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# SEMI-SIMPLICIAL COMPLEXES AND SINGULAR HOMOLOGY

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A simplicial complex has the following two properties: (a) each $q$-simplex determines $q + 1$ faces of dimension $q - 1$, (b) the faces of a simplex determine the simplex.

Recent work in singular homology theory of topological spaces [1] and homology and cohomology theory of abstract groups [2] have led to abstract complexes which satisfy (a) without satisfying (b). We shall give a general definition of this class of complexes and show how the various constructions of homology theory (including homology with local coefficients, cup-products, etc.) can be carried out just as for simplicial complexes.

The chief example of such a "semi-simplicial" complex is the singular complex $S(X)$ of a topological space $X$. Although this complex is very "large" it is possible to find subcomplexes of $S(X)$ which contain all the information that $S(X)$ carries but which are stripped of everything superfluous from the point of view of homotopy. The existence and uniqueness of such minimal subcomplexes is established. These minimal complexes are the main tool in the paper of S. Eilenberg and S. MacLane [4] immediately following.

## 1. Semi-simplicial complexes

A semi-simplicial complex $K$ is a collection of elements $\{\sigma\}$ called *simplexes* together with two functions. The first function associates with each simplex $\sigma$ an integer $q \geqq 0$ called the dimension of $\sigma$; we then say that $\sigma$ is a $q$-simplex. The second function associates with each $q$-simplex $\sigma$ $(q > 0)$ of $K$ and with each integer $0 \leqq i \leqq q$ a $(q-1)$-simplex $\sigma^{(i)}$ called the $i^{\text{th}}$ *face* of $\sigma$, subject to the condition

$$[\sigma^{(j)}]^{(i)} = [\sigma^{(i)}]^{(j-1)} \tag{1.1}$$

for $i < j$ and $q > 1$.

We observe that this definition does not exclude the possibility of two distinct $q$-simplexes $\sigma$ and $\tau$ with $\sigma^{(i)} = \tau^{(i)}$ for $i = 0, \cdots, q$.

We may pass to lower dimensional faces of $\sigma$ by iteration. If $0 \leqq i_1 < \cdots < i_n \leqq q$ then we define inductively

$$\sigma^{(i_1, \cdots, i_n)} = [\sigma^{(i_2, \cdots, i_n)}]^{(i_1)}.$$

This is a $(q - n)$-simplex. If $0 \leqq j_0 < \cdots < j_{q-n} \leqq q$ is the set complementary to $\{i_1, \cdots, i_n\}$ then we also write

$$\sigma^{(i_1, \cdots, i_n)} = \sigma_{(j_0, \cdots, j_{q-n})}.$$

In particular $\sigma_{(i)}$ for $0 \leqq i \leqq q$ is a 0-simplex called the $i^{\text{th}}$ vertex of $\sigma$. We shall also refer to $\sigma_{(0)}$ as the *leading vertex* and $\sigma_{(0,1)}$ as the *leading edge*.

A *subcomplex* $L$ of $K$ is a subcollection of simplexes of $K$ with the property that if $\sigma \in L$ then all the faces of $\sigma$ are in $L$.

The group of $q$-dimensional (integral) chains $C_q(K)$ is defined as the free abelian group with the $q$-simplexes of $K$ as free generators. The *boundary homomorphism*

$$\partial : C_q(K) \to C_{q-1}(K)$$

is defined by setting for each generator

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \sigma^{(i)}.$$

One verifies readily that $\partial\partial = 0$. This leads directly to the definition of cycles, boundaries, and of the homology group $H_q(K)$. Following the usual procedure we may also define homology groups $H_q(K, G)$ with an arbitrary abelian coefficient group $G$ as well as relative homology groups $H_q(K, L, G)$ modulo a subcomplex $L$.

A $q$-dimensional cochain $f \in C^q(K, G)$ may be defined either as a homomorphism $f: C_q(K) \to G$ or as a function $f(\sigma)$ defined on the set of $q$-simplexes of $K$ with values in $G$. The coboundary is defined by

$$(\delta f)(\sigma) = \sum_{i=0}^{q+1} (-1)^i f(\sigma^{(i)})$$

for each $(q + 1)$-simplex $\sigma$ of $K$. This leads to cohomology groups $H^q(K, G)$. Relative cohomology groups $H^q(K, L, G)$ are obtained by considering cochains which are zero on every simplex of the subcomplex $L$.

The cup-products for cohomology may be defined by the Alexander formula just as in the case of a simplicial complex with ordered vertices. Let the groups $G_1$ and $G_2$ be paired to the group $G$ and let cochains $f_1 \in C^p(K, G_1)$, $f_2 \in C^q(K, G_2)$ be given. Define the cochain $f_1 \smile f_2 \in C^{p+q}$ $(G)$ by setting

$$(f_1 \smile f_2)(\sigma) = f_1(\sigma_{(0,\dots,p)})f_2(\sigma_{(p,\dots,p+q)})$$

for every $(p + q)$-simplex $\sigma$ of $K$. The usual coboundary formula

$$\delta(f_1 \smile f_2) = (\delta f_1) \smile f_2 + (-1)^p f_1 \smile \delta f_2$$

is then valid, and the pairing of the cohomology groups $H^p(K, G_1)$ and $H^q(K, G_2)$ to the group $H^{p+q}(K, G)$ is defined.

Following the same procedure all the formal definitions and results of the $\smile_i$-product theory of Steenrod [6] can be carried out in a semi-simplicial complex.

A simplicial map $T: K \to K_1$ of a semi-simplicial complex $K$ into another such complex $K_1$ is a function which to each $q$-simplex $\sigma$ of $K$ assigns a $q$-simplex $\tau = T(\sigma)$ of $K_1$ in such a fashion that $\tau^{(i)} = T(\sigma^{(i)})$, $i = 0, \cdots, q$. Clearly $T$ induces homomorphisms of the homology groups of $K$ into those of $K_1$ and of the cohomology groups of $K_1$ into those of $K$. The latter homomorphisms preserve the products.

## 2. Local coefficients

Steenrod's theory of homology and cohomology with local coefficients [5] can be built very conveniently in a semi-simplicial complex. Let $K$ be such a complex. For the sake of brevity we shall discuss cohomology only.

A local system $G = \{G(\alpha), \gamma(\beta)\}$ of groups in $K$ consists of two functions; the first assigns to each vertex (i.e. 0-simplex) $\alpha$ of $K$ a group $G(\alpha)$, the second assigns to each edge (i.e. 1-simplex) $\beta$ of $K$ an isomorphism

$$\gamma(\beta): G(\beta_{(1)}) \to G(\beta_{(0)})$$

subject to the condition

$$\gamma(\sigma_{(0,1)})\gamma(\sigma_{(1,2)}) = \gamma(\sigma_{(0,2)})$$

for each 2-simplex $\sigma$.

