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Semi-pseudo Ricci symmetric manifold

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Abstract

Semi-pseudo Ricci symmetric manifold has been defined and studied.

Key words: Semi-pseudo Ricci symmetric manifold $(SPRS)_n$, Einstein $(SPRS)_n$, conformal curvature tensor of $(SPRS)_n$, quarter symmetric metric connection on $(SPRS)_n$.

1. Introduction

In a recent paper[1], Chaki introduced pseudo-Ricci symmetric manifold $(PRS)_n$, *i.e.*, non-flat n-dimensional Riemannian manifold whose Ricci tensor s satisfies

$$(\nabla_x s)(y,z) = 2\pi(x)s(y,z) + \pi(y)s(x,z) + \pi(z)s(x,y)$$

where π is a 1-form, ρ is a particular vector field such that

$$\pi(x) = g(x, \rho)$$

and ∇ is the covariant differentiation.

Consider a non-flat n-dimensional Riemannian manifold with its metric g, whose Ricci tensor s is such that

$$(\nabla_x s)(y,z) = \pi(y)s(x,z) + \pi(z)s(x,y) \tag{1}$$

where ∇, ρ and π are already defined. Such a manifold shall be called semi-pseudo Ricci symmetric n-dimensional manifold and will be denoted by $(SPRS)_n$.

The existence of such a structure on a Riemannian manifold is first established. It is shown that, on such an $(SPRS)_n$, the scalar curvature is zero. Some conditions satisfied by the Ricci tensor with respect to the vector ρ are established and it is shown that an $(SPRS)_n$ cannot be conformally flat. Also, a particular type of quarter

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symmetric metric connection $\bar{D}$ has been introduced on $(SPRS)_n$. The curvature tensor $\bar{R}$, the Ricci tensor $\bar{S}$ and the scalar curvature tensor $\bar{r}$ with respect to $\bar{D}$ have been derived in the last section.

2. Existence of an $(SPRS)_n$

For the existence of such structure, defined in (1), consider a Riemannian manifold M^n with metric tensor g which admits a linear connection D defined by

$$D_x y = \nabla_x y + \pi(y)x \tag{2}$$

and

$$(D_x s)\ (y,z) = 0. \tag{3}$$

Then, from (2) and (3), we can have,

$$(\nabla_x s)\ (y,z) = \pi(y)s(x,z) + \pi(z)s(x,y). \tag{4}$$

Hence, $\nabla s \neq 0$, since D is not identical at ∇. Therefore, structure (1) exists on a Riemannian manifold if it admits a linear connection which satisfies (2) and (3).

3. Preliminaries for $(SPRS)_n$

From (1), we can have

$$(\nabla_x s)(y,z) - (\nabla_y s)(x,z) = \pi(y)s(x,z) - \pi(x)s(y,z). \tag{5}$$

Contracting (5), with respect to y and z, we get

$$dr(x) = 2\pi(s'x) - 2\pi(x)r \tag{6}$$

where s' is the symmetric endomorphism of the tangent space at each point of (M^n, g) corresponding to the Ricci tensor s.

Next, contracting (1) with respect to y and z we get

$$dr(x) = 2\pi\ (s'x). \tag{7}$$

Hence, from (6) and (7), we get

$$\pi(x)r = 0.$$

Hence, $r=0$, since $\pi(x) \neq 0$.

Thus, we can state

Theorem 1: The scalar curvature is zero on $(SPRS)_n$

4. Ricci tensor and the vector ρ on an $(SPRS)_n$

Since $r=0$ on $(SPRS)_n$, we get from (6),

$$\pi(s'x) = 0. \tag{8}$$

Hence,

$$g(s'x,\ \rho) = 0,$$

that is

$$s(x,\rho) = 0, \tag{9}$$

Now,

$$(\nabla_x s)\ (y,z) = x\ s(y,z) - s(\nabla_x y,z) - s(y,\nabla_x z).$$

Taking $z=\rho$ in the above equation, we get by virtue of (9)

$$(\nabla_x s)(y,\rho) = -s(y,\nabla_x \rho).$$

By virtue of (1) the above equation takes the form

$$\pi(\rho)s(x,y) + s(y,\nabla_x\rho) = 0. \tag{10}$$

Now, let ρ be a torse-forming vector field[5] given by

$$\nabla_x\rho = ax + \omega(x)\rho \tag{11}$$

where a is a non-zero scalar and ω is a 1-form.

