta \end{array}} \qquad \textit{1 choice}$$

$$(2)\quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ & \diagdown & \\ \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} \\ & & \diagdown \\ \alpha & = & \beta \end{array}} \qquad \langle \text{vlen}(\delta,\beta,r-1)\rangle - \langle \text{vlen}(\delta,\beta,r)\rangle \ \textit{choices}$$

$$(3)\quad \boxed{\begin{array}{ccc} \gamma & = & \delta \\ & \diagdown & \\ \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} \\ & & \diagdown \\ \alpha & \lessdot_{|s|} & \beta \end{array}} \qquad \langle \text{vlen}'(\gamma,\beta,s-1)\rangle - \langle \text{vlen}'(\gamma,\beta,s)\rangle \ \textit{choices}$$

$$(4)\quad \boxed{\begin{array}{ccc} \gamma & \lessdot_{|r|} & \delta \\ & \diagdown & \\ \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\vee} & \rotatebox{90}{$\le$}_{\mathrm{e}} \\ & & \diagdown \\ \alpha & \lessdot_{|s|} & \beta \end{array}} \qquad \textit{where } g_{k-1} \not\leq_{\mathrm{e}} f_k$$

We have $r > s$ for all solutions of this last case, even in semi-primary lattices that are not q-regular. For ones that are, choose f_{k-1} in one of

$$\langle \text{vlen}(\delta,\beta,r-1)\rangle - \langle \text{vlen}(\delta,\beta,r)\rangle$$

ways, and then g_{k-1} in one of the following number of ways;

$$\begin{cases} \langle \text{vlen}'(\gamma,\beta,s-1)\rangle - \langle \text{vlen}'(\gamma,\beta,s)\rangle & \textit{if } \delta/\alpha \textit{ has a horizontal brick;} \\ q^{\text{vlen}'(\delta,\beta,s-1)} & \textit{if } \delta \geq_\vee \alpha \textit{ and } r-1 = s; \\ q^{\text{vlen}'(\delta,\beta,s-1)} - q^{\text{vlen}'(\delta,\beta,s)} & \textit{if } \delta \geq_\vee \alpha \textit{ and } r-1 > s. \end{cases}$$

In cases (1), (2), (4) we have $C_{g_{k-1}} f_k = g_k$, and in cases (1), (3), (4) we have $A_{f_k} g_{k-1} = f_{k-1}$.

Proof. Consider a solution f_{k-1} and g_{k-1} to one of these configurations. In (1) and (2), we have $g_{k-1} = g_k$, so $[g_{k-1}, f_k]$ is elementary; thus, $C_{g_{k-1}} f_k = g_{k-1} = g_k$. In (1) and (3), $f_{k-1} = f_k$ so $[g_{k-1}, f_k]$ is elementary, so $A_{f_k} g_{k-1} = f_k = f_{k-1}$. In (4), the interval $[g_k, f_k]$ is elementary, so $g_k \geq C f_k$. We have $C_{g_{k-1}} f_k = C f_k \vee g_{k-1}$, which is at least g_{k-1} and at most g_k (because $C_{g_{k-1}} f_k \leq C_{g_k} f_k = g_k$). It's not g_{k-1} because $[g_{k-1}, f_k]$ is given as not elementary, whence $C_{g_{k-1}} f_k = g_k$. Similarly in (4), $A_{f_k} g_{k-1} = f_{k-1}$.

Now we enumerate the solutions to each case.

(1) Since $\gamma = \delta$ and $\alpha = \beta$, we have $f_{k-1} := f_k$ and $g_{k-1} := g_k$, in one way.

(2) Set $g_{k-1} := g_k$ in one way.

Since $\gamma \lessdot_{|r|} \delta$, we have that $f_{k-1} \geq f_k[r-1]$ but $f_{k-1} \not\geq f_k[r]$. Also, f_{k-1} is a coatom of $[g_k, f_k]$. Thus, f_{k-1} is any coatom of $I_1 = \big[f_k[r-1] \vee g_k, f_k\big]$

not also in $I_2 = \Big[f_k[r] \vee g_k, f_k\Big]$. These are both elementary intervals, as they are contained in the elementary interval $[g_k, f_k]$, so altogether there are $\langle\rho(I_1)\rangle - \langle\rho(I_2)\rangle$ choices of f_{k-1}. We must find the ranks of I_1 and I_2.

Since $[g_k, f_k]$ is elementary, there is a hereditary decomposition of g_k and f_k by Theorem 4.69, and so by Theorem 4.62, $\operatorname{type}(f_k[r-1] \vee g_k) = \delta[r-1] \vee \beta$ and $\operatorname{type}(f_k[r] \vee g_k) = \delta[r] \vee \beta$. Thus $\rho(I_1) = \operatorname{vlen}(\delta, \beta, r-1)$ and $\rho(I_2) = \operatorname{vlen}(\delta, \beta, r)$.

(3) Set $f_{k-1} := f_k$ in one way.

We will compute

$$N(g_k, f_{k-1}, s) = \#\Big\{ g_{k-1} \lessdot g_k \,:\, g_{k-1} \le_{\mathrm{e}} f_{k-1} \text{ and } \operatorname{type} g_{k-1} \lessdot_{|s|} \operatorname{type} g_k \Big\}.$$

Since g_{k-1} is a lower cover of g_k with type $\alpha \lessdot_{|s|} \beta$, we have $g_{k-1} \geq g_k[s-1]$ but $g_{k-1} \not\geq g_k[s]$. Since $[g_{k-1}, f_{k-1}]$ is elementary, $g_{k-1} \geq C f_{k-1}$.

Thus, g_{k-1} is any coatom of $I_3 = \Big[C f_{k-1} \vee g_k[s-1], g_k\Big]$ not also in $I_4 = \Big[C f_{k-1} \vee g_k[s], g_k\Big]$. These intervals are both elementary, because they are contained in the elementary interval $[C f_{k-1}, f_{k-1}]$, so there are $\langle\rho(I_3)\rangle - \langle\rho(I_4)\rangle$ choices of g_{k-1}. We must find the ranks of the intervals.

Since $[g_k, f_{k-1}]$ is elementary, there is a hereditary decomposition of g_k and f_k, so we may compute types as follows: $\operatorname{type}(C f_{k-1} \vee g_k[s-1]) = \mathcal{C}\gamma \vee \beta[s-1]$ and $\operatorname{type}(C f_{k-1} \vee g_k[s]) = \mathcal{C}\gamma \vee \beta[s]$. Thus $\rho(I_3) = \operatorname{vlen}'(\gamma, \beta, s-1)$ and $\rho(I_4) = \operatorname{vlen}'(\gamma, \beta, s)$, so $N(g_k, f_{k-1}, s) = \langle \operatorname{vlen}'(\gamma, \beta, s-1)\rangle - \langle \operatorname{vlen}'(\gamma, \beta, s)\rangle$.

(4) Given a solution f_{k-1} and g_{k-1}, the values of r and s are

$$\begin{aligned} r &= 1 + \max\Big\{ t \,:\, A^t\hat{0} \wedge f_{k-1} = A^t\hat{0} \wedge f_k \Big\} \\ s &= 1 + \max\Big\{ t \,:\, A^t\hat{0} \wedge g_{k-1} = A^t\hat{0} \wedge g_k \Big\}. \end{aligned}$$

If $A^t\hat{0} \wedge g_{k-1} = A^t\hat{0} \wedge g_k$, apply A_{f_k} to both sides to obtain $A^{t+1}\hat{0} \wedge A g_{k-1} \wedge f_k = A^{t+1}\hat{0} \wedge A g_k \wedge f_k$, which simplifies to $A^{t+1}\hat{0} \wedge f_{k-1} = A^{t+1}\hat{0} \wedge f_k$. Thus, $r > s$.

Select f_{k-1} as in case (2) in one of $\langle \operatorname{vlen}(\delta, \beta, r-1)\rangle - \langle \operatorname{vlen}(\delta, \beta, r)\rangle$ ways.

If δ/α has a horizontal brick, then $[g_{k-1}, f_k]$ is not elementary for all $g_{k-1} \lessdot g_k$ of type α, by Proposition 4.67. So choose g_{k-1} in the same fashion as in case (3).

Now assume δ/α is a vertical strip.

If $[g_{k-1}, f_k]$ is elementary, then so is the subinterval $[g_{k-1}, f_{k-1}]$. Thus, the number of choices of g_{k-1} is

$$\begin{aligned} N(g_k, f_{k-1}, s) - N(g_k, f_k, s) = &\Big(\langle \operatorname{vlen}'(\gamma, \beta, s-1)\rangle - \langle \operatorname{vlen}'(\gamma, \beta, s)\rangle\Big) \\ &- \Big(\langle \operatorname{vlen}'(\delta, \beta, s-1)\rangle - \langle \operatorname{vlen}'(\delta, \beta, s)\rangle\Big). \end{aligned}$$

For some j we have $\delta/\gamma = (j, r)$, and then $\delta_j \gtrdot \gamma_j = \beta_j = \alpha_j = r-1 \geq s$. Thus, $\operatorname{vlen}'(\gamma, \beta, s-1) \gtrdot \operatorname{vlen}'(\delta, \beta, s-1)$, while $\operatorname{vlen}'(\beta, \gamma, s) \geq \operatorname{vlen}'(\delta, \beta, s)$ with equality when $r-1 = s$ and covering when $r-1 > s$. Note that $\langle M+1\rangle - \langle M\rangle = q^M$ for all M, so when $r-1 = s$ there are $q^{\operatorname{vlen}'(\delta,\beta,s-1)}$ choices of g_{k-1}, and when $r-1 > s$ there are $q^{\operatorname{vlen}'(\delta,\beta,s-1)} - q^{\operatorname{vlen}'(\delta,\beta,s)}$ choices. □

Definition 8.8. Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\boldsymbol{Q} = (\mu^{(l)}, \ldots, \mu^{(h)})$ be skew tableau such that $\lambda^{(k)} \gtrdot_\vee \mu^{(k)}$ for $l < k \le h$. Let $\boldsymbol{P}^{(k)}$ be a skew tableau of shape $\lambda^{(h)}/\mu^{(l)}$ in which entries $e \le k$ are in the same cells as in $\boldsymbol{Q}$; entries $e > k$ are in the same cells as in $\boldsymbol{P}$; and the vertical strip $\lambda^{(k)}/\mu^{(k)}$ is filled with $\star$. Define

$$\begin{aligned} E(\boldsymbol{P}, \boldsymbol{Q}; q) &\stackrel{\text{def}}{=} \prod_{k \in \boldsymbol{P}} \Big(\Big\langle \text{vlen}(\boldsymbol{P}^{(k-1)}, k) + 1\Big\rangle - \Big\langle \text{vlen}_1(\boldsymbol{P}^{(k-1)}, k)\Big\rangle\Big) \\ &\times \prod_{\substack{k \in \boldsymbol{Q} \\ k \notin \boldsymbol{P}}} \Big(\Big\langle \text{vlen}(\boldsymbol{P}^{(k)}, k)\Big\rangle - \Big\langle \text{vlen}_1(\boldsymbol{P}^{(k)}, k)\Big\rangle\Big) \\ &\times \prod_{\substack{k \in \boldsymbol{Q} \\ k \in \boldsymbol{P}}} \begin{cases} \Big(\Big\langle \text{vlen}'(\boldsymbol{P}^{(k)}, k) + 1\Big\rangle - & \text{if } k \text{ slides} \\ \quad \Big\langle \text{vlen}'_1(\boldsymbol{P}^{(k)}, k) + 1\Big\rangle\Big) & \quad \text{horizontally one cell;} \\ q^{\text{vlen}'(\boldsymbol{P}^{(k)}, k)} & \quad \text{one column to a different row;} \\ q^{\text{vlen}'(\boldsymbol{P}^{(k)}, k)} - q^{\text{vlen}'_1(\boldsymbol{P}^{(k)}, k)} & \quad \text{more than one column.} \end{cases} \end{aligned}$$

This is the product of $T(\mu^{(k-1)}, \mu^{(k)}, \lambda^{(k-1)}, \lambda^{(k)})$ over all k. In configurations (2) and (4), we have $\text{vlen}(\delta, \beta, r-1) = \text{vlen}(\boldsymbol{P}^{(k-1)}, k) + 1$ and $\text{vlen}(\delta, \beta, r) = \text{vlen}_1(\boldsymbol{P}^{(k-1)}, k)$ because $\star$'s form the strip γ/α in $\boldsymbol{P}^{(k-1)}$, which differs from δ/β only in one cell in column $r = \text{col}(\boldsymbol{P}^{(k-1)}, k)$. In (4), we have $\text{vlen}'(\delta, \beta, s-1) = \text{vlen}'(\boldsymbol{P}^{(k)}, k)$ because the vertical strip of $\star$'s in the tableau is also δ/β, and the columns we examine are the same. The value in (3) is similar. Finally in (4), one cell of δ/β in column r is deleted to form the strip γ/β, and since $r > s$, this deleted cell is counted by both $\text{vlen}'(\gamma, \beta, s-1) = \text{vlen}'(\boldsymbol{P}^{(k)}, k) + 1$ and $\text{vlen}'(\gamma, \beta, s) = \text{vlen}'_1(\boldsymbol{P}^{(k)}, k) + 1$.

Theorem 8.9. *For all multisaturated flags f in a semi-primary lattice,* $\text{ftype}_{\hat{0}} Af = \widetilde{\mathcal{L}}(\text{ftype}_{\hat{0}} f)$ *and* $\text{ftype}_{\hat{0}} Cf = \widetilde{\mathcal{R}}(\text{ftype}_{\hat{0}} f)$.

Now consider a q-regular semi-primary lattice of type λ. Let $\boldsymbol{P}$ be a skew tableau of outer shape λ.

Fix $\boldsymbol{Q} = \widetilde{\mathcal{R}}\boldsymbol{P}$ of outer shape μ. The number of flags f for which $\text{ftype}_{\hat{0}} f = \boldsymbol{P}$ *and* $\text{ftype}_{\hat{0}} Cf = \boldsymbol{Q}$ *is $g^{\lambda}_{\mu,(m)'}(q) \cdot E(\boldsymbol{P}, \boldsymbol{Q}; q)$, where $m = |\lambda| - |\mu|$. Flags f with* $\text{ftype}_{\hat{0}} f = \boldsymbol{P}$ *are uniformly mapped by C to flags g from f_l to $C_{f_l} f_h$ with* $\text{ftype}_{\hat{0}} g = \mathcal{R}\boldsymbol{P}$.

Fix $\boldsymbol{Q} = \widetilde{\mathcal{L}}\boldsymbol{P}$. The number of flags g for which $\text{ftype}_{\hat{0}} f = \boldsymbol{P}$ *and* $\text{ftype}_{\hat{0}} Af = \boldsymbol{Q}$ *is $E(\boldsymbol{Q}, \boldsymbol{P}; q)$. Flags f with* $\text{ftype}_{\hat{0}} f = \boldsymbol{P}$ *are uniformly mapped by A to flags g from $A_{f_h} f_l$ to f_h with* $\text{ftype}_{\hat{0}} g = \mathcal{L}\boldsymbol{P}$.

Proof. If f and g are flags with $g = Cf$, then we have $g_l = C_{f_l} f_l = g_l$ and $g_k = C_{f_l} f_k = Cf_k \vee f_l = Cf_k \vee Cf_{k-1} \vee f_l = Cf_k \vee g_{k-1}$. The intervals $[g_{k-1}, f_{k-1}]$ and $[g_k, f_k]$ are elementary, so by Proposition 4.67, $\text{type}\, g_{k-1} \le_\vee \text{type}\, f_{k-1}$ and $\text{type}\, g_k \le_\vee \text{type}\, f_k$. Since $f_{k-1} \le f_k$, we have $Cf_k \le f_{k-1}$. Also $g_{k-1} \le f_{k-1}$, so $g_k \le f_{k-1}$. Thus $\text{type}\, g_k \le \text{type}\, f_{k-1}$. Also, $[g_k, f_{k-1}]$ is a subinterval of the elementary interval $[g_k, f_k]$, so it is elementary, and $\text{type}\, g_k \le_\vee \text{type}\, f_{k-1}$.

Consider a semi-primary lattice of type λ. Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ be a standard tableau of outer shape λ, and fix $\boldsymbol{Q} = \widetilde{\mathcal{R}}\boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(h)})$. To find all pairs of flags (f, g) with $g = Cf$ and $\text{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\text{ftype}_{\hat{0}} Cf = \boldsymbol{Q}$, set $f_h := \hat{1}$ in one way, and choose $g_h \le_{\text{e}} f_h$ so that $\text{type}\, g_h = \mu^{(h)}$ in one of $g^{\lambda^{(h)}}_{\mu^{(h)},(m(h))'}(q)$ ways (where $m(k) = |\lambda(k)| - \Big|\mu^{(k)}\Big|$). Then for $k = h, h-1, \ldots, l+1$, successively find pairs

(f_{k-1}, g_{k-1}) of weak lower covers of f_k and g_k with specified types $\lambda^{(k-1)}$ and $\mu^{(k-1)}$ such that $g_k = Cf_k \vee g_{k-1}$. Cases (1), (2), and (4) of Lemma 8.7 give all possibilities: if $\lambda^{(k-1)} = \lambda^{(k)}$ then $f_{k-1} = f_k$ so applying C maintains equality, and we are in case (1), not (3). Suppose $\lambda^{(k-1)} \lessdot \lambda^{(k)}$. If $\mu^{(k-1)} = \mu^{(k)}$, we are in case (2). If $\mu^{(k-1)} \lessdot \mu^{(k)}$, then for any solution we find, we have $g_k = Cf_k \vee g_{k-1} \gtrdot g_{k-1}$ so $g_{k-1} \not\geq Cf_k$, whence $g_{k-1} \not\leq_{\mathrm{e}} f_k$. Thus, we are in case (4).

In a q-regular semi-primary lattice, multiply the numbers of choices together to obtain

$$g^{\lambda^{(h)}}_{\mu^{(h)},(m(h))'}(q) \cdot E(\boldsymbol{P}, \boldsymbol{Q}; q)$$

flags f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_{\hat{0}}\, Cf = \boldsymbol{Q}$. Denote this product $p(\boldsymbol{P}, \boldsymbol{Q}; q)$. This product is a monic polynomial in q: the Hall polynomial is monic, because its leading coefficient is the number of ways to fill a Littlewood-Richardson tableau of specified skew shape with just 1's, which can only be done in one way, and $E(\boldsymbol{P}, \boldsymbol{Q}; q)$ is a product of monic polynomials.

We show that $\mathcal{R}$ is the generic game by inducting on $\mathrm{high}(\boldsymbol{P}) - \mathrm{low}(\boldsymbol{P})$. Note that $d\mathcal{R}\boldsymbol{P} = \mathcal{R}d\boldsymbol{P}$, and by the inductive hypothesis, $\mathcal{R}d\boldsymbol{P}$ is the generic value of $\widetilde{\mathcal{R}}d\boldsymbol{P}$. Thus, $p(d\boldsymbol{P}, d\mathcal{R}\boldsymbol{P}; q)$ is a monic polynomial whose degree is the same as that of $F^{d\boldsymbol{P}}(q)$. If we show that the degrees of $p(\boldsymbol{P}, \mathcal{R}\boldsymbol{P}; q)$ and $F^{\boldsymbol{P}}(q)$ are the same, then all but $O(q^{-1})$ flags f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ have $\mathrm{ftype}_{\hat{0}}\, Cf = \mathcal{R}\boldsymbol{P}$, so $\mathcal{R}$ gives the generic value.

Consider the quotient

$$\frac{p(\boldsymbol{P}, \mathcal{R}\boldsymbol{P}; q)}{p(d\boldsymbol{P}, d\mathcal{R}\boldsymbol{P}; q)} = \frac{g^{\lambda^{(h)}}_{\mu^{(h)},(m(h))'(q)}}{g^{\lambda^{(h-1)}}_{\mu^{(h-1)},(m(h-1))'}(q)} \cdot T(\mu^{(h-1)}, \mu^{(h)}, \lambda^{(h-1)}, \lambda^{(h)})$$

We must show the degree of the ratio is $n(\mathrm{osh}\, \boldsymbol{P}/\, \mathrm{osh}\, d\boldsymbol{P}) = n(\delta/\gamma)$. We evaluate the difference in degrees of the Hall polynomials, and the degree of T, for cases (1), (2), and (4) of the preceding lemma. Note that the Hall polynomial $g^{\lambda}_{\mu,\nu}(q)$ has degree $n(\lambda) - n(\mu) - n(\nu)$, and in this case, ν is always a single column.

In (1), the ratio equals 1 and also $F^{\boldsymbol{P}}(q)/F^{d\boldsymbol{P}}(q) = 1$, so $\mathcal{R}$ is generic.

In (2), the ratio of Hall polynomials has degree $\delta'_r - |\delta/\gamma|$, and the factor T has degree $\mathrm{vlen}(\delta, \beta, r-1) - 1$. When configuration (2) arises in $\mathcal{R}$, the vertical strip δ/β is wholly contained in columns $r, r+1, \ldots$. Thus, $\mathrm{vlen}(\delta, \beta, r-1) = |\delta/\beta|$, so the total ratio has degree $\delta'_r - 1 = n(\delta/\gamma)$, as required.