Let $G = \{G(\alpha), \gamma(\beta)\}$ be a local system of abelian groups in $K$. A $q$-cochain $f$ of $K$ over $G$ is a function which to each $q$-simplex $\sigma$ of $K$ assigns an element $f(\sigma)$ of the group $G(\sigma_{(0)})$ associated with the leading vertex $\sigma_{(0)}$ of $\sigma$. The $q$-cochains form an abelian group $C^q(K, G)$. The coboundary $\delta f$ is a $(q + 1)$-cochain defined by

$$(\delta f)(\sigma) = \gamma(\sigma_{(0,1)})f(\sigma^{(0)}) + \sum_{i=1}^{q+1} (-1)^i f(\sigma^{(i)}).$$

It is easy to verify that $\delta\delta f = 0$. The group $Z^q(K, G)$ of cocyles is then defined as the kernel of $\delta: C^q \to C^{q+1}$ while the group $B^q(K, G)$ of coboundaries is the image group of $\delta: C^{q-1} \to C^q$. The $q^{\text{th}}$ cohomology group of $K$ over $G$ is

$$H^q(K, G) = Z^q(K, G)/B^q(K, G).$$

Let $T: K_1 \to K$ be a simplicial map. From the given local system $G$ in $K$ we define a local system $T^*G$ of groups in $K_1$ as follows: $T^*G = \{G(T(\alpha)), \gamma(T(\beta))\}$ for vertices $\alpha$ and edges $\beta$ in $K_1$. If $f \in C^q(K, G)$ we define

$$(T^*f)(\sigma) = f(T\sigma)$$

for $q$-simplexes $\sigma$ of $K_1$, and find that $T^*f \in C^q(K_1, T^*G)$. Clearly $\delta(T^*f) = T^*(\delta f)$ so that a homomorphism

$$T^*: H^q(K, G) \to H^q(K_1, T^*G)$$

is obtained.

## 3. Singular homology

Typical examples of semi-simplicial complexes are encountered in singular homology theory.

Select for each dimension $q$ a fixed Euclidean $q$-simplex $\Delta_q$ with ordered vertices $d_q^0 < \cdots < d_q^q$. Consider the simplicial maps

$$e_q^i: \Delta_{q-1} \to \Delta_q \qquad i = 0, \cdots, q$$

which are order preserving and map $\Delta_{q-1}$ onto the face of $\Delta_q$ opposite $d_q^i$.

A map $T: \Delta_q \to X$ of $\Delta_q$ into a topological space $X$ is called a singular $q$-simplex in $X$. The faces of $T$ are defined as

$$T^{(i)} = Te_q^i : \Delta_{q-1} \to X.$$

The singular simplexes of $X$ thus make up a semi-simplicial complex $S(X)$ called the total singular complex of $X$. If $A$ is a subspace of $X$, $S(A)$ is a subcomplex of $S(X)$; the homology and cohomology groups of the pair $(X, A)$ are defined to be those of the pair $(S(X), S(A))$.

From now on we shall assume that $X$ is arcwise connected and that a fixed point $x^*$ of $X$ has been selected as base point. A singular simplex $T: \Delta_q \to X$ such that $T(\Delta_q) = (x^*)$ will be called *collapsed*. We denote by $S_n(X)$ the subcomplex of $S(X)$ consisting of all singular simplexes $T$ such that all faces of $T$ of dimension $< n$ are collapsed. Thus for dimensions $q < n$, $S_n(X)$ contains only one $q$-simplex, namely the collapsed one.

With reference to local coefficients it should be remarked that a local system of groups on the space $X$ yields a local system of groups on the complex $S(X)$ and vice-versa. It also determines a local coefficient system on each of the complexes $S_n(X)$. On the complex $S_1(X)$ the local system reduces to one group $G = G(x^*)$, and each 1-simplex of $S_1(X)$ defines an automorphism of this group. These automorphisms determine, and are determined by, the fashion in which the fundamental group $\pi_1(X)$ (with $x^*$ as base point) operates on $G$. In the complexes $S_n(X)$, $n > 1$, the automorphisms are all identity maps and the local coefficient system collapses.

## 4. Minimal complexes

Two singular $q$-simplexes $T_0$ and $T_1$ in a space $X$ are called *compatible* if their faces coincide: $T_0^{(i)} = T_1^{(i)}$ for $i = 0, \cdots, q$. If in addition $T_0$ and $T_1$ are members of a continuous one parameter family $T_t$, $0 \leqq t \leqq 1$, of singular $q$-simplexes, all of which are compatible, we say that $T_0$ and $T_1$ are homotopic. For $q = 0$ any two simplexes are compatible, and since $X$ is assumed to be arcwise connected, they are also homotopic.

A subcomplex $M$ of $S(X)$ will be called minimal provided:

(4.1) *For each $q$ the collapsed $q$-simplex $T: \Delta_q \to x^*$ is in $M$.*

(4.2) *If $T$ is a singular $q$-simplex all of whose faces are in $M$, then $M$ contains a unique singular $q$-simplex $T'$ compatible with and homotopic to $T$.*

To show that minimal subcomplexes exist we proceed by induction. Assume that a subcomplex $M^{(n)}$ of $S(X)$ has been defined containing only simplexes of dimension $\leqq n$ and satisfying (4.1) and (4.2) for $q \leqq n$. Consider all $(n + 1)$-simplexes in $S(X)$ all of whose faces are in $M^{(n)}$. Divide these simplexes into equivalence classes, counting two simplexes as equivalent if they are compatible and homotopic. Select one simplex out of each equivalence class with the provision that the collapsed $(n + 1)$-simplex be one of those selected. Define $M^{(n+1)}$ by adjoining to $M^{(n)}$ these selected $(n + 1)$-simplexes. This yields a "dimension by dimension" construction of a minimal complex $M$.

Since every two 0-simplexes in $S(X)$ are compatible and homotopic, $M$ contains only one 0-simplex, namely the collapsed one. This implies

(4.3) *Every minimal subcomplex $M$ of $S(X)$ is a subcomplex of $S_1(X)$.*

More generally we have

(4.4) *If the homotopy group $\pi_{n-1}(X)$ vanishes, then for every minimal subcomplex $M$ of $S(X)$ we have $M \cap S_{n-1}(X) \subset S_n(X)$.*

PROOF. Let $T$ be an $(n - 1)$-simplex of $M \cap S_{n-1}(X)$. Since all the faces of $T$ are collapsed, $T$ is compatible with the collapsed $(n - 1)$-simplex $T_0$, and since $\pi_{n-1}(X) = 0$, $T$ is homotopic to $T_0$. Thus by (4.2) $T = T_0$ and $T$ is collapsed.