By virtue of (10) one can have

$$\{a + \pi(\rho)\}\ s(x,y) = 0. \tag{12}$$

Since $s \neq 0$ it follows that

$$a + \pi(\rho) = 0.$$

Thus, we can state,

Theorem 2: If on an $(SPRS)_n$ the vector ρ is a torse-forming vector field given by (11), then, the scalar a must be equal to $-\pi(\rho)$.

5. Einstein $(SPRS)_n$

It is known that in an Einstein space (M^n,g) $(n>2)$ the scalar curvature r is constant and the Ricci tensor is given by

$$s(x,y) = \frac{r}{n}\ g(x,y).$$

Since on $(SPRS)_n$, $r = 0$, we have from above

$$s(x,y) = 0$$

which contradicts the hypothesis of the definition of $(SPRS)_n$. Thus, we state,

Theorem 3: An $(SPRS)_n$ $(n>2)$ cannot be an Einstein manifold.

6. Conformal curvature tensor of $(SPRS)_n$

It is known[2] that in a conformally flat manifold

$$(\nabla_x s)(y,z) - (\nabla_z s)(x,y) = \frac{1}{n(n-1)} \{dr(x)g(y,z) - dr(z)g(x,y)\}.$$

Using Theorem 1, we get

$$(\nabla_x s)(y,z) - (\nabla_z s)(x,y) = 0.$$

Thus, the Ricci tensor is of Codazzi type[2].

By virtue of (1), one gets from the above

$$\pi(z)s(x,y) = \pi(x)s(z,y).$$

Taking $x=\rho$, in the above equation, we get on using (9)

$$\pi(\rho)s(y,z) = 0.$$

Since $\pi(\rho)\neq 0$, we have $s=0$. Thus, we can state,

Theorem 4: An $(SPRS)_n$ $(n>3)$ cannot be conformally flat.

Theorem 5: The Ricci tensor of $(SPRS)_n$ $(n>3)$ cannot be of Codazzi type.

Further, it is known[2] that on a Riemannian manifold

$$(\text{div } c)\,(x,y,z) = \frac{n-3}{n-2}\{(\nabla_x s)(y,z) - (\nabla_z s)(y,x)\} + $$
$$+ \frac{1}{n(n-1)}\{g(x,y)dr(z) - g(y,z)dr(x)\}$$

where c is the conformal curvature tensor of the manifold.

Now, if the conformal curvature tensor of the manifold is conservative[3], then since $r=0$ in $(SPRS)_n$, we have,

$$(\nabla_x s)(y,z) - (\nabla_z s)(y,x) = 0.$$

Using Theorem 5, we can state,

Theorem 6: An $(SPRS)_n$ cannot be of conservative conformal curvature tensor.

7. Quarter symmetric metric connection on $(SPRS)_n$

Consider a Riemannian manifold M^n with its Levi–Civita connection ∇ and quarter symmetric metric connection[4] $\bar{D}$. Then, the torsion tensor $\bar{T}$ is given by

$$\bar{T}(x,y) = \pi(y)s'x - \pi(x)s'y. \tag{13}$$

Let,

$$\bar{D}_x y = \nabla_x y + H(x,y); \tag{14}$$

then, since $(\bar{D}_x g)(y,z) = 0$, we can have

$$g(H(x,y),z) + g(H(x,z),y) = 0. \tag{15}$$

From (13) and (14), one can have

$$H(x,y) - H(y,x) = \pi(y)s'x - \pi(x)s'y. \tag{16}$$

It is easy to see from (15) and (16) that

$$H(x,y) = \pi(y)s'x - s(x,y)\rho$$

So that, from (14) one can write

$$\bar{D}_x y = \nabla_x y + \pi(y)s'x - s(x,y)\rho. \tag{17}$$

Let

$$\bar{R}\ (x,y,z) = \bar{D}_x\bar{D}_y z - \bar{D}_y\bar{D}_x z - \bar{D}_{[x,y]}z$$

be the curvature tensor with respect to the quarter symmetric metric connection $\bar{D}$. Then from (17) one can have

$$\begin{aligned}\bar{R}(x,y,z) = {} & R(x,y,z) + (\nabla_x\pi)(z)s'y - (\nabla_y\pi)(z)s'x + \\ & + \pi(z)\{(\nabla_x s')y - (\nabla_y s')x\} - \{(\nabla_x s)(y,z) - (\nabla_y s)(x,z)\}\rho \\ & - s(y,z)\{\nabla_x\rho + \pi(\rho)s'x\} + s(x,z)\{\nabla_y\rho + \pi(\rho)s'y\}.\end{aligned}$$