In (4), the ratio of Hall polynomials has degree $\delta'_r - \beta'_s$, and the factor T has degree $\mathrm{vlen}(\delta, \beta, r-1) - 1 + \mathrm{vlen}'(\delta, \beta, s-1)$. When configuration (4) arises in $\mathcal{R}$ (and $\mathcal{L}$), the vertical strip δ/β has no cells in columns $s+1, s+2, \ldots, r-1$, so rows $\beta'_s + 1, \beta'_s + 2, \ldots, \delta'_s$ have length at least $s-1$ and no cells of the vertical strip. Thus, $\mathrm{vlen}'(\delta, \beta, r-1) = \mathrm{vlen}'(\delta, \beta, s-1) - (\delta'_s - \beta'_s)$, so the factor T has degree

$$(\mathrm{vlen}'(\delta, \beta, s-1) + \mathrm{vlen}(\delta, \beta, s-1)) - 1 - \delta'_s + \beta'_s = \delta'_s - 1 - \delta'_s + \beta'_s = \beta'_s - 1.$$

The degree of the total ratio is $(\delta'_r - \beta'_s) + (\beta'_s - 1) = \delta'_r - 1 = n(\delta/\gamma)$, as required.

Now we show that $\mathcal{R}$ is uniform: we show that all but $O(q^{-1})$ of the flags g with $\mathrm{ftype}_{\hat{0}}\, g = \mathcal{R}\boldsymbol{P}$ and $g_h \leq_{\mathrm{e}} \hat{1}$ have $q^{n(\mathrm{sh}\, \boldsymbol{P})}(1 + O(q^{-1}))$ inverses f with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $g = Cf$. Begin with any $g_h \leq_{\mathrm{e}} \hat{1}$ with $\mathrm{type}\, g_h = \mu^{(h)}$. Suppose we have chosen $g_h, \ldots, g_k$ and $f_h, \ldots, f_k$, and we want to choose g_{k-1} and f_{k-1}. In cases (1) and

(2), we set $g_{k-1} := g_k$ and choose any of $T(\alpha,\beta,\gamma,\delta)$ values for f_{k-1}; this count is independent of g. In case (4), we saw that the degree of T in the generic game $\mathcal{R}$ is $\beta'_s - 1$, which is the same as the degree of the number of lower covers of g_k of type α. Thus, all but a fraction $O(q^{-1})$ of partial flags g can be extended so that the inverse f has an extension. For each extension of g, the extension of f is unique: $f_{k-1} := A_{f_k} g_{k-1}$. Thus, by case (4), only a fraction $O(q^{-1})$ of g with $\operatorname{ftype}_{\hat{0}} g = \mathcal{R}\boldsymbol{P}$ are not invertible (cases (1) and (2) impose no restrictions), and by case (2), there are $q^{n(\operatorname{sh}\boldsymbol{P})}(1+O(q^{-1}))$ inverses of each (cases (1) and (4) yield unique choices of f_{k-1} while (2) contributes many choices).

The proof for the game $\widetilde{\mathcal{L}}$ is similar. The roles of f and g are reversed from the lemma; we denote the configurations $(1')$, etc., to indicate that all f's and g's in the lemma should be swapped. We set $(\alpha,\beta,\gamma,\delta) = (\lambda^{(k-1)}, \lambda^{(k)}, \mu^{(k-1)}, \mu^{(k)})$ so that the partitions agree with those in the lemma. For pairs of flags (f,g) with $g = Af$, we have $f_h = g_h$, and $g_k = A_{f_h} f_k = Af_k \wedge f_h = Af_k \wedge g_h$, so $g_{k-1} = Af_{k-1} \wedge g_h = Af_{k-1} \wedge (Af_k \wedge g_h) = Af_{k-1} \wedge g_k$. So a local part of (f,g) has form $(1')$, $(3')$, or $(4')$, whence $\operatorname{ftype}_{\hat{0}} Af = \widetilde{\mathcal{L}}(\operatorname{ftype}_{\hat{0}} f)$.

We compute the number of flags f with $\operatorname{ftype}_{\hat{0}} f = \boldsymbol{P}$ and $\operatorname{ftype}_{\hat{0}} Af = \boldsymbol{Q}$ as follows. The flag g will equal Af during the construction. Let $f_h := g_h := \hat{1}$ in one way. Then for each $k = h, h-1, \dots, l+1$, choose g_{k-1} and f_{k-1} in one of $T(\lambda^{(k-1)}, \lambda^{(k)}, \mu^{(k-1)}, \mu^{(k)})$ ways; multiply all the values of T together to obtain the total number $E(\boldsymbol{Q}, \boldsymbol{P}; q)$.

We show that $\mathcal{L}$ is the generic value of $\widetilde{\mathcal{L}}$. We must show for each k that the degree of the polynomial T equals $n(\beta/\alpha)$, the degree of the total number of weak lower covers of f_k of type α. In $(1')$ there is one choice of (f_{k-1}, g_{k-1}) and one weak lower cover of f_k of type α. In $(4')$, the generic value of the number of choices of (f_{k-1}, g_{k-1}) has degree $\beta'_s - 1$, by the same proof used for $\mathcal{R}$ above; this is also the degree of the number of lower covers of f_k of type α. When configuration $(3')$ arises in the game $\mathcal{L}$, the vertical strip δ/β is wholly contained in columns $1, \dots, s$, so the degree of T is $N_2(\gamma, \beta, s-1) - 1 = \beta'_s - 1$, which again is the degree of the number of lower covers of f_k of type α. Thus, the generic value of $\widetilde{\mathcal{L}}$ is $\mathcal{L}$.

Now we show that the generic value is uniform. Let $\boldsymbol{P}$ be a tableau of ordinary shape. Since $Af = (C(f^*))^*$ for all flags f, the generic value of $\operatorname{ftype}((C(f^*))^*)$ is a composition $\operatorname{ev}(\mathcal{C}(\operatorname{ev}(\boldsymbol{P})))$ of uniform games. Thus, $\mathcal{L}\boldsymbol{P} = \operatorname{ev}(\mathcal{C}(\operatorname{ev}(\boldsymbol{P})))$, and this is uniform. When $\boldsymbol{P}$ is skew with inner shape of size N, fill in the inner shape of $\boldsymbol{P}$ with a standard tableau on $l-N+1, l-N+2, \dots, l$ to obtain a tableau $\boldsymbol{R}$ of ordinary shape. We have $\boldsymbol{P} = \partial^N \boldsymbol{R}$. Also, $\mathcal{L}\boldsymbol{P} = \partial^N \mathcal{L}\boldsymbol{R}$, and this composition of uniform games is uniform. □

Corollary 8.10. *For all multisaturated flags f in a semi-primary lattice,* $\operatorname{ftype} Af = \mathcal{C}\widetilde{\mathcal{L}}(\operatorname{ftype} f)$*, and uniformly,* $\operatorname{ftype} Af = \mathcal{C}\mathcal{L}(\operatorname{ftype} f)$.

8.3. Relation of vertical strip games to evacuation and *jeu de taquin*.

Theorem 8.11. *Let $\boldsymbol{P}$ be a standard tableau of shape $\lambda \vdash n$ on entries $1, \dots, n$. Form a tableau $\boldsymbol{Q}$ as follows: for each $r > 0$, take the entries that vanish in the game $\mathcal{L}^{r-1}\boldsymbol{P} \to \mathcal{L}^r \boldsymbol{P}$, complement them by subtracting each from $n+1$, and place the complements in column r of $\boldsymbol{Q}$ in increasing order from top to bottom. Then $\boldsymbol{Q} = \operatorname{ev} \boldsymbol{P}$.*

Proof. We can compute the generic cotype of flags f with ftype $f = \boldsymbol{P}$ in two ways. On the one hand, by Theorem 5.14, it is ev $\boldsymbol{P}$. Alternately, the differences in successive ranks of $f_k, Af_k, A^2 f_k, \ldots,$ are the parts of Atype$[f_k, f_n]$. We evaluate the generic values of these ranks over all k by the playing games $\boldsymbol{P}, \mathcal{L}\boldsymbol{P}, \mathcal{L}^2\boldsymbol{P}, \ldots$. If k is deleted in the transition $\mathcal{L}^{r-1}\boldsymbol{P} \to \mathcal{L}^r\boldsymbol{P}$, then generically, $A^{r-1}f_k \gtrdot A^{r-1}f_{k-1}$ and $A^r f_k = A^{r-1}f_{k-1}$, so the cotypes of f_{k-1} and f_k differ in column r. Thus, the complement $n+1-k$ to $k-1$ is in column r of the cotype tableau. This determines the columns of all entries in the cotype tableau, and sorting the entries in each column determines their positions. $\square$

Example 8.12.

$$\boldsymbol{P} = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & 7 & \\ 8 & & \end{array} \qquad \mathcal{L}\boldsymbol{P} = \begin{array}{ccc} \bullet & 2 & 4 \\ \bullet & 3 & \\ \bullet & 5 & \\ \bullet & & \end{array} \qquad \mathcal{L}^2\boldsymbol{P} = \begin{array}{ccc} \bullet & \bullet & 3 \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array} \qquad \mathcal{L}^3\boldsymbol{P} = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

The entries that disappear in the first step are 1,6,7,8; in the second step, 2,4,5; and in the third step, 3. Subtracting them from 9, the first column of ev $\boldsymbol{P}$ has 8,3,2,1; the second, 7,5,4; and the third, 6. So

$$\text{ev}\,\boldsymbol{P} = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & 7 & \\ 8 & & \end{array}.$$

Theorem 8.13. *Let $\boldsymbol{P}$ be a skew tableau on distinct entries. Form a Young tableau $\boldsymbol{Q}$ whose entries in the rth column are the entries that vanish in the game $\mathcal{R}^{r-1}\boldsymbol{P} \to \mathcal{R}^r\boldsymbol{P}$. Then $\boldsymbol{Q} = j(\boldsymbol{P})$.*

Proof. We can compute the generic value of ftype f over flags with ftype$_{\hat{0}}\, f = \boldsymbol{P}$ in two ways. On the one hand, by Theorem 6.3, it is $j(\boldsymbol{P})$. Alternately, the differences in successive ranks of $f_k, C_{f_l}f_k, C_{f_l}^2 f_k, \ldots,$ are the parts of Ctype$[f_l, f_k]$. We evaluate the generic values of these ranks over all k by the playing games $\boldsymbol{P}, \mathcal{R}\boldsymbol{P}, \mathcal{R}^2\boldsymbol{P}, \ldots$. If k is deleted in the transition $\mathcal{R}^{r-1}\boldsymbol{P} \to \mathcal{R}^r\boldsymbol{P}$, then generically, $C_{f_l}^{r-1}f_k \gtrdot C_{f_l}^{r-1}f_{k-1}$ and $C_{f_l}^r f_k = C_{f_l}^{r-1}f_{k-1}$, so type$[f_l, f_{k-1}]$ and type$[f_l, f_k]$ differ in column r. Thus, k is in column r of ftype f. This determines the columns of all entries in ftype f, and sorting the entries in each column determines their positions. $\square$

Example 8.14.

$$\boldsymbol{P} = \begin{array}{ccc} \bullet & \bullet & 3 \\ \bullet & \bullet & 4 \\ \bullet & 1 & 5 \\ 2 & & \\ 6 & & \end{array} \qquad \mathcal{R}\boldsymbol{P} = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & 3 \\ 4 & \end{array} \qquad \mathcal{R}^2\boldsymbol{P} = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} \qquad j(\boldsymbol{P}) = \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \\ 6 & \end{array}$$

8.4. A-statistics of flags. Let $f = (f_0 \lessdot \cdots \lessdot f_n)$ be a saturated flag in a modular lattice and $\vec{r} = (r_0, \ldots, r_n)$ be a sequence of nonnegative integers. Define

$$\begin{aligned} A^{\vec{r}} f &\overset{\text{def}}{=} A^{r_0} f_0 \wedge A^{r_1} f_1 \wedge \cdots \wedge A^{r_n} f_n \\ \rho^{\vec{r}}(f) &\overset{\text{def}}{=} \rho(A^{\vec{r}} f) \end{aligned}$$

If some $A^{r_i} f_i$ with $i < n$ is omitted from the expression for $A^{\vec{r}} f$, an equivalent expression containing it can be formed by inserting $A^{r_n + n - i} f_i$ because $A^{n-i} f_i \geq f_n$. If the lattice has finite length and the term $A^{r_n} f_n$ is missing, we may insert $A^{\operatorname{arank} \mathcal{L}} f_n$ because that quantity equals $\hat{1}$.

Definition 8.15. For an arbitrary modular lattice, $\vec{r} \in \mathbb{N}^{n+1}$ is **reduced** when $r_0 \geq r_1 \geq \cdots r_n$. For a modular lattice of finite height, we further impose the conditions $r_n = 0$ and $r_0 \leq \operatorname{arank} \mathcal{L}$.

Suppose $\vec{r}$ is not reduced. If $r_i < r_{i+1}$ then $A^{r_i} f_i \wedge A^{r_{i+1}} f_{i+1} = A^{r_i} f_i = A^{r_i} f_i \wedge A^{r_i} f_{i+1}$, so we can replace r_{i+1} by r_i. If $r_i - r_{i+1} > 1$, then since $A f_i \geq f_{i+1}$, we see that $A^{r_i} f_i \wedge A^{r_{i+1}} f_{i+1} = A^{r_{i+1}} f_{i+1} = A^{r_{i+1}+1} f_i \wedge A^{r_{i+1}} f_{i+1}$. Substitutions of these forms always lower exponents, so repeated substitutions of these forms eventually terminate in a reduced sequence $\vec{r}^*$ with $A^{\vec{r}} f = A^{\vec{r}^*} f$ for all flags f. Explicitly,

$$r_i^* \overset{\text{def}}{=} \min_{k=i}^{n} \left(k - i + \min_{j=0}^{k} r_j \right).$$

If we are only considering maximum length saturated flags with $f_0 = \hat{0}$ and $f_1 = \hat{1}$, then $A^{r_n} \hat{1} = \hat{1}$ for all r_n, so we can take r_n to be 0, while $A^{r_0} \hat{0} = \hat{1}$ iff $r_0 \geq \operatorname{arank} \mathcal{L}$, so we can take $r_0 \leq \operatorname{arank} \mathcal{L}$.

Now we restrict our attention to semi-primary lattices. The reduced sequences for a semi-primary lattice of type $\lambda \vdash n$ have the form $0 = r_n \leq r_{n-1} \leq \cdots \leq r_0 \leq \lambda_1$, and there are $\sum_{k=0}^{\lambda_1} \binom{n}{k}$ of them. The quantities $\rho^{\vec{r}}(f)$ are interesting because they generalize the notions of Littlewood-Richardson sequences and interval type tables, and an extension of it generalizes relative positions. It generalizes the notion of Littlewood-Richardson sequences, since $\operatorname{ALR}_{[f_i, f_k]}(f_j) = ((\operatorname{type}[f_i, A^r_{f_k} f_j])')_{r \geq 0}$ and in turn,

$$\operatorname{type}[f_i, A^r_{f_k} f_j] = (\rho(A^{s-1} f_i \wedge A^r f_j \wedge f_k, A^s f_i \wedge A^r f_j \wedge f_k))_{s \geq 1},$$

which is derivable from such quantities. It also generalizes the notion of interval type tables, since

$$\operatorname{type}[f_i, f_j] = \rho(A^{s-1} f_i \wedge f_j, A^s f_i \wedge f_j)$$

is derivable from such quantities.

Definition 8.16.

$$\begin{aligned} R_i(\vec{r}) &\overset{\text{def}}{=} (r_0, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_n) \\ R(\vec{r}) &\overset{\text{def}}{=} (r_0 + 1, \ldots, r_n + 1) \\ L_i(\vec{r}) &\overset{\text{def}}{=} (r_0, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_n) \end{aligned}$$

Proposition 8.17. *The following are necessary conditions for a specification of $\rho^{\vec{r}}$'s to be realizable in a semi-primary lattice. For all $\vec{r} \in \mathbb{N}^{n+1}$,*

(1) $\rho^{\vec{r}} = \rho^{\vec{r}^*}$.
(2) $\rho^{\vec{r}} \lessapprox \rho^{R_i(\vec{r})}$ *when* $r_{i-1} > r_i = r_{i+1}$, *for* $0 < i < n$;
(3) $\rho^{R(R_i(\vec{r}))} - \rho^{R(\vec{r})} \le \rho^{R_i(\vec{r})} - \rho^{\vec{r}}$, *for* $0 \le i \le n$;
(4) $\rho^{R_i(\vec{r})} - \rho^{\vec{r}} \ge \rho^{R_iR_j(\vec{r})} - \rho^{R_j(\vec{r})}$, *for* $0 \le i, j \le n$ *and* $i \ne j$.

Conditions (2)–(4) only need to be checked for reduced $\vec{r}$.

Proof.

(1) This was shown when $\vec{r}^*$ was constructed.
(2) $A^{\vec{r}}f$ and $A^{R_i(\vec{r})}f$ differ by replacing $A^{r_i}f_i$ in the former with $A^{r_i+1}f_i$ in the latter. Let $r = r_{i-1}$. We have $A^r f_{i-1} \le A^r f_i$ and $A^r f_{i-1} \ge A^{r-1}f_i$. Therefore,

$$\begin{aligned} A^r f_{i-1} \wedge A^{r-1}f_i \wedge A^{r-1}f_{i+1} &= A^{r-1}f_i \;=\; A^r f_{i-1} \wedge A^r f_i \wedge A^{r-1}f_i \\ &\lessapprox A^r f_{i-1} \wedge A^{r-1}f_i \wedge A^{r-1}f_{i+1}. \end{aligned}$$

The initial expression is three terms of $A^{R_i(\vec{r})}f$, and the final expression is the replacement terms in $A^{\vec{r}}f$. Separately meet these with $A^{R_i(\vec{r})}f$ to obtain $A^{\vec{r}}f \lessapprox A^{R_i(\vec{r})}f$.

(3)–(4) By Theorem 4.19, the lattice polynomials At and $t \wedge A^{r_j}f_j$ in t shrink interval lengths. Apply the former to the interval $[A^{\vec{r}}f, A^{R_i(\vec{r})}f]$ to obtain a weakly shorter interval $[A^{R(\vec{r})}f, A^{RR_i(\vec{r})}]$. Apply the latter to the interval $[A^{R_iR_j(\vec{r})}f, A^{R_j(\vec{r})}f]$ to obtain a weakly shorter interval $[A^{R_i(\vec{r})}f, A^{\vec{r}}f]$. □

Definition 8.18. For a flag $f = (f_l, \ldots, f_h)$ and $l \le k \le h$, let

$$A_k f \overset{\text{def}}{=} Ad^{h-k} = (A_{f_k}f_l, \ldots, A_{f_k}f_k).$$

For a tableau $\boldsymbol{P}$ and $\text{low}(\boldsymbol{P}) \le k \le \text{high}(\boldsymbol{P}) = h$, define $\widetilde{\mathcal{L}}_k\boldsymbol{P} = \widetilde{\mathcal{L}}d^{h-k}\boldsymbol{P}$, and similarly define $\mathcal{L}_k$.

Proposition 8.19. *Let* $\vec{r}$ *be a reduced sequence. Let* $s_i = \min\{\, j : r_j = i \,\}$ *for* $i = 0, \ldots, k$ *where* $k = r_0$. *Then* $\rho^{\vec{r}}(f) = \left|\widetilde{\mathcal{L}}_{s_k} \cdots \widetilde{\mathcal{L}}_{s_1}\widetilde{\mathcal{L}}_{s_0}\boldsymbol{P}\right|$ *for some possible outcome of the games* $\widetilde{\mathcal{L}}$.

Proof. Consider

$$\begin{aligned} A_{s_0}f &= (Af_j \wedge f_{s_0})_{0\le j\le s_0} \\ &\quad\ \text{ending in } Af_{s_0} \wedge f_{s_0} = f_{s_0}; \\ A_{s_1}A_{s_0}f &= (A(Af_j \wedge f_{s_0}) \wedge (Af_{s_1} \wedge f_{s_0}))_{0\le j\le s_1} \\ &= (A^2f_j \wedge Af_{s_1} \wedge f_{s_0})_{0\le j\le s_1} \\ &\quad\ \text{ending in } A^2f_{s_1} \wedge Af_{s_1} \wedge f_{s_0} = Af_{s_1} \wedge f_{s_0} \\ &\ \vdots \\ A_{s_m}\cdots A_{s_0}f &= (A^mf_j \wedge A^{m-1}f_{s_{m-1}} \wedge \cdots \wedge f_{s_0})_{0\le j\le f_{s_m}} \\ &\quad\ \text{ending in } A^mf_{s_m} \wedge \cdots \wedge f_{s_0} \\ &\ \vdots \\ A_{s_k}\cdots A_{s_0}f &= (A^kf_0 \wedge A^{k-1}f_{s_{k-1}} \wedge \cdots \wedge f_{s_0}) \\ &\quad\ \text{because } s_k = 0. \end{aligned}$$

The possible types are obtained by replacing each A_{s_i} with $\widetilde{\mathcal{L}}_{s_i}$. Ultimately, a tableau with one partition (μ) is obtained, and and the required rank is $|\mu|$. $\square$

Conjecture 8.20. *Let $\boldsymbol{P}$ be a standard tableau. Let $\rho_g^{\vec{r}}$ be the generic ranks for flags of type $\boldsymbol{P}$, obtained by replacing $\widetilde{\mathcal{L}}$ with $\mathcal{L}$ in Proposition* 8.19. *Then $\rho_g^{\vec{r}} \le \rho^{\vec{r}}(f)$ for all $\vec{r}$ and flags f.*

Applied to interval type tables, the following corollary to the conjecture empirically holds for all the interval type tables in Appendix A.

