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Semi-slant and bi-slant submanifolds of almost contact metric 3-structure manifolds

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Abstract: In this paper we introduce the notions of semi-slant and bi-slant submanifolds of an almost contact 3-structure manifold. We give some examples and characterization theorems about these submanifolds. Moreover, the distributions of semi-slant submanifolds of 3-cosymplectic and 3-Sasakian manifolds are studied.

Key words: Almost contact 3-structure manifold, semi-slant and bi-slant submanifold, 3-Sasakian manifold

1. Introduction

After slant submanifolds of complex manifolds were introduced by Chen [6], the properties of slant submanifolds became an interesting subjects in differential geometry, both in complex geometry and in contact geometry. Lotta [9] introduced this notion in contact manifolds and Cabrerizo et al. [4] studied widely in this area and found many interesting results, especially on slant submanifolds of Sasakian manifolds. On the other hand, Papaghiuc [12] defined semi-slant submanifolds as a generalization of slant and CR-submanifolds. Carriazo [5] generalized these notions by introducing bi-slant submanifolds. Moreover, in [3], the authors investigated bi-slant and semi-slant submanifolds of Sasakian manifolds. From then on, many authors have studied these types of submanifolds when the ambient manifolds have been endowed with other structures such as trans-Sasakian and Kenmotsu [1, 14, 15, 17]. In fact, one of the important reasons for studying slant and semi-slant submanifolds is that they are a generalization of invariant, anti-invariant, semi-invariant, and totally real submanifolds.

Recently, Sahin [13] studied slant and semi-slant submanifolds of quaternion Kaehler manifolds, generalizing semi-invariant and QR-submanifolds of quaternion Kaehler manifolds.

The authors introduced the notion of 3-slant submanifolds of an almost contact metric 3-structure in [10]. Almost contact 3-structures have been defined by Kuo and Udrişte independently [8, 16]. Because of important properties and applications [7, 11], 2 types of these manifolds, 3-Sasakian and 3-cosymplectic manifolds, are very interesting for geometricians and physicians. This motivated us to study semi-slant and bi-slant submanifolds of these manifolds. In these structures there exist 3 $(1,1)$-tensor fields $(\phi_i)_{i=1,2,3}$ and the vector fields should be slant or invariant with respect to all of the ϕ_i's. Therefore, it is a generalization of invariant, anti-invariant, slant, semi-slant, and bi-slant submanifolds in almost contact metric 3-structures and we denote them by 3-semi-slant and 3-bi-slant submanifolds. Following the approaches of [3, 13], we characterized 3-bi-slant and 3-semi-slant submanifolds and studied geometric properties of distributions of these submanifolds where the ambient manifolds are 3-Sasakian or 3-cosymplectic. It should be noted that, in the definition of semi-slant

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submanifolds in the sense of Sahin, the distributions are on the normal bundle of the manifold but in this paper they are on the tangent bundle.

In this paper we review the basic definitions and information in Section 2; next, we define 3-semi-slant and 3-bi-slant submanifolds and show the existence of them by introducing some non-trivial examples and then we characterize them in Section 3. Finally, in Section 4, we investigate the geometry of distributions of 3-semi-slant submanifolds of 3-Sasakian and 3-cosymplectic manifolds.

2. Preliminaries

Definition 1 *[2] Let $\tilde{M}$ be a $(2m+1)$ dimensional manifold and ϕ, ξ, η be a tensor field of type (1,1), a vector field, a $1-$form on $\tilde{M}$, respectively. If ϕ, ξ, and η satisfy*

$$\eta(\xi) = 1$$

$$\phi^2(X) = -X + \eta(X)\xi \tag{1}$$

for any vector field X on $\tilde{M}$, then $\tilde{M}$ is said to have an almost contact structure (ϕ, ξ, η).

Definition 2 *[8] $(\tilde{M}, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is called an almost contact 3-structure manifold if there exist 3 almost contact structures (ξ_i, η_i, ϕ_i), $i = 1, 2, 3$, on $\tilde{M}$ such that*

$$\eta_i(\xi_j) = 0, \quad \phi_i\xi_j = -\phi_j\xi_i = \xi_k, \quad \eta_i(\phi_j) = -\eta_j(\phi_i) = \eta_k, \tag{2}$$

$$\phi_i o\phi_j - \eta_j \otimes \xi_i = -\phi_j o\phi_i + \eta_i \otimes \xi_j = \phi_k, \tag{3}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

The vector fields ξ_i's are called structure vector fields. Moreover, if $\tilde{M}$ admits a Riemannian metric g satisfying

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \ \forall X, Y \in T\tilde{M}, \tag{4}$$

then $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is said to be an almost contact metric 3-structure manifold. It is easy to show that (4) implies

$$g(\phi_i X, Y) = -g(X, \phi_i Y). \tag{5}$$

An almost contact metric 3-structure $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a 3-cosymplectic manifold if

$$\tilde{\nabla}\phi_i = 0, \tag{6}$$

and a 3-Sasakian manifold if

$$(\tilde{\nabla}_X\phi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X, \ \forall X, Y \in T\tilde{M}, \tag{7}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{M}$. By using (6) and (7), one can obtain

$$\tilde{\nabla}\xi_i = 0 \text{ and } \tilde{\nabla}\xi_i = -\phi_i, \tag{8}$$

in 3-cosymplectic and 3-Sasakian manifolds, respectively.

For an isometrically immersed submanifold M of a Riemannian manifold $\tilde{M}$, we denote its induced Riemannian metric by the same symbol g and the Levi-Civita connection of M by ∇. Let TM and $(TM)^{\perp}$ be the tangent bundle and normal bundle of M, respectively. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \tilde{\nabla}_X V = D_X V - A_V X, \tag{9}$$

for $X, Y \in TM$ and $V \in (TM)^{\perp}$, where D is the connection in the normal bundle, and B is the second fundamental form related to A by the following equation:

$$g(A_V X, Y) = g(B(X,Y), V). \tag{10}$$

M is called totally geodesic if and only if B vanishes identically on TM.

3. Slant submanifolds in almost contact metric 3-structure manifolds

Let M be a submanifold of $(\tilde{M}, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$. Then for all $X \in TM$ and $V \in (TM)^{\perp}$, we have

$$\phi_i X = T_i X + N_i X \quad \text{and} \quad \phi_i V = t_i V + n_i V, \tag{11}$$

such that $T_i X$ (resp. $N_i X$) is the tangential (resp. normal) component of $\phi_i X$, and $t_i V$ (resp. $n_i V$) is the tangential (resp. normal) component of $\phi_i V$. M is called an invariant submanifold of $\tilde{M}$ if N_i vanishes identically, that is, $\phi_i(T_pM) \subset T_pM$, and it is called an anti-invariant submanifold if $T_i = 0$, which means $\phi_i(T_pM) \subset (T_pM)^{\perp}$, for all $p \in M$ and $i = 1, 2, 3$.

As a generalization of invariant and anti-invariant submanifolds, the authors have introduced slant submanifolds of almost contact metric 3-structure manifolds:

Definition 3 *[10] Let M be a submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. M is called a 3-slant submanifold if for all $i \in \{1,2,3\}$, the angle between $\phi_i X$ and T_pM is constant θ, for each $p \in M$ and each non-zero vector $X \in T_pM$, linearly independent of ξ_i. It means that the angle between $\phi_i X$ and $T_j X$ is θ for all $i, j \in \{1,2,3\}$.*

Now we introduce the notion of semi-slant and bi-slant submanifolds of almost contact metric 3-structures. These notions generalize the concept of semi-slant and bi-slant submanifolds in almost contact manifolds.

Definition 4 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then we say M is a 3-semi-slant submanifold of $\tilde{M}$, if it is endowed with 3 orthogonal distributions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, where $\mathcal{D}_3 = span < \xi_1, \xi_2, \xi_3 >$ and the following conditions satisfy:*

(a) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$,

(b) *The distribution $\mathcal{D}_1$ is an invariant distribution, i.e. $\phi_i(\mathcal{D}_1) = \mathcal{D}_1$, $\forall i \in \{1,2,3\}$,*

(c) *The distribution $\mathcal{D}_2$ is 3-slant with slant angle $\theta \neq 0$, i.e. for each non-zero vector $X \in \mathcal{D}_2$ at any point $p \in M$, the angle between $\phi_i(X)$, $i = 1,2,3$ and $\mathcal{D}_2$ is constant and it is independent of the choice of $X \in \mathcal{D}_2$ and $p \in M$.*

Definition 5 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. We say M is a 3-bi-slant submanifold of $\tilde{M}$, if there exist 3 orthogonal distributions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ on M, such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ and for $i = 1,2$, $\mathcal{D}_i$ is 3-slant with slant angle θ_i and $\mathcal{D}_3 = span < \xi_1, \xi_2, \xi_3 >$.*

Thus a 3-semi-slant submanifold is a 3-bi-slant, in which $\theta_1 = 0$. In both Definitions 4 and 5, TM can be decomposed to $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$. Therefore, for any $X \in TM$, we can write $X = P_1X + P_2X + \sum_{i=1}^{3} \eta_i(X)\xi_i$, where $P_\alpha X$ is the projection of X in $\mathcal{D}_\alpha$, $\alpha = 1,2$.

Moreover, for all $X \in TM$ and $V \in (TM)^\perp$ we put

$$\phi_i X = T_{1i}X + T_{2i}X + N_iX \quad \text{and} \quad \phi_i V = t_{1i}V + t_{2i}V + n_iV, \tag{12}$$

where $T_{\alpha i}X$ (resp. $t_{\alpha i}V$) is the tangential component of ϕ_iX (resp. ϕ_iV) on $\mathcal{D}_\alpha$ and N_iX (resp. n_iV) is the normal component of ϕ_iX (resp. ϕ_iV), for $i \in \{1,2,3\}$ and $\alpha = 1,2$. In fact, by virtue of (11), it is easy to see that $T_{\alpha i}X = P_\alpha \circ T_iX$ and $t_{\alpha i}V = P_\alpha \circ t_iV$, in which P_α is the projection on $\mathcal{D}_\alpha$.

Now, we give some non-trivial examples of 3-semi-slant and 3-bi-slant submanifolds of an almost contact metric 3-structure manifold.

Example 1 *We consider an almost contact metric 3-structure $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ as follows:*

$$\tilde{M} = \mathbb{R}^{15}, g((x_i)_{i=\overline{1,15}}, (y_i)_{i=\overline{1,15}}) = \sum_{i=1}^{15} x_iy_i,\ \xi_1 = \partial x_{13},\ \xi_2 = \partial x_{14},\ \xi_3 = \partial x_{15},$$

$$\phi_1((x_i)_{i=\overline{1,15}}) = (-x_3, x_4, x_1, -x_2, ..., -x_{11}, x_{12}, x_9, -x_{10}, 0, -x_{15}, x_{14}),$$

$$\phi_2((x_i)_{i=\overline{1,15}}) = (-x_4, -x_3, x_2, x_1, ..., -x_{12}, -x_{11}, x_{10}, x_9, x_{15}, 0, -x_{13}),$$

$$\phi_3((x_i)_{i=\overline{1,15}}) = (-x_2, x_1, -x_4, x_3, ..., -x_{10}, x_9, -x_{12}, x_{11}, -x_{14}, x_{13}, 0),$$

and η_i is the dual of ξ_i for $i \in \{1,2,3\}$.

Let $M = (-u_1 - u_4, u_1 - u_4, -u_2 + u_3, -u_2 - u_3, v_1sin\theta, v_2sin\theta, v_2sin\theta, v_2sin\theta,$ $v_1cos\theta, 0, 0, 0, t_1, t_2, t_3)$ for $\theta \in (0, \frac{\pi}{2})$. Then M is a 9-dimensional submanifold of $\tilde{M}$ and TM is spanned by

$$X_1 = -\partial x_1 + \partial x_2,\ X_2 = -\partial x_3 - \partial x_4,\ X_3 = \partial x_3 - \partial x_4,\ X_4 = -\partial x_1 - \partial x_2,$$

$$X_5 = sin\theta\partial x_5 + cos\theta\partial x_9,\ X_6 = sin\theta(\partial x_6 + \partial x_7 + \partial x_8)$$

$$X_7 = \partial x_{13},\ X_8 = \partial X_{14},\ X_9 = \partial x_{15}.$$

We put $\mathcal{D}_1 =< X_1, X_2, X_3, X_4 >$, $\mathcal{D}_2 =< X_5, X_6 >$ and $\mathcal{D}_3 =< X_{13}, X_{14}, X_{15} >$. It is easy to see that $\mathcal{D}_1$ is invariant with respect to ϕ_1, ϕ_2, and ϕ_3. Moreover,

$$\phi_1(X_5) = \frac{1}{3}[X_6 + sin\theta(2\partial x_7 - \partial x_6 - \partial x_8)] + cos\theta\partial x_{11}$$

$$\Rightarrow T_{21}(X_5) = \frac{1}{3}X_6,$$

$$\phi_1(X_6) = -sin^2\theta X_5 - cos^2\theta sin\theta\partial x_5 + sin^2\theta cos\theta\partial x_9 + sin\theta(-\partial x_8 + \partial x_6)$$

$$\Rightarrow T_{21}(X_6) = -sin^2\theta X_5,$$

$$\phi_2(X_5) = \frac{1}{3}[X_6 + sin\theta(2\partial x_8 - \partial x_6 - \partial x_7)] + cos\theta\partial x_{12}$$

$$\Rightarrow T_{22}(X_5) = \frac{1}{3}X_6,$$

$$\phi_2(X_6) = -sin^2\theta X_5 - cos^2\theta sin\theta\partial x_5 + sin^2\theta cos\theta\partial x_9 + sin\theta(-\partial x_6 + \partial x_7)$$

$$\Rightarrow T_{22}(X_6) = -sin^2\theta X_5.$$

Thus we have

$$cos\beta = \frac{g(\phi_i X, T_{2j}X)}{|\phi_i X|.|T_{2j}X|} = \frac{sin\theta}{\sqrt{3}},\ \forall X \in \mathcal{D}_2 \text{ and } i,j \in \{1,2,3\}.$$

Therefore, $\mathcal{D}_2$ *is a 3-slant with slant angle* $\beta = cos^{-1}(\frac{sin\theta}{\sqrt{3}})$. M *is a 3-semi-slant submanifold of* $\tilde{M}$.

Example 2 *Let* $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ *be the almost contact metric 3-structure manifold in the previous example and* M *be a submanifold of* $\tilde{M}$ *given by the following equations:*

$$x_1 = x_2 = x_3 = u_1,\ x_4 = u_2 cos\theta, x_{12} = u_2 sin\theta,$$

$$x_5 = x_6 = x_7 = v_1,\ x_8 = x_9 = x_{10} = v_2, x_{11} = 0,$$

$$x_{13} = t_1, x_{14} = t_2, x_{15} = t_3.$$

By defining

$$\mathcal{D}_1 =< \partial x_1 + \partial x_2 + \partial x_3, cos\theta\partial x_4 + sin\theta\partial x_{12} >,$$

$$\mathcal{D}_2 =< \partial x_5 + \partial x_6 + \partial x_7, \partial x_8 + \partial x_9 + \partial x_{10} >,$$

$$\mathcal{D}_3 =< \partial x_{13}, \partial x_{14}, \partial x_{15} >,$$

we have $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ *and* $\mathcal{D}_1$, $\mathcal{D}_2$ *are 3-slant with slant angles* $cos^{-1}(\frac{cos\theta}{\sqrt{3}})$ *and* $cos^{-1}(\frac{1}{3})$, *respectively. Thus* M *is a 3-bi-slant submanifold of* $\tilde{M}$.

Example 3 *Let* $\tilde{M} = \mathbb{R}^{11}$ *be endowed with the following almost contact metric 3-structure manifold:*

$$\phi_1((x_i)_{i=\overline{1,11}}) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, 0, -x_{11}, x_{10}),$$

$$\phi_2((x_i)_{i=\overline{1,11}}) = (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, x_{11}, 0, -x_9),$$

$$\phi_3((x_i)_{i=\overline{1,11}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, 0),$$

$g((x_i)_{i=\overline{1,11}}, (y_i)_{i=\overline{1,11}}) = \sum_{i=1}^{11} x_i y_i$, $\xi_1 = \partial x_9$, $\xi_2 = \partial x_{10}$, $\xi_3 = \partial x_{11}$ *and* η_i *'s be the dual of* ξ_i *'s.*

Let $M = (v_1 cos\theta, v_1 cos\theta, v_1 cos\theta, v_2 sin\theta + u_1 cos\theta, v_2 cos\theta - u_1 sin\theta, u_2 sin\theta, u_2 sin\theta, u_2 sin\theta, t_1, t_2, t_3)$. *By taking*

$$\mathcal{D}_1 =< cos\theta(\partial x_1 + \partial x_2 + \partial x_3), sin\theta\partial x_4 + cos\theta\partial x_5) >,$$

$$\mathcal{D}_2 =< sin\theta(\partial x_6 + \partial x_7 + \partial x_8), cos\theta\partial x_4 - sin\theta\partial x_5) >,$$

$$\mathcal{D}_3 =< \partial x_9, \partial x_{10}, \partial x_{11} >,$$

we have $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$. By some computations in the same way as in Example 1, it can be verified that $\mathcal{D}_1$ and $\mathcal{D}_2$ are 3-slant with slant angles $cos^{-1}(\frac{sin\theta}{\sqrt{3}})$ and $cos^{-1}(\frac{cos\theta}{\sqrt{3}})$, respectively. Therefore, M is a 3-bi-slant submanifold of $\tilde{M}$.

It should be noted that in a 3-bi-slant submanifold M, if the slant angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal, then M is not necessarily a 3-slant submanifold. For example, by taking $cos\theta = \frac{1}{\sqrt{3}}$ and $cos\theta = \frac{1}{\sqrt{2}}$, in Examples 2 and 3 respectively, M is 3-bi-slant but not 3-slant. A sufficient condition for a 3-bi-slant submanifold with slant angles $\theta_1 = \theta_2$ to be a 3-slant submanifold is the following

$$g(X, \phi_i Y) = 0, \forall X \in \mathcal{D}_1, \ \forall Y \in \mathcal{D}_2 \text{ and } i \in \{1, 2, 3\}.$$

Theorem 1 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ such that TM can be decomposed to three orthogonal distributions $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi_1, \xi_2, \xi_3 >$. Then M is a 3-bi-slant submanifold if and only if for $\alpha = 1, 2$, there exists a constant $\lambda_\alpha \in [-1, 0]$ such that:*

$$T_i T_j X = \lambda_\alpha X, \quad \forall X \in \mathcal{D}_\alpha \text{ and } i, j \in \{1, 2, 3\}. \tag{13}$$

Moreover, in that case $\lambda_\alpha = -cos^2\theta_\alpha$, in which θ_α is the slant angle of distribution $\mathcal{D}_\alpha$.

Proof Let $X \in \mathcal{D}_\alpha$ and β_α and θ_α be the angles $(\widehat{\phi_i X, T_j X})$ and $(\widehat{\phi_j X, T_j X})$, respectively. Using (4), (5), and (11), implies

$$cos\beta_\alpha = \frac{g(\phi_i X, T_j X)}{|\phi_i X|.|T_j X|} = -\frac{g(X, \phi_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|}, \tag{14}$$

$$cos\theta_\alpha = \frac{g(\phi_j X, T_j X)}{|\phi_j X|.|T_j X|} = -\frac{g(X, T_j T_j X)}{|X||T_j X|}. \tag{15}$$

Therefore, if (13) is satisfied, then the angles are equal. On the other hand, we have

$$cos\theta_\alpha = \frac{g(\phi_j X, T_j X)}{|\phi_j X|.|T_j X|} = \frac{g(T_j X, T_j X)}{|X||T_j X|} = \frac{|T_j X|}{|X|}, \tag{16}$$

and then from (15) and (16) it follows that

$$cos^2\theta_\alpha = -\frac{g(X, T_j T_j X)}{|X|^2}. \tag{17}$$

Thus, if $T_i T_j X = \lambda_\alpha X$, then $\lambda_\alpha = -cos^2\theta_\alpha$ and θ_α is constant. Conversely if M is a 3-bi-slant submanifold then β_α and θ_α are equal and constant. Thus, (17) is satisfied and it implies $T_i T_j X = -cos^2\theta_\alpha X$. □