As a corollary of (4.4) we have

(4.5) *If the homotopy groups $\pi_i(X)$ vanish for $i < n$ then every minimal subcomplex $M$ of $S(X)$ is a subcomplex of $S_n(X)$.*

## 5. The main homotopy

We shall consider prisms

$$\Pi_q = \Delta_{q-1} \times I, \qquad q > 0$$

where $\Delta_{q-1}$ is the $(q - 1)$-simplex used to define singular $(q - 1)$-simplexes while $I$ is the closed interval $0 \leqq t \leqq 1$. The maps

$$e^i_{q-1}:\Delta_{q-2} \to \Delta_{q-1}, \qquad i = 0, \cdots, q - 1$$

define maps

$$p^i_q:\Pi_{q-1} \to \Pi_q$$

by setting $p^i_q(x, t) = (e^i_{q-1}(x), t)$.

We further have the maps

$$b^t_q:\Delta_{q-1} \to \Pi_q, \qquad 0 \leqq t \leqq 1$$

defined by $b^t_q(x) = (x, t)$. The maps $b^0_q$ and $b^1_q$ are of special interest.

A continuous mapping

$$P:\Pi_q \to X$$

is called a *singular q-prism* in $X$. The singular $(q - 1)$-prism

$$P^{(i)} = Pp^i_q:\Pi_{q-1} \to X$$

is called the $i^{\text{th}}$ face of $P$, $i = 0, \cdots, q - 1$. The singular $(q - 1)$-simplexes

$$P(t) = P\, b^t_q:\Delta_{q-1} \to X, \qquad 0 \leqq t \leqq 1$$

will be considered, in particular $P(0)$ and $P(1)$ will be called the lower and the upper base of the singular prism $P$.

(5.1) *Let $X$ be an arcwise connected space and let $M$ be a minimal subcomplex of $X$ relative to some base point $x^* \in X$. There is then a function $\{P_T\}$ which to each* *singular $q$-simplex $T$ in $X$ assigns a singular $(q + 1)$-prism $P_T$ in $X$ subject to the following conditions*

(i) $P_{T^{(i)}} = P_T^{(i)}$,

(ii) $P_T(0) = T$,

(iii) *$P_T(1)$ is in the subcomplex $M$,*

(iv) *If $T \in M$ then $P_T(t) = T$ for all $0 \leqq t \leqq 1$.*

A function $\{P_T\}$ satisfying conditions (i)–(iv) will be called a *homotopy* deforming $S(X)$ into $M$.

We begin the construction of $P_T$ with the dimension zero. A 0-simplex $T$ in $X$ is represented by a point $x \in X$; we then select $P_T$ to be a path joining $x^*$ and $x$, with the provision that $P_T$ is the collapsed path if $x = x^*$.

Suppose, by induction, that $P_T$ is defined for simplexes $T$ of dimension $<q$ so that (i)–(iv) hold. Let $T$ be a singular $q$-simplex in $X$. If $T$ is in $M$ we define $P_T$ by (iv). We may thus assume that $T$ is not in $M$. Consider the following subsets of $\Pi_{q+1}$

$$A = \Delta_q \times (0) \cup \dot{\Delta}_q \times I,$$

$$B = A \cup \Delta_q \times (1),$$

where $\dot{\Delta}_q$ denotes the boundary of $\Delta_q$. Conditions (i) and (ii) define a mapping

$$f:A \to X$$

such that

$$f\, b_{q+1}^0 = T, \qquad fp_{q+1}^i = P_{T^{(i)}} \text{ for } i = 0, \cdots, q.$$

Since $A$ is a retract of $\Pi_{q+1}$, there is an extension

$$f_1 : \Pi_{q+1} \to X.$$

The map $T_1 = f_1\, b_{q+1}^1 : \Delta_q \to X$ is a singular $q$-simplex. The $i^{\text{th}}$ face of $T_1$ is $P_{T^{(i)}}(1)$ and thus in $M$. Since the faces of $T_1$ are in $M$, there is in $M$ a unique $q$-simplex $T_2$ which is compatible and homotopic with $T_1$. We define

$$f_2 : B \to X$$

so that $f_2 = f_1 = f$ on $A$ and $f_2 b_{q+1}^1 = T_2$. The map $f_2$ is then homotopic with $f_1$ (considered only on $B$). Since $f_1$ is defined all over $\Pi_{q+1}$, there is an extension $f_3 : \Pi_{q+1} \to X$ of $f_2$. Define $P_T = f_3$; conditions (i)–(iv) are then easily verified.

If we denote

$$\varphi_t T = P_T(t), \qquad 0 \leqq t \leqq 1,$$

then for every singular $q$-simplex $T$ in $X$, $(\varphi_t T)\ (p)$ is continuous simultaneously in $0 \leqq t \leqq 1$ and $p \in \Delta_q$, and conditions (i)–(iv) can be rewritten as follows.

(i)′ $\varphi_t : S(X) \to S(X)$ is simplicial,

(ii)′ $\varphi_0$ is the identity,

(iii)′ $\varphi_1 T \in M$,

(iv)′ $\varphi_t T = T$ for $T \in M$ and $0 \leqq t \leqq 1$.

Thus in a sense $\varphi_1$ is a retraction of $S(X)$ onto $M$ while $\{\varphi_t\}$ is a homotopy connecting this retraction with the identity map. Thus we may say that $M$ is a deformation retract of $S(X)$.

## 6. Applications of the main homotopy

We shall use the main homotopy to compare the homology and cohomology of the complex $S(X)$ with those of a minimal subcomplex $M$.

By a suitable simplicial subdivision of $\Pi_n$ one can define (see [1, §16]) a function which to each singular prism $P: \Pi_q \to X$ assigns a $q$-dimensional chain $c(P)$ in $S(X)$ such that

$$\partial c(P) = P(1) - P(0) - \sum (-1)^i c(P^{(i)}).$$

Now let $P_T$ be the main homotopy of §5 and define $DT = c(P_T) \in C^{q+1}(S(X))$ for each $q$-simplex $T$ of $S(X)$. There result homomorphisms

$$D: C^q(S(X)) \to C^{q+1}(S(X))$$

such that

$$\partial DT + D\partial T = \varphi_1 T - T.$$

Consequently we have

(6.1) *The inclusion simplicial map $i:M \to S(X)$ and the simplicial map $\varphi_1 : S(X) \to M$ are such that the composition $\varphi_1 i:M \to M$ is the identity while the composition $i\varphi_1 : S(X) \to S(X)$ is chain homotopic to the identity.*

A corollary of (6.1) is

(6.2) *The inclusion map $i:M \to S(X)$ induces isomorphisms of the homology and cohomology groups of the space $X$ with those of the minimal complex $M$.*

From the properties of $P_T$ and $c(P_T)$ it follows easily that if $T$ is a $q$-simplex in $S_n(X)$ then $D(T)$ is a $(q + 1)$-chain in $S_n(X)$. Thus (6.1) and (6.2) may be restated with $S(X)$ and $M$ replaced by $S_n(X)$ and $M \cap S_n(X)$ respectively.