Definition 8.21. Let λ and μ be partitions. Then λ **(row) dominates** μ, written $\lambda \trianglerighteq \mu$, iff $\lambda_1 + \cdots + \lambda_k \ge \mu_1 + \cdots + \mu_k$ for all $k > 0$. Also, λ column dominates μ, written $\lambda \trianglerighteq' \mu$, when $\lambda' \trianglerighteq \mu'$. These differ from the usual definition of dominance in which $|\lambda| = |\mu|$ is a requirement; in the usual definition, $\lambda \trianglerighteq \mu$ iff $\mu \trianglerighteq' \lambda$, but this does not hold in the extended definition.

Let $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\boldsymbol{Q} = (\mu^{(l)}, \ldots, \mu^{(h)})$ be tableaux. Then $\boldsymbol{P} \trianglerighteq \boldsymbol{Q}$ iff $\lambda^{(k)} \trianglerighteq \mu^{(k)}$ for $k = l, \ldots, h$, and similarly for $\trianglerighteq'$.

Corollary 8.22. *Let $f = (f_0, \ldots, f_n)$ be a flag with* $\operatorname{ftype} f = \boldsymbol{P}$. *Then* $\operatorname{ftype} \partial^i f \trianglerighteq' \Delta^i \boldsymbol{P}$ *for $0 \le i \le n$.*

Proof, given the conjecture. Since $\operatorname{ftype} \partial^i f = (\operatorname{type}[f_i, f_j])_{j=i}^n$, the number of squares in the first k columns of the jth partition is $\rho(f_i, A^k f_i \wedge f_j) = \rho(A^k f_i \wedge f_j) - i$. If the conjecture holds, the generic value of $\rho(A^k f_i \wedge f_j)$ is weakly lower than the one this f realizes. The generic value is the number of squares in the first k columns of the jth partition in $\Delta^i \boldsymbol{P}$, so the realized value dominates the generic value. $\square$

Example 8.23. Note that $\widetilde{\Delta}^i \boldsymbol{P} \trianglerighteq' \Delta^i \boldsymbol{P}$ does not always hold, Consider

$$\boldsymbol{P} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \Delta \boldsymbol{P} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \quad \Delta^2 \boldsymbol{P} = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}$$

$$\widetilde{\Delta} \boldsymbol{P} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \quad \widetilde{\Delta}^2 \boldsymbol{P} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}$$

The shapes of $\widetilde{\Delta}^2 \boldsymbol{P}$ and $\Delta^2 \boldsymbol{P}$ are $\begin{array}{|c|c|}\hline \, & \, \\ \hline \end{array}$ and $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \end{array}$, and $\begin{array}{|c|c|}\hline \, & \, \\ \hline \end{array} \not\trianglerighteq' \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \end{array}$. However, no flag with $\operatorname{ftype} f = \boldsymbol{P}$ has $\operatorname{ftype} \partial^2 f = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}$ because

$$\boldsymbol{P} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} + \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}$$

so by Theorem 4.76, $\operatorname{ftype} \partial^2 f = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}$.

8.5. Counting the number of flags with given A-statistics. The method developed in Section 5.5 for enumerating the number of flags realizing an interval type table generalizes to counting the number of flags realizing some specification of the $\rho^{\vec{r}}$'s. Let $\Lambda = (\rho^{\vec{r}})_{\vec{r}}$ be a collection of numbers satisfying Proposition 8.17, where the indicies run over reduced sequences $\vec{r}$. The **A-statistics** of a flag f are $\Lambda(f) = (\rho^{\vec{r}}(f))_{\vec{r}}$.

Fix k with $0 < k < n$. Consider two k-adjacent flags f, f'. Let $\vec{r}$ be a reduced sequence. If $r_k > r_{k+1}$ then

$$A^{r_k} f_k \wedge A^{r_{k+1}} f_{k+1} = A^{r_{k+1}} f_{k+1} = A^{r_k} f'_k \wedge A^{r_{k+1}} f'_{k+1}$$

so $A^{\vec{r}}f = A^{\vec{r}}f'$ and $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f')$. If $r_{k-1} = r_k$ then

$$A^{r_{k-1}}f_{k-1} \wedge A^{r_k}f_k = A^{r_{k-1}}f_{k-1} = A^{r_{k-1}}f'_{k-1} \wedge A^{r_k}f'_k.$$

so again, $A^{\vec{r}}f = A^{\vec{r}}f'$ and $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f')$. Thus, we say $\vec{r}$ is **k-free** when $r_{k-1} = r_k$ or $r_k \gtrdot r_{k+1}$, and is **k-dependent** when $r_{k-1} \gtrdot r_k = r_{k+1}$. When $\vec{r}$ is k-free, we may substitute for f_k any atom of $[f_{k-1}, f_{k+1}]$ without changing the value of $A^{\vec{r}}f$. In particular, when $\vec{r}$ is reduced and k-dependent, both $R_k(\vec{r})$ and $(L_{k-1}(\vec{r}))^*$ are k-free, so $A^{R_k(\vec{r})}f$ and $A^{L_{k-1}(\vec{r})}f$ do not depend on f_k.

Consider k-dependent $\vec{r}$. We have $A^{L_{k-1}(\vec{r})}f \leq A^{\vec{r}}f \leq A^{R_k(\vec{r})}f$: the first cover is obtained by meeting $A^{r_k}f_{k-1} \leq A^{r_k}f_k$ with $A^{\vec{r}}f$, and the second by Proposition 8.17. The left and right expressions have no dependence on f_k, so they are the same if we substitute f' for f. If $\rho^{L_{k-1}(\vec{r})} = \rho^{R_k(\vec{r})}$ then both weak covers are equality, so $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f') = \rho^{L_{k-1}(\vec{r})}$. If $\rho^{R_k(\vec{r})} - \rho^{L_{k-1}}(\vec{r}) = 2$ then $\rho^{\vec{r}}(f) = \rho^{\vec{r}}(f') \gtrdot \rho^{L_{k-1}(\vec{r})}$. We say that Λ is **$\vec{r}$-specializable** when $\rho^{R_k(\vec{r})} - \rho^{L_{k-1}}(\vec{r}) = 1$, that it is **$\vec{r}$-special** when $\rho^{\vec{r}} = \rho^{R_k(\vec{r})}$, and that it is **$\vec{r}$-generic** when $\rho^{\vec{r}} = \rho^{L_{k-1}(\vec{r})}$. A flag is $\vec{r}$-specializable, $\vec{r}$-special, or $\vec{r}$-generic when $\Lambda(f)$ is.

Let $\rho^{\vec{r}}$ be specializable. If $f \neq f'$ are k-adjacent, they cannot both be $\rho^{\vec{r}}$-special, for suppose they are. Then $A^{\vec{r}}f = A^{\vec{r}}f' = A^{R_k(\vec{r})}f$, but

$$\begin{aligned}
A^{\vec{r}}f \wedge A^{\vec{r}}f' &= A^{R_k(\vec{r})}f \wedge A^{r_k}f_k \wedge A^{r_k}f'_k \\
&= A^{R_k(\vec{r})}f \wedge A^{r_k}(f_k \wedge f'_k) \\
&= A^{R_k(\vec{r})}f \wedge A^{r_k}f_{k-1} \\
&= A^{L_{k-1}(\vec{r})}f \\
&\lessdot A^{R_k(\vec{r})}f,
\end{aligned}$$

a contradiction. So when an adjacency class is $\vec{r}$-specializable for a particular $\vec{r}$, at most one flag in it is $\vec{r}$-special.

In fact, exactly one flag is. Let $\vec{r}$ be reduced and k-dependent, and f be $\vec{r}$-specializable. Let f' be the k-adjacent flag with $f'_k = f_{k-1} \vee C^{r_k}A^{R_k(\vec{r})}f$. We must verify that f'_k is an atom of $[f_{k-1}, f_{k+1}]$ and that $\rho(A^{\vec{r}}f') = \rho(A^{R_k(\vec{r})}f')$. Clearly $f'_k \geq f_{k-1}$. Let $r = r_{k+1} = r_k \lessdot r_{k-1}$. Then $C^{r_k}A^{R_k(\vec{r})}f \leq C^rA^rf_{k+1} \leq f_{k+1}$ so $f'_k \leq f_{k-1} \vee f_{k+1} = f_{k+1}$. Thus $f_{k-1} \leq f'_k \leq f_{k+1}$. Since

$$\begin{aligned}
\rho(f'_k) &= \rho(f_{k-1}) + \rho(A^{R_k(\vec{r})}f) - \rho(A^rf_{k-1} \wedge A^{R_k(\vec{r})}f) \\
&= \rho(f_{k-1}) + \Big(\rho(A^{R_k(\vec{r})}f) - \rho(A^{L_{k-1}(\vec{r})}f)\Big) \\
&= \rho(f_{k-1}) + 1,
\end{aligned}$$

f_{k-1} is an atom of $[f_{k-1}, f_{k+1}]$. Finally, let

$$\begin{aligned}
y &= A^{R_k(\vec{r})}f' \\
z &= A^{\vec{r}}f' = y \wedge A^{r_k}f'_k = y \wedge A^{r_k}(f_{k-1} \vee C^{r_k}y).
\end{aligned}$$

Then $y \leq A^{r_k}C^{r_k}y \leq A^{r_k}(f_{k-1} \vee C^{r_k}y)$ so $z = y$. Thus, f' is $\vec{r}$-special.

We have built up a generalization of the framework designed for interval type tables in Section 5.5. A similar technique could be applied to enumerate the number N_Λ of flags realizing each A-statistic Λ. List all Λ's satisfying Proposition 8.17 and split them into k-adjacency classes over all k. If a class lacks an $\vec{r}$-generic or an $\vec{r}$-special Λ for some specializable $\vec{r}$, delete all members of that class from the list. A class

$\{\Lambda_0, \ldots, \Lambda_m\}$ in which Λ_0 is $\vec{r}$-generic for all specializable $\vec{r}$, and with exactly one Λ_i that's $\vec{r}$-special for each specializable $\vec{r}$, yields the equation $N_{\Lambda_0}/(q+1-m) = N_{\Lambda_1} = \cdots = N_{\Lambda_i}$. It would be interesting to pursue this further. If the equations turn out to be solvable, it would provide a refinement of the polynomials that enumerate how many flags realize specified interval type tables.

The theory of hereditary flags in Section 5.6 computed the A-statistics of hereditary flags, and then only used the portion of those statistics necessary to construct the interval type table. For hereditary flags of composition tableau type $\boldsymbol{P}$, the A-statistics are given by substituting the operator $\mathcal{A}$ of that section for $\widetilde{\mathcal{L}}$ in Proposition 8.19. Let $\Lambda(\boldsymbol{P})$ be those statistics. In Theorem 5.52, it is shown that $f_{\mathrm{itype}\,\boldsymbol{P}}(q) = q^{2n(\lambda)-\mathrm{dom}(\boldsymbol{P})} M_\lambda(q^{-1})$; in fact, this also is the number of flags with A-statistics $\Lambda(\boldsymbol{P})$.

8.6. Further statistics and future problems. Consider the C-statistics of a flag, given as

$$\rho_{\vec{r}}(f) = \rho(C^{r_0} f_0 \vee \cdots \vee C^{r_n} f_n)$$

where $0 = r_0 \leq r_1 \leq \cdots \leq r_n \leq \lambda_1$ are the reduced sequences in semi-primary lattices of type λ. Let $C_m(f) = C\partial^{m-\mathrm{low}(f)}(f) = (C_{f_m} f_m, \ldots, C_{f_m} f_h)$ and $\mathcal{R}_m(\boldsymbol{P}) = \mathcal{R}\partial^{m-\mathrm{low}(\boldsymbol{P})}(\boldsymbol{P})$. Let $s_i = \max\{\, j \,:\, r_j = i \,\}$ for $i = 0, \ldots, k$, where $k = r_n$. Then for flags of type $\boldsymbol{P}$, we have $\rho_{\vec{r}}(f) = |\mathcal{R}_{s_k} \cdots \mathcal{R}_{s_0}(\boldsymbol{P})|$, analogously to Proposition 8.19. The enumeration theory is similar to that of the A-statistics.

We can also consider multiple flags, Combinations of the games $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{D}}_c$ can be used to compute possible values of $\rho(A^{\vec{r}} f \wedge A^{\vec{s}} g \wedge \cdots)$. We illustrate for two and three flags.

First define the operator $A_k^{\#}(f) = (A_{f_k} f_l, A_{f_k} f_{l+1}, \ldots, A_{f_k} f_k, f_{k+1}, \ldots, f_h)$. If f is a multisaturated chain, so is $A_k^{\#}(f)$, because A_{f_k} maintains weak covers in the l through kth terms; $A_{f_k} f_k = f_k \leq f_{k+1}$; and the $k+1$ through hth terms already form a multisaturated chain. Also, for $\boldsymbol{P} = (\lambda^{(l)}, \ldots, \lambda^{(h)})$ and $\widetilde{\mathcal{L}}_k \boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(k)})$, define $\widetilde{\mathcal{L}}_k^{\#} \boldsymbol{P} = (\mu^{(l)}, \ldots, \mu^{(k)}, \lambda^{(k+1)}, \ldots, \lambda^{(h)})$. In terms of tableau, we apply $\widetilde{\mathcal{L}}$ to the subtableau of $\boldsymbol{P}$ whose entries are at most k, and leave all the entries larger than k in place. Similarly define $\mathcal{L}_k^{\#}$. Uniformly we have

$$\mathcal{L}_k^{\#}, \widetilde{\mathcal{L}}_k^{\#} : \mathrm{ftype}_{\hat{0}}\, f \longmapsto \mathrm{ftype}_{\hat{0}}\, A_k^{\#}(f).$$

Let $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_{\hat{0}}\, g = \boldsymbol{Q}$. Let $\vec{r}$ and $\vec{s}$ be reduced sequences and set $r_i' = \min\{\, j \,:\, r_j = i \,\}$ and $s_i' = \min\{\, j \,:\, s_j = i \,\}$. Let

$$\begin{aligned} f^{\#} &= (A^{r_i} f_i \wedge A^{r_{i+1}} f_{i+1} \wedge \cdots \wedge A^{r_n} f_n) \\ &= \cdots A_{r_1'}^{\#} A_{r_0'}^{\#}(f), \\ \boldsymbol{P}^{\#} &= \cdots \widetilde{\mathcal{L}}_{r_1'}^{\#} \widetilde{\mathcal{L}}_{r_0'}^{\#}(\boldsymbol{P}). \end{aligned}$$

Then $\mathrm{ftype}_{\hat{0}}\, f^{\#} = \boldsymbol{P}^{\#}$, with the uniform type found by replacing $\widetilde{\mathcal{L}}_{r_i'}^{\#}$ by $\mathcal{L}_{r_i'}^{\#}$. Similarly define $g^{\#}$ and $\boldsymbol{Q}^{\#}$.

Next, perform a skew Robinson-Schensted algorithm on $(\boldsymbol{P}^{\#}, \boldsymbol{Q}^{\#})$. Let $k_1 < \cdots < k_m$ be the entries of $\boldsymbol{Q}^{\#}$ and $c_1, \ldots, c_m$ be the cells of $\boldsymbol{Q}^{\#}$ with $k_1, \ldots, k_m$. Then

$\mathrm{ftype}_{\hat{0}}(f^{\#} \wedge A^{\vec{s}}g) = \widetilde{\mathcal{D}}_{c_1} \cdots \widetilde{\mathcal{D}}_{c_m}(\boldsymbol{P}^{\#})$, so the inner shape of this skew tableau is the value of $\mathrm{type}(A^{\vec{r}}f \wedge A^{\vec{s}}g)$.

For three flags f, g, h of types $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$, and reduced sequences $\vec{r}, \vec{s}, \vec{t}$, define $f^{\#}$, $g^{\#}$, $h^{\#}$ and $\boldsymbol{P}^{\#}, \boldsymbol{Q}^{\#}, \boldsymbol{R}^{\#}$ analogously to the two flag case. Let $k_1 < \cdots < k_m$ be the entries of $\boldsymbol{R}^{\#}$ and $c_1, \ldots, c_m$ their cells. The games $\boldsymbol{P}^{\#\#} = \mathrm{ftype}_{\hat{0}}(f^{\#} \wedge A^{\vec{t}}h) = \widetilde{\mathcal{D}}_{c_1} \cdots \widetilde{\mathcal{D}}_{c_m}(\boldsymbol{P})$ and $\boldsymbol{Q}^{\#\#} = \mathrm{ftype}_{\hat{0}}(g^{\#} \wedge A^{\vec{t}}h) = \widetilde{\mathcal{D}}_{c_1} \cdots \widetilde{\mathcal{D}}_{c_m}(\boldsymbol{Q})$ each result in tableaux of the same shape (the inner shape of $\boldsymbol{R}^{\#}$). Perform the skew Robinson-Schensted deletion algorithm on $(\boldsymbol{P}^{\#\#}, \boldsymbol{Q}^{\#\#})$, and the final P-tableau obtained is $\mathrm{ftype}_{\hat{0}}(f^{\#} \wedge A^{\vec{s}}g \wedge A^{\vec{t}}h)$, so its inner shape is $\mathrm{type}(A^{\vec{r}}f \wedge A^{\vec{s}}g \wedge A^{\vec{t}}h)$.

For two flags f and g, the A-statistics simultaneously encode the relative positions of the flags and their interval type tables. In Section 7, we were not able to enumerate the number of pairs of flags in given relative positions by the techniques used in Section 5.5 for interval type tables, because our statistics were insufficient to determine the types of length 2 intervals. Since A-statistics resolve that difficulty, they should be explored further to see if the number of pairs of flags realizing particular A-statistics is in fact enumerable, exactly or asymptotically in the generic case.

The following is a generalization of Conjecture 8.20 to multiple flags.

Conjecture 8.24. *Let $\boldsymbol{P}, \boldsymbol{Q}, \ldots$ be standard tableaux, and $\vec{r}, \vec{s}, \ldots$ be reduced sequences. Compute the generic values $\rho_{\mathrm{gen}}^{\vec{r},\vec{s},\cdots}$ of $A^{\vec{r}}f \wedge A^{\vec{r}}g \wedge \cdots$ by using $\mathcal{L}$ and $\mathcal{D}_c$ in place of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{D}}_c$ in the procedure described above. Then for all flags $f, g, \ldots$, we have*

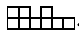
$$\rho_{\mathrm{gen}}^{\vec{r},\vec{s},\cdots} \leq \rho(A^{\vec{r}}f \wedge A^{\vec{s}}g \wedge \cdots).$$

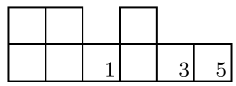
Corollary 8.25. *Let $(\boldsymbol{P}, \boldsymbol{Q})$ be a pair of standard tableau of shape λ. Let $\boldsymbol{P} = \boldsymbol{P}_n, \ldots, \boldsymbol{P}_0 = \emptyset$ be the sequence of tableau obtained in the inverse Robinson-Schensted algorithm. Let f and g be flags with $\mathrm{ftype}_{\hat{0}}\, f = \boldsymbol{P}$ and $\mathrm{ftype}_{\hat{0}}\, g = \boldsymbol{Q}$ in a semi-primary lattice of type λ. Then $\mathrm{ftype}_{\hat{0}}(f \wedge g_k) \trianglerighteq' \boldsymbol{P}_k$.*

Conjectural Proof. We assume the previous conjecture, and assume that compositions of games $\mathcal{D}_c$ give generic values. The jth partition in $\mathrm{ftype}(f \wedge g_k)$ is $\mathrm{type}(f_j \wedge g_k)$. The first i columns of it have a total of $\rho(A^i f_0 \wedge f_j \wedge g_k)$ squares. By the conjecture, this is minimized by the generic value, which is the first i columns of the jth partition of $\boldsymbol{P}_k$. $\square$

Part IV. Strongly modular lattices

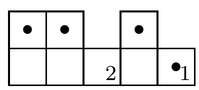
9. Strongly modular lattices

9.1. The Fibonacci lattices $Fib(r)$ and $Z(r)$. Let r be a positive integer, and $W(r)$ be the set of all finite length words formed from the alphabet $\{1_1, 1_2, \ldots, 1_r, 2\}$, including the empty word $\emptyset$. A digit is "a 1" if it is any 1_i. When $r = 1$, the alphabet may be abbreviated $\{1, 2\}$. As with partitions, an arbitrary number of 0's may be appended to the end of a word when convenient. Fomin [**5**] introduced a pictorial diagram for words of $W(1)$ that is a rotation of the usual composition diagram; the word 221211 has **snake diagram** . We may expand this to words of $W(r)$ by placing an i in the lower right corner of the cell representing 1_i, so $221_1 21_3 1_5$ has snake diagram

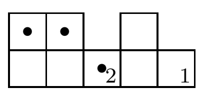


In [**28**], two different partial orders on $W(r)$ are defined by defining the cover relations and then extending to a partial order by transitivity.

