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# SEMI-STABLE PROCESSES ON LOCAL FIELDS

KUMI YASUDA

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**Abstract.** Some characters of semi-stable stochastic processes on local fields such as epochs, spans, and indices are given, and differences in nature from the corresponding objects for Euclidean spaces are clarified. Criteria for the recurrence and for the polarity of one point sets are given, and it is shown that semi-stable processes are characterized as limits of suitably scaled sums of independent identically distributed random variables.

**1. Introduction.** Since 1980's $p$-adic stochastic analysis has been discussed in connection with $p$-adic physics. In particular the fundamental solutions to $p$-adic heat equations are given by transition densities of rotation-symmetric $p$-adic semi-stable processes, and with respect to this point research of stochastic processes is expected to contribute to physical problems. Recent developments in $p$-adic mathematical physics are summarized by Vladimirov-Volovich-Zelenov ([14]), and Kochubei ([4]) indicates some remarkable relations between $p$-adic stochastic analysis and physics.

Rotation-symmetric additive processes on the $p$-adics, including rotation-symmetric semi-stable processes, were constructed by Albeverio-Karwowski ([1]), and their properties were investigated by the present author ([15]). She also showed in [16] limit theorems on groups, and gave a characterization of infinitely divisible and semi-selfdecomposable distributions as limits of sums of infinitesimal independent random variables. In particular, $p$-adic valued rotation-symmetric semi-stable processes are limits of suitably scaled sums of rotation-symmetric independent identically distributed random variables.

This article aims at characterizing semi-stable processes on local fields, generalizing the results in [15] and [16] even to non-symmetric case. Section 2 is a characterization of semi-stable processes, where the range of epochs and the indices of processes are given in contrast with semi-stable processes on Euclidean $n$-space $\boldsymbol{R}^n$. The theory of $\boldsymbol{R}^n$-valued semi-selfsimilar processes have been established by Sato, Maejima, Watanabe et al. (e.g., [6, 7, 9, 12, 13]). Section 3 is devoted to criteria for the recurrence and for the polarity of one point sets. In Section 4 we give several limit theorems. Theorem 4.1 claims that transition probabilities of semi-stable processes are characterized as limit distributions of sums of i.i.d. random variables. This assertion is essentially equivalent to the result given by Kochubei ([3]). Maejima-Shah ([8]) deals with a more general concept, i.e., operator semi-stable measures, where a corresponding limit theorem is given. Our main objective is Theorem 4.5, which

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gives a convergence in a stronger sense, namely semi-stable processes are realized as scaling limits of sums of i.i.d. random variables in the space of right continuous paths with left limits.

Throughout this article, $K$ denotes a local field of characteristic 0. Namely, $K$ is a finite algebraic extension of the $p$-adic field $\boldsymbol{Q}_p$ for some prime $p$. Let $N_{K/\boldsymbol{Q}_p} : K \to \boldsymbol{Q}_p$ be the norm map of the field extension $K/\boldsymbol{Q}_p$, and let $|\ |_p$ be the $p$-adic norm on $\boldsymbol{Q}_p$. Then

$$\|x\| := |N_{K/\boldsymbol{Q}_p}(x)|_p\,, \quad x \in K\,,$$

defines a norm on the field $K$. We denote the ring of integers in $K$ by $R := \{x \in K \mid \|x\| \leq 1\}$. Then $P := \{x \in K \mid \|x\| < 1\}$ is the unique prime ideal of the ring $R$. Let $q$ be the module of $K$ (i.e., $q$ is the cardinality of the residue field $R/P$), and take a complete set of representatives $\mathcal{A}$ of the classes of $R$ modulo $P$. If we fix a prime element $\omega$, then any non-zero element $x$ of $K$ is uniquely represented by a series

$$x = \sum_{i=m}^{\infty} a_i \omega^i\,, \tag{1}$$

with $m \in \boldsymbol{Z}$ and $a_i \in \mathcal{A}$, $a_m \neq 0$. For those elements $x$ of $K$ having a representation (1), it holds that $\|x\| = q^{-m}$.

We denote by $dx$ the Haar measure on $K$ normalized so that $\int_R 1dx = 1$, and by $\mathrm{vol}(\cdot)$ the volume with respect to the Haar measure ; $\mathrm{vol}(B) := \int_B 1dx$, $B$ being a measurable subset of $K$. We write $B_m := \omega^{-m}R = \{x \in K \mid \|x\| \leq q^m\}$ for integers $m$. Then it can be verified that $\mathrm{vol}(B_m) = q^m$.

We fix a character $\chi_1$ of $K$ with rank 0. Namely, $\chi_1$ is a homomorphism on the additive group $K$ to the multiplicative group $S^1 = \{w \in \boldsymbol{C} \mid |w| = 1\}$, such that $\chi_1(B_0) = \{1\}$ and $\chi_1(B_1) \neq \{1\}$. For each $y$ in $K$, $\chi_y(\cdot) := \chi_1(y\cdot)$ gives a character on $K$, and by means of this correspondence $y \mapsto \chi_y$, the additive group $K$ is isomorphic to its character group. The formula

$$\int_{B_m} \chi_y(x)dx = \begin{cases} q^m\,, & \|y\| \leq q^{-m}\,, \\ 0\,, & \|y\| > q^{-m}\,, \end{cases} \tag{2}$$

(see e.g., [5]) will be frequently used in the subsequent sections.

The *characteristic function* $\hat{\mu}$ of a probability measure $\mu$ on $K$ is the complex-valued function on $K$, defined by

$$\hat{\mu}(y) := \int_K \chi_y(x)\mu(dx)\,.$$

**2. Semi-stable processes and characteristic functions.** A stochastic process $X(t)$, $t \geq 0$, on $K$ with $X(0) = 0$ is a Lévy process if it is stochastically continuous, temporally homogeneous, and if it has independent increments and right continuous paths with left limits. Semi-stable processes on $K$ are $K$-valued Lévy processes $X(t)$ satisfying the semi-selfsimilarity $\{X(at)\} = \{bX(t)\}$ in law for some $a > 0, \neq 1$, and $b \neq 0$ in $K$.

Let us fix a semi-stable process $X(t) \not\equiv 0$ on $K$, and denote by $\mu_t$ its transition probability. Then there are $a > 0$ and $b \in K$ such that $\{X(at)\} = \{bX(t)\}$ in law, but not unique.

Indeed, we can see by iteration that $\{X(a^n t)\} = \{b^n X(t)\}$ holds for any integer $n$. Furthermore, even for a fixed $a$, there may correspond some distinct $b$'s. The following proposition describes the structure of the set of such $a$'s, as well as a relation between $a$ and $b$. The assertion and the proof are analogous to known results in $\boldsymbol{R}^n$ (Lemma 13.8 and Theorem 13.11 in [12]), except some differences indicated in the subsequent Remark.

We set $\Gamma = \Gamma(X) := \{a > 0 \mid \{X(at)\} = \{bX(t)\}$ in law for some $b \in K\}$.

PROPOSITION 2.1. (i) *If $b$ and $b'$ in $K$ satisfy $\{bX(t)\} = \{b'X(t)\}$ in law, then* $\|b\| = \|b'\|$.

(ii) *Put $a_0 := \inf(\Gamma \cap (1, \infty))$. Then $a_0 > 1$ and $\Gamma$ is the cyclic group generated by* $a_0$.

(iii) *There exists $\alpha > 0$ such that for any $a \in \Gamma$ and any corresponding $b \in K$, it holds that $a = \|b\|^\alpha$.*

PROOF. (i) Suppose $\|b\| > \|b'\|$ and put $\tilde{b} = b'/b$. Then $\{X(t)\} = \{\tilde{b}X(t)\}$ in law. By iteration we have $\{X(t)\} = \{\tilde{b}^n X(t)\}$ for any $n \geq 1$, and since $\|\tilde{b}\| < 1$, we obtain a contradiction $X(t) \equiv 0$.

(ii) We shall show that $\Gamma$ is a proper closed subgroup of $(0, +\infty)$. Suppose $\Gamma = (0, +\infty)$, fix $a \in \Gamma$, and take $b \in K$ such that $\{X(at)\} = \{bX(t)\}$. Put $\|b\| = q^n$. Then by the assumption we have $a' := a^{1/(2n)} \in \Gamma$, and can take $b' \in K$ such that $\{X(a't)\} = \{b'X(t)\}$. Since $\{(b')^{2n}X(t)\} = \{X((a')^{2n}t)\} = \{X(at)\} = \{bX(t)\}$, (i) implies $\|b'\| = \|b\|^{1/(2n)} = q^{1/2}$, which is impossible. Therefore we have $\Gamma \neq (0, +\infty)$.

It is evident that $1 \in \Gamma$. Suppose $a$ and $a'$ belong to $\Gamma$. Then there exist $b$ and $b'$ in $K$ such that $\{X(at)\} = \{bX(t)\}$ and $\{X(a't)\} = \{b'X(t)\}$ in law. Then it holds that $\{X(aa't)\} = \{bX(a't)\} = \{bb'X(t)\}$, and therefore $aa'$ belongs to $\Gamma$. For $a \in \Gamma$ and $b \in K$ such that $\{X(at)\} = \{bX(t)\}$, we have $\{b^{-1}X(t)\} = \{X(a^{-1}t)\}$ and hence $a^{-1} \in \Gamma$. Assume that $a_n \in \Gamma$, $n = 1, 2, \ldots$, satisfy $a_n \to a \in (0, +\infty)$, and take $b_n \in K$ such that $\{X(a_n t)\} = \{b_n X(t)\}$. Since $X(t)$ is stochastically continuous, $X(a_n t)$ converges to $X(at)$ in law. Suppose $\{b_n\}$ is non-compact in $K$, and take a subsequence $\{n(k)\}_{k=1,2,\ldots}$ of $\boldsymbol{N}$ so that $\|b_{n(k)}\| \to +\infty$. Then we obtain $\hat{\mu}_t(y) = \hat{\mu}_{a_{n(k)}t}(b_{n(k)}^{-1}y) \to \hat{\mu}_{at}(0) = 1$ for any $y \in K$, which contradicts that $X(t) \not\equiv 0$. Therefore there exists an accumulation point $b \in K$ of $\{b_n\}$. If we take a subsequence $\{n(k)\}$ so that $b_{n(k)} \to b$, then we have $\{X(at)\} = \{\lim_{k\to\infty} b_{n(k)}X(t)\} = \{bX(t)\}$, and hence $a$ belongs to $\Gamma$. Thus we have proved that $\Gamma$ is a proper closed group of $\Gamma$.

We can take $c \in (0, +\infty)\setminus\Gamma$, and since $\Gamma$ is closed, it holds that $(r^{-1}c, rc) \subset (0, +\infty)\setminus \Gamma$ for some $r > 1$. If we suppose $a_0 = 1$, we can take $\theta \in (1, r^2) \cap \Gamma$, namely we have $0 < \log\theta < 2\log r$. Then $\log c - \log r < n\log\theta < \log c + \log r$ holds for some $n \in \boldsymbol{N}$. Therefore it holds that $r^{-1}c < \theta^n < rc$. Since $\Gamma$ is a group, we have $\theta^n \in \Gamma$, a contradiction.

We have $\{a_0^n\}_{n\in\boldsymbol{Z}} \subset \Gamma$, since $\Gamma$ is a closed subgroup of $(0, +\infty)$. Suppose that there exists $c \in \Gamma \setminus \{a_0^n\}$ and take $n_0 \in \boldsymbol{Z}$ such that $a_0^{n_0} < c < a_0^{n_0+1}$. Then we get a contradiction $ca_0^{-n_0} \in \Gamma$, $1 < ca_0^{-n_0} < a_0$.

(iii) Take $b_0 \in K$ for which $\{X(a_0 t)\} = \{b_0 X(t)\}$ holds, and put $\alpha := \log a_0 / \log \|b_0\|$. Any $a$ in $\Gamma$ is represented as $a = a_0^n$ with some integer $n$, and if $b \in K$ is such that $\{X(at)\} = \{bX(t)\}$, then we have $\{bX(t)\} = \{X(a_0^n t)\} = \{b_0^n X(t)\}$. Hence (i) implies $\|b\| = \|b_0^n\| = a^{1/\alpha}$. □

We call an element $a$ of $\Gamma$ an *epoch*, a corresponding $b \in K$ a *span*, and the $\alpha$ the *index* of the semi-stable process $X(t)$.

REMARK 2.2. The indices of $\boldsymbol{R}^d$-valued semi-stable processes take values in $(0, 2]$ (Theorem 13.15 in [12]), while for any positive number $\alpha$, there exists a $K$-valued semi-stable process having index $\alpha$. Indeed, for every $\alpha > 0$, there is a rotation-symmetric semi-stable process on $K$ of index $\alpha$ having characteristic function $\hat{\mu}_t(y) = \exp(-t\|y\|^\alpha)$ (Proposition 4.1 in [15]).

For a $\boldsymbol{R}^d$-valued semi-stable process, it may occur that $a_0 = 1$ and $\Gamma = (0, \infty)$, in which case the process is called stable. As seen in (ii) of the above proposition, there exists no stable process on $K$.

Let $\{\mu_t\}_{t \geq 0}$ be a one-parameter convolution semigroup of probability measures on $K$ which converges weakly to the $\delta$-measure at the origin as $t \to 0$. Since the field $K$ is totally disconnected, the characteristic function has a canonical representation

$$\hat{\mu}_t(y) = \chi_y(x_0) \exp\left( t \int_K (\chi_y(x) - 1)\nu(dx) \right), \tag{3}$$

where $x_0 \in K$, and $\nu$ is a $\sigma$-finite measure with $\nu(N^c) < \infty$ for any neighborhood $N$ of the origin and $\int_K (1 - \mathrm{Re}\, \chi_y(x))\nu(dx) < \infty$ for any $y \in K$. The $x_0$ and $\nu$ are uniquely determined by $\{\mu_t\}_{t \geq 0}$ (Theorem 10.1 and Remark 1 following Corollary 7.1 in [10]).

LEMMA 2.3. *A Lévy process $X(t)$ on $K$ is semi-stable with an epoch $a$ and the corresponding span $b$ if and only if in the canonical representation* (3) *of its transition probability $\mu_t$, it holds that $x_0 = 0$ and $\nu(b^{-1}dx) = a\nu(dx)$.*

PROOF. This is immediate from $X_0 = 0$ a.s., $\hat{\mu}_{at}(y) = \hat{\mu}_t(by)$, and the uniqueness of the representation (3). □

**3. Absolute continuity, recurrence, and polarity.** In this section we will investigate some properties of semi-stable processes. We begin with showing the absolute continuity of the transition probabilities.

PROPOSITION 3.1. *The transition probability $\mu_t$ of a semi-stable process $X(t) \not\equiv 0$ on $K$ is absolutely continuous relative to the Haar measure, and the Radon-Nikodým derivative is given by*

$$p_t(x) = \int_K \chi_x(-y)\hat{\mu}_t(y)dy\,.$$

According to general theory of Fourier analysis on groups, the proposition is proved if we can show that $\hat{\mu}_t$ is integrable (see, e.g., Section 1.7 in [11]).

PROOF. Let $a < 1$ be an epoch and $b$ a corresponding span, and put $\|b\| = q^{-k}$, $k > 0$. For $y \in K$ with $\|y\| = q^{-ks-l}$, $s \in \mathbf{Z}$, $0 \le l \le k-1$, by the canonical representation (3) and Lemma 2.3 it holds that

$$
\begin{aligned}
|\hat{\mu}_t(y)| &= \exp\Big( - t \int_K (1 - \operatorname{Re} \chi_y(x))\nu(dx)\Big) \\
&= \exp\Big( - a^s t \int_K (1 - \operatorname{Re} \chi_y(b^{-s}x))\nu(dx)\Big) .
\end{aligned}
$$

Since $1 - \operatorname{Re} \chi_y(b^{-s}x) \ge 0$, we proceed to

$$
|\hat{\mu}_t(y)| \le \exp\Big( - a^s t \int_{\|x\|=q^{l+1}} (1 - \operatorname{Re} \chi_y(b^{-s}x))\nu(dx)\Big) .
$$

On the circle $\{\|x\| = q^{l+1}\}$, there exists $c < 1$ such that $\operatorname{Re} \chi_y(b^{-s}x) \le c$. On the other hand, there exists $0 \le l_0 \le k-1$ such that $\nu(\|x\| = q^{l_0+1}) > 0$. Indeed, supposing the contrary, we have $\nu(q \le \|x\| \le q^k) = 0$, and by the self-similarity of $\nu$, $\nu(K \setminus \{0\}) = \sum_{i=-\infty}^{\infty} \nu(q \le \|b^i x\| \le q^k) = \sum_{i=-\infty}^{\infty} a^i \nu(q \le \|x\| \le q^k) = 0$, which contradicts that $X(t) \not\equiv 0$. Hence we have

$$
\begin{aligned}
\int_K |\hat{\mu}_t(y)| dy &\le \sum_{s=-\infty}^{\infty} \sum_{l=0}^{k-1} \exp\big( - a^s t(1-c)\nu\big(\|x\| = q^{l+1}\big)\big) \operatorname{vol}\big(\{\|y\| = q^{-ks-l}\}\big) \\
&\le \sum_{s=-\infty}^{\infty} \exp\big( - a^s t(1-c)\nu\big(\|x\| = q^{l_0+1}\big)\big)(1-q^{-1})q^{-ks-l_0} \\
&< \infty .
\end{aligned}
$$

□

The next proposition gives a criterion for the recurrence and the polarity of one point sets according to the index.

PROPOSITION 3.2. *A semi-stable process $X(t)$ of index $\alpha$ is recurrent if and only if $\alpha \ge 1$. It visits almost every point with probability* 1 *if and only if $\alpha > 1$.*

PROOF. Take an epoch $a < 1$, and let $b$ be a corresponding span with $\|b\| = a^{1/\alpha} =: q^{-k} < 1$. Recall Formula (2), and put

$$
h_m(t, y) := q^m \hat{\mu}_t(y) \mathbf{1}_{B_{-m}}(y) = \int_{B_m} \chi_x(-y)\hat{\mu}_t(y) dx .
$$

As in the proof of Proposition 3.1, we can take $0 \le l_0 \le k-1$ such that $\nu\big(\|x\| = q^{l_0+1}\big) > 0$.

First of all, we assume that $\alpha < 1$. Let $\|y\| = q^n \le q^{-m}$, and put $-n = ks + l$ with $s \in \mathbf{Z}$ and $0 \le l \le k-1$. Then, similarly as in the proof of Proposition 3.1, we have

$$\begin{aligned}
&\int_0^\infty dt \int_K |h_m(t,y)|dy \\
&\quad = q^m \int_0^\infty dt \int_{B_{-m}} \left|\hat{\mu}_t(y)\right| dy \\
&\quad \le q^m \int_0^\infty dt \sum_{s=[m/k]}^{\infty} \exp\big(-a^s t(1-c)\nu\big(\|x\| = q^{l_0+1}\big)\big)(1-q^{-1})q^{-ks-l_0}\,.
\end{aligned}$$

By Fubini's theorem we proceed to

$$\begin{aligned}
&\int_0^\infty dt \int_K |h_m(t,y)|dy \\
&\quad \le (1-q^{-1})q^{m-l_0}(1-c)^{-1}\nu\big(\|x\| = q^{l_0+1}\big)^{-1} \sum_{s=[m/k]}^{\infty} p^{-k(1-\alpha)s} < \infty\,.
\end{aligned}$$

Therefore, by Proposition 3.1 and Fubini's theorem, we obtain

$$\begin{aligned}
\int_0^\infty \mu_t(B_m)dt &= \int_0^\infty dt \int_{B_m} dx \int_K \chi_{-x}(y)\hat{\mu}_t(y)dy \\
&= \int_0^\infty dt \int_K h_m(t,y)dy < \infty\,,
\end{aligned}$$

and thus $X(t)$ is transient provided $\alpha < 1$.

Let $\alpha$ be arbitrary. By Proposition 3.1 and Fubini's theorem, we have for $\lambda > 0$,

$$\begin{aligned}
&\int_0^\infty dt \int_K \exp(-\lambda t)|h_m(t,y)|dy \\
&\quad = q^m \int_0^\infty dt \int_{B_{-m}} \exp(-\lambda t)|\hat{\mu}_t(y)|dy \\
&\quad \le q^m \int_0^\infty dt \sum_{s=[m/k]}^{\infty} (1-q^{-1})q^{-ks-l_0} \exp\big(-t\big(\lambda + a^s(1-c)\nu\big(\|x\| = p^{l_0+1}\big)\big)\big) \\
&\quad = (1-q^{-1})q^{m-l_0} \sum_{s=[m/k]}^{\infty} \frac{q^{-ks}}{\lambda + a^s(1-c)\nu\big(\|x\| = p^{l_0+1}\big)} \\
&\quad \le (1-q^{-1})q^{m-l_0}\lambda^{-1} \sum_{s=[m/k]}^{\infty} q^{-ks} < \infty\,.
\end{aligned}$$

Put $\varphi(y) := \int_K (\chi_y(x) - 1)\nu(dx)$. Then Lemma 2.3 implies $\varphi(by) = a\varphi(y)$. Applying the canonical representation (3), Proposition 3.1, and Fubini's theorem, we obtain that

$$
\begin{aligned}
V^\lambda(B_m) &:= \int_0^\infty \exp(-\lambda t)\mu_t(B_m)dt \\
&= \int_0^\infty dt \int_K \exp(-\lambda t) h_m(t, y)dy \\
&= q^m \int_{B_{-m}} dy \int_0^\infty \exp(-\lambda t)\hat{\mu}_t(y)dt \\
&= q^m \int_{B_{-m}} (\lambda - \varphi(y))^{-1} dy\,.
\end{aligned} \tag{4}
$$

Now we assume that $\alpha \geq 1$. Then it holds that

$$
\begin{aligned}
\int_{B_{-m}} \operatorname{Re}(-\varphi(y))^{-1} dy &\geq \sum_{s=[m/k]+1}^{\infty} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-ks-l}} \operatorname{Re}(-\varphi(y))^{-1} dy \\
&= \sum_{s=[m/k]+1}^{\infty} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} \operatorname{Re}(-\varphi(b^s y))^{-1} q^{-ks} dy \\
&= \sum_{s=[m/k]+1}^{\infty} q^{(\alpha-1)ks} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} \operatorname{Re}(-\varphi(y))^{-1} dy \\
&= \infty\,.
\end{aligned}
$$

Since $V^\lambda(B_m)$ is real, the monotone convergence theorem and Fatou's lemma imply that

$$
\begin{aligned}
\int_0^\infty \mu_t(B_m)dt &= \lim_{\lambda \downarrow 0} V^\lambda(B_m) \\
&= \lim_{\lambda \downarrow 0} q^m \int_{B_{-m}} \operatorname{Re}(\lambda - \varphi(y))^{-1} dy \\
&\geq q^m \int_{B_{-m}} \operatorname{Re}(-\varphi(y))^{-1} dy = \infty\,,
\end{aligned}
$$

and hence $X(t)$ is recurrent in case $\alpha \geq 1$.

By the above formula (4), we have

$$
\frac{V^\lambda(B_m)}{\operatorname{vol}(B_m)} = \int_{B_{-m}} \operatorname{Re}(\lambda - \varphi(y))^{-1} dy\,.
$$

If $-n = ks + l$, $s \in \mathbf{Z}$, $0 \leq l \leq k-1$, then Lemma 2.3 implies that

$$
\begin{aligned}
&\int_{\|y\|=q^n} \operatorname{Re}(\lambda - \varphi(y))^{-1} dy \\
&\quad = q^{-ks} \int_{\|y\|=q^{-l}} \frac{\lambda - q^{-\alpha ks} \operatorname{Re}\varphi(y)}{\left(\lambda - q^{-\alpha ks}\operatorname{Re}\varphi(y)\right)^2 + \left(q^{-\alpha ks}\operatorname{Im}\varphi(y)\right)^2} dy\,.
\end{aligned} \tag{5}
$$

Since $\operatorname{Re}\varphi(y) \le 0$, it holds for $m < 0$ that

$$\begin{aligned}
&\frac{V^{\lambda}(B_m)}{\operatorname{vol}(B_m)} \\
&\quad \le \sum_{s=[m/k]}^{\infty} \sum_{l=0}^{k-1} q^{-ks} \int_{\|y\|=q^{-l}} (\lambda - q^{-\alpha ks} \operatorname{Re}\varphi(y))^{-1} dy \\
&\quad \le \sum_{s=[m/k]}^{-1} q^{-ks} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} (q^{-\alpha ks} \operatorname{Re}\varphi(y))^{-1} dy + \sum_{s=0}^{\infty} q^{-ks} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} \lambda^{-1} dy \\
&\quad = \sum_{s=[m/k]}^{-1} q^{(\alpha-1)ks} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} (\operatorname{Re}\varphi(y))^{-1} dy + \lambda^{-1}(1-q^{-k}) \sum_{s=0}^{\infty} q^{-ks} .
\end{aligned}$$

If $\alpha > 1$, the last line converges as $m \to -\infty$, and hence a one-point set is not essentially polar. On the other hand, we derive from (5) that

$$\begin{aligned}
\frac{V^{\lambda}(B_m)}{\operatorname{vol}(B_m)} &\ge \sum_{s=[m/k]+1}^{0} q^{(\alpha-1)ks} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} \frac{q^{\alpha ks}\lambda - \operatorname{Re}\varphi(y)}{\left(q^{\alpha ks}\lambda - \operatorname{Re}\varphi(y)\right)^2 + \left(\operatorname{Im}\varphi(y)\right)^2} dy \\
&\ge \sum_{s=[m/k]+1}^{0} q^{(\alpha-1)ks} \sum_{l=0}^{k-1} \int_{\|y\|=q^{-l}} \frac{-\operatorname{Re}\varphi(y)}{\left(\lambda - \operatorname{Re}\varphi(y)\right)^2 + \left(\operatorname{Im}\varphi(y)\right)^2} dy .
\end{aligned}$$

If we suppose $\operatorname{Re}\varphi(y) = 0$ a.e. on $q^{-(k-1)} \le \|y\| \le 1$, then, since $\varphi(by) = a\varphi(y)$ and $|\chi_y(x)| \equiv 1$, we obtain $\varphi(y) = 0$ a.e. on $K$. This contradicts the non-degeneracy of $X(t)$, and hence the last integral is strictly positive. Then we obtain that, in case $\alpha \le 1$, $V^{\lambda}(B_m)/\operatorname{vol}(B_m)$ diverges as $m \to -\infty$, and as a consequence, a one-point set is essentially polar. □

**4. Limit theorem.** This section is devoted to establishing limit theorems for semi-stable processes. We show first that one dimensional distributions of semi-stable processes are characterized as limit distributions of suitably scaled sums of independent identically distributed random variables. We make a further investigation, in what condition the limit gives a convergence in paths space, and conclude that semi-stable processes are realized as limits of scaled sums in the space of right continuous paths with left limits.

Let $\xi_i$, $i = 1, 2, \dots$, be independent identically distributed $K$-valued random variables. For positive numbers $a_n$ with $\lim_{n\to\infty} a_n = +\infty$, non-zero elements $b_n$ of $K$ $(n = 1, 2, \dots)$, and $t > 0$, we consider a scaled sum $Y_n(t) := b_n \sum_{i=1}^{[a_n t]} \xi_i$. Here, for $h \in \boldsymbol{R}$, $[h]$ denotes the maximum integer which does not exceed $h$.

For a random variable $\xi$, we denote its law by $\mathcal{L}(\xi)$.

THEOREM 4.1. *A Lévy process $X(t)$ on $K$ is semi-stable if and only if there exist independent identically distributed random variables $\xi_i$, $i = 1, 2, \dots$, positive numbers $a$, $a_n$, $n = 1, 2, \dots$, and non-zero elements $b, b_n$, $n = 1, 2, \dots$, in $K$, such that $a_n \to +\infty$,*

$\lim_{n\to\infty} a_{n+1}/a_n = a^{-1}$, $\lim_{n\to\infty} b_{n+1}/b_n = b$, *and for any* $t > 0$ *the random variable* $Y_n(t)$ *converges weakly to* $X(t)$ *as* $n \to \infty$.

PROOF. We assume first the existence of such $\xi_i$, $a_n$, $b_n$, and prove the semi-selfsimilarity of the limit process $X(t)$. If we put $\rho$ for the law of $\xi_i$, then by the assumption we have $\hat{\rho}(b_n y)^{[a_n t]} = \hat{\mathcal{L}}(b_n \sum_{i=1}^{[a_n t]} \xi_i)(y) \to \hat{\mu}_t(y)$ as $n \to \infty$, for every $y \in K$ and $t > 0$. Since $b_{n+1}/b_n \to b$, we can verify that

$$\hat{\rho}(b_{n+1} y)^{[a_n t]} = \hat{\rho}(b_n(b_{n+1}/b_n)y))^{[a_n t]} \to \mu_t(by)\,.$$

On the other hand, the assumptions $a_n \to +\infty$ and $a_{n+1}/a_n \to a^{-1}$ imply that

$$\hat{\rho}(b_{n+1} y)^{[a_n t]} = (\hat{\rho}(b_{n+1} y)^{[a_{n+1} t]})^{[a_n t]/[a_{n+1} t]} \to \mu_t(y)^a = \mu_{at}(y)\,.$$

Therefore we obtain $\mu_t(by) = \mu_{at}(y)$ for every $y$, and hence $X(t)$ is semi-stable.

Conversely, suppose that $X(t)$ is semi-stable, and take an epoch $a < 1$ and a corresponding span $b$. Let $\xi_i$ be an i.i.d. random variables such that $\mathcal{L}(\xi_i) = \mu_1$, and put $a_n := a^{-n}$, $b_n := b^n$. According to the canonical representation (3), it holds that $\hat{\mu}_t(y) = \exp(t\varphi(y))$ with $\varphi(y) := \int_K (\chi_y(x) - 1)\nu(dx)$, and $\varphi(by) = a\varphi(y)$. Then

$$\begin{aligned}\hat{\mathcal{L}}(Y_n(t))(y) &= \hat{\mu}_1(b^n y)^{[a^{-n} t]}\\ &= \exp(\varphi(b^n y))^{[a^{-n} t]}\\ &= \exp(\varphi(y))^{a_n [a^{-n} t]}\\ &\to \mu_t(y), \ \text{as } n \to \infty\,,\end{aligned}$$

and hence for each $t > 0$, $Y_n(t)$ converges to $X(t)$ in law. □

For a metric space $S$, let $\mathcal{D}_S$ denote the space of right-continuous $S$-valued functions on $[0, \infty)$ with left limits. We give a sufficient condition for the relative compactness of the random sequence $Y_n$ in $\mathcal{P}(\mathcal{D}_K)$, the space of probability measures on $\mathcal{D}_K$. As a consequence, we will see that Theorem 4.1 remains valid if the convergence of one-dimensional distributions is substituted by the weak convergence of the random variables $Y_n$ on $\mathcal{D}_K$.

PROPOSITION 4.2. *Suppose there exist* $C > 0$ *and* $0 < r < 1$ *such that*

$$\sup_n a_n P(\|\xi_i\| \geq \|b_n\|^{-1} q^l) \leq C r^l \tag{6}$$

*holds for any integer* $l$. *Then the sequence* $\{Y_n\}_{n\geq 1}$ *is relatively compact in* $\mathcal{P}(\mathcal{D}_K)$.

For a proof, we apply the following criterion (Theorems 8.6 and 8.8 in [2]).

LEMMA 4.3. *Let* $(S, \|\ \|)$ *be a complete separable metric space, and* $\{Z_\alpha(t)\}$ *a family of processes with sample paths in* $\mathcal{D}_S$. *Suppose that*

(i) *for every* $\varepsilon > 0$ *and rational* $t \geq 0$, *there exists a compact set* $F = F(\varepsilon, t)$ *such that*

$$\inf_\alpha P(Z_\alpha(t) \in F^\varepsilon) \geq 1 - \varepsilon\,, \tag{7}$$

*where* $F^\varepsilon := \{x \in S \mid \inf_{y\in F} \|x - y\| < \varepsilon\}$,

(ii) *for any $T > 0$, there exist $\beta > 0$, $C' > 0$, and $\theta > 1$ such that*

$$E((\|Z_\alpha(t+h) - Z_\alpha(t)\| \wedge 1)^{\beta/2}(\|Z_\alpha(t) - Z_\alpha(t-h)\| \wedge 1)^{\beta/2}) \le C'h^\theta \tag{8}$$

*holds for all $\alpha$, $0 \le t \le T+1$, $0 \le h \le t$, and such that*

$$\lim_{\delta \to 0} \sup_\alpha E((\|Z_\alpha(\delta) - Z_\alpha(0)\| \wedge 1)^\beta) = 0\,. \tag{9}$$

*Then $\{Z_\alpha\}$ is relatively compact in $\mathcal{P}(\mathcal{D}_S)$.*

PROOF (Proposition 4.2). We shall verify the conditions in Lemma 4.3. By the non-archimedean property, $\|\xi_i\| < R$, $i = 1, \ldots, n$, implies that $\sum_{i=1}^n \|\xi_i\| < R$, for $R > 0$ and $n \ge 1$. Therefore for any integer $l$,

$$\begin{aligned} P(\|Y_n(t)\| \ge q^l) &= P\left(\left\| \sum_{i=1}^{[a_n t]} \xi_i \right\| \ge \|b_n\|^{-1} q^l\right) \\ &\le P(\|\xi_i\| \ge \|b_n\|^{-1} q^l, 1 \le \exists i \le [a_n t]) \\ &= 1 - P(\|\xi_i\| < \|b_n\|^{-1} q^l)^{[a_n t]} \\ &\le [a_n t] P(\|\xi_i\| \ge \|b_n\|^{-1} q^l) \\ &\le [a_n t] a_n^{-1} C r^l\,, \end{aligned} \tag{10}$$

where we used the inequality $1 - x^n \le n(1-x)$ for $n \ge 1$ and $x \le 1$. For any $\varepsilon > 0$ and rational $t > 0$, take an integer $l = l(\varepsilon, t)$ large enough so that $Cr^l < \varepsilon t^{-1}$ and $q^l > \varepsilon$. Put $F = B_l$ and assume $x \in F^\varepsilon$. Then we can take $y \in F$ for which $\|x - y\| < \varepsilon$, and the non-archimedean inequality implies $\|x\| \le \max(\|y\|, \|x - y\|) \le q^l$. Namely, we obtain $F^\varepsilon = F$, and hence

$$\begin{aligned} P\left(Y_n(t) \in F^\varepsilon\right) &\ge 1 - P(\|Y_n(t)\| \ge q^l) \\ &\ge 1 - [a_n t] a_n^{-1} \varepsilon t^{-1} > 1 - \varepsilon\,. \end{aligned}$$

Thus the condition (i) in Lemma 4.3 is cleared.

For the condition (ii), take $\beta > -2 \log r / \log q$. Since the process $Y_n(t)$ has independent increments, it follows from (10) that

$$\begin{aligned} &E((\|Y_n(t+h) - Y_n(t)\| \wedge 1)^{\beta/2}(\|Y_n(t) - Y_n(t-h)\| \wedge 1)^{\beta/2}) \\ &\quad = E((\|Y_n(t+h) - Y_n(t)\| \wedge 1)^{\beta/2}) E((\|Y_n(t) - Y_n(t-h)\| \wedge 1)^{\beta/2}) \\ &\quad \le \sum_{m=0}^\infty q^{-\beta m/2} P(\|Y_n(t+h) - Y_n(t)\| \ge q^{-m}) \\ &\qquad\qquad \times \sum_{m=0}^\infty q^{-\beta m/2} P(\|Y_n(t) - Y_n(t-h)\| \ge q^{-m}) \\ &\quad \le ([a_n(t+h)] - [a_n t])([a_n t] - [a_n(t-h)]) \left( \sum_{m=0}^\infty q^{-\beta m/2} a_n^{-1} C r^{-m} \right)^2 \end{aligned}$$

for $0 \le h \le t$. In case $2a_n h < 1$, we have either $[a_n(t+h)] = [a_n t]$ or $[a_n t] = [a_n(t-h)]$, and the above expectation is 0. If $2a_n h \ge 1$, then we see that

$$([a_n(t+h)] - [a_n t])([a_n t] - [a_n(t-h)]) \le (a_n(t+h) - a_n(t-h) + 1)^2 \le (4a_n h)^2 ,$$

and then

$$\begin{aligned} &E((\|Y_n(t+h) - Y_n(t)\| \wedge 1)^{\beta/2} (\|Y_n(t) - Y_n(t-h)\| \wedge 1)^{\beta/2}) \\ &\le 16C^2 \biggl( \sum_{m=0}^{\infty} (q^{-\beta/2} r^{-1})^m \biggr)^2 h^2 . \end{aligned}$$

By the assumption $\beta > -2\log r / \log q$, this implies (8). Furthermore, (10) leads to

$$\begin{aligned} E((\|Y_n(\delta)\| \wedge 1)^{\beta}) &\le \sum_{m=0}^{\infty} q^{-\beta m} P(\|Y_n(\delta)\| \ge q^{-m}) \\ &\le \sum_{m=0}^{\infty} q^{-\beta m} [a_n \delta] a_n^{-1} C r^{-m} \\ &\le C\delta \sum_{m=0}^{\infty} (q^{-\beta} r^{-1})^m , \end{aligned}$$

and hence (9) follows immediately. □

REMARK 4.4. Suppose $a_{n+1}/a_n \to a^{-1}$, $b_{n+1}/b_n \to b$, and Condition (6). Then, by Proposition 4.2, there exists a subsequence $n(k)$ of $\boldsymbol{N}$ such that $\{Y_{n(k)}(t)\}_{k=1,2,\dots}$ has a limit in $\mathcal{D}_K$, and Theorem 4.1 implies that the limit $X(t)$ is a semi-stable process. It should be noted that the original sequence $\{Y_n(t)\}$ does not necessarily converge, and as is shown in the following example, it may have distinct accumulation points more than one.

Suppose that $X(t)$ is semi-stable with an epoch $a < 1$ and a corresponding span $b$. Let $\xi_i$ be identically distributed as $X(1)$, and put $b_n := b^n$,

$$a_n := \begin{cases} a^{-n} , & n = 3^k , \\ 3^{-k} n a^{-n} , & 3^k < n \le 2 \cdot 3^k , \\ (4 - 3^{-k} n) a^{-n} , & 2 \cdot 3^k < n < 3^{k+1} , \quad k \in \boldsymbol{N} . \end{cases}$$

It can be verified that $a_{n+1}/a_n \to a^{-1}$, $b_{n+1}/b_n \to b$, and Condition (6) is fulfilled. If we take $n(k) = 3^k$, then as $k \to \infty$ we have

$$\hat{\mathcal{L}}(Y_{n(k)}(t))(y) = \hat{\mu}_1(b^{3^k} y)^{[a^{-3^k} t]} = \hat{\mu}_1(y)^{a^{3^k} [a^{-3^k} t]} \to \hat{\mu}_t(y) .$$

On the other hand, taking $n'(k) = 2 \cdot 3^k$, we obtain

$$\hat{\mathcal{L}}(Y_{n'(k)}(t)(y) = \hat{\mu}_1(b^{2 \cdot 3^k} y)^{[2a^{-2 \cdot 3^k}]} = \hat{\mu}_1(y)^{a^{2 \cdot 3^k} [2a^{-2 \cdot 3^k} t]} \to \hat{\mu}_{2t}(y) .$$

Thus $Y_{n(k)}(t)$ converges to $X(t)$, while $Y_{n'(k)}(t)$ goes to $X(2t)$.

THEOREM 4.5. *Let $X(t)$ be a semi-stable process on $K$ with an epoch $a$ and a span $b$. If $\xi_i$ is identically distributed as $X(1)$, $a_n = a^{-n}$, and $b_n = b^n$, then $Y_n$ weakly converges to $X$ in $\mathcal{D}_K$.*

PROOF. Recalling the proof of Theorem 4.1, it is clear that for every $t$, $Y_n(t)$ converges to $X(t)$. Since $Y_n(t)$ has independent increments, the convergence of every finite dimensional distribