SEMI-PRIMARY QF-3 RINGS

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A ring R (with identity) is *semi-primary* if it contains a nilpotent ideal N with R/N semi-simple with minimum condition. R is called a left QF-3 ring if it contains a faithful projective injective left ideal. If R is semi-primary and left QF-3, then there is a faithful projective injective left ideal of R which is a direct summand of every faithful left R-module [5], in agreement with the definition of QF-3 algebra given by R.M. Thrall [6]. Let $Q(M)$ denote the injective envelope of a (left) R-module M. We call R left QF-3^+ if $Q(R)$ is projective. J.P. Jans showed that among rings with minimum condition on left ideals, the classes of QF-3 and QF-3^+ rings coincide [5].

In this note we determine the class of semi-primary rings in which the notions of QF-3 and QF-3^+ coincide. Next, we show that the characterization of QF-3^+ rings given by Wu, Mochizuki, and Jans [7] for rings with the property that direct products of projective modules are projective, can be used to characterize semi-primary QF-3 rings. Finally, we give some results relating the notions of torsionless and torsion-free modules as defined by H. Bass [1] and A.W. Goldie [3]. In particular we show that if R is semi-primary, these notions coincide if and only if R is left QF-3 and has zero left singular ideal.

S. Eilenberg has given the following characterization of projective modules for semi-primary rings [2].

PROPOSITION 1. *If R is semi-primary and P is a projective R-module, then $P = \oplus \sum P_\alpha$ where each P_α is isomorphic to an indecomposable direct summand of ${}_RR$.*

PROPOSITION 2. *If R is semi-primary then R is left QF-3^+ if and only if R is left QF-3 and the left socle of R is the direct sum of a finite number of simple left ideals of R.*

Received June 12, 1967.

Proof. If R is left QF-3^+, then $Q(R) \cong \oplus \sum P_\alpha$ where each P_α is an indecomposable direct summand of ${}_R R$. Since the restriction of this isomorphism to R is given by multiplication of an element of $\oplus \sum P_\alpha$, the image of R is contained in the sum of only finitely many summands. Since R is essential in $Q(R)$, the sum, $\oplus \sum P_\alpha$, is a finite sum. Since each P_α is indecomposable and injective, the socle of each P_α is simple so the socle of R is the direct sum of a finite number of simple left ideals. Also, if $P_{\alpha_1}, \cdots, P_{\alpha_t}$ is one of each isomorphism class of the P_α, then $P_{\alpha_1} \oplus \cdots \oplus P_{\alpha_t}$ is a faithful projective injective left ideal of R. Conversely, suppose R is left QF-3 and the socle of R has the form $S_1 \oplus \cdots \oplus S_t$ with each S_k simple. Then $Q(S_1) \oplus \cdots \oplus Q(S_t) = Q(R)$. Let I be a faithful projective injective left ideal of R. For each i, $S_i I \neq (0)$ so S_i is isomorphic to a submodule of I. Since I is injective, $Q(S_i)$ is isomorphic to a direct summand of I so is projective. Hence $Q(R)$ is projective.

EXAMPLE. Let D and D_1 be division rings and let M be a (D, D_1)-bimodule such that $[M : D] = \infty$. Let

$$R = \left\{ \begin{pmatrix} d & 0 & 0 \\ x & d_1 & 0 \\ d'' & m & d' \end{pmatrix} \middle| \, d, d', d'' \in D, \; m \in M, \; x \in \operatorname{Hom}_D(M, D) \right\}.$$

M. Harada has shown that R is semi-primary and left QF-3 but is not right QF-3 [4]. One computes that the left socle of R consists of all elements of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d'' & m & d' \end{bmatrix}$$

and is an infinite direct sum of simple left ideals. Hence R is not left QF-3^+.

An R-module M is *torsionless* if for every $x \in M$, there exists $f \in \operatorname{Hom}_R(M, R)$ such that $f(x) \neq 0$ [1]. Denote the class of all torsionless left R-modules by $\mathfrak{L}$ and the class of all left R-modules M with $\operatorname{Hom}_R(M, R) = 0$ by $\mathfrak{T}$. Then $\mathfrak{L}$ is closed under taking submodules and direct products and $\mathfrak{T}$ is closed under taking factors, extensions by elements of $\mathfrak{T}$, and direct sums. Also, any element of $\mathfrak{L}$ is isomorphic to a submodule of a direct product of copies of R.