In $Fib(r)$, the cover relation $v \gtrdot_F u$ holds when u is obtained from v either by changing a single 2 to a 1 anywhere in v, or by deleting the last digit of v if it is a 1. When extended transitively to a full partial order, $v \geq_F u$ if u may be obtained from v by changing some 2's to 1's, and deleting any number of digits from the right end of the word. Thus in $Fib(2)$, the word $221_2 21_1$ covers $1_1 21_2 21_1$, $1_2 21_2 21_1$, $21_1 1_2 21_1$, $21_2 1_2 21_1$, $221_2 1_1 1_1$, $221_2 1_2 1_1$, and $221_2 2$. Pictorially, we may delete any of the marked cells, and if they in the upper row, we may introduce any subscript in the lower cell.



The cover relation $v \gtrdot_Z u$ holds in $Z(r)$ when either of two conditions hold. Either the leftmost 1 of v is deleted to obtain u, denoted $v \gtrdot_1 u$, or a 2 in v preceded only by 2's is changed to a 1, denoted $v \gtrdot_2 u$. So in $Z(2)$, we have $221_2 21_1 \gtrdot_1 2221_1$ and $221_2 21_1 \gtrdot_2 1_1 21_2 21_1$, $1_2 21_2 21_1$, $21_1 1_2 21_1$, $21_2 1_2 21_1$. Pictorially, we may delete any marked cell, and when it is a 2, add any subscript to the lower cell.



The total number of upper covers of u is $\kappa^+(u) = (k+1)r$ or $(k+1)r + 1$, where $u = 2^k$ or $2^k 1_i x$, respectively, for arbitrary $k \geq 0$ and $x \in Z(r)$. This is because any of r 1's may be inserted before the first 2 or following any of the first k 2's, or if there is a 1, the first 1 may be changed to a 2. Similarly, the total number of lower covers of v is $\kappa^-(v) = kr$ or $kr + 1$, where $v = 2^k$ or $2^k 1x$, respectively; any one of the first k 2's can replaced by any of r 1's, or the first 1 (if present) can be deleted.

Both $Fib(r)$ and $Z(r)$ are graded lattices with rank function $\rho(a_1 a_2 \cdots a_k) = a_1 + a_2 + \cdots + a_k$, where we add the a_i's as integers by ignoring the subscripts on the 1's. These lattices are called **Fibonacci lattices** because the number of elements of rank n when $r = 1$ is the Fibonacci number F_{n+1} (where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$).

We will evaluate the operators A and C in $Z(r)$ in Theorems 9.4 and 9.7, and establish in Theorem 9.2 a new criterion for determining whether two elements of $Z(r)$

are comparable under $\geq_Z$ without having to exhibit a chain of elements satisfying the cover relation $\gtrdot_Z$.

Definition 9.1. Let u and v be two elements of $Z(r)$ and factor them $u = \mu\tau$ and $v = \nu\tau$, where τ is as large as possible; that is, one or both of μ and ν is empty, or they are both nonempty but end in different digits. Let μ_1, μ_2, and μ_{12} respectively denote the number of 1's of all kinds in μ; the number of 2's in μ; and the total number of digits in μ, and define similar subscripted quantities for u, v, τ, ν, and other variables in $Z(r)$ that are introduced later. When dealing with multiple values of u and v, all of μ, ν, τ and their subscripted forms may be expressed as functions of u and v, so that $u = \mu(u, v)\tau(u, v)$ and $v = \nu(u, v)\tau(u, v)$.

Theorem 9.2. *Let $v, u \in Z(r)$. Then $v \geq_Z u$ iff $\nu_2 \geq \mu_{12}$.*

Proof. Suppose $\nu_2 \geq \mu_{12}$. First, prepend to u a total of $\nu_2 - \mu_{12}$ 2's by alternately prepending 1_1 and then changing the 1_1 to 2. The successive elements obtained increase by alternate cover relations $\gtrdot_1$ and $\gtrdot_2$. So now we have the word $2^{\nu_2-\mu_{12}}\mu\tau$. Next, change the 1's in the μ part to 2's, sequentially from left to right, to obtain $2^{\nu_2}\tau$; each successive change is a $\gtrdot_2$ relation in $Z(r)$. Finally, by a sequence of $\gtrdot_1$ relations, insert into this 1_i's from right to left with the same positions and subscripts as in ν, to obtain $v = \nu\tau$. Thus $v \geq_Z u$.

Conversely, suppose $v \geq_Z u$. Any saturated chain from u to v must pass through an element of the form $2^m\tau$ because the digit preceding τ is eventually changed, and to change a digit requires all preceding digits to be 2's. We must have $m \geq \mu_{12}$ because the number of digits in a word weakly increases as we go up in $Z(r)$ (so comparing u and $2^m\tau$, this number increases from $\mu_{12} + \tau_{12}$ to $m + \tau_{12}$), and also we must have $\nu_2 \geq m$ because the number of 2's in a word weakly increases as we go up in $Z(r)$ (so comparing $2^m\tau$ and v, this increases from $m + \tau_2$ to $\nu_2 + \tau_2$). So $\nu_2 \geq \mu_{12}$. $\square$

Example 9.3. Let

$$\begin{aligned} u &= && 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\ v &= 2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \end{aligned}$$

Factor these as $u = \mu\tau$ and $v = \nu\tau$ with τ as large as possible.

$$\begin{aligned} \mu &= && 2\ 1_2 1_1 2\ 2\ 1_3 1_2 \\ \nu &= 2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2 \\ \tau &= && 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \end{aligned}$$

Since $\nu_2 = 10 \geq 7 = \mu_{12}$, we should have $v \geq_Z u$. Starting with u, first prepend three 2's (since $\nu_2 - \mu_{12} = 3$) by prepending 1_1 and changing it to 2, yielding alternating

$\gtrdot_1$ and $\gtrdot_2$ cover relations.

$$\begin{array}{r}
2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
1_1 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
1_1 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
1_1 2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2
\end{array}$$

Then change the 1_i's of μ into 2's, yielding a succession of $\gtrdot_2$ cover relations.

$$\begin{array}{r}
2\ 2\ 2\ 2\ 1_2 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 1_1 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_3 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_2 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2
\end{array}$$

Finally, insert the 1_i's of ν into this from right to left, yielding a succession of $\gtrdot_1$ cover relations.

$$\begin{array}{r}
2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2 \\
2\ 2\ 1_1 2\ 2\ 1_4 2\ 1_1 2\ 2\ 1_2 2\ 2\ 2\ 2\ 1_1 1_2 2\ 2\ 1_3 1_4 2
\end{array}$$

We will see below that $\mathrm{Atype}[u,v] = 2^3 1^8$ and $\mathrm{Ctype}[u,v] = 22121^7$.

Theorem 9.4. *If $v \geq_Z u$ then $A_v u$ is determined as follows:*

$2u$	*if* $\mu_{12} < \nu_2$;
Change the leftmost 1 *of* u *to* 2	*if* $\mu_2 < \mu_{12} = \nu_2$;
$\mu 1_i \tau$ *where* ν *ends in* 1_i	*if* $\mu_2 = \mu_{12} = \nu_2 < \nu_{12}$;
u	*if* $\mu_2 = \mu_{12} = \nu_2 = \nu_{12} = 0$.

The only exception occurs when $r = 1$, $u = \emptyset$, and $\nu_2 > 0$; in this case, $A_v u = 1$.

Proof. If $\mu_{12} < \nu_2$ then $u <_Z 2u \leq_Z v$ because $\nu_2(2u,v) - \mu_{12}(2u,v) \lessdot \nu_2(u,v) - \mu_{12}(u,v)$; either τ and ν stay the same and μ_{12} goes up by 1, or $\tau(u,v) = u$, $\tau(2u,v) = 2u$, $\mu(u,v) = \mu(2u,v) = \emptyset$, $\nu(u,v) = 2\nu(2u,v)$ so that ν_2 decreases by 1 and μ_{12} stays 0. Further, all $\kappa^+(u)$ upper covers of u are lower covers of $2u$, and unless $r = 1$ and $u = \emptyset$, there is more than one upper cover, so their join Au is $2u$. Since $2u \leq v$, also $A_v u = Au \wedge v = Au = 2u$. When $r = 1$ and $u = \emptyset$, there is only one upper cover and $A_v u = Au = 1$.

So assume $\mu_{12} = \nu_2$. Suppose $\mu_2 < \mu_{12}$. What are the atoms u' of $[u,v]$? There is at least one 1 in μ because $\mu_2 < \mu_{12}$, so any $u' \gtrdot u$ has $\tau(u',v) = \tau(u,v)$, and the altered digit is in the μ portion. We cannot insert a 1 because then $\mu_{12}(u',v) > \nu_2(u',v)$ so

$u' \not\leq_Z v$. So $u' >_2 u$, and the only such u' is obtained by changing the first 1 in u to a 2.

Now suppose $\mu_2 = \mu_{12} = \nu_2$. Then $\mu = 2^{\mu_2}$. If $\nu_2 = \nu_{12}$ then $\mu = \nu$, but since μ and ν by definition do not end in the same digit, this requires $\mu = \nu = \emptyset$, so $u = v$ and $A_v u = u$. Only the case $\mu_2 = \mu_{12} = \nu_2 < \nu_{12}$ remains, and ν must end in some 1; say it ends in 1_i. Then $v \geq_Z \mu 1_i \tau >_Z u$. Are there any other atoms u' of $[u, v]$? Consider another upper cover u' of u in $Z(r)$. What are $\tau(u', v)$, $\mu(u', v)$, and $\nu(u', v)$ as compared with $\tau(u, v)$, $\mu(u, v)$, and $\nu(u, v)$? All digits of τ left of the altered one in $u <_Z u'$ are moved to μ and ν (if the alteration was in τ), thereby adding at least as many digits to μ as 2's are added to ν; next, μ gains a 1 if $u' >_1 u$, or μ gains a 2 and ν a 1 if $u' >_2 u$, thereby adding a digit to μ but not adding any 2's to ν. So at least one more digit is added to μ than 2's are added to ν, yielding $\mu_{12}(u', v) \geq \nu_2(u', v) + 1$. Thus, $u' \not\leq_Z v$. So $\mu 1_i \tau$ is the unique atom of $[u, v]$. $\square$

Corollary 9.5. $\mathrm{Atype}[u, v] = 2^{\nu_2 - \mu_{12}} 1^{\mu_1 + \nu_1}$, *unless* $r = 1$ *and* $u = \emptyset$ *and* $\nu_2 > 0$, *in which case* $\mathrm{Atype}[u, v] = 12^{\nu_2 - 1} 1^{1+\nu_1}$ *if* v *ends in* 2, *and* $\mathrm{Atype}[u, v] = 12^{\nu_2} 1^{\nu_1 - 1}$ *if* v *ends in* 1.

Proof. Except in the exceptional case, the chain of elements from u to v built in the proof of Theorem 9.2 is precisely $u, A_v u, A_v^2 u, \ldots$ by the preceding theorem; first we prepend $\nu_2 - \mu_{12}$ 2's to u, and then change μ_1 1's to 2's, and then insert ν_1 1's, and sequentially listing the changes in rank yields $\mathrm{Atype}[u, v] = 2^{\nu_2 - \mu_{12}} 1^{\mu_1 + \nu_1}$. In the exceptional case $r = 1$, $u = \emptyset$, and $\nu_2 > 0$, we have $A_v u = 1$ instead of 2, but the next $\nu_2 - 1$ or ν_2 (depending on whether v ends in 2 or 1) applications of A_v to this do prepend 2's. If v ends in 2, the next application of A_v changes the terminal 1 to a 2. Finally, we insert the remaining 1's into ν. $\square$

Example 9.6. In Example 9.3 above, $\mathrm{Atype}[u, v] = 2^3 1^8$. For exceptional cases, $\mathrm{Atype}[\emptyset, 221] = 122$ if $r = 1$ but $\mathrm{Atype}[\emptyset, 221_1] = 221$ if $r > 1$, and $\mathrm{Atype}[\emptyset, 22] = 121$ if $r = 1$ but $\mathrm{Atype}[\emptyset, 22] = 22$ if $r > 1$.

Theorem 9.7. *If* $v \geq_Z u$ *then* $C_u v$ *is determined as follows:*

Delete the first digit of v	*if* $\nu_2 > \mu_{12}$, *or if* $\nu_2 = \mu_{12}$ *and* v *starts with a* 1;
Delete the leftmost 1 *of* v	*if* $\nu_{12} > \nu_2 = \mu_{12}$;
$\nu' 1_i \tau$, *where* $\nu = \nu' 2$ *and* $\mu = \mu' 1_i$	*if* $\nu_{12} = \nu_2 = \mu_{12} > \mu_2$;
u	*if* $\nu_{12} = \nu_2 = \mu_{12} = \mu_2$.

The only exception occurs when $r = 1$; *in this case,* $C_\emptyset 2 = 1$.

Proof. If $\nu_2 > \mu_{12}$ and we delete the first digit of v to obtain v', we will decrease ν_1 or ν_2 by 1 but keep μ_{12} the same, so $v >_Z v' \geq_Z u$. If this first digit is a 1, then v' is the unique lower cover of v in $Z(r)$. If $v = 2v'$, then all $\kappa^-(v)$ lower covers of v are upper covers of v', and there is more than one lower cover unless $r = 1$ and $v = 2$, so the meet of all lower covers of v is $Cv = v' \geq u$. Further, $C_u v = Cv \vee u = v'$. If $v = 2$ and $r = 1$ then $u = \emptyset$ (as $1 = \nu_2 > \mu_{12} \geq 0$) and $C_u v = 1$.

Now suppose $\nu_2 = \mu_{12}$. If $\nu_{12} > \nu_2$ then ν has 1's in it. Consider lower covers v' of v. If $v >_2 v'$ then some 2 of $\nu(u, v)$ is changed to a 1 to obtain $\nu(u, v')$, but $\mu(u, v') = \mu(u, v)$, so μ_{12} stays the same and ν_2 goes down by 1, yielding $\nu_2(u, v') < \mu_{12}(u, v')$, so $u \not\leq_Z v'$. So $v \gtrdot_1 v'$, which uniquely determines v' by deleting the first 1 in v. So $C_u v = v'$.

Now suppose $\nu_{12} = \nu_2 = \mu_{12}$. If $\mu_{12} = \mu_2$ then μ and ν have the same length and are all 2's, and since they must not end in the same digit, $\mu = \nu = \emptyset$ and $u = v$, so $C_u v = u$. So we are left with the case $\mu_{12} > \mu_2$. All digits of ν are 2, so μ has the form $\mu = \mu' 1_i$. Write $\nu = \nu' 2$. Then $\nu' 1_i \tau$ satisfies $u \leq_Z \nu' 1_i \tau \lessdot_2 v$. Do any other lower covers v' of v satisfy $v' \geq_Z u$? To compute $\mu(u, v')$ and $\nu(u, v')$ from μ and ν, we add to both μ and ν all digits of τ left of the one that changes in $v \gtrdot_Z v'$ (if the change is in τ), and then we add a 1 to μ if $v \gtrdot_1 v'$ or a 2 to μ and 1 to ν if $v \gtrdot_2 v'$. Either way, the number of digits added to μ goes up by at least one more than the number of 2's added to ν, so $\mu_{12}(u, v') > \nu_2(u, v')$, so $v' \not\geq_Z u$. So the only lower cover of v that exceeds u is $\nu' 1_i \tau$, whence $C_u v = \nu' 1_i \tau$. □

Corollary 9.8. Ctype$[u, v]$ *is found as follows. Factor* $\nu = st$ *where* $t_2 = \mu_{12}$*; the factorization isn't unique, as there may be some* 1*'s that can be placed either at the right of* s *or the left of* t*. Form* s' *from* s *by dropping the subscripts on all the* 1*'s. Then* Ctype$[u, v] = s' 1^{t_1 + \mu_1}$*, except when* $r = 1$*,* $u = \emptyset$*, and* $\nu_2 > 0$*. When* $r = 1$*,* Ctype$[\emptyset, v] = v$ *unless it ends in* 2*, in which case the final* 2 *is changed to* 11*.*

Proof. We will consider $\mu(u, v')$ and $\nu(u, v')$ as v' takes on successive values v, $C_u v$, $C_u^2 v, \ldots$. Initially set v' to v. The first s_{12} applications of C_u simply remove initial digits s of v; remove initial digits one by one as long as doing so maintains $\nu_2(u, v') \geq \mu_{12}(u, v')$. Then remove 1's from left to right as long as $\nu_{12} > \mu_{12}$ (a total of t_1 1's). Now u and v' have the same number of digits, and where they disagree, u has some 1_i and v' has a 2, so from right to left, lower the 2's of v' to the 1_i's found in u (a total of μ_1 times). Finally we obtain u.

Each digit removed or lowered is precisely an application of C_u by the preceding theorem, except for the exceptional case. List the successive drops in rank to obtain Ctype$[u, v]$ as stated above: removing a 1 or 2 decreases the rank by 1 or 2, respectively, and lowering a 2 to a 1 decreases the rank by 1.

The only exceptional case is when $r = 1$, $u = \emptyset$, and v ends in a 2. When $r = 1$ and $u = \emptyset$, successive applications of $C_u = C$ remove leading digits of v one at a time, until all that is left is the terminal digit, and then $C2 = 1$ and $C1 = \emptyset$, instead of $C2 = \emptyset$ as happens with $r > 1$. □

Theorem 9.9. *Let* $u, v \in Z(r)$*. Without loss of generality,* $\mu_2 \leq \nu_2$*. Let* $m = \max\{0, \mu_{12} - \nu_2\}$*. Then* $u \wedge v$ *is obtained from* u *by deleting the first* m 1*'s.*

Without loss of generality, $\mu_{12} \leq \nu_{12}$*. Let* $m = \max\{0, \mu_{12} - \nu_2\}$*. Then* $u \vee v$ *is obtained from* v *by changing the first* m 1*'s to* 2*'s.*

Finally,

$$\begin{aligned} \rho(u \wedge v) &= \min\{\rho(u), \rho(v), \mu_2 + \nu_2 + \rho(\tau)\}, \\ \rho(u \vee v) &= \max\{\rho(v), \rho(v), \mu_{12} + \nu_{12} + \rho(\tau)\}. \end{aligned}$$

Proof. We compute the meet; the join is computed similarly. Let α be the result of deleting the first m 1's from u. Then $\alpha \leq_Z u$. Either $m = 0$, or $m = \mu_{12} - \nu_2 \leq \mu_{12} - \mu_2 = \mu_1$, so these 1's are all contained in μ. Let β be the result of deleting the first m 1's from μ, so $\alpha = \beta\tau$. To prove $\alpha \leq_Z v$, we show $\beta \leq_Z \nu$, for if this holds, append τ to both sides to obtain $\alpha \leq_Z v$. If β and ν do not end in the same digit then $\beta_{12} = \mu_{12} - m = \min\{\mu_{12}, \nu_2\} \le \nu_2$, so $\beta \le \nu$. If they do end in the same digit then it is a 2 and we deleted all the 1's from μ, because μ and ν do not end in the same digit. So $\beta = 2^{\mu_2}$, and ν has at least μ_2 2's (since $\mu_2 \le \nu_2$), so $\beta \le_Z \mu$. Thus, $\alpha \le_Z u \wedge v$.

If $\alpha = u$ or v then α is certainly the meet, so assume $\alpha <_Z u, v$. Since α is obtained from u by deleting 1's, the only upper cover α^+ of α with $\alpha^+ \le_Z u$ is $A_u\alpha$, obtained by reinserting the rightmost deleted 1 in the β portion. Then $\nu(\alpha^+, v) = \nu$, $\tau(\alpha^+, v) = \tau$, and $\mu(\alpha^+, v) = \beta^+$ is β with a 1 suitably inserted. We have $\beta_{12} = \min\{\mu_{12}, \nu_2\} < \mu_{12}$ because if $\beta_{12} = \mu_{12}$, inserting a 1 would yield more digits than u has. Thus, $\beta_{12} = \nu_2$ and $\beta^+_{12} = \nu_2 + 1 \not\le \nu_2$, so $\alpha^+ \not\le_Z v$, and we conclude $\alpha = u \wedge v$.

The rank of the meet is

$$\begin{aligned}
\rho(\alpha) &= \rho(u) - m \\
&= \min\{\rho(u), \rho(u) - \mu_{12} + \nu_2\} \\
&= \min\{\rho(u), \mu_2 + \mu_{12} - \mu_{12} + \nu_2\} \\
&= \min\{\rho(u), \mu_2 + \nu_2\}\,.
\end{aligned}$$

Combining this with the like expression for the case $\mu_2 \ge \nu_2$ gives the value stated in the theorem. □

9.2. Strongly modular lattices. We now introduce a new class of lattices.

Theorem 9.10. *The following conditions on a lattice are equivalent.*

(1) *The lattice is modular, and all complemented intervals have length at most 2.*
(2) *For any set S of at least two elements of the lattice, the meet of all the elements of S is covered by each element in S iff each element in S is covered by the join of all elements of S.*

Proof.