Using (5) and also the relation

$$(\nabla_x s')y - (\nabla_y s')x = \pi(y)s'x - \pi(x)s'y$$

we get from above

$$\begin{aligned}\bar{R}(x,y,z) = {} & R(x,y,z) + \{(\nabla_x\pi)(z) - \pi(x)\pi(z) + \tfrac{1}{2}s(x,z)\pi(\rho)\}s'y \\ & -\{(\nabla_y\pi)(z) - \pi(y)\pi(z) + \tfrac{1}{2}s(y,z)\pi(\rho)\}s'x + \\ & + s(x,z)\{\nabla_y\rho - \pi(y)\rho + \tfrac{1}{2}\pi(p)s'y\} - s(y,z)\ \{\nabla_x\rho - \\ & - \pi(x)\rho + \tfrac{1}{2}\pi(\rho)s'x\}.\end{aligned}$$

Let us write

$$\lambda(x,z) = (\nabla_x\pi)(z) - \pi(x)\pi(z) + \tfrac{1}{2}s(x,z)\pi(\rho) = g(Lx,z). \tag{18}$$

Hence, we can have

$$\bar{R}(x,y,z) = R(x,y,z) + \lambda(x,z)s'y - \lambda(y,z)s'x + s(x,z)Ly - s(y,z)Lx. \tag{19}$$

Contracting (19) with respect to x, we get, on using Theorem 1,

$$\bar{s}(y,z) = s(y,z) + \lambda(s'y,z) + \lambda(y,s'z) - a\ s(y,z) \tag{20}$$

where

$$a = \text{trace } L = \text{div } \pi + \frac{r-2}{2}\pi(\rho). \tag{21}$$

Contracting (20) and using Theorem 1, we get

$$\bar{r} = r + \lambda(s'x,x) + \lambda(x,s'x). \tag{22}$$

Using (8), we get from (18)

$$\lambda(x,s'y) = (\nabla_x\pi)s'y + \frac{1}{2}\pi(\rho)s(x,s'y)$$

$$\lambda(s'x,y) = (\nabla_{s'x}\pi)y + \frac{1}{2}\pi(\rho)s(s'x,y).$$

Also, on using (4) and (8), we get

$$(\nabla_x\pi)s'y = -\pi(\rho)s(x,y).$$

Consequently (22) reduces to, as

$$\bar{r} = (\nabla_{s'x}\pi)x + \pi(\rho)s(x,s'x). \tag{23}$$

Thus, we can state

Theorem 7: If an $(SPRS)_n$ admits a quarter symmetric metric connection $\bar{D}$, then we have (19), (20) and (23).

Theorem 8: On an $(SPRS)_n$ with quarter symmetric metric connection $\bar{D}$, the necessary and sufficient condition for $\lambda(x,y)$ defined by (18) to be symmetric is that π be closed.

Theorem 9: On an $(SPRS)_n$ with quarter symmetric metric connection $\bar{D}$ the necessary and sufficient condition for $\bar{R} = R$ is that

$$\lambda(x,z)s'y - \lambda(y,z)s'x + s(x,z)Ly - s(y,z)Lx = 0.$$

Corollary 1: On an $(SPRS)_n$ with a quarter symmetric metric connection $\bar{D}$, if $\bar{R} = R$ then, we have

$$a\ s(y,z) = \lambda(s'y,z) + \lambda(y,s'z)$$

$$\lambda(x,s'x) + \lambda(s'x,x) = 0, \text{ and}$$

$$(\nabla_{s'x}\pi)x = -\pi(p)s(x,s'x).$$

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References

1. Chaki, M. C. On pseudo Ricci symmetric manifolds, *Bulg. J. Phys.*, 1988, **15**, 6.
2. Eisenhart, L. P. *Riemannian geometry*, 1967, Princeton University Press.
3. Hicks, N. J. *Notes on differential geometry*, 1969, Affiliated East West Press.
4. Mishra, R. S. *Structures on a differentiable manifold and their application*, 1984, Chandrama Prakashan, Allahabad.
5. Schouten, J. A. *Ricci calculus*, 1954, Springer–Verlag.