(6.3) *If the homotopy group $\pi_{n-1}(X)$ vanishes then the inclusion map $j:S_n(X) \to S_{n-1}(X)$ induces isomorphisms of the homology and cohomology groups of $S_n(X)$ with those of $S_{n-1}(X)$. In particular this always applies to the map $S_1(X) \to S(X)$.*

Proof. Consider the inclusion maps

$$M \cap S_n(X) \xrightarrow{i_n} S_n(X) \xrightarrow{j} S_{n-1}(X).$$

Since by (4.4) we have $M \cap S_n(X) = M \cap S_{n-1}(X)$, it follows that $ji_n = i_{n-1}$ is the inclusion map $i_{n-1} : M \cap S_{n-1}(X) \to S_{n-1}(X)$. Since both $i_n$ and $i_{n-1}$ induce isomorphisms of the homology and cohomology groups, the same applies to $j$.

As a corollary of (6.3) we have

(6.4) *If the homotopy groups $\pi_i(X)$ vanish for $i < n$ then the inclusion map $S_n(X) \to S(X)$ induces isomorphisms of the homology and cohomology groups of $S(X)$ with those of $S_n(X)$.*

All the isomorphisms asserted in (6.2)–(6.4) are also valid for cohomology groups with local coefficients. As an example we shall indicate the reasoning leading to the analogue of (6.2).

Let $G$ be a local coefficient system in $X$ (i.e. in $S(X)$) and let $G'$ be the induced local system in $M$. The inclusion map $i:M \to S(X)$ induces homomorphisms:

$$i^*: H^q(X, G) \to H^q(M, G').$$

To define an inverse map proceed as follows. Let $T$ be any $q$-simplex in $X$ with leading vertex $T_{(0)}$. Then $P_{T_{(0)}}$ may be regarded as a 1-simplex in $X$ which yields an isomorphism

$$\rho(T) = \gamma[\varphi_t(T_{(0)})] : G(\varphi_1 T_{(0)}) \to G(T_{(0)}).$$

Now for any cochain $f \in C^q(M, G')$ we define a cochain $\psi f \in C^q(X, G)$ by setting

$$(\psi f)(T) = \rho(T) f (\varphi_1 T).$$

It is easy to verify that $\delta\psi f = \psi\delta f$ and thus $\psi$ induces homomorphisms

$$\psi^*: H^q(M, G') \to H^q(X, G).$$

The composition $i^* \psi^*$ is evidently the identity map. To examine the composition $\psi^* i^*$ we must examine the definition of the chain $DT = c(P_T) \in C^{q+1}(S(X))$ and observe that $DT$ is a linear combination of simplexes all of which have the same leading vertex as the simplex $T$. Thus for each $f \in C^q(X, G)$ we may define $D^*f \in C^{q-1}(X, G)$ by setting

$$D^*f(T) = f(DT)$$

for every singular $(q - 1)$-simplex $T$. The coboundary formula

$$\delta D^*f + D^*\delta f = \psi^* i^* f - f$$

then follows by computation. Thus if $f$ is a cocycle it follows that $\psi^* i^* f - f$ is a coboundary, proving that $\psi^* i^*$ is the identity.

As an example of another application of the main homotopy we briefly discuss the maps of a simplicial complex into the space $X$.

Let $K$ be a simplicial complex with ordered vertices. For each $q$-simplex $s$ of $K$ there is a unique simplicial map $T_s : \Delta_q \to K$ which maps $\Delta_q$ simplicially onto $s$ and preserves the order of the vertices.

Let $f:K \to X$ be a continuous map and let $M$ be a minimal subcomplex of $S(X)$. The map $f$ will be called *minimal* if for every simplex $s$ of $K$ the singular simplex $fT_s$ is in $M$.

(6.5) *Every map $f:K \to X$ is homotopic to a minimal map. Further the homotopy $\psi_t$ may be so chosen that if $L$ is a subcomplex of $K$ on which $f$ is minimal then $\psi_t(y) = f(y)$ for every point $y \in L$.*

PROOF. Let $\{\varphi_t\}$ be the homotopy retracting $S(X)$ onto $M$ as defined in §5. For each simplex $s$ of $K$ consider the homotopy

$$\psi_{t,s} = (\varphi_t f T_s) T_s^{-1}.$$

The homotopies $\psi_{t,s}$ defined on each simplex $s$, together yield the desired homotopy $\psi_t$.

## 7. Uniqueness of minimal complexes

Suppose that in the arcwise connected space $X$ we have, in addition to the minimal complex $M$, another minimal complex $M_1$ (constructed relative to some base point $x_1^*$).

(7.1) *The simplicial map $\varphi_1$ of* §5 *maps the minimal complex $M_1$ isomorphically onto the minimal complex $M$.*

Use will be made of the following elementary lemma, the proof of which is left to the reader.

(7.2) *If $P_1$ and $P_2$ are two $q$-prisms in $X$ such that $P_1^{(i)} = P_2^{(i)}$ for $i = 0, \cdots, q - 1$, then $P_1(0)$ and $P_2(0)$ are homotopic if and only if $P_1(1)$ and $P_2(1)$ are homotopic.*

In order to establish (7.1) we shall prove by induction that $\varphi_1$ maps the $q$-skeleton $M_1^{(q)}$ of $M_1$ isomorphically onto the $q$-skeleton $M^{(q)}$ of $M$. For $q = 0$ the proposition is obvious. Suppose inductively that the proposition is valid for $q - 1$.

Let $T_1$ and $T_2$ be two $q$-simplexes in $M_1$ and suppose that $\varphi_1 T_1 = \varphi_1 T_2 = T$. Then $\varphi_1(T_1^{(i)}) = T^{(i)} = \varphi_1(T_2^{(i)})$. Thus by the inductive hypothesis $T_1^{(i)} = T_2^{(i)}$ for $i = 0, \cdots, q$ i.e., $T_1$ and $T_2$ are compatible. Since $P_{T_1}^{(i)} = P_{T_2}^{(i)}$ for $i = 0, \cdots, q$ and $P_{T_1}(1) = P_{T_2}(1)$ it follows from (7.2) that $P_{T_1}(0)$ and $P_{T_2}(0)$ are homotopic. Thus $T_1$ and $T_2$ are homotopic, and since they are both in the minimal complex $M_1$, it follows that $T_1 = T_2$.

Let now $T$ be any $q$-simplex of $M$. By the inductive hypothesis there is for each $i = 0, \cdots, q$ a unique $(q - 1)$-simplex $T_i$ in $M_1$ such that $\varphi_1 T_i = T^{(i)}$. Consider the subset $A = \Delta_q \times (1) \cup \dot{\Delta}_q \times I$ of $\Pi_{q+1}$ where $\dot{\Delta}_q$ is the boundary of $\Delta_q$. The singular prisms $P_{T_i}$ and the singular simplex $T$ together define a map

$$f: A \to X$$

such that

$$f b_{q+1}^1 = T, f p_{q+1}^i = P_{T_i} \quad \text{for } i = 0, \cdots, q.$$

Since $A$ is a retract of $\Pi_{q+1}$ the map $f$ can be extended to a singular prism $\bar{P}$. Consider the singular $q$-simplex $\bar{T} = \bar{P}(0)$. Since $\bar{T}^{(i)} = T_i$, the faces of $\bar{T}$ are in $M_1$ and therefore there is in $M_1$ a (unique) $q$-simplex $T'$ compatible and homotopic with $\bar{T}$. For the prisms $\bar{P}$ and $P_{T'}$ we then have $\bar{P}^{(i)} = P_{T'}^{(i)}$ for $i = 0, \cdots, q$ and $\bar{P}(0)$ and $P_{T'}(0)$ are homotopic. It follows that $\bar{P}(1) = T$ and $P_{T'}(1) = \varphi_1 T'$ are homotopic. Since they are both in $M$, we conclude that $T = \varphi_1 T'$. This concludes the proof.

## 8. Complete semi-simplicial complexes

We write $[m]$ for the ordered set $(0, 1, \cdots, m)$, where $m$ is an integer $\geqq 0$. By a map $\alpha: [m] \to [n]$ will always be meant a *weakly monotone* function from $[m]$ to $[n]$. A map which is not strictly monotone will be called *degenerate*.

A map $\alpha: [m] \to [n]$ induces a simplicial map $\bar{\alpha}: \Delta_m \to \Delta_n$. Thus for every singular simplex $T: \Delta_n \to X$ the composition $T\alpha$ is defined; we shall write $T\alpha$

instead of $T\tilde{\alpha}$. This operation $T\alpha$ could be used, in place of the concept of the $i^{\text{th}}$ face, as a starting point of the definition of $S(X)$. The corresponding abstract theory will now be outlined.

A *complete semi-simplicial complex* $K$ is a collection of "simplexes" $\sigma$, to each of which is attached a dimension $q \geqq 0$, such that for each $q$-simplex $\sigma$ and each map $\alpha:[m] \to [q]$, where $m \geqq 0$, there is defined an $m$-simplex $\sigma\alpha$ of $K$, subject to the conditions

(8.1) *If $\varepsilon_q$ is the identity map $[q] \to [q]$, then $\sigma\, \varepsilon_q = \sigma$.*

(8.2) *If $\beta: [n] \to [m]$, then $(\sigma\alpha)\beta = \sigma(\alpha\beta)$.*

Whenever the relation $\sigma = \tau\alpha$ subsists, we say that $\sigma$ *lies on* $\tau$.

A complete semi-simplicial complex is semi-simplicial (§1) in the following sense. Let

$$\varepsilon_q^i:[q - 1] \to [q] \qquad\qquad 0 \leqq i \leqq q$$

be the map which covers all of $[q]$ except the integer $i$. Then the definition $\sigma^{(i)} = \sigma\varepsilon_q^i$ where $\sigma$ is a $q$-simplex, turns $K$ into a semi-simplicial complex; moreover, one verifies readily that the simplex $\sigma_{(j_0, \dots, j_{q-n})}$ of §1 may be written $\sigma\rho$, where $\rho:[q - n] \to [q]$ is defined by

$$\rho i = j_i \qquad\qquad 0 \leqq i \leqq q - n.$$

With this understanding, all of the discussion of §1 and 2 applies, except that the requirements on a subcomplex and a simplicial map become more stringent: $L \subset K$ is a subcomplex if $\sigma \in L$ implies $\sigma\alpha \in L$; and $T:K \to K_1$ is simplicial if $T(\sigma\alpha) = (T\sigma)\alpha$. Here $\sigma$ is any simplex, and $\alpha$ any map of $[m]$ into $[q]$, where dim $\sigma = q$.

For every integer $m \geqq 0$ introduce a complete semi-simplicial complex $K[m]$ as follows. A $q$-simplex of $K[m]$ is any map $\sigma: [q] \to [m]$. For every map $\alpha:[n] \to [q]$ the simplex $\sigma\alpha$ is defined as the composite map.

A $q$-simplex $\sigma$ is *degenerate* if it has a factorization $\tau\alpha$, where $\alpha$ is degenerate.

(8.3) *A $q$-simplex $\sigma$ of a complete semi-simplicial complex $K$ has a unique "minimal" factorization $\tau\alpha$, where $\alpha$ is a map onto, and $\tau$ is a non-degenerate simplex. The dimension of $\tau$ will be called the rank of $\sigma$.*

PROOF. Because of (8.1), $\sigma$ has at least one factorization $\tau'\alpha'$ where $\alpha': [q] \to [m']$, $m' \leqq q$ and $\tau'$ is an $m'$-simplex. Let $m = m(\sigma)$ be the smallest such $m'$ and let $\tau\alpha$ be the corresponding factorization. We assert that $\tau$ is non-degenerate and $\alpha$ is onto. Indeed if $\tau$ is degenerate then $\tau = \tau'\beta$ where $\beta:[m] \to [m']$ is degenerate. Then $\beta$ may be factored into $\beta = \delta\gamma$ where $\gamma:[m] \to [m'']$, $\delta:[m''] \to [m']$ with $m'' < m$. Consequently $\sigma = \tau\alpha = \tau'\beta\alpha = (\tau'\delta)(\gamma\alpha)$ where $\gamma\alpha:[q] \to [m'']$, contrary to the definition of $m$. Similarly if $\alpha$ is not onto then $\alpha = \gamma\beta$ where $\beta:[q] \to [m']$, $\gamma:[m'] \to [m]$ and $m' < m$. Hence $\sigma = \tau(\gamma\beta) = (\tau\gamma)\beta$ with $\beta:[q] \to [m']$ for $m' < m$. This shows the existence of a factorization asserted in (8.3).