THEOREM 1. *The following are equivalent:*

1. $\mathfrak{L}$ *is closed under taking essential extensions.*
2. a) $\mathfrak{L}$ *is closed under taking extensions by elements of* $\mathfrak{L}$, *and*

   b) $\mathfrak{T}$ *is closed under taking submodules.*

Proof. Assume condition 1 and suppose that B is an extension of A by C with $A, C \in \mathfrak{L}$. By 1, $Q(A)$, $Q(C) \in \mathfrak{L}$ and since $Q(A)$ is injective we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \downarrow & \overset{\mu}{\swarrow} & \downarrow{\scriptstyle\lambda} & & \downarrow & & \\ 0 & \longrightarrow & Q(A) & \longrightarrow & Q(A) \oplus Q(C) & \longrightarrow & Q(C) & \longrightarrow & 0, \end{array}$$

where λ is given by $\lambda(b) = (\mu(b), \pi(b))$. Since $\mathfrak{L}$ is closed under taking direct products and submodules, $B \in \mathfrak{L}$. Next suppose $A \subset B$ and $0 \neq f \in \operatorname{Hom}_R(A, R)$. Form the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ 0 & \longrightarrow & R & \longrightarrow & Q(R), \end{array}$$

where $\bar{f}$ exists since $Q(R)$ is injective. By 1, $Q(R) \in \mathfrak{L}$ and so $Q(R)$ is a submodule of a direct product of copies of R. Hence $\bar{f} \neq 0$ implies $\operatorname{Hom}_R(B, R) \neq 0$. Conversely, assume condition 2, let $A \in \mathfrak{L}$, and suppose B is an essential extension of A. Let

$$K = \bigcap_{f \in \operatorname{Hom}(B, R)} \operatorname{Ker} f, \text{ and } K' = \bigcap_{g \in \operatorname{Hom}(K, R)} \operatorname{Ker} g.$$

Then the sequence

$$0 \longrightarrow K/K' \longrightarrow B/K' \longrightarrow B/K \longrightarrow 0,$$

is exact with K/K', $B/K \in \mathfrak{L}$. By 2a, $B/K' \in \mathfrak{L}$. It follows that $K' = K$ so $K \in \mathfrak{T}$. If $K \neq (0)$, then since A is essential in B, $(0) \neq K \cap A \in \mathfrak{T}$ by 2b, contradicting $\mathfrak{L} \cap \mathfrak{T} = (0)$. Hence $K = (0)$ and $B \in \mathfrak{L}$.

Wu, Mochizuki, and Jans [7] proved that for rings with the property that direct products of projective modules are projective, R is left QF-3^+ if and only if condition 2 of Theorem 1 holds. In order to prove the corresponding result for semi-primary rings, one must replace QF-3^+ by QF-3.

THEOREM 2. *Let R be a semi-primary ring. The following are equivalent:*

1. $\mathfrak{L}$ *is closed under taking essential extensions.*
2. R *is left* QF-3.
3. $Q(R) \in \mathfrak{L}$.

Proof. Assume condition 1 and let S be a simple left ideal of R. Then $Q(S) \in \mathfrak{L}$. Let $f \in \operatorname{Hom}_R(Q(S), R)$ with $f(S) \neq 0$. Since $\operatorname{Ker} f \cap S = 0$ and S is essential in $Q(S)$, $\operatorname{Ker} f = 0$. Hence $Q(S)$ is isomorphic to a direct summand of R and so is projective. It follows that R is left QF-3 with faithful projective injective left ideal $Q(S_1) \oplus \cdots \oplus Q(S_t)$ where $S_1, \cdots, S_t$ is one of each isomorphism class of simple left ideals of R. Next assume that R is left QF-3. The injective envelope of each simple left ideal of R is projective and hence torsionless. Thus $Q(R)$ is a submodule of a direct product of torsionless modules so is torsionless. Finally, assume condition 3 and let $A \in \mathfrak{L}$. There exists a monomorphism

$$k : A \longrightarrow \Pi R.$$

If B is an essential extension of A, then B is isomorphic to a submodule of $\Pi Q(R)$. Since $\mathfrak{L}$ is closed under taking direct products and submodules, $B \in \mathfrak{L}$.

If M is a left R-module, $Z(M) = Z^1(M) = \{x \in M \mid Ix = 0$ for some essential left ideal I of $R\}$ is the *singular submodule* of M. Inductively, $Z^{n+1}(M) = \{x \in M \mid Ix \subset Z^n(M)$ for some essential left ideal I of $R\}$. For any ring, $Z^2(M) = Z^3(M)$ and if $Z(R) = 0$, $Z(M) = Z^2(M)$ [3]. Clearly, $Z^2(M) = 0$ if and only if $Z(M) = 0$. We define classes $\mathfrak{L}_1$ and $\mathfrak{T}_1$ by $\mathfrak{L}_1 = \{M \mid Z(M) = 0\}$ and $\mathfrak{T}_1 = \{M \mid Z^2(M) = M\}$. $\mathfrak{L}_1$ is closed under taking submodules, direct products, extensions by elements of $\mathfrak{L}_1$, and essential extensions. $\mathfrak{T}_1$ is closed under taking factors, submodules, and direct sums.