(1) ⇒ (2): The function Au is defined for all u even when the lattice has infinite length: if there are an infinite number of elements covering some u and their join is unbounded, a finite subcollection of them yields joins of any finite rank, and the interval from u to these joins is atomic and hence complemented. Thus, complemented intervals of all finite ranks exist, violating (1).

Let S be a set of at least two elements of the lattice such that the meet u of all elements of S is covered by each element of S. Then Au is the join of all the elements of S and possibly other elements of the lattice. Since there are at least two elements in the join, Au is strictly larger than either, and $[u, Au]$ is an atomic (hence complemented) interval of length at least 2, so by (1), its length is exactly 2. Since Au is larger than all elements of S and has rank 1 larger, it covers them all.

The converse statement in (2) is proven dually, using C instead of A.

(2) ⇒ (1): If a lattice satisfies (2), it is modular, for given any two elements, they cover their meet iff they are covered by their join. Further, suppose $[x, y]$ is a complemented interval of length greater than 1. There must be at least two atoms, for if there is only one atom, v, it has no complement: the only element disjoint with it is x, but $x \vee v = v < \hat{1}$. The meet of all the atoms of $[x, y]$ is x, which is a lower cover of every atom; thus, by (2), the join of all atoms must cover each atom, so $[x, A_y x]$ has length 2. If $A_y x < y$ then $A_y x$ has no complement, because the only element disjoint from it is x, and $x \vee A_y x = A_y x < \hat{1}$. So $A_y x = y$, and $[x, y]$ has length 2. $\square$

Definition 9.11. A **strongly modular lattice** is a lattice satisfying either of the above equivalent conditions.

A semi-primary lattice of type λ with $\lambda'_1 \leq 2$ is a strongly modular lattice.

Theorem 9.12 (see [28, Prop. 5.4]**).** *$Z(r)$ is a strongly modular lattice.*

Proof. If S is a set of at least two elements whose meet x is a lower cover of each element of S, then $Ax = 2x$ is an upper cover of each element of S. If S is a set of at least two elements whose join y is an upper cover of each element of S, then y has the form $2x$ (since words not beginning with 2 have at most one lower cover), and x is a lower cover of each element of S. $\square$

Theorem 9.13. *Let $\mathcal{L}$ be a strongly modular lattice. For any closed interval $[x, y]$, both* $\mathrm{Atype}[x, y]$ *and* $\mathrm{Ctype}[x, y]$ *are in $W(1)$.*

Proof. The parts of $\mathrm{Atype}[x, y]$ are the successive lengths of the intervals $[x, A_y x]$, $[A_y x, A_y^2 x]$, $[A_y^2 x, A_y^3 x], \ldots$. Each such interval is atomic and hence complemented, so for i with $A_y^i x < y$, the interval $[A_y^i x, A_y^{i+1} x]$ has length 1 or 2, while all further intervals have length 0. Dualize this argument for Ctype. $\square$

Theorem 9.14. *If $x \lessdot v \leq y$ in $\mathcal{L}$ then* $\mathrm{Atype}[v, y] \lessdot_Z \mathrm{Atype}[x, y]$ *in $Z(1)$. Dually, if $x \leq v \lessdot y$, then* $\mathrm{Ctype}[x, v] \lessdot_Z \mathrm{Ctype}[x, y]$ *in $Z(1)$.*

Proof. We show the first cover relation by inducting on the length of the interval $[x, y]$.

Suppose $\mathrm{Atype}[x, y]$ begins with a 1. Then $A_y x$ is the unique atom of $[x, y]$, which must be v, so $\mathrm{Atype}[x, y]$ is $\mathrm{Atype}[v, y]$ with a 1 prepended.

Suppose $\mathrm{Atype}[x, y]$ and $\mathrm{Atype}[v, y]$ both begin with a 2. Since $v \geq x \Rightarrow A_y v \geq A_y x$ and v has rank one greater than x and A_y increases both ranks by 2, we must have $A_y v \gtrdot A_y x$. By induction, $\mathrm{Atype}[A_y v, y] \lessdot_Z \mathrm{Atype}[A_y x, y]$, and we prepend 2 to both sides to obtain $\mathrm{Atype}[v, y] \lessdot_Z \mathrm{Atype}[x, y]$.

Finally, if $\mathrm{Atype}[x, y]$ begins with 2 and $\mathrm{Atype}[v, y]$ begins with 1, then $A_y v \geq A_y x$, and both sides have the same rank, so they are equal. So all other entries in the types are the same. $\square$

Theorem 9.15. *Let $x \lessdot v \leq y$. If* $\mathrm{Atype}[v, y] \lessdot_1 \mathrm{Atype}[x, y]$ *with a* 1 *inserted after the first k 2's, then $v = C^k A_y^{k+1} x$. If* $\mathrm{Atype}[v, y] \lessdot_2 \mathrm{Atype}[x, y]$ *with the* 1 *changed to a* 2 *in the kth position, then $x = C^k A_y^k v$.*

Dually, let $x \leq v \lessdot y$. If $\mathrm{Ctype}[x, v] \lessdot_1 \mathrm{Ctype}[x, y]$ *with a* 1 *inserted after the first k 2's, then $v = A^k C_x^{k+1} y$. If* $\mathrm{Ctype}[x, v] \lessdot_2 \mathrm{Ctype}[x, y]$ *with the* 1 *changed to a* 2 *in the kth position, then $y = A^k C_x^k v$.*

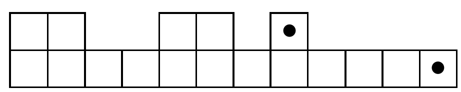
Only uniqueness of v, x, or y in each case is asserted, not existence.

Proof. Let $x \lessdot v \leq y$. Suppose $\mathrm{Atype}[v,y] \lessdot_1 \mathrm{Atype}[x,y]$ with a 1 inserted after the first k 2's. Then $A_y^k x \lessdot A_y^k v$ since each side has rank $2k$ more than in $x \lessdot v$. Now $A_y^k v \leq A_y(A_y^k x) = A_y^{k+1} x$ because the middle term is a join of elements which includes the left term. The left and right terms both have the same rank by examining $\mathrm{Atype}[v,y]$ and $\mathrm{Atype}[x,y]$, so in fact, $A_y^k v = A_y^{k+1} x$. Next, $C_v^k A_y^k v \leq v$ by Theorem 3.3(2), and also the left side is less than v because C_v is bounded below by v. So $v = C_v^k A_y^k v = C_v^k A_y^{k+1} x$. This is bounded below by $C^k A_y^{k+1} x$. Each of the k applications of C_v must have decreased the rank by 2, and so the applications of C can decrease the rank no further. Thus, $v = C_x^k A_y^{k+1} x$.

Now suppose $\mathrm{Atype}[v,y] \lessdot_2 \mathrm{Atype}[x,y]$ with the 1 changed to a 2 in the kth position. Then $A_y^k x = A_y^k v$ since the right side is a weak upper bound for the left, and both have the same rank (by adding the first k digits of the types to the ranks of $x \lessdot v$). Then $x = C_x^k A_y^k x = C_x^k A_y^k v$, and since each application of C_x reduces the rank by 2, it could be replaced by C, to obtain $x = C^k A_y^k v$.

The proofs for Ctype are dual. $\square$

To examine the relation of $\mathrm{Ctype}[x,y]$ and $\mathrm{Ctype}[v,y]$ when v is an atom of $[x,y]$ requires a new graded poset Z_*. Its elements are those of $W(1)$, and the cover relations are $u \lessdot_{1_*} u1$ and $u11^m \lessdot_{2_*} u21^m$, where $u \in W(1)$ and $m \geq 0$ are arbitrary. In other words, $u \lessdot_{Z_*} v$ when either a 1 is inserted in u in a position followed only by 1's (equivalently, when a 1 is appended to u), or when a 1 followed only by 1's is changed to a 2. In terms of snake diagrams, this may be visualized by the following example, in which v is shown, and either marked cell may be removed to obtain some $u \lessdot_{Z_*} v$.



Although we will use Z_* in comparing the types as stated above, we will not have any other use for Z_*. Note that Z_* is not a lattice, since upper bounds may not exist; for example, 12 and 2 have no upper bound, because in Z_*, all words larger than 12 start with 12 and all words larger than 2 start with 2. It is a meet semilattice: given two words $u = xz$ and $v = xy$ where z or y is empty, or they are both nonempty and start with different digits, then $u \wedge v = x1^m$ where m is the smaller of the number of digits of y or z.

Theorem 9.16. *If $x \lessdot v \leq y$ in $\mathcal{L}$, then* $\mathrm{Ctype}[v,y] \lessdot_{Z_*} \mathrm{Ctype}[x,y]$*; further, if* $\mathrm{Ctype}[v,y] \lessdot_{1_*} \mathrm{Ctype}[x,y]$*, then $v = C_x^{k-1} y$, where $k = \mathrm{arank}[x,y]$. Dually, if $x \leq v \lessdot y$, then* $\mathrm{Atype}[x,v] \lessdot_{Z_*} \mathrm{Atype}[x,y]$*, and if* $\mathrm{Atype}[x,v] \lessdot_{1_*} \mathrm{Atype}[x,y]$*, then $v = A_y^{k-1} x$, where $k = \mathrm{arank}[x,y]$.*

Proof. We induct on the length of the interval $[x,y]$, and set $\alpha = \mathrm{Ctype}[x,y]$ and $\beta = \mathrm{Ctype}[v,y]$.

If α and β begin with the same digit, then $C_x y = C_v y$, and if we replace y by $C_x y$, we get a smaller interval with the first digit of α and β deleted; the theorem holds in the smaller interval, and by prepending the common deleted digit, it holds in $[x,y]$.

So assume α and β do not start with the same digit. Then $C_x y \neq C_v y = v \vee C_x y$, so $v \not\leq C_x y$. Join v elementwise with the chain $C_x y \geq C_x^2 y \geq \cdots$ to obtain $C_v y \geq C_v^2 y \geq \cdots$; the ranks of the elements in the second chain are one higher than in the first, because the chains are in $[x,y]$ and v is an atom of $[x,y]$, with v not smaller than the top element of the chain. Thus, successive differences in ranks in the two chains are the same, so only the first digit of α and β differ. If $\alpha = 1$ and $\beta = \emptyset$ we are done. If $\alpha = 1\gamma$ and $\beta = 2\gamma$ for some word γ, then $\rho(\alpha) \lessdot \rho(\beta)$; however, $[x, y]$ is an interval of length one higher than $[v, y]$, so this cover relation should be reversed. Thus, this situation doesn't occur. We are left with $\alpha = 2\gamma$ and $\beta = 1\gamma$. Can any digit of β be 2? If the jth digit of β is 2, then $C_v^j y = C^j y$ has rank 2 lower than $C_v^{j-1} y$, and $v \leq C^j y \leq C_x^j y$, violating the fact that $v \not\leq C_x^j y$. So all digits of β are 1, and all digits of α but the first are 1.

Note that if $\alpha = \beta 1$ then v is uniquely determined by Theorem 3.8. $\square$

9.3. Interval type tables in strongly modular lattices. Let $f = (f_0 \lessdot f_1 \lessdot \cdots \lessdot f_n)$ be a saturated flag in a strongly modular lattice.

The interval Ctype table of a saturated flag f in a strongly modular lattice is the triangular array $(\mathrm{Ctype}[f_i, f_j])_{0 \leq i \leq j \leq n}$. The interval Atype table is defined dually. We will be interested in determining the possible interval type tables of a flag given just some of the entries of the table, such as the first row, $(\mathrm{Ctype}[f_0, f_j])_{0 \leq j \leq n}$, and more generally, any sequence $\mathrm{Ctype}[f_{i_0}, f_{j_0}], \ldots, \mathrm{Ctype}[f_{i_n}, f_{j_n}]$ where $i_0 = j_0$, $i_n = 0$, $j_n = n$, and for each $k > 0$, either $i_k = i_{k-1} - 1$ and $j_k = j_{k-1}$, or $i_k = i_{k-1}$ and $j_k = j_{k-1} + 1$.

We will develop necessary (but not sufficient) conditions for a triangular array $\Lambda = (\lambda^{(ij)})_{0 \leq i \leq j \leq n}$ of words of $W(1)$ to be realizable as the interval type table of some flag in an arbitrary strongly modular lattice, and then will focus on $Z(r)$.

By Theorem 9.14, $\lambda^{(ij)}$ increases in $Z(1)$ for fixed i and increasing j. By Theorem 9.16, it increases in Z_* for fixed j and decreasing i.

Let $i < j$ and consider the types of the intervals determined by the four elements $f_i \lessdot f_{i+1} \leq f_{j-1} \lessdot f_j$. The types are denoted as shown in this table, and have the cover relations shown by Theorems 9.14 and 9.16.

$$\begin{array}{c|ccc}
\mathrm{Ctype}[f_a, f_b] & b = j-1 & & b = j \\ \hline
a = i & \gamma & \lessdot_Z & \delta \\
 & \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} & & \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} \\
a = i+1 & \alpha & \lessdot_Z & \beta
\end{array}$$

Proposition 9.17. *The local configurations permitted in an interval Ctype table are as follows, where $x \in Z(1)$ and $k, m \geq 0$ are arbitrary. The label at the center of each is for later reference.*

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & \delta \\ \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} & & \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} \\ \alpha & \lessdot_Z & \beta \end{array}}
\qquad
\boxed{\begin{array}{ccc} 2^k x1 & \lessdot_1 & 2^k 1x1 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} & (1) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k x & \lessdot_1 & 2^k 1x \end{array}}
\qquad
\boxed{\begin{array}{ccc} 2^k 2 & \lessdot_1 & 2^k 21 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (2) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k 1 & \lessdot_2 & 2^k 2 \end{array}}$$

$$\boxed{\begin{array}{ccc} 2^k x21^m & \lessdot_1 & 2^k 1x21^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (3) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k x11^m & \lessdot_1 & 2^k 1x11^m \end{array}}
\qquad
\boxed{\begin{array}{ccc} 2^k 21^m & \lessdot_1 & 2^k 211^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (4) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k 11^m & \lessdot_1 & 2^k 111^m \end{array}}
\qquad
\boxed{\begin{array}{ccc} 2^k 1x21^m & \lessdot_2 & 2^k 2x21^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (5) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k 1x11^m & \lessdot_2 & 2^k 2x11^m \end{array}}$$

$$\boxed{\begin{array}{ccc} 2^k 211^m & \lessdot_2 & 2^k 221^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (6) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k 111^m & \lessdot_2 & 2^k 211^m \end{array}}
\qquad
\boxed{\begin{array}{ccc} 2^k 1x1 & \lessdot_2 & 2^k 2x1 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} & (7) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k 1x & \lessdot_2 & 2^k 2x \end{array}}
\qquad
\boxed{\begin{array}{ccc} 2^k 1 & \lessdot_2 & 2^k 2 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} & (8) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k & \lessdot_1 & 2^k 1 \end{array}}$$

Proof. In the left column of a configuration, given γ, there is at most one α with $\alpha \lessdot_{1_*} \gamma$ (namely, if γ ends in 1 then remove it), and at most one α with $\alpha \lessdot_{2_*} \gamma$ (if γ has a 2, change the last 2 to 1). In the right column, there are similar possibilities for obtaining β from δ. In the top row, if $\gamma \lessdot_1 \delta$ then γ is uniquely determined from δ by removing the first 1, while if $\gamma \lessdot_2 \delta$ then δ is uniquely determined from γ by changing the first 1 to 2. A similar relationship holds in the bottom row. This allows us to explicitly list all forms this table may take on by considering the kind of cover relation we have in each of the four positions. Given these four cover relations, if the top row has $\lessdot_1$ then δ determines all three of the other entries of the table, while if it has $\lessdot_2$ then γ determines the other three. Exhaustively listing all possibilities gives the configurations in the statement of the theorem, except for (2) and (8).

In place of (2), we actually obtain

$$\boxed{\begin{array}{ccc} 2^k21^m & \lessdot_1 & 2^k211^m \\ \gtrdot_{2_*} & (2') & \gtrdot_{1_*} \\ 2^k11^m & \lessdot_2 & 2^k21^m \end{array}}$$

Assume we have elements in configuration $(2')$. We show $m = 0$, which is configuration (2). Let $z = C_{f_i}^{k+1} f_j$. Since β and δ both start with 2^{k+1}, the first $k+1$ applications of C_{f_i} and $C_{f_{i+1}}$ to f_j agree, and $z = C_{f_{i+1}}^{k+1} f_j \geq f_{i+1}$. Since the remaining digits of δ are 1's, $\mathrm{Ctype}[f_i, z] = 1^{m+1}$, so $[f_i, z]$ is a chain and f_{i+1} is its atom. Next, since $\delta = \gamma 1$ and γ has $k+m+1$ digits, the first $k+m+1$ applications of C_{f_i} to $f_{j-1} \lessdot f_j$ maintain a cover relation: $C_{f_i}^t f_{j-1} \lessdot C_{f_i}^t f_j$ for $0 \leq t \leq k+m+1$. When $t \leq k+m$, both sides are strictly larger than f_i, hence weakly larger than the unique cover f_{i+1} of f_i, so we can replace C_{f_i} with $C_{f_{i+1}}$. Thus, the first $k+m$ digits of α, β, γ, δ should all agree. But when $m > 0$, the $(k+1)$th digit is 1 in α and 2 in the others, a contradiction. So only $m = 0$ is possible.

In place of (8), we actually obtain

$$\boxed{\begin{array}{ccc} 2^k11^m & \lessdot_2 & 2^k21^m \\ \gtrdot_{1_*} & (8') & \gtrdot_{2_*} \\ 2^k1^m & \lessdot_1 & 2^k11^m \end{array}}$$

Assume we have elements in this configuration and that $m > 0$. There are at least $k+1$ digits in each of $\alpha, \beta, \gamma, \delta$. We have $C_{f_i}^{k+1} f_{j-1} \leq C_{f_i}^{k+1} f_j$, and the ranks of both sides are the same since the first $k+1$ digits of γ and δ differ in sum by 1, as do the ranks of f_{j-1} and f_j. So $C_{f_i}^{k+1} f_{j-1} = C_{f_i}^{k+1} f_j$. Join both sides with f_{i+1} to obtain $C_{f_{i+1}}^{k+1} f_{j-1} = C_{f_{i+1}}^{k+1} f_j$. However, the first $k+1$ digits of α and β agree, so this should be $\lessdot$, a contradiction. Thus, we cannot have $m > 0$, so only $m = 0$ is conceivable. $\square$

Whether these configurations of types are realized in a particular strongly modular lattice depends on the lattice.

Next, we examine determination of β or γ from the other three types.

Given $\alpha \lessdot_Z \gamma \lessdot_{Z_*} \delta$, there is always some β consistent with them (if configuration (2) is removed, the rest yield unique solutions covering all cases), and it is usually unique. Similarly, given $\alpha \lessdot_{Z_*} \beta \lessdot_Z \delta$, there is always some γ consistent with them (omit configurations (2) and (6) to get unique solutions covering all cases), usually unique. Usually (odd numbered configurations), both vertical covers $\alpha \lessdot_{Z_*} \gamma$ and $\beta \lessdot_{Z_*} \delta$ entail changing a digit in the same position from the right end of the word, and both horizontal covers $\alpha \lessdot_Z \beta$ and $\gamma \lessdot_Z \delta$ entail changing the same position from the left end of the word. When α has the form $2\cdots 21\cdots 1$ (even numbered configurations), however, the positions near the central 21 are accessible for alterations by both $\lessdot_Z$ and $\lessdot_{Z_*}$.

Examination of configurations (1)–(8) yields the following as the only possibilities for β or γ not being uniquely determined by the other three types.

$$\boxed{\begin{array}{ccc} 2^k2 & \lessdot_1 & 2^k21 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} & (9) & \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} \\ 2^k1 & \lessdot_Z & \beta \end{array}}$$

(4) $\beta = 2^k11$; generic
(2) $\beta = 2^k2$; degenerate: $f_{i+1} = C^{k+1}f_j$

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & 2^k21 \\ \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} & (10) & \rotatebox[origin=c]{90}{$\lessdot$}_{1_*} \\ 2^k1 & \lessdot_2 & 2^k2 \end{array}}$$

(7) $\gamma = 2^k11$; generic
(2) $\gamma = 2^k2$; degenerate: $f_{j-1} = A^{k+1}f_i$

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & 2^k221^m \\ \rotatebox[origin=c]{90}{$\lessdot$}_{Z_*} & (11) & \rotatebox[origin=c]{90}{$\lessdot$}_{2_*} \\ 2^k111^m & \lessdot_2 & 2^k211^m \end{array}}$$

(6) $\gamma = 2^k211^m$; more likely in $Z(r)$
(5) $\gamma = 2^k121^m$; less likely in $Z(r)$

Given f_i, f_j with $\mathrm{Ctype}[f_i, f_j] = 2^k21$, the only f_{i+1} with $\mathrm{Ctype}[f_{i+1}, f_j] = 2^k2$ is $f_{i+1} = C^{k+1}f_j$, and the only f_{j-1} with $\mathrm{Ctype}[f_i, f_{j-1}] = 2^k2$ is $f_{j-1} = A^{k+1}f_i$. Thus, in (9) and (10), both these conditions holding simultaneously is "degenerate," while one holding and the other element being chosen freely from among potentially many possibilities is "generic." Of course, the number of "generic" possibilities depends on the lattice. In $Z(r)$, the generic possibilities truly do occur with higher frequency than the degenerate ones. Choose any f_i, f_j with $\mathrm{Ctype}[f_i, f_j] = 2^k21$. Then $f_j = 2^k2x$ for some $x \in Z(r)$. To obtain configuration (9), let $f_{j-1} = 2^k2f_i$, and f_{i+1} be any upper cover of f_i except for x. There are $\kappa^+(f_i) - 1$ choices, and $\kappa^+(f_i) - 1 \geq r - 1$. To obtain configuration (10), let $f_{i+1} = x$ and f_{j-1} be any lower cover of f_j except for 2^k2f_i. There are $\kappa^-(f_j) - 1 \geq (k+1)r - 1$ choices.