To prove the uniquenesss of this factorization, suppose that $\sigma = \tau_1\alpha_1$ with $\tau_1$ non-degenerate and $\alpha_1$ a mapping of $[q]$ onto $[m_1]$. We shall show that $\tau_1 = \tau$ and $\alpha_1 = \alpha$. Since both $\alpha$ and $\alpha_1$ are onto there exist maps

$$\beta:[m] \to [q], \qquad \beta_1:[m_1] \to [q]$$

with

$$\alpha\beta = \varepsilon_m, \qquad \alpha_1\beta_1 = \varepsilon_{m_1}.$$

Since $\tau_1\alpha_1 = \tau\alpha$ we have $\tau_1\alpha_1\beta = \tau\alpha\beta = \tau\varepsilon_m = \tau$ and similarly $\tau\alpha\beta_1 = \tau_1$. Since both $\tau$ and $\tau_1$ are non-degenerate the maps $\alpha_1\beta$ and $\alpha\beta_1$ are non-degenerate. Consequently $m_1 = m$ and $\alpha_1\beta = \alpha\beta_1 = \varepsilon_m$. This implies $\tau = \tau_1$. Suppose that for some $i \in [q]$ we have $\alpha(i) \neq \alpha_1(i)$. We may select $\beta$ so that $\beta\alpha(i) = i$, then $\alpha_1\beta\alpha i = \alpha_1 i \neq \alpha i$ contradicting $\alpha_1\beta = \varepsilon_m$. Consequently $\alpha = \alpha_1$.

Notice that if $\alpha:[m] \to [q]$, then $\alpha \in K[q]$ and $\alpha^{(i)} = \alpha\varepsilon_m^i$ is the $i^{\text{th}}$ face of $\alpha$. Therefore for any $q$-simplex $\sigma$ we have $(\sigma\alpha)^{(i)} = (\sigma\alpha)\varepsilon_m^i = \sigma(\alpha\varepsilon_m^i) = \sigma\alpha^{(i)}$. Thus

$$(8.4) \qquad (\sigma\alpha)^{(i)} = \sigma\alpha^{(i)}.$$

Suppose now that $\sigma = \tau\alpha$ is the minimal factorization of $\sigma$ with $\alpha:[q] \to [m]$ onto and $\tau$ non-degenerate. If $\alpha^{(i)}$ is onto then $\tau\alpha^{(i)} = \sigma^{(i)}$ is the minimal factorization of $\sigma^{(i)}$ and rank $\sigma^{(i)} =$ rank $\sigma$. If $\alpha^{(i)}:[q-1] \to [m]$ is not onto then $\alpha^{(i)} = \varepsilon_m^{\alpha i}\beta$ where $\beta:[q-1] \to [m-1]$ is onto. Thus

$$\sigma^{(i)} = \tau\alpha^{(i)} = \tau\varepsilon_m^{\alpha i}\beta = \tau^{(\alpha i)}\beta$$

and since $\beta$ is onto it follows that rank $\sigma^{(i)} \leqq \dim \tau^{(\alpha i)} < \dim \tau =$ rank $\sigma$. Summarizing we have

(8.5) *If $\sigma = \tau\alpha$ is the minimal factorization given by* (8.3) *then either $\alpha^{(i)}$ is onto, and then $\tau\alpha^{(i)}$ is the minimal factorization of $\sigma^{(i)}$, or $\alpha^{(i)}$ is not onto and then* rank $\sigma^{(i)} <$ rank $\sigma$.

As an application of (8.5) we prove

(8.6) *If $\sigma$ and $\sigma_1$ are degenerate $q$-simplexes such that $\sigma^{(i)} = \sigma_1^{(i)}$ for $i = 0, \cdots, q$, then $\sigma = \sigma_1$.*

Proof. Let $\sigma = \tau\alpha$, $\sigma_1 = \tau_1\alpha_1$ be minimal factorizations with $\alpha:[q] \to [m]$, $\alpha_1:[q] \to [m_1]$. Since $\alpha$ is degenerate and onto there exist at least two indices $j$ such that $\alpha^{(j)}$ is onto. Then by (8.5)

$$m = \text{rank } \sigma = \text{rank } \sigma^{(j)} = \text{rank } \sigma_1^{(j)} \leqq \text{rank } \sigma_1 = m_1.$$

Similarly we prove that $m_1 \leqq m$. Thus $m = m_1$. Consequently rank $\sigma_1^{(j)} =$ rank $\sigma_1$ and therefore $\alpha_1^{(j)}$ is onto for the same values of $j$ for which $\alpha^{(j)}$ is onto. For such a value of $j$ the simplex $\sigma^{(j)} = \sigma_1^{(j)}$ has two minimal factorizations $\tau\alpha^{(j)}$ and $\tau_1\alpha_1^{(j)}$. Thus $\tau = \tau_1$ and $\alpha^{(j)} = \alpha_1^{(j)}$. This implies $\alpha(i) = \alpha_1(i)$ for $i \neq j$. Since this is true for at least two indices $j$, it follows that $\alpha = \alpha_1$. Thus $\sigma = \sigma_1$.

## 9. Complete minimal complexes

As was already remarked early in §8, the total singular complex $S(X)$ of a space $X$ may be regarded as a complete semi-simplicial complex. Turning to minimal subcomplexes it is natural to require that the minimal complex $M$, be a subcomplex of $S(X)$ regarded as a complete semi-simplicial complex. Thus an additional condition has to be imposed:

(9.1) *If $T$ is a $q$-simplex of $M$ and $\alpha:[m] \to [q]$ then $T\alpha$ is an $m$-simplex of $M$.*

It will be shown that a slight change in the construction of $M$ described in

§4 will insure this additional property. First observe that in view of (8.6) two degenerate simplexes of $S(X)$ that are compatible are equal. Thus the equivalence classes used in defining $M^{(n+1)}$ in §4 each contains at most one degenerate simplex. We shall require that the degenerate simplex be selected whenever there is one in the equivalence class. We must now prove that (9.1) holds. The case of $\alpha$ non-degenerate is trivial since then $T\alpha = T^{(i_1,\dots,i_{q-m})}$ for a suitable choice of $i_1 < i_2 \cdots < i_{q-m}$. Assume then that $\alpha$ and therefore also $T\alpha$ are degenerate. The proposition is valid for $m = 0$ since then $\alpha$ is not degenerate. Suppose, by induction, that the proposition holds for $m - 1$. Then $(T\alpha)^{(i)} = T\alpha^{(i)}$ is in $M$ and therefore in $M^{(m-1)}$. Thus $T\alpha$ is a degenerate simplex with all of its faces in $M^{(m-1)}$. By our modification of the construction it follows that $T\alpha$ is in $M$.

Turning to the main homotopy of §5, we modify the construction so as to insure that each map $\varphi_t$ be a simplicial map in the sense of complete semi-simplicial complexes. Thus we must replace the condition

$$(*) \qquad \varphi_t(T^{(i)}) = (\varphi_t T)^{(i)}$$

by the stronger condition

$$(**) \qquad \varphi_t(T\alpha) = (\varphi_t T)\alpha,$$

where $T$ is a singular $p$-simplex and $\alpha:[m] \to [p]$.

Suppose then that $\varphi_t T$ has been defined for all singular simplexes $T$ of dimension $< q$ and that it satisfies the conditions of §5 as well as condition (**) for $m < q$ and $p < q$. First consider the degenerate singular $q$-simplexes $T$. Let $T = \tau\alpha$ be the "minimal" factorization given by (8.3) where $\tau$ is a non-degenerate $m$-simplex and $\alpha:[q] \to [m]$ is onto with $m < q$. Define

$$\varphi_t T = (\varphi_t \tau)\alpha.$$

After $\varphi_t T$ has been defined for all degenerate $q$-simplexes define $\varphi_t T$ for the non-degenerate $q$-simplexes exactly as in §5. Thus (*) is assured for all non-degenerate $q$-simplexes. We shall show that (*) holds also for the degenerate $q$-simplexes. Indeed we have

$$\varphi_t(T^{(i)}) = \varphi_t(\tau\alpha^{(i)}) = (\varphi_t\tau)\alpha^{(i)} = (\varphi_t T)^{(i)}.$$

We now prove (**) for $q$-simplexes $T\alpha$, $\alpha:[m] \to [p]$, $m \leqq q, p \leqq q$. First consider the case when $\alpha$ is non-degenerate. In this case (**) follows from (*) since $T\alpha$ is $T^{(i_1,\dots,i_{q-m})}$ for suitable indices $i_1 < i_2 < \cdots < i_{q-m}$. Next consider the case when $\alpha$ is onto. If $T = \tau\beta$ is the minimal factorization of $T$ then $T\alpha = \tau(\beta\alpha)$ is the minimal factorization of $T\alpha$ and therefore

$$\varphi_t(T\alpha) = \varphi_t(\tau\beta\alpha) = (\varphi_t\tau)\beta\alpha = (\varphi_t T)\alpha.$$

Since any $\alpha:[q] \to [m]$ may be factored into $\alpha = \alpha_1\alpha_2$ where $\alpha_1$ is nondegenerate and $\alpha_2$ is onto, it follows that (**) holds for all $\alpha$.

## 10. Normalization

Let $K$ be a complete semi-simplicial complex and $G = \{G(\alpha), \gamma(\beta)\}$ a local system of abelian groups in $K$. A cochain $f \epsilon C^q(K, G)$ will be called normalized provided that $f(\sigma) = 0$ for every degenerate $q$-simplex $\sigma$ of $K$. It will be shown in the sequel that the coboundary of a normalized cochain is again normalized. Let $H^q_n(K, G)$ denote the cohomology group obtained using only normalized cochains. Explicitly let $H^q_n(K, G) = Z^q_n(K, G)/B^q_n(K, G)$, where $Z^q_n$ is the group of normalized $q$-cocycles while $B^q_n$ is the group of coboundaries of normalized $(q - 1)$-cochains. The inclusions $Z^q_n \subset Z^q$ and $B^q_n \subset B^q$ then yield a natural homomorphism $H^q_n \rightarrow H^q$. The main result of this section is

(10.1) *The homomorphism* $H^q_n(K, G) \rightarrow H^q(K, G)$ *is an isomorphism onto:*

$H^q_n(K, G) \approx H^q(K, G)$.

As will be shown in [4] this theorem is a generalization of the normalization theorem in cohomology theory of groups [3, §6] [2]. The proof of (10.1) that follows is a direct generalization of that of [3].

In addition to the identity map $\varepsilon_q:[q] \rightarrow [q]$ and its faces $\varepsilon^i_q:[q - 1] \rightarrow [q]$, $i = 0, \cdots, q$ we shall also consider the maps

$$\eta^i_q:[q] \rightarrow [q - 1] \qquad\qquad i = 0, \cdots, q - 1$$

defined by $\eta^i_q j = j$ for $j \leqq i$ and $\eta^i_q j = j - 1$ for $i < j$. We note that $\{\eta^i_q\}$ is the totality of all the maps of $[q]$ *onto* $[q - 1]$.

The following identities will constantly be used

$$\begin{aligned}
\eta^i_q\, \varepsilon^j_q &= \varepsilon^j_{q-1}\, \eta^{i-1}_{q-1} \quad \text{for} \quad j < i \\
\eta^i_q\, \varepsilon^i_q &= \varepsilon_{q-1} = \eta^i_q\, \varepsilon^{i+1}_q \\
\eta^i_q\, \varepsilon^j_q &= \varepsilon^{j-1}_{q-1}\, \eta^i_{q-1} \quad \text{for} \quad i + 1 < j \\
\eta^i_{q-1}\, \eta^j_q &= \eta^j_{q-1}\, \eta^{i+1}_q \quad \text{for} \quad j \leqq i \\
\eta^i_{q-1}\, \eta^j_q &= \eta^{j-1}_{q-1}\, \eta^i_q \quad \text{for} \quad i < j.
\end{aligned}$$

(10.2) *A $q$-simplex $\sigma$ of $K$ is degenerate if and only if it has the form $\sigma = \tau\eta^i_q$ for some $i = 0, \cdots, q - 1$.*

Proof. Since $\eta^i_q$ is degenerate, $\tau\eta^i_q$ must be degenerate. If $\sigma$ is degenerate then it factors into $\sigma = \tau\alpha$ where $\alpha:[q] \rightarrow [m]$ is degenerate. Thus $\alpha(i) = \alpha(i + 1)$ for some $i = 0, \cdots, q - 1$. It follows that $\alpha$ factors into $\alpha = \beta\eta^i_q$ where $\beta:[q - 1] \rightarrow [m]$. Thus $\sigma = (\tau\beta)\eta^i_q$ as desired.

(10.3) *If $\beta$ is a degenerate 1-simplex of $K$ then $\beta_{(1)} = \beta_{(0)}$ and $\gamma(\beta)$ is the identity map $G(\beta_{(1)}) \rightarrow G(\beta_{(0)})$.*

Proof. By (10.2) $\beta$ must have the form $\beta = \alpha\eta^0_1$ where $\alpha$ is a 0-simplex. The identities then imply $\beta_{(0)} = \alpha = \beta_{(1)}$. Consider the 2-simplex $\beta\eta^0_2 = \alpha\eta^0_1\eta^0_2$. By definition of a local system, we have

$$\gamma(\alpha\eta^0_1\eta^0_2\varepsilon^2_2)\gamma(\alpha\eta^0_1\eta^0_2\varepsilon^0_2) = \gamma(\alpha\eta^0_1\eta^0_2\varepsilon^1_2).$$

It follows from the identities that the expressions in parentheses all give $\alpha\eta_1^0 = \beta$. Thus $\gamma(\beta) =$ identity.

A cochain $f \in C^q(K, G)$ will be called $i$-normalized ($i = 0, \cdots, q$) if

$$f(\sigma\eta_q^j) = 0 \qquad \text{for all } j < i.$$

Every cochain is 0-normalized. The $q$-normalized cochains are in view of (10.2) the normalized cochains.

(10.4) *If $f$ is $i$-normalized, then so is $\delta f$.*

PROOF. Let $k < i$. Then

$$(\delta f)(\sigma\eta_{q+1}^k) = \gamma(\beta)f(\sigma\eta_{q+1}^k\varepsilon_{q+1}^0) + \sum_{j=1}^{q+1} (-1)^j f(\sigma\eta_{q+1}^k\,\varepsilon_{q+1}^j)$$

where $\beta$ is the leading edge of $\sigma\eta_{q+1}^k$. For $j < k$ we have $f(\sigma\eta_{q+1}^k\varepsilon_{q+1}^j) = f(\sigma\varepsilon_q^j\eta_q^{k-1}) = 0$ since $f$ is $i$-normalized. Similarly $f(\sigma\eta_{q+1}^k\varepsilon_{q+1}^j) = f(\sigma\varepsilon_q^{j-1}\eta_q^k) = 0$ for $k + 1 < j$. For $j = k, k + 1$ we have

$$f(\sigma\eta_{q+1}^k\varepsilon_{q+1}^k) = f(\sigma) = f(\sigma\eta_{q+1}^k\varepsilon_{q+1}^{k+1}).$$

Thus we obtain that $(\delta f)(\sigma\eta_{q+1}^k) = 0$ if $k > 0$. For $k = 0$ there remains the expression $\gamma(\beta)f(\sigma) - f(\sigma)$, where $\beta$ is the leading edge of $\sigma\eta_{q+1}^0$. Since $\beta$ is degenerate, (10.3) implies that $\gamma(\beta)$ is the identity and the two terms cancel out.

(10.5) *For every cochain $f \in C^q(K, G)$ such that $\delta f$ is normalized, there is a cochain $g \in C^{q-1}(K, G)$ such that $f - \delta g$ is normalized.*

PROOF. The proof depends on the following algorithm. Starting with the cochain $f \in C^q$ construct cochains $f_0, \cdots, f_q \in C^q$ and $g_0, \cdots, g_{q-1} \in C^{q-1}$ by induction as

$$f_0 = f, \qquad f_{i+1} = f_i - \delta g_i, \qquad g_i(\sigma) = (-1)^i f_i(\sigma\eta_q^i)$$

for $i = 0, \cdots, q - 1$. Then clearly $\delta f_i = \delta f_0 = \delta f$ for all $i$. Since $f_q = f - \delta g$, where $g = g_0 + \cdots + g_{q-1}$ it suffices to show that $f_q$ is normalized. We shall prove by induction that $f_i$ is $i$-normalized. This is clearly valid for $i = 0$. We proceed by induction and assume that $f_i$ is $i$-normalized. Since for $j < i$ we have

$$g_i(\sigma\eta_{q-1}^j) = (-1)^i f_i(\sigma\eta_{q-1}^j\eta_q^i) = (-1)^i f_i(\sigma\eta_{q-1}^{i-1}\eta_q^j) = 0,$$

$g_i$ is $i$-normalized. Thus by (10.4) $\delta g_i$ is $i$-normalized and therefore $f_{i+1}$ is $i$-normalized. To show that $f_{i+1}$ is $(i + 1)$-normalized we use the identity

$$f_{i+1}(\sigma\eta_q^i) = (-1)^i\delta f_i(\sigma\eta_q^i\eta_{q+1}^i)$$

which follows by straightforward computation using the identities and the fact that $f_i$ is $i$-normalized. Since $\delta f_i = \delta f$ is normalized, the right hand side is zero. Thus $f_{i+1}(\sigma\eta_q^i) = 0$ and $f_{i+1}$ is $(i + 1)$-normalized. This concludes the proof of (10.5).

Theorem (10.1) is an immediate consequence of (10.5). Indeed if $f \in Z^q$ then $\delta f = 0$ is normalized and (10.5) yields a cochain $g \in C^{q-1}$ such that $f - \delta g$ is normalized. Since $f - \delta g \in Z_n^q$ this shows that the homomorphism $H_n^q \to H^q$ is onto. Assume now that $h \in Z^q_n$ and that $h = \delta f$ for some $f \in C^{q-1}$. Since $\delta f = h$ is normalized there is a $g \in C^{q-2}$ such that $f - \delta g$ is normalized. Thus $h$ is the coboundary of the normalized cochain $f - \delta g$, showing that $H^q_n \to H^q$ is an isomorphism.

If we limit ourselves to simple coefficients, we can give theorem (10.1) a more intuitive interpretation. Call a chain of $K$ degenerate if it is a linear combination of degenerate simplexes. The same argument as in (10.4) shows that the boundary of a degenerate simplex is a degenerate chain. Thus the degenerate chains form a chain complex $D$ which is a subcomplex of $K$ (regarded as a chain complex and not as a semi-simplicial complex). The normalized cohomology groups $H^q_n(K, G)$ are then nothing else than the relative cohomology groups $H^q(K, D, G)$. If we examine the cohomology sequence

$$\cdots \to H^q(K, D, G) \to H^q(K, G) \to H^q(D, G) \to H^{q+1}(K, D, G) \to \cdots$$

the exactness (kernel = image property) implies that $H^q(K, D, G) \to H^q(K, G)$ is an isomorphism onto for all values of $q$ if and only if $H^q(D, G) = 0$ for all dimensions $q$. It follows from well known universal coefficient theorems that this holds for all $G$ if and only if all the integral homology groups $H_q(D)$ vanish. Since $D$ contains no non-trivial zero chains, this is equivalent with the statement that $D$ is acyclic. Thus (10.1) stated only for simple coefficients is equivalent with

(10.6) *The degenerate chain complex $D$ of a complete semi-simplicial complex is acyclic.*

A direct proof of (10.6) could be given, but then (10.1) would still have to be proved for local coefficients.

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