PROPOSITION 3. *If R is semi-primary then $M \in \mathfrak{L}_1$ if and only if the socle of M is projective.*

Proof. Let E denote the left socle of R. Then E is the unique minimal essential left ideal of R. Suppose $M \in \mathfrak{L}_1$ and let C be a simple submodule of M. Since $EC \neq 0$, there exists a simple left ideal $S \subset R$ with $SC \neq 0$. Then $SC = C$ so $S^2 \neq 0$ and $S \cong C$. Hence S is a direct summand of R and so is projective. Conversely, suppose the socle of M is projective. If $EC = 0$ for a simple submodule C of M, then C is not isomorphic to a direct summand of R and is not projective. Hence $EC \neq 0$. But if $Z(M) \neq 0$, it contains a simple submodule. Thus $Z(M) = 0$.

COROLLARY. *If R is semi-primary and left QF-3, then $\mathfrak{L}_1 \subset \mathfrak{L}$ and $\mathfrak{T} \subset \mathfrak{T}_1$.*

Proof. If $M \in \mathfrak{L}_1$ then the socle of M is projective by Proposition 3 so is in $\mathfrak{L}$. Thus, by Theorem 2, $M \in \mathfrak{L}$. If $M \in \mathfrak{T}$, then $M/Z^2(M) \in \mathfrak{L}_1 \subset \mathfrak{L}$ so, since $\mathfrak{T}$ is closed under taking homomorphic images and $\mathfrak{T} \cap \mathfrak{L} = (0)$, $M = Z^2(M)$. Hence $M \in \mathfrak{T}_1$.

PROPOSITION 4. *The following are equivalent.*

1. $R \in \mathfrak{L}_1$.
2. $\mathfrak{L} \subset \mathfrak{L}_1$.
3. $\mathfrak{T}_1 \subset \mathfrak{T}$.

Proof. Condition 2 follows from 1 since $\mathfrak{L}_1$ is closed under taking direct products and submodules and any torsionless R-module is a submodule of a direct product of copies of R. Assume condition 2. If $M \in \mathfrak{T}_1$, then since $R \in \mathfrak{L} \subset \mathfrak{L}_1$, $Z^2(M) = Z(M) = M$. If $f \in \mathrm{Hom}\,(M, R)$ and $x \in M$, let I be an essential left ideal of R with $Ix = 0$. Then $If(x) = f(Ix) = 0$ so since $Z(R) = 0$, $f(x) = 0$. Hence $M \in \mathfrak{T}$. Finally, if $Z(R) \neq 0$ then $Z^2(R) \neq 0$ and $Z^2(R) \in \mathfrak{T}_1$ but $Z^2(R) \notin \mathfrak{T}$. Thus 3 implies 1.

THEOREM 3. *Let R be a semi-primary ring. The following are equivalent:*

1. *R is left QF-3 and $Z(R) = 0$.*
2. $\mathfrak{L} = \mathfrak{L}_1$.
3. $\mathfrak{T} = \mathfrak{T}_1$.

Proof. Condition 1 implies conditions 2 and 3 by the Corollary and Proposition 4. Assume condition 2. Since $\mathfrak{L}_1$ is closed under taking essential extensions and $\mathfrak{L}_1 = \mathfrak{L}$, R is left QF-3 by Theorem 2 and $Z(R) = 0$ by Proposition 4. Thus 2 implies 1. Assume condition 3. By Proposition

4, $Z(R)=0$. Let S be a simple left ideal of R. Since $Z(S)=0$ and S is essential in $Q(S)$, $Z(Q(S))=0$. Hence $\mathrm{Hom}_R(Q(S), R)\neq 0$. Let $0\neq f\in \mathrm{Hom}_R(Q(S), R)$. If $\mathrm{Ker}\, f\neq 0$ then $\mathrm{Ker}\, f$ is essential in $Q(S)$ so $Q(S)/\mathrm{Ker}\, f \in \mathfrak{T}_1=\mathfrak{T}$. From this contradiction we conclude that f is a monomorphism and $Q(S)$ is torsionless. Hence $Q(R)\in \mathfrak{L}$ and R is left QF-3 by Theorem 2.

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