Last, we consider configuration (11). Choose any f_i, f_j with $\mathrm{Ctype}[f_i, f_j] = 2^k221^m$. Since δ begins with 2^k22, we have $f_j = 2^k22x$ with $x = C^{k+2}f_j \geq f_i$. Choose any $f_i \lessdot f_{i+1} \leq f_j$ with $\mathrm{Ctype}[f_i, f_j] = \delta$ and $\mathrm{Ctype}[f_{i+1}, f_j] = \beta$. Since δ begins with 2^k22, we can write $f_j = 2^k22x$, with $x \geq f_i$ and $\mathrm{Ctype}[f_i, x] = 1^m$. Since β begins 2^k21, we have $2x \geq f_{i+1}$ but $x \not\geq f_{i+1}$, and $\mathrm{Ctype}[f_{i+1}, 2x] = 11^m$. All lower covers of f_{j-1} exceed f_{i+1} because they all exceed 2^k2x. Since $\alpha = 2^k111^m$, the form of f_{j-1} is $2^kx'$ where $\mathrm{Ctype}[f_{i+1}, x'] = 111^m$. Also, since $C^{k+2}_{f_i}f_j = C^{k+2}_{f_i}f_{j-1}$ by rank considerations, $C^2x' = x$. Either $\mathrm{Ctype}[x, x'] = 12$ so that $x' = 1_s2x$ in one of r ways, or $\mathrm{Ctype}[x, x'] = 21$, so that x' is obtained from x by prepending 2 to any of the $\kappa^+(x)$ upper covers of x. In the first case, $\gamma = 2^k121^m$, and in the second, $\gamma = 2^k211^m$ unless $x' = 22C_{f_i}x$, when $\gamma = 2^k221^{m-1}$. In the second case, $\kappa^+(x) > r$ unless $x = \emptyset$, so that $f_i = \emptyset$, $m = 0$, and $\kappa^+(x) = r$; since $m = 0$, the exception $\gamma = 2^k221^{m-1}$ cannot occur. Thus for all f_i, f_{i+1}, f_j, there are at least as many choices of f_{j-1} with $\gamma = 2^k211^m$ as there are with $\gamma = 2^k121^m$. Note that the ratio $\kappa^+(x)/r$ is asymptotically an integer as $r \to \infty$, rather than a power of r, so this more frequent choice is not generic in the sense that term has been used elsewhere.

Theorem 9.18. *Let $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$ be a triangular array whose local configurations satisfy Proposition* 9.17*. There is a polynomial $p_\Lambda(r)$ such that the number of flags $\hat{0} = f_0 \lessdot \cdots \lessdot f_n$ in $Z(r)$ with* $\mathrm{Ctype}[f_i, f_j] = \lambda^{(ij)}$ *for all i, j is $p_\Lambda(r)$ when $r > 1$. When configuration (2) is not present, this polynomial is monic, with degree equal to the number of digits of $\lambda^{(0n)}$. When configuration (2) is present, the degree is lower.*

Proof. A necessary condition for Λ to be the interval Ctype table of a flag is that Proposition 9.17 applies to every local portion of Λ.

Let $\boldsymbol{P} = (\emptyset = \lambda^{(0)} \lessdot \cdots \lessdot \lambda^{(n)})$ be a chain in $Z(1)$. Let $J = \left\{ j : \lambda^{(j-1)} \lessdot_1 \lambda^{(j)} \right\}$. Let $\pi = \{B_1, \ldots, B_k\}$ be a set partition of J, and $\vec{c} = (c_1, \ldots, c_k)$ be distinct numbers in $\{1, \ldots, r\}$. Given π, there are $r(r-1)\cdots(r-k+1)$ choices of $\vec{c}$. Let $f(\boldsymbol{P}, \pi, \vec{c})$ be the flag $\emptyset = f_0 \lessdot \cdots \lessdot f_n$ in which $\mathrm{Ctype}[\emptyset, f_j] = \lambda^{(j)}$ and when $j \in B_m$, a 1_{c_m} is inserted in f_{j-1} to obtain f_j. The specification of $\boldsymbol{P}$, π, and $\vec{c}$ uniquely determines f. Conversely, all flags are expressible in this form.

The computation of $\mathrm{Ctype}[f_i, f_j]$ in Corollary 9.8 only depends on knowing $\lambda^{(i)}$, $\lambda^{(j)}$, and whether subscripts on particular 1's agree or disagree, which is all encoded in $\boldsymbol{P}$ and π. Thus, $\Lambda(f(\boldsymbol{P}, \pi, \vec{c}))$ is independent of $\vec{c}$, so we can call it $\Lambda(\boldsymbol{P}, \pi)$.

Now take any Λ locally satisfying Proposition 9.17, and let $\boldsymbol{P} = (\lambda^{(00)} \lessdot \cdots \lessdot \lambda^{(0n)})$. Then

$$p_\Lambda(r) = \sum_{\{\, \pi \,:\, \Lambda(\boldsymbol{P}, \pi) = \Lambda \,\}} r(r-1)\cdots(r-k+1)$$

where π is a partition of J and k is the number of blocks of π.

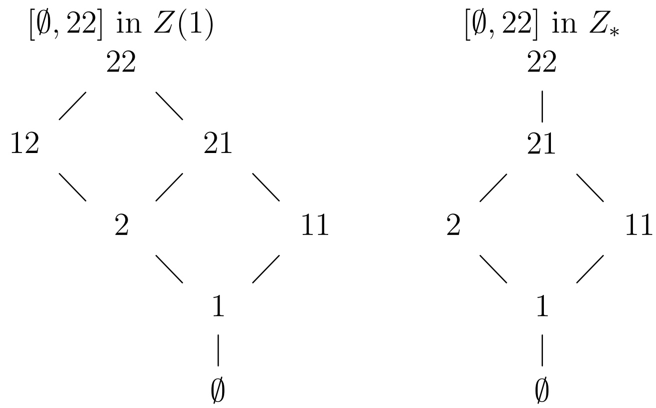
The total number of flags with $\mathrm{Ctype}[\emptyset, f_j] = \lambda^{(j)}$ for all j is $r^{\#J}$, obtained by making independent choices of which 1 to insert when $f_{j-1} \lessdot_1 f_j$. The discussion before this theorem showed that for all but $O(r^{-1})$ flags, configuration (2) does not arise in the interval Ctype table, and that without configuration (2), all other entries are uniquely determined by specifying the top row $\boldsymbol{P}$. Thus, for each $\boldsymbol{P}$, there is some Λ with $\deg(p_\Lambda) = \#J$; this can only be $\Lambda = \Lambda(\boldsymbol{P}, \pi)$, where π is the partition of J into singleton blocks, and p_Λ is monic because all other π with $\Lambda = \Lambda(\boldsymbol{P}, \pi)$ have fewer blocks and hence contribute only smaller degree terms. $\square$

Note.

(1) The restriction $r > 1$ is necessary for the trivial reason that $C2 = 1$ in $Z(1)$ and $C2 = \emptyset$ for $r > 1$.

(2) Recall that ev is an involution on chains in Young's lattice, and generically gives the cotype of a flag of given type. The preceding theorem similarly gives the generic value of $(\mathrm{Ctype}[f_i, f_n])_{0 \le i \le n}$ when $(\mathrm{Ctype}[f_0, f_i])_{0 \le i \le n}$ is specified, for $\emptyset = f_0 \lessdot \cdots \lessdot f_n$ in $Z(r)$. However, it does not give a bijection between saturated chains in $Z(1)$ and in Z_* from $\emptyset$ to the same maximum element: configuration (11) gives α, δ with two different γ necessarily mapping to the same β, so we cannot specify a chain in Z_* and apply certain of configurations (1)–(8) to recover a unique chain in $Z(1)$. In fact, there usually aren't the same number of saturated chains in $Z(1)$ and Z_* from $\emptyset$ to $x \in W(1)$. For

example, if $x = 22$, there are 3 chains in the first and 2 in the second.



(3) Stanley [**28**, Prop. 5.4] showed that the number of saturated chains from to x in $Z(r)$ is the same as in $Fib(r)$. In the same spirit as ev and the preceding theorem, it is tempting to try to form a triangular array $\Lambda = (\lambda^{(ij)})_{0 \le i \le j \le n}$ of words of $W(r)$ of rank $j - i$, increasing in $Z(r)$ as j increases and increasing in $Fib(r)$ as i decreases. We initialize $\lambda^{(ii)} = \emptyset$, and then $\lambda^{(0j)}$'s to a chain in $Z(r)$, or $\lambda^{(in)}$'s to a chain in $Fib(r)$, and have a propagation rule on local 2×2 configurations to determine the remaining entries. Unfortunately, this scheme does not work. One problematic configuration is

$$\boxed{\begin{array}{ccc} \gamma & \lessdot_Z & \delta \\ \dot{\vee}_F & & \dot{\vee}_F \\ \alpha & \lessdot_Z & \beta \end{array}} = \boxed{\begin{array}{ccc} 2^k 2 2^l x & \lessdot_Z & 2^k 2 2^l 1_1 x \\ \dot{\vee}_F & & \dot{\vee}_F \\ 2^k 1_2 2^l x & \lessdot_Z & ? \end{array}}$$

When $x = (1_1)^m$, the only possible solution is $\beta = 2^k 2 2^l x$. When x is not of this form, there are no solutions.

9.4. Fomin's Robinson-Schensted algorithm for $Z(1)$. Fomin [**4**] developed an analogue of the Robinson-Schensted correspondence for $Z(1)$: a bijection between pairs of saturated chains from $\emptyset$ to the same endpoint, and permutations in S_n. See Fomin [**6**] and Roby [**21**] for a unified treatment of these and similar correspondences. The bijection is as follows.

Form an array $\Lambda = (\lambda^{(ij)})_{0 \le i,j \le n}$. A local configuration in it is

$$\Lambda^{(ij)} = \boxed{\begin{array}{ccc} \lambda^{(i,j-1)} & \dot{\leq}_Z & \lambda^{(ij)} \\ \dot{\vee}\!\mid_Z & & \dot{\vee}\!\mid_Z \\ \lambda^{(i-1,j-1)} & \dot{\leq}_Z & \lambda^{(i-1,j)} \end{array}} = \boxed{\begin{array}{ccc} \gamma & \dot{\leq}_Z & \delta \\ \dot{\vee}\!\mid_Z & & \dot{\vee}\!\mid_Z \\ \alpha & \dot{\leq}_Z & \beta \end{array}}$$

Fomin's Robinson-Schensted correspondence in $Z(1)$. Let $\sigma \in S_n$. Form the array Λ as follows. Initialize the left and bottom edges of Λ to $\emptyset$: $\lambda^{(i0)} = \lambda^{(0i)} = \emptyset$

for $i = 0, \ldots, n$. Working in some order of increasing i and j, set

$$\delta = \begin{cases} 1\alpha & \text{if } \alpha = \beta = \gamma \text{ and } \sigma(j) = i \\ \alpha & \text{if } \alpha = \beta = \gamma \text{ and } \sigma(j) \neq i \\ 2\alpha & \text{if } \alpha \lessdot_Z \beta, \gamma \\ \beta & \text{if } \beta \gtrdot_Z \gamma = \alpha \\ \gamma & \text{if } \gamma \gtrdot_Z \beta = \alpha \end{cases}$$

As we construct $\lambda^{(ij)}$, it is a weakly increasing function of i and j, and $\rho(\lambda^{(ij)}) = \#\{\, 0 < j' \leq j \,:\, \sigma(j') \leq i \,\}$. Finally, the chains on the right and top edges of Λ are $\boldsymbol{P} = (\lambda^{(0n)}, \ldots, \lambda^{(nn)})$ and $\boldsymbol{Q} = (\lambda^{(n0)}, \ldots, \lambda^{(nn)})$.

The inverse correspondence. Given saturated chains $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $Z(1)$ from $\emptyset$ to the same top element, form the array $\Lambda_{\mathrm{FRS}}(\boldsymbol{P}, \boldsymbol{Q})$ as follows. Initialize the chains on the right and top edges to $(\lambda^{(0n)}, \ldots, \lambda^{(nn)}) = \boldsymbol{P}$ and $(\lambda^{(n0)}, \ldots, \lambda^{(nn)}) = \boldsymbol{Q}$. Working in some order of decreasing i and j, set

$$\alpha = \begin{cases} \beta & \text{if } \beta \lessdot_Z \gamma = \delta \\ \gamma & \text{if } \gamma \lessdot_Z \beta = \delta \\ \text{delete first digit of } \delta & \text{otherwise} \end{cases}$$

We have $\lambda^{(n,j-1)} \lessdot_Z \lambda^{(nj)}$ and $\lambda^{(0,j-1)} = \lambda^{(0j)}$. If $\lambda^{(i,j-1)} = \lambda^{(ij)}$, the same is true upon replacing i with $i-1$, so for each j there is a unique i with $\alpha = \beta = \gamma \lessdot \delta$, and we set $\sigma(j) = i$.

9.5. Relative positions of flags in strongly modular lattices. Let $f = (f_0 \lessdot \cdots \lessdot f_n)$ and $f' = (f'_0 \lessdot \cdots \lessdot f'_n)$ be two saturated flags in a strongly modular lattice with the same endpoints, $f_0 = f'_0$, $f_n = f'_n$, where f_0 has finite rank. Let $x_{ij} = f_i \wedge f'_j$. We consider the meet type table $\Lambda(f, f') = (\lambda^{(ij)})$, where $\lambda^{(ij)} = \mathrm{Ctype}[\hat{0}, x_{ij}]$ for $0 \leq i, j \leq n$. Since $x_{i-1,j-1} = x_{i-1,j} \wedge x_{i,j-1}$, we may examine a local portion of Λ:

$$\Lambda^{(ij)} = \begin{array}{l|ccc} \mathrm{Ctype}[\emptyset, f_a \wedge f'_b] & b = j-1 & & b = j \\ \hline a = i & \gamma & \lessdot_Z & \delta \\ & \gtrdot_Z & & \gtrdot_Z \\ a = i-1 & \alpha & \lessdot_Z & \beta \end{array}$$

We seek possible values of α given the other three.

If $x_{i,j-1}$ or $x_{i-1,j}$ equals x_{ij}, we obtain one of the following configurations.

$$(1)\ \boxed{\begin{array}{ccc} \gamma & \lessdot & \delta \\ \parallel & & \parallel \\ (\gamma) & \lessdot & \delta \end{array}} \qquad (2)\ \boxed{\begin{array}{ccc} \delta & = & \delta \\ \gtrdot & & \gtrdot \\ (\beta) & = & \beta \end{array}} \qquad (3)\ \boxed{\begin{array}{ccc} \gamma & = & \gamma \\ \parallel & & \parallel \\ (\gamma) & = & \gamma \end{array}}$$

These all occur in Λ_{FRS} too.

We consider configurations with $\beta, \gamma \lessdot_Z \delta$. If $x_{i-1,j}$ and $x_{i,j-1}$ are distinct lower covers of x_{ij}, the interval $[x_{i-1,j} \wedge x_{i,j-1}, x_{ij}]$ is complemented of length 2, so $Cx_{ij} =$

$x_{i-1,j-1}$, and $\delta = 2\alpha$.

$$(4)\ \boxed{\begin{array}{ccc} \gamma & \lessdot_Z & 2\alpha \\ & \diagdown & \\ \rotatebox{90}{$\lessdot$}_Z & \neq & \rotatebox{90}{$\lessdot$}_Z \\ & & \diagdown \\ (\alpha) & \lessdot_Z & \beta \end{array}} \qquad (5)\ \boxed{\begin{array}{ccc} \gamma & \lessdot_2 & 2\alpha \\ & \diagdown & \\ \rotatebox{90}{$\lessdot$}_1 & = & \rotatebox{90}{$\lessdot$}_2 \\ & & \diagdown \\ (\alpha) & \lessdot_1 & \gamma \end{array}}$$

Either $\beta \neq \gamma$, or we have two distinct lower covers of equal type. By Theorem 9.15, when $\gamma \lessdot_1 \delta$, there is at most one lower cover of x_{ij} of type γ, so to have two distinct lower covers requires $\gamma \lessdot_2 \delta$. This restriction on a local configuration does not exist in Λ_{FRS}.

Finally, if $x_{ij} \gtrdot x_{i-1,j} = x_{i,j-1}$, we have

$$(6)\ \boxed{\begin{array}{ccc} \gamma & \lessdot_1 & \delta \\ \shortparallel & & \rotatebox{90}{$\lessdot$}_1 \\ (\gamma) & = & \gamma \end{array}} \qquad (7)\ \boxed{\begin{array}{ccc} \gamma & \lessdot_2 & \delta \\ \shortparallel & & \rotatebox{90}{$\lessdot$}_2 \\ (\gamma) & = & \gamma \end{array}}$$

Unlike Λ_{FRS}, this does not require $\delta = 1\gamma$.

In all cases, if $\alpha = \beta \wedge \gamma$ then any solutions of $\alpha \leq \beta \leq \delta$ and $\alpha \leq \gamma \leq \delta$ fall into one of these configurations. The only other case is $\alpha \lessdot \beta = \gamma \lessdot \delta$, when we are restricted to (5) and not $\alpha \lessdot_2 \beta = \gamma \lessdot_1 \delta$.

Given the chains along the right and top boundaries of Λ, if we propagate in some order of decreasing i and j to fill Λ using rules (1)–(7), the only ambiguity arises when $\gamma = \beta \lessdot_2 \delta$, since either (5) or (7) can be used; all other arrangements of β, γ, δ admit exactly one choice of α, namely $\alpha = \beta \wedge \gamma$. Let $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ be the array obtained by using (5) when the ambiguity arises, and $\sigma_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ be the associated permutation. This array is not the same as $\Lambda_{\mathrm{FRS}}(\boldsymbol{P}, \boldsymbol{Q})$ because their local rules differ. Using rule (7) instead of rule (5) would give $\lambda^{(ij)} = \lambda^{(nj)} \wedge \lambda^{(in)}$ with associated permutation $\sigma(\boldsymbol{P}, \boldsymbol{Q})$, and provides a lower bound of the relative position in the Bruhat order by Theorem 7.7.

Note. When $\boldsymbol{P}$ and $\boldsymbol{Q}$ consist only of words of the form $2^k 1^m$, so do all entries of Λ, no matter whether (5) or (7) is used. In fact, this is exactly the same as the local rule version of the ordinary Robinson-Schensted algorithm in Young's lattice when we interpret $2^k 1^m \in Z(1)$ as the two row partition $(k+m, m)$ in Young's lattice, for each word in $\boldsymbol{P}$, $\boldsymbol{Q}$, and Λ. Rule (5) gives the generic choice and rule (7) the degenerate choice for Young's lattice as well.

Theorem 9.19. *Let f, f' be saturated chains with the same endpoints in a strongly modular lattice, and $\boldsymbol{P} = (\mathrm{Ctype}[f_0, f_i])_{0 \leq i \leq n}$, $\boldsymbol{Q} = (\mathrm{Ctype}[f'_0, f'_j])_{0 \leq j \leq n}$. Then $\sigma(\boldsymbol{P}, \boldsymbol{Q}) \leq \sigma(f, f') \leq \sigma_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ in the Bruhat order.*

Proof. The lower bound holds by Theorem 7.7.

Let Λ be a meet type table. We show how to alter certain $\lambda^{(ij)}$ with $0 \leq i, j < n$ so that these $\lambda^{(ij)}$ decrease in $Z(1)$, and all other $\lambda^{(ij)}$ stay intact. Such alterations can be carried out iff configuration (7) appears somewhere in Λ, so the stable table obtained at the end is $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$, and is componentwise less than Λ in $Z(1)$; thus, the permutation Λ defines is smaller than the one $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P},\boldsymbol{Q})$ defines in the Bruhat order.

So suppose $\Lambda^{(i',j')}$ is in configuration (7). Then we can write $\delta = 2x$. Define a new meet type table $M = (\mu^{(ij)})_{0\le i,j\le n}$ by

$$\mu^{(ij)} = \begin{cases} \lambda^{(ij)} & \text{if } i \ge i' \text{ or } j \ge j' \\ \lambda^{(ij)} \wedge x & \text{if } i < i' \text{ and } j < j' \end{cases}$$

We show that all local configurations of M are of the forms (1)–(7), and that $\mu^{(ij)} = \lambda^{(ij)}$ or $\mu^{(ij)} \lessdot_1 \lambda^{(ij)}$ (denoted $\mu^{(ij)} \underline{\lessdot}_1 \lambda^{(ij)}$) for all i,j. The local configurations are denoted as follows.

$$M^{(ij)} = \boxed{\begin{array}{ccc} \mu^{(i,j-1)} & \underline{\lessdot}_Z & \mu^{(ij)} \\ \rotatebox{90}{$\underline{\lessdot}_Z$} & & \rotatebox{90}{$\underline{\lessdot}_Z$} \\ \mu^{(i-1,j-1)} & \underline{\lessdot}_Z & \mu^{(i-1,j)} \end{array}} = \boxed{\begin{array}{ccc} \tilde\gamma & \underline{\lessdot}_Z & \tilde\delta \\ \rotatebox{90}{$\underline{\lessdot}_Z$} & & \rotatebox{90}{$\underline{\lessdot}_Z$} \\ \tilde\alpha & \underline{\lessdot}_Z & \tilde\beta \end{array}}$$

First, since $x \lessdot_1 \lambda^{(i'-1,j'-1)} \lessdot_2 \lambda^{(i',j')}$, we have $\mu^{(i'-1,j'-1)} = x \lessdot_1 \lambda^{(i'-1,j'-1)}$.

For some $i < i'$ and $j < j'$, let $\beta = \lambda^{(i-1,j)}$, $\delta = \lambda^{(ij)}$, $\tilde\beta = \mu^{(i-1,j)}$, $\tilde\delta = \mu^{(ij)}$. Suppose $\tilde\delta \underline{\lessdot}_1 \delta$. We show $\mu^{(i-1,j)} \underline{\lessdot}_1 \lambda^{(i-1,j)}$, that is, $\tilde\beta \underline{\lessdot}_1 \beta$. We have

$$\tilde\beta = x \wedge \beta = x \wedge (\delta \wedge \beta) = (x \wedge \delta) \wedge \beta = \tilde\delta \wedge \beta.$$

$\tilde{\boldsymbol\delta} = \boldsymbol\delta$: Then $\tilde\beta = \delta \wedge \beta = \beta$.
$\boldsymbol\beta = \tilde{\boldsymbol\delta} \lessdot_1 \boldsymbol\delta$: Then $\tilde\beta = \tilde\delta \wedge \beta = \beta$.
$\tilde{\boldsymbol\delta} \lessdot_1 \boldsymbol\delta = \boldsymbol\beta$: Then $\tilde\beta = \tilde\delta \lessdot_1 \delta = \beta$.
$\tilde{\boldsymbol\delta} \lessdot_1 \boldsymbol\delta$ and $\tilde{\boldsymbol\delta} \neq \boldsymbol\beta \lessdot \boldsymbol\delta$: We have $\delta = 2y \gtrdot_1 \tilde\delta \gtrdot_2 y$ and $\delta \gtrdot_2 \beta \gtrdot_1 y$ (there's only one $\gtrdot_1$ lower cover, and $\tilde\delta \neq \beta$), so $\tilde\beta = y \lessdot_1 \beta$.

Similarly, $\mu^{(i,j-1)} \underline{\lessdot}_1 \lambda^{(i,j-1)}$, so we can propagate to obtain $\mu^{(ij)} \underline{\lessdot}_1 \lambda^{(ij)}$ for all i and j.

Next, we verify that all $M^{(ij)}$ are valid configurations.

When $i > i'$ or $j > j'$ we have $M^{(ij)} = \Lambda^{(ij)}$, so the configuration is valid.

Let $i \le i'$ and $j = j'$, so $\alpha = \beta = \tilde\beta$ and $\gamma = \delta = \tilde\delta$. Then

$$\tilde\alpha = \alpha \wedge x = (\gamma \wedge \alpha) \wedge x = (\alpha \wedge x) \wedge \gamma = (\beta \wedge x) \wedge \gamma = \beta \wedge \tilde\gamma = \tilde\beta \wedge \tilde\gamma,$$

so $M^{(ij)}$ is in a valid configuration.

Similarly, $i = i'$ and $j \le j'$ yields $M^{(ij)}$ in a valid configuration.

Finally, consider $i < i'$ and $j < j'$. If $\Lambda^{(ij)}$ is not in configuration (5) then $\alpha = \beta \wedge \gamma$ and so $\tilde\alpha = \tilde\beta \wedge \tilde\gamma$ so $M^{(ij)}$ is valid. If $\Lambda^{(ij)}$ is in configuration (5), either $\tilde\delta = \delta$ so $M^{(ij)} = \Lambda^{(ij)}$ is valid, or else $\tilde\delta \lessdot_1 \delta$. Then $\tilde\delta \neq \gamma$ since $\gamma \lessdot_2 \delta$, so $\tilde\alpha = \tilde\beta = \tilde\gamma = \alpha$ and $M^{(ij)}$ is in configuration (7) (since $\alpha \lessdot_2 \tilde\delta \lessdot_1 \delta = 2\alpha$). □

Corollary 9.20. *Conjecture* 7.6 *holds in semi-primary lattices whose type has two rows.*

Proof. That conjecture for semi-primary lattices is identical to the preceding theorem for strongly modular lattices; they coincide for semi-primary lattices whose type has two rows. □

Theorem 9.21. *Let $\Lambda = (\lambda^{(ij)})_{0\le i,j\le n}$ be a meet type table. There is a polynomial $p_\Lambda(r)$ such that the number of pairs of saturated chains f, f' in $Z(r)$ with $f_0 = f'_0 = \emptyset$ and $f_n = f'_n$ and $\mathrm{Ctype}[\emptyset, f_i \wedge f'_j] = \lambda^{(ij)}$ for all $0 \le i, j \le n$, is $p_\Lambda(r)$ when $r > 1$. The degree is the number of configuration (6)'s appearing in Λ, and the maximum degree n is obtained precisely when Λ has the form $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$ for saturated chains $\boldsymbol{P}$, $\boldsymbol{Q}$ in $Z(1)$ of length n from $\emptyset$ to the same endpoint.*

Proof. We form a matrix $X = (x_{ij})_{0\le i,j\le n}$ of entries in $Z(r)$ that increase in $Z(r)$ as i and j do; have $\mathrm{Ctype}[\emptyset, x_{ij}] = x_{ij}$; and have $x_{i-1,j} \wedge x_{i,j-1} = x_{i-1,j-1}$ for $0 < i, j \le n$. Let $x_{i0} = x_{0i} = \emptyset$ for $0 \le i \le n$. Propagate from the perimeter for increasing i and j to compute the remaining entries of X. When $\Lambda^{(ij)}$ is in local configuration (m), propagate according to rule (m') below.

$$(1')\ \boxed{\begin{array}{ccc} x & \lessdot & (y) \\ \parallel & & \parallel \\ x & \lessdot & y \end{array}} \qquad (2')\ \boxed{\begin{array}{ccc} y & = & (y) \\ \rotatebox{-90}{$\lessdot$} & & \rotatebox{-90}{$\lessdot$} \\ x & = & x \end{array}} \qquad (3')\ \boxed{\begin{array}{ccc} x & = & (x) \\ \parallel & & \parallel \\ x & = & x \end{array}}$$

$$(4', 5')\ \boxed{\begin{array}{ccc} z & \lessdot & (2x) \\ & \diagdown & \\ \rotatebox{-90}{$\lessdot$} & \neq & \rotatebox{-90}{$\lessdot$} \\ & & \diagdown \\ x & \lessdot & y \end{array}} \quad (6')\ \boxed{\begin{array}{ccc} 2^k x & \lessdot & 2^k(1_{c_j} x) \\ \parallel & & \rotatebox{-90}{$\lessdot$} \\ 2^k x & = & 2^k x \end{array}} \quad (7)\ \boxed{\begin{array}{ccc} 2^k 1_m x & \lessdot_2 & (2^k 2x) \\ \parallel & & \rotatebox{-90}{$\lessdot$}_2 \\ 2^k 1_m x & = & 2^k 1_m x \end{array}}$$

In $(1')$, $(2')$, $(3')$, $(4')$, and $(7')$, the value given in the upper right corner is clearly the unique solution of x_{ij} given the other three values and $\mathrm{Ctype}[\emptyset, x_{ij}] = \delta$. In $(6')$, the subscript c_j may be chosen arbitrarily from $\{1, \dots, r\}$. In $(4')$, since $\beta \ne \gamma$, we already have $x_{i-1,j} \ne x_{i,j-1}$. In $(5')$, we require $x_{i-1,j} \ne x_{i,j-1}$, meaning that the subscripts on their first 1's are different; these words were given to us from earlier propogation steps, so it's a bit late to impose this restriction. Proceed with the construction by ignoring this restriction, and we'll fix it up later. All conditions we require on the entries of X are met except that $x_{i,j-1} \wedge x_{i-1,j} = x_{i-1,j-1}$ may fail when $\Lambda^{(ij)}$ is in configuration (5).

Let $J = \Big\{ j : \Lambda^{(ij)} \text{ is in configuration (6) for some } i \Big\}$. There is a subscript c_j for each $j \in J$. The restrictions from configuration (5) say that certain c_j's should be unequal; let $R = \{ \{j_1, j_2\} : (5) \Rightarrow c_{j_1} \ne c_{j_2} \}$. Then

$$p_\Lambda(r) = \sum_\pi r(r-1)\cdots(r-k+1)$$

where π runs over all set partitions of J in which no block contains both elements of a restricted pair in R, and k is the number of blocks of π. Assign values $1, \dots, r$ to the c_j's so that $c_{j_1} = c_{j_2}$ iff j_1 and j_2 are in the same block of π; the number of ways to do this assignment for a given π is $r(r-1)\cdots(r-k+1)$.

The partition into singleton blocks yields the term of maximum degree in the sum, so $\deg(p_\Lambda) = \#J$.

The maximum degree p_Λ can have is n, when $J = \{1, \dots, n\}$. The total number of configuration (6) and (7)'s in Λ is n. For any $\boldsymbol{P}$ and $\boldsymbol{Q}$, the unique Λ with right and top perimeter chains $\boldsymbol{P}$ and $\boldsymbol{Q}$ and n configuration (6)'s (hence no (7)'s) is $\Lambda_{\mathrm{ZRS}}(\boldsymbol{P}, \boldsymbol{Q})$. $\square$

Appendix A. Statistical summary of interval type tables computed with Mathematica®

The **weight** $f_\Lambda(q)$ of an interval type table Λ is the number of flags realizing it in a q-regular semi-primary lattice; it is the polynomial of Conjecture 5.25. Empirically, it usually has a factorization involving powers of q and the following functions.

$$\begin{aligned}
[n] &= (q-1)(q-2)\cdots(q-n) \\
\langle n\rangle &= 1+q+\cdots+q^{n-1} = \frac{q^n-1}{q-1} \\
\langle n\rangle! &= \langle 1\rangle\langle 2\rangle\cdots\langle n\rangle \\
\langle 0\rangle! &= 1 \\
M_\lambda(q) &= \prod_j \langle m_j(\lambda)\rangle!
\end{aligned}$$

For q-regular semi-primary lattices of various types, the weights of all interval type tables were computed by the method developed in Section 5.5 with Mathematica® (a symbolic mathematics program, ©1993 Wolfram Research, Inc.). We present a brief summary and a longer summary of the results. In all cases, the parameter q_0 associated with these polynomials has value 1, so the polynomials enumerate the interval type tables in all q-regular semi-primary lattices. The Mathematica computations also show that for the lattice types listed, if $f_\Lambda(q)$ is identically 0 as a polynomial, then Λ is not realized even in irregular semi-primary lattices.

A.1. Brief summary of results.

Key:

- λ Type of lattice. Let $n = |\lambda|$.
- N_r Number of interval type tables realized over all lattices of type λ.
- N_s Number of realized interval type tables that are symmetric.
- N_h Number of realized interval type tables that are hereditary, equal to the number of unordered set partitions of $\{1,\dots,n\}$ of type λ. The order total is the nth Bell number.
- N_{sh} Number of realized interval type tables that are symmetric and hereditary, equal to the number of unordered set partitions of $\{1,\dots,n\}$ of type λ invariant under the permutation $i \mapsto n+1-i$.
- min Minimum degree of weights of interval type tables, equal to

$$n(\lambda) - n_0(\lambda) = \sum_i (i-1)\lambda_i - \sum_j \binom{m_j(\lambda)}{2}.$$

- max Maximum degree of weights of interval type tables, equal to

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

- f^λ Number of standard Young tableaux of shape λ.

λ	N_r	N_s	N_h	N_{sh}	min	max	$M_\lambda(q)$	f^λ
$\emptyset$	1	1	1	1	0	0	1	1
Order 0	**1**	**1**	**1**	**1**				
(1)	1	1	1	1	0	0	1	1
Order 1	**1**	**1**	**1**	**1**				
(2)	1	1	1	1	0	0	1	1
$(1,1)$	1	1	1	1	1	1	$\langle 2\rangle$	1
Order 2	**2**	**2**	**2**	**2**				
(3)	1	1	1	1	0	0	1	1
$(2,1)$	3	1	3	1	0	1	1	2
(1^3)	1	1	1	1	3	3	$\langle 3\rangle!$	1
Order 3	**5**	**3**	**5**	**3**				
(4)	1	1	1	1	0	0	1	1
$(3,1)$	5	1	4	0	0	1	1	3
$(2,2)$	3	3	3	3	1	2	$\langle 2\rangle$	2
$(2,1,1)$	6	2	6	2	1	3	$\langle 2\rangle$	3
(1^4)	1	1	1	1	6	6	$\langle 4\rangle!$	1
Order 4	**16**	**8**	**15**	**7**				
(5)	1	1	1	1	0	0	1	1
$(4,1)$	7	1	5	1	0	1	1	4
$(3,2)$	12	2	10	2	0	2	1	5
$(3,1,1)$	15	3	10	2	1	3	$\langle 2\rangle$	6
$(2,2,1)$	15	3	15	3	1	4	$\langle 2\rangle$	5
$(2,1^3)$	10	2	10	2	3	6	$\langle 3\rangle!$	4
(1^5)	1	1	1	1	10	10	$\langle 5\rangle!$	1
Order 5	**61**	**13**	**52**	**12**				
(6)	1	1	1	1	0	0	1	1
$(5,1)$	9	1	6	0	0	1	1	5
$(4,2)$	26	6	15	3	0	2	1	9
$(4,1,1)$	28	4	15	3	1	3	$\langle 2\rangle$	10
$(3,3)$	12	6	10	4	1	3	$\langle 2\rangle$	5
$(3,2,1)$	87	3	60	0	0	4	1	16
$(3,1^3)$	35	3	20	0	3	6	$\langle 3\rangle!$	10
(2^3)	15	7	15	7	3	6	$\langle 3\rangle!$	5
$(2,2,1,1)$	45	9	45	9	2	7	$\langle 2\rangle^2$	9
$(2,1^4)$	15	3	15	3	6	10	$\langle 4\rangle!$	5
(1^6)	1	1	1	1	15	15	$\langle 6\rangle!$	1
Order 6	**274**	**44**	**203**	**31**				

λ	N_r	N_s	N_h	N_{sh}	min	max	$M_\lambda(q)$	f^λ
(7)	1	1	1	1	0	0	1	1
$(6,1)$	11	1	7	1	0	1	1	6
$(5,2)$	44	4	21	3	0	2	1	14
$(5,1,1)$	45	5	21	3	1	3	$\langle 2\rangle$	15
$(4,3)$	57	3	35	3	0	3	1	14
$(4,2,1)$	240	6	105	3	0	4	1	35
$(4,1^3)$	84	4	35	3	3	6	$\langle 3\rangle!$	20
$(3,3,1)$	128	10	70	4	1	5	$\langle 2\rangle$	21
$(3,2,2)$	148	10	105	9	1	6	$\langle 2\rangle$	21
$(3,2,1,1)$	357	9	210	6	1	7	$\langle 2\rangle$	35
$(3,1^4)$	70	6	35	3	6	10	$\langle 4\rangle!$	15
$(2^3,1)$	105	7	105	7	3	9	$\langle 3\rangle!$	14
$(2,2,1^3)$	105	9	105	9	4	11	$\langle 3\rangle!\,\langle 2\rangle$	14
$(2,1^5)$	21	3	21	3	10	15	$\langle 5\rangle!$	6
(1^7)	1	1	1	1	21	21	$\langle 7\rangle!$	1
Order 7	**1417**	**79**	**877**	**59**				
(8)	1	1	1	1	0	0	1	1
$(7,1)$	13	1	8	0	0	1	1	7
$(6,2)$	66	8	28	4	0	2	1	20
$(6,1,1)$	66	6	28	4	1	3	$\langle 2\rangle$	21
$(5,3)$	145	9	56	0	0	3	1	28
$(5,2,1)$	493	5	168	0	0	4	1	64
$(5,1^3)$	165	5	56	0	3	6	$\langle 3\rangle!$	35
$(4,4)$	57	19	35	11	1	4	$\langle 2\rangle$	14
$(4,3,1)$	769	7	280	0	0	5	1	70
$(4,2,2)$	580	38	210	18	1	6	$\langle 2\rangle$	56
$(4,2,1,1)$	1224	30	420	12	1	7	$\langle 2\rangle$	90
$(4,1^4)$	210	10	70	6	6	10	$\langle 4\rangle!$	35
$(3,3,2)$	475	27	280	16	1	7	$\langle 2\rangle$	42
$(3,3,1,1)$	725	37	280	16	2	8	$\langle 2\rangle^2$	56
$(3,2,2,1)$	1394	18	840	0	1	9	$\langle 2\rangle$	70
$(3,2,1^3)$	1092	12	560	0	3	11	$\langle 3\rangle!$	64
$(3,1^5)$	126	6	56	0	10	15	$\langle 5\rangle!$	21
(2^4)	105	25	105	25	6	12	$\langle 4\rangle!$	14
$(2^3,1,1)$	420	28	420	28	4	13	$\langle 3\rangle!\,\langle 2\rangle$	28
$(2,2,1^4)$	210	18	210	18	7	16	$\langle 4\rangle!\,\langle 2\rangle$	20
$(2,1^6)$	28	4	28	4	15	21	$\langle 6\rangle!$	7
(1^8)	1	1	1	1	28	28	$\langle 8\rangle!$	1
Order 8	**8365**	**315**	**4140**	**164**				

λ	N_r	N_s	N_h	N_{sh}	min	max	$M_\lambda(q)$	f^λ
(9)	1	1	1	1	0	0	1	1
$(6,3)$	278	6	84	4	0	3	1	48
$(5,4)$	303	7	126	6	0	4	1	42
$(5,1^4)$	495	15	126	6	6	10	$\langle 4\rangle!$	70
$(4,1^5)$	462	10	126	6	10	15	$\langle 5\rangle!$	56
(3^3)	475	27	280	16	3	9	$\langle 3\rangle!$	42
$(2^4,1)$	945	25	945	25	6	16	$\langle 4\rangle!$	42
(1^9)	1	1	1	1	36	36	$\langle 9\rangle!$	1
Order 9	**incomplete**							
(10)	1	1	1	1	0	0	1	1
$(5,5)$	303	45	126	16	1	5	$\langle 2\rangle$	42
$(5,1^5)$	1287	15	252	0	10	15	$\langle 5\rangle!$	126
(1^{10})	1	1	1	1	45	45	$\langle 10\rangle!$	1
Order 10	**incomplete**							

A.2. Special classes of types. For hooks $\lambda = (i, 1^j)$ (with $i > 0$ and $j \geq 0$), the formula $N_r = \binom{2i+j-2}{j}$ empirically holds when $1 \leq i \leq 5$ and $0 \leq j \leq 5$.

For two column partitions $\lambda = (i,j)'$ (with $i \geq j \geq 0$), all interval type tables are hereditary by Theorem 5.53, so $N_r = N_h = n!/(j!(i-j)!2^j)$.

For two row partitions (i,j) (with $i \geq j \geq 0$), the following values of N_r were computed.

$j \backslash i$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1		1	3	5	7	9	11
2			3	12	26	44	66
3				12	57	145	278
4					57	303	n/a
5						303	n/a

A.3. Extended summary of results. We now list all the weights occurring for each type. The individual interval type tables Λ are not listed, but counts of how many Λ share common characteristics are given.

- $f_\Lambda(q)$ Number of flags realizing interval type table Λ in a q-regular semi-primary lattice.
- d Degree of $f_\Lambda(q)$.
- N_r Number of realized interval type tables with this weight.
- N_s Number of realized interval type tables with this weight that are symmetric.

The tables below have the following components.

	$M_\lambda(q)$		λ
degree of $f_\Lambda(q)$	$f_\Lambda(q)/M_\lambda(q)$	# of Λ of this form	# of symmetric Λ
degree of $f_\Lambda(q)$	$f_\Lambda(q)/M_\lambda(q)$	# of Λ of this form	# of symmetric Λ
...	...	...	...

When $M_\lambda(q) = 1$, it is omitted.

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
			$\emptyset$
0	1	1	1
			(1)
0	1	1	1
			(2)
0	1	1	1
	$\langle 2\rangle$		$(1,1)$
1	1	1	1
			(3)
0	1	1	1
			$(2,1)$
1	q	2	0
0	1	1	1
	$\langle 3\rangle!$		(1^3)
3	1	1	1
			(4)
0	1	1	1
			$(3,1)$
1	q	2	0
1	$[1]$	1	1
0	1	2	0
	$\langle 2\rangle$		$(2,2)$
2	q	2	2
1	1	1	1
	$\langle 2\rangle$		$(2,1,1)$
3	q^2	3	1
2	q	2	0
1	1	1	1
	$\langle 4\rangle!$		(1^4)
6	1	1	1
			(5)
0	1	1	1

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
			$(4,1)$
1	q	2	0
1	$[1]$	2	0
0	1	3	1
			$(3,2)$
2	q^2	3	1
2	$q[1]$	2	0
1	q	6	0
0	1	1	1
	$\langle 2\rangle$		$(3,1,1)$
3	q^2	3	1
3	$q[1]$	3	1
2	q	4	0
2	$[1]$	2	0
1	1	3	1
	$\langle 2\rangle$		$(2,2,1)$
4	q^3	5	1
3	q^2	6	0
2	q	3	1
1	1	1	1
	$\langle 3\rangle!$		$(2,1^3)$
6	q^3	4	0
5	q^2	3	1
4	q	2	0
3	1	1	1
	$\langle 5\rangle!$		(1^5)
10	1	1	1
			(6)
0	1	1	1
			$(5,1)$
1	q	2	0
1	$[1]$	3	1
0	1	4	0

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
			$(4,2)$
2	q^2	3	1
2	$q[1]$	5	1
2	$[2]$	1	1
1	q	8	0
1	$[1]$	5	1
0	1	4	2
	$\langle 2\rangle$		$(4,1,1)$
3	q^2	3	1
3	$q[1]$	6	0
3	$[1]^2$	1	1
2	q	6	0
2	$[1]$	6	0
1	1	6	2
	$\langle 2\rangle$		$(3,3)$
3	q^2	3	1
3	$q[1]$	2	2
2	q	6	2
1	1	1	1
			$(3,2,1)$
4	q^4	8	0
4	$q^3[1]$	8	0
3	q^3	20	0
3	$q^2[1]$	12	2
2	q^2	20	0
2	$q[1]$	6	0
1	q	10	0
1	$[1]$	1	1
0	1	2	0
	$\langle 3\rangle!$		$(3,1^3)$
6	q^3	4	0
6	$q^2[1]$	6	2
5	q^2	6	0
5	$q[1]$	6	0
4	q	6	0
4	$[1]$	3	1
3	1	4	0

M_λ			λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 3\rangle!$		(2^3)
6	q^3	5	3
5	q^2	6	2
4	q	3	1
3	1	1	1
	$\langle 2\rangle^2$		$(2,2,1,1)$
7	q^5	9	3
6	q^4	13	1
5	q^3	12	2
4	q^2	7	1
3	q	3	1
2	1	1	1
	$\langle 4\rangle!$		$(2,1^4)$
10	q^4	5	1
9	q^3	4	0
8	q^2	3	1
7	q	2	0
6	1	1	1
	$\langle 6\rangle!$		(1^6)
15	1	1	1
			(7)
0	1	1	1
			$(6,1)$
1	q	2	0
1	$[1]$	4	0
0	1	5	1
			$(5,2)$
2	q^2	3	1
2	$q[1]$	8	0
2	$[1]^2$	1	1
2	$[2]$	2	0
1	q	10	0
1	$[1]$	12	0
0	1	8	2

M_λ			λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 2\rangle$		$(5,1,1)$
3	q^2	3	1
3	$q[1]$	9	1
3	$[1]^2$	3	1
2	q	8	0
2	$[1]$	12	0
1	1	10	2
			$(4,3)$
3	q^3	4	0
3	$q^2[1]$	8	0
3	$q[2]$	2	0
2	q^2	16	2
2	$q[1]$	12	0
1	q	14	0
0	1	1	1
			$(4,2,1)$
4	q^4	8	0
4	$q^3[1]$	19	1
4	$q^2[1]^2$	4	0
4	$q[1]^3$	2	0
4	$q^2[2]$	2	0
3	q^3	26	0
3	$q^2[1]$	44	0
3	$q[1]^2$	8	0
3	$q[2]$	4	0
2	q^2	37	1
2	$q[1]$	38	0
2	$[1]^2$	2	0
2	$[2]$	1	1
1	q	26	0
1	$[1]$	11	1
0	1	8	2

M_λ			λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 3\rangle!$		$(4,1^3)$
6	q^3	4	0
6	$q^2[1]$	12	0
6	$q[1]^2$	4	0
5	q^2	9	1
5	$q[1]$	18	0
5	$[1]^2$	3	1
4	q	12	0
4	$[1]$	12	0
3	1	10	2
	$\langle 2\rangle$		$(3,3,1)$
5	q^4	7	1
5	$q^3[1]$	9	1
5	$q\,(q^2-q+1)$		
	$\times[1]$	1	1
5	$q^2[1]^2$	4	0
4	q^3	22	0
4	$q^2[1]$	20	0
4	$q[1]^2$	5	1
3	q^2	22	0
3	$q[1]$	17	3
2	q	16	2
2	$[1]$	2	0
1	1	3	1
	$\langle 2\rangle$		$(3,2,2)$
6	q^5	10	2
6	$q^4[1]$	10	0
6	$q^3[1]^2$	1	1
5	q^4	27	1
5	$q^3[1]$	20	0
4	q^3	37	3
4	$q^2[1]$	10	0
3	q^2	23	1
3	$q[1]$	2	0
2	q	7	1
1	1	1	1

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 2\rangle$		$(3,2,1,1)$
7	q^6	15	1
7	$q^5[1]$	20	0
6	q^5	40	0
6	$q^4[1]$	42	0
5	q^4	54	2
5	$q^3[1]$	46	2
4	q^3	54	0
4	$q^2[1]$	26	0
3	q^2	30	2
3	$q[1]$	11	1
2	q	14	0
2	$[1]$	2	0
1	1	3	1
	$\langle 4\rangle!$		$(3,1^4)$
10	q^4	5	1
10	$q^3[1]$	10	2
9	q^3	8	0
9	$q^2[1]$	12	0
8	q^2	9	1
8	$q[1]$	9	1
7	q	8	0
7	$[1]$	4	0
6	1	5	1
	$\langle 3\rangle!$		$(2^3,1)$
9	q^6	14	0
8	q^5	28	2
7	q^4	28	0
6	q^3	20	2
5	q^2	10	2
4	q	4	0
3	1	1	1
	$\langle 3\rangle!\langle 2\rangle$		$(2,2,1^3)$
11	q^7	14	2
10	q^6	22	0
9	q^5	24	2
8	q^4	21	1
7	q^3	13	1
6	q^2	7	1
5	q	3	1
4	1	1	1

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 5\rangle!$		$(2,1^5)$
15	q^5	6	0
14	q^4	5	1
13	q^3	4	0
12	q^2	3	1
11	q	2	0
10	1	1	1
	$\langle 7\rangle!$		(1^7)
21	1	1	1
			(8)
0	1	1	1
			$(7,1)$
1	q	2	0
1	$[1]$	5	1
0	1	6	0
			$(6,2)$
2	q^2	3	1
2	$q[1]$	11	1
2	$[1]^2$	3	1
2	$[2]$	3	1
1	q	12	0
1	$[1]$	21	1
0	1	13	3
	$\langle 2\rangle$		$(6,1,1)$
3	q^2	3	1
3	$q[1]$	12	0
3	$[1]^2$	6	2
2	q	10	0
2	$[1]$	20	0
1	1	15	3

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
			$(5,3)$
3	q^3	4	0
3	$q^2[1]$	14	2
3	$q[1]^2$	3	1
3	$q[2]$	6	0
3	$[3]$	1	1
2	q^2	18	0
2	$q[1]$	37	1
2	$[2]$	9	1
1	q	26	0
1	$[1]$	19	3
0	1	8	0
			$(5,2,1)$
4	q^4	8	0
4	$q^3[1]$	30	0
4	$q^2[1]^2$	16	0
4	$q[1]^3$	4	0
4	$q^2[2]$	4	0
4	$q[1][2]$	2	0
3	q^3	32	0
3	$q^2[1]$	88	2
3	$q[1]^2$	32	0
3	$[1]^3$	1	1
3	$q[2]$	10	0
2	q^2	58	0
2	$q[1]$	96	0
2	$[1]^2$	10	0
2	$[2]$	2	0
1	q	52	0
1	$[1]$	30	2
0	1	18	0
	$\langle 3\rangle!$		$(5,1^3)$
6	q^3	4	0
6	$q^2[1]$	18	2
6	$q[1]^2$	12	0
6	$[1]^3$	1	1
5	q^2	12	0
5	$q[1]$	36	0
5	$[1]^2$	12	0
4	q	20	0
4	$[1]$	30	2
3	1	20	0

d	f_Λ/M_λ	N_r	N_s
	M_λ		λ
	$\langle 2\rangle$		$(4,4)$
4	q^3	4	2
4	$q^2[1]$	8	2
4	$q[2]$	2	2
3	q^2	16	2
3	$q[1]$	12	4
2	q	14	6
1	1	1	1

d	f_Λ/M_λ	N_r	N_s
			$(4,3,1)$
5	q^5	10	0
5	$q^4[1]$	27	1
5	$q^2\,(q^2-q+1)\times[1]$	2	0
5	$q^3[1]^2$	18	0
5	$q^2[1]^3$	5	1
5	$q^3[2]$	4	0
5	$q\,(q^2-2q+2)\times[1]^2$	2	0
5	$q^2[1][2]$	2	0
4	q^4	48	0
4	$q^3[1]$	100	0
4	$q\,(q^2-q+1)\times[1]$	2	0
4	$q^2[1]^2$	46	0
4	$q[1]^3$	8	0
4	$q^2[2]$	6	0
4	$q[1][2]$	6	0
3	q^3	88	0
3	$q^2[1]$	124	4
3	$q[1]^2$	44	0
3	$q[2]$	6	0
2	q^2	80	0
2	$q[1]$	84	0
1	q	50	0
1	$[1]$	3	1
0	1	4	0

d	f_Λ/M_λ	N_r	N_s
	M_λ		λ
	$\langle 2\rangle$		$(4,2,2)$
6	q^5	10	2
6	$q^4[1]$	26	2
6	$q^3[1]^2$	11	1
6	$q^2[1]^3$	3	1
6	$(q^3-2q^2+q-1)\times q[1]$	1	1
6	$q^3[2]$	3	1
6	$q^2[1][2]$	2	0
5	q^4	35	1
5	$q^3[1]$	79	1
5	$q^2[1]^2$	23	1
5	$q[1]^3$	6	2
5	$q^2[2]$	12	0
4	q^3	63	5
4	$q^2[1]$	100	2
4	$q[1]^2$	15	1
4	$q[2]$	10	2
3	q^2	60	2
3	$q[1]$	61	3
3	$[1]^2$	2	2
3	$[2]$	2	0
2	q	33	5
2	$[1]$	14	0
1	1	9	3

d	f_Λ/M_λ	N_r	N_s
	M_λ		λ
	$\langle 2\rangle$		$(4,2,1,1)$
7	q^6	15	1
7	$q^5[1]$	46	2
7	$q^4[1]^2$	18	0
7	$q^2\,(q^2-q+1)\times[1]^2$	1	1
7	$q^3[1]^3$	7	1
7	$q^4[2]$	3	1
6	q^5	52	0
6	$q^4[1]$	133	1
6	$q^3[1]^2$	46	0
6	$q^2[1]^3$	11	1
6	$q^3[2]$	8	0
5	q^4	93	3
5	$q^3[1]$	190	2
5	$q^2[1]^2$	48	2
5	$q[1]^3$	2	0
5	$q^2[2]$	12	2
4	q^3	116	0
4	$q^2[1]$	156	2
4	$q[1]^2$	22	0
4	$q[2]$	4	0
3	q^2	87	5
3	$q[1]$	72	0
3	$[1]^2$	5	1
3	$[2]$	1	1
2	q	44	0
2	$[1]$	19	1
1	1	13	3

d	f_Λ/M_λ	N_r	N_s
	$\langle 4\rangle!$		$(4,1^4)$
10	q^4	5	1
10	$q^3[1]$	20	0
10	$q^2[1]^2$	10	2
9	q^3	12	0
9	$q^2[1]$	36	0
9	$q[1]^2$	12	0
8	q^2	18	2
8	$q[1]$	36	0
8	$[1]^2$	6	2
7	q	20	0
7	$[1]$	20	0
6	1	15	3

M_λ			λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 2\rangle$		$(3,3,2)$
7	q^6	12	2
7	$q^5[1]$	18	2
7	$q^3\,(q^2-q+1)$		
	$\times[1]$	3	1
7	$q^4[1]^2$	9	1
6	q^5	54	2
6	$q^4[1]$	54	0
6	$q^3[1]^2$	18	2
5	q^4	81	3
5	$q^3[1]$	72	4
4	q^3	87	5
4	$q^2[1]$	18	0
3	q^2	36	2
3	$q[1]$	3	1
2	q	9	1
1	1	1	1

M_λ			λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 2\rangle^2$		$(3,3,1,1)$
8	q^6	12	2
8	$q^5[1]$	24	2
8	$q^3\,(q^2-q+1)$		
	$\times[1]$	2	0
8	$q^4[1]^2$	17	3
8	$q^3[1]^3$	1	1
7	q^5	42	2
7	$q^4[1]$	70	0
7	$q^2\,(q^2-q+1)$		
	$\times[1]$	4	0
7	$q^3[1]^2$	39	3
6	q^4	65	3
6	$q^3[1]$	105	3
6	$q\,(q^2-q+1)$		
	$\times[1]$	2	0
6	$q^2[1]^2$	32	2
6	$q[1]^3$	1	1
5	q^3	75	3
5	$q^2[1]$	78	0
5	$q[1]^2$	18	2
4	q^2	49	1
4	$q[1]$	45	3
4	$[1]^2$	1	1
3	q	31	3
3	$[1]$	6	0
2	1	6	2

M_λ			λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 2\rangle$		$(3,2,2,1)$
9	q^8	30	0
9	$q^7[1]$	38	2
9	$q^6[1]^2$	2	0
8	q^7	102	0
8	$q^6[1]$	110	0
8	$q^5[1]^2$	4	0
7	$q^4\,(2q-1)$		
	$\times[1]$	1	1
7	q^6	176	0
7	$q^5[1]$	154	6
7	$q^4[1]^2$	2	0
6	q^5	208	0
6	$q^4[1]$	132	2
5	q^4	168	0
5	$q^3[1]$	73	3
4	q^3	98	0
4	$q^2[1]$	30	2
3	q^2	44	0
3	$q[1]$	7	1
2	q	12	0
2	$[1]$	1	1
1	1	2	0
	$\langle 3\rangle!$		$(3,2,1^3)$
11	q^8	24	0
11	$q^7[1]$	40	0
10	q^7	66	0
10	$q^6[1]$	97	3
9	q^6	100	0
9	$q^5[1]$	132	0
8	q^5	122	0
8	$q^4[1]$	123	5
7	q^4	112	0
7	$q^3[1]$	80	0
6	q^3	74	0
6	$q^2[1]$	41	3
5	q^2	40	0
5	$q[1]$	16	0
4	q	18	0
4	$[1]$	3	1
3	1	4	0

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s

	$\langle 5\rangle!$		$(3,1^5)$
15	q^5	6	0
15	$q^4[1]$	15	3
14	q^4	10	0
14	$q^3[1]$	20	0
13	q^3	12	0
13	$q^2[1]$	18	2
12	q^2	12	0
12	$q[1]$	12	0
11	q	10	0
11	$[1]$	5	1
10	1	6	0

	$\langle 4\rangle!$		(2^4)
12	q^6	14	6
11	q^5	28	6
10	q^4	28	4
9	q^3	20	4
8	q^2	10	2
7	q	4	2
6	1	1	1

	$\langle 3\rangle!\langle 2\rangle$		$(2^3,1,1)$
13	q^9	28	4
12	q^8	64	2
11	q^7	88	6
10	q^6	87	3
9	q^5	68	4
8	q^4	45	3
7	q^3	24	2
6	q^2	11	3
5	q	4	0
4	1	1	1

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s

	$\langle 4\rangle!\langle 2\rangle$		$(2,2,1^4)$
16	q^9	20	4
15	q^8	33	1
14	q^7	39	3
13	q^6	39	1
12	q^5	33	3
11	q^4	22	2
10	q^3	13	1
9	q^2	7	1
8	q	3	1
7	1	1	1

	$\langle 6\rangle!$		$(2,1^6)$
21	q^6	7	1
20	q^5	6	0
19	q^4	5	1
18	q^3	4	0
17	q^2	3	1
16	q	2	0
15	1	1	1

	$\langle 8\rangle!$		(1^8)
28	1	1	1

			(9)
0	1	1	1

			$(6,3)$
3	q^3	4	0
3	$q^2[1]$	20	0
3	$q[1]^2$	10	0
3	$q[2]$	10	0
3	$[1][2]$	2	0
3	$[3]$	2	0
2	q^2	21	1
2	$q[1]$	66	0
2	$[1]^2$	10	2
2	$[2]$	20	0
1	q	38	0
1	$[1]$	54	0
0	1	21	3

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s

			$(5,4)$
4	q^4	5	1
4	$q^3[1]$	20	0
4	$q^2[1]^2$	5	1
4	$q^2[2]$	10	0
4	$q[3]$	2	0
3	q^3	30	0
3	$q^2[1]$	70	0
3	$q[2]$	20	0
2	q^2	60	4
2	$q[1]$	50	0
1	q	30	0
0	1	1	1

	$\langle 4\rangle!$		$(5,1^4)$
10	q^4	5	1
10	$q^3[1]$	30	2
10	$q^2[1]^2$	30	2
10	$q[1]^3$	5	1
9	q^3	16	0
9	$q^2[1]$	72	0
9	$q[1]^2$	48	0
9	$[1]^3$	4	0
8	q^2	30	2
8	$q[1]$	90	2
8	$[1]^2$	30	2
7	q	40	0
7	$[1]$	60	0
6	1	35	3

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 5\rangle !$		$(4,1^5)$
15	q^5	6	0
15	$q^4[1]$	30	0
15	$q^3[1]^2$	20	0
14	q^4	15	1
14	$q^3[1]$	60	0
14	$q^2[1]^2$	30	2
13	q^3	24	0
13	$q^2[1]$	72	0
13	$q[1]^2$	24	0
12	q^2	30	2
12	$q[1]$	60	0
12	$[1]^2$	10	2
11	q	30	0
11	$[1]$	30	0
10	1	21	3
	$\langle 3\rangle !$		(3^3)
9	q^6	12	2
9	$q^5[1]$	18	2
9	$q^3\,(q^2-q+1)$ $\times[1]$	3	1
9	$q^4[1]^2$	9	1
8	q^5	54	2
8	$q^4[1]$	54	0
8	$q^3[1]^2$	18	2
7	q^4	81	3
7	$q^3[1]$	72	4
6	q^3	87	5
6	$q^2[1]$	18	0
5	q^2	36	2
5	$q[1]$	3	1
4	q	9	1
3	1	1	1

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 4\rangle !$		$(2^4,1)$
16	q^{10}	42	2
15	q^9	120	0
14	q^8	180	4
13	q^7	195	3
12	q^6	165	3
11	q^5	117	5
10	q^4	70	2
9	q^3	35	3
8	q^2	15	1
7	q	5	1
6	1	1	1
	$\langle 9\rangle !$		(1^9)
36	1	1	1
			(10)
0	1	1	1
	$\langle 2\rangle$		$(5,5)$
5	q^4	5	1
5	$q^3[1]$	20	4
5	$q^2[1]^2$	5	1
5	$q^2[2]$	10	2
5	$q[3]$	2	2
4	q^3	30	2
4	$q^2[1]$	70	6
4	$q[2]$	20	4
3	q^2	60	6
3	$q[1]$	50	10
2	q	30	6
1	1	1	1

	M_λ		λ
d	f_Λ/M_λ	N_r	N_s
	$\langle 5\rangle !$		$(5,1^5)$
15	q^5	6	0
15	$q^4[1]$	45	3
15	$q^3[1]^2$	60	0
15	$q^2[1]^3$	15	3
14	q^4	20	0
14	$q^3[1]$	120	0
14	$q^2[1]^2$	120	0
14	$q[1]^3$	20	0
13	q^3	40	0
13	$q^2[1]$	180	4
13	$q[1]^2$	120	0
13	$[1]^3$	10	2
12	q^2	60	0
12	$q[1]$	180	0
12	$[1]^2$	60	0
11	q	70	0
11	$[1]$	105	3
10	1	56	0
	$\langle 10\rangle !$		(1^{10})
45	1	1	1

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