Since 3-semi-slant submanifolds are 3-bi-slant submanifolds with $\theta_1 = 0$, the previous theorem is true for 3-semi-slant submanifolds too.
The following theorem generalizes Theorem 5.1 of [3], for 3-semi-slant submanifolds of an almost contact metric 3-structure.

Theorem 2 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ such that all the structure vector fields are tangent to M. Then M is a 3-semi-slant submanifold if and only if $\exists\lambda \in [-1, 0)$ such that for $i, j \in \{1, 2, 3\}$:*

(a) $\mathcal{D} = \{X \in TM \setminus <\xi_1, \xi_2, \xi_3> \mid T_iT_jX = \lambda X\}$ is a distribution.

(b) $\forall X \in TM$, orthogonal to $\mathcal{D}$, $N_iX = 0$.

Moreover, in that case $\lambda = -cos^2\theta$, in which θ is the slant angle of M.

Proof If M is 3-semi-slant, then by taking $\lambda = -cos^2\theta$ and using Theorem 1, we get $\mathcal{D} = \mathcal{D}_2$. On the other hand, since $\mathcal{D}_1$ is invariant, $\forall X \in TM$, orthogonal to $\mathcal{D}$, $N_iX = 0$. Conversely, if we take $TM = \mathcal{D}^\perp \oplus \mathcal{D} \oplus <\xi_1, \xi_2, \xi_3>$, then (b) implies that $\mathcal{D}^\perp$ is invariant. Using (a) and by the same way as in the proof of Theorem 1, it can be proved that $\mathcal{D}$ is 3-slant with slant angle θ satisfying $\lambda = -cos^2\theta$. Thus, M is 3-semi-slant. □

Corollary 1 *Let M be a 3-semi-slant submanifold of $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ with slant angle θ. Then for all $X, Y \in TM$ we have*

$$g(T_iX, T_jP_2Y) = cos^2\theta g(X, P_2Y), \tag{18}$$

$$g(N_iX, N_jP_2Y) = -g(X, \phi_kP_2Y) - cos^2\theta g(X, P_2Y), \tag{19}$$

$$g(N_iX, N_iP_2Y) = sin^2\theta g(X, P_2Y). \tag{20}$$

Proof Using (5) and statement (a) of Theorem 2, implies

$$g(T_iX, T_jP_2Y) = -g(X, T_iT_jP_2Y) = cos^2\theta g(X, P_2Y).$$

Since $\mathcal{D}_2$ is orthogonal to structure vector fields, by (3), (5), (11), and (18) we have

$$\begin{aligned} -g(X, \phi_kP_2Y) &= g(\phi_iX, \phi_jP_2Y) = g(T_iX + N_iX, T_jP_2Y + N_jP_2Y) \\ &= cos^2\theta g(X, P_2Y) + g(N_iX, N_jP_2Y). \end{aligned}$$

By using (4), (11), and (18), equation (20) can be easily proved. □

Theorem 3 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ and $\xi_1, \xi_2, \xi_3 \in TM$. Then M is a 3-semi-slant submanifold if and only if $\exists\lambda \in [-1, 0)$ such that for $i, j \in \{1, 2, 3\}$:*

(a) $\mathcal{D} = \{X \in TM \setminus <\xi_1, \xi_2, \xi_3> \mid t_jN_iX = -T_kX - \lambda X\}$ is a distribution.

(b) $\forall X \in TM$, orthogonal to $\mathcal{D}$, $N_iX = 0$.

Proof Let $X \in TM \setminus <\xi_1, \xi_2, \xi_3>$. Applying ϕ_j to (11) implies

$$-\phi_kX = T_jT_iX + t_jN_iX + N_jT_iX + n_jN_iX. \tag{21}$$

Taking tangential and normal parts of (21),

$$-T_kX = T_jT_iX + t_jN_iX, \quad -N_kX = N_jT_iX + n_jN_iX. \tag{22}$$

If M is 3-semi-slant, then by putting $\mathcal{D} = \mathcal{D}_2$ and using (22) and statement (a) of Theorem 2, we obtain $t_jN_iX = -T_kX + cos^2\theta X$ and also for all $X \in \mathcal{D}^\perp$, $N_iX = 0$. Conversely, by virtue of (22) and (a), we have

$$T_jT_iX = -T_kX - t_jN_iX = \lambda X,$$

thus by Theorem 2, M is 3-semi-slant. □

4. 3-Semi-slant submanifolds of 3-cosymplectic and 3-Sasakian manifolds

Now, we study some geometric properties of the distributions of a 3-semi-slant submanifold when the ambient manifold is a 3-cosymplectic or a 3-Sasakian manifold.

Let M be a 3-semi-slant submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ and $X \in TM\setminus <\xi_1, \xi_2, \xi_3>$. Then if $\tilde{M}$ is a 3-cosymplectic manifold, using (8) it follows that $g([\xi_i, \xi_j], X) = g(\tilde{\nabla}_{\xi_j}\xi_i - \tilde{\nabla}_{\xi_i}\xi_j, X) = 0$, and if $\tilde{M}$ is a 3-Sasakian manifold, then $g([\xi_i, \xi_j], X) = g(-\phi_i\xi_j + \phi_j\xi_i, X) = -2g(\xi_k, X) = 0$. Therefore, the distribution $\mathcal{D}_3 =<\xi_1, \xi_2, \xi_3>$ is integrable in both cases.

On the other hand, if $\tilde{M}$ is a 3-cosymplectic manifold, then we have $0 = \tilde{\nabla}_{\xi_j}\xi_i = \nabla_{\xi_j}\xi_i + B(\xi_i, \xi_j)$. Thus, $B(\xi_i, \xi_j) = 0$. Furthermore, if $\tilde{M}$ is a 3-Sasakian manifold, $\nabla_{\xi_j}\xi_i + B(\xi_i, \xi_j) = \tilde{\nabla}_{\xi_j}\xi_i = -\phi_i\xi_j = -\xi_k$. Thus, $B(\xi_i, \xi_j) = 0$ and so the distribution $\mathcal{D}_3 =<\xi_1, \xi_2, \xi_3>$ is totally geodesic in both cases. Therefore, we can state the following theorem.

Theorem 4 *Let M be a 3-semi-slant submanifold of a 3-cosymplectic or a 3-Sasakian manifold. Then the distribution $\mathcal{D}_3 =<\xi_1, \xi_2, \xi_3>$ is integrable and totally geodesic.*

Theorem 5 *Let M be a 3-semi-slant submanifold of an almost contact metric 3-structure manifold $(\tilde{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. If $\tilde{M}$ is a 3-cosymplectic manifold, then the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2$ is integrable. However, if $\tilde{M}$ is 3-Sasakian, then the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2$ is not integrable.*

Proof Let $\tilde{M}$ be a 3-cosymplectic manifold. Then for all $X, Y \in \mathcal{D}_1 \oplus \mathcal{D}_2$ and $i \in \{1,2,3\}$, (8) implies $g([X,Y], \xi_i) = g(\tilde{\nabla}_YX - \tilde{\nabla}_XY, \xi_i) = -g(X, \tilde{\nabla}_Y\xi_i) + g(Y, \tilde{\nabla}_X\xi_i) = 0$. Thus, $\mathcal{D}_1 \oplus \mathcal{D}_2$ is integrable.

If $\tilde{M}$ is a 3-Sasakian manifold, then (8) implies for all $X, Y \in \mathcal{D}_1 \oplus \mathcal{D}_2$

$$g([X,Y], \xi_i) = -g(X, \tilde{\nabla}_Y\xi_i) + g(Y, \tilde{\nabla}_X\xi_i) = -2g(X, \phi_iY) = -2g(X, T_iY), \tag{23}$$

which shows that $[X, Y]$ is not in $\mathcal{D}_1 \oplus \mathcal{D}_2$. □

Note that if $X, Y \in \mathcal{D}_1$ or $X, Y \in \mathcal{D}_2$, then (23) is satisfied too, and so if M is a 3-semi-slant submanifold of a 3-Sasakian manifold, then the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are not integrable in general. Moreover, from (23) it follows that if $\mathcal{D}_2$ is integrable then the slant angle of this distribution is $\theta = \frac{\pi}{2}$.

Proposition 1 *Let M be a 3-semi-slant submanifold of a 3-cosymplectic or a 3-Sasakian manifold $\tilde{M}$. Then for all $X,Y \in \mathcal{D}_1$*

$$P_1(\nabla_X \phi_i)Y = 0, \ \forall i \in \{1,2,3\}. \tag{24}$$

Proof First, we show that if M is a 3-semi-slant submanifold, then $\phi_i(\mathcal{D}_1^\perp) \subset \mathcal{D}_1^\perp$. Let $Z \in \mathcal{D}_1^\perp$ and $X \in \mathcal{D}_1$. Since $\mathcal{D}_1$ is invariant, using (5) implies

$$g(\phi_i Z, X) = -g(Z, \phi_i X) = 0.$$

Now, let $\tilde{M}$ be a 3-cosymplectic manifold. Then, by Gauss formula, we obtain

$$(\tilde{\nabla}_X \phi_i)Y = (\nabla_X \phi_i)Y + B(X, \phi_i Y) - \phi_i B(X,Y) = 0, \tag{25}$$

for all $X,Y \in \mathcal{D}_1$. Since $B(X,\phi_i Y) - \phi_i B(X,Y) \in \mathcal{D}_1^\perp$, applying P_1 on (25) it follows that

$$P_1(\nabla_X \phi_i)Y = 0.$$

If $\tilde{M}$ is a 3-Sasakian manifold, then by (7)

$$(\tilde{\nabla}_X \phi_i)Y = (\nabla_X \phi_i)Y + B(X, \phi_i Y) - \phi_i B(X,Y) = g(X,Y)\xi_i, \tag{26}$$

for all $X,Y \in \mathcal{D}_1$, since $\eta_i(Y) = 0$. Applying P_1 to (26) completes the proof. □

Equation (24) shows that if $\mathcal{D}_1$ is an integrable distribution, then the ambient 3-cosymplectic or 3-Sasakian manifold induces a 3-cosymplectic structure on $\mathcal{D}_1$. Now the following theorem shows an interesting geometric property of distribution $\mathcal{D}_1$.

Theorem 6 *Let M be a 3-semi-slant submanifold of a 3-cosymplectic or a 3-Sasakian manifold $\tilde{M}$. Then the distribution $\mathcal{D}_1$ is integrable if and only if $\mathcal{D}_1$ is totally geodesic.*

Proof By taking the normal parts, both equalities (25) and (26) imply

$$N_i P_2 \nabla_X Y = -B(X, \phi_i Y) + N_i B(X,Y), \forall X,Y \in \mathcal{D}_1. \tag{27}$$

By interchanging the role of X and Y in (27), since $B(X,Y) = B(Y,X)$, we obtain

$$N_i P_2 [X,Y] = B(X, \phi_i Y) - B(\phi_i X, Y), \forall X,Y \in \mathcal{D}_1. \tag{28}$$

Equation (28) shows that $\mathcal{D}_1$ is integrable if and only if

$$B(X, \phi_i Y) = B(\phi_i X, Y). \tag{29}$$

On the other hand, from (3) and (29), we get $B(\phi_i X, Y) = B(X, \phi_i Y) = B(X, \phi_j \phi_k Y) = B(\phi_j X, \phi_k Y) = B(\phi_k \phi_j X, Y) = -B(\phi_i X, Y)$. It follows that $B(X,Y) = 0, \forall X,Y \in \mathcal{D}_1$ and thus $\mathcal{D}_1$ is totally geodesic. Conversely, if $\mathcal{D}_1$ is totally geodesic, (28) implies $[X,Y] \in \mathcal{D}_1$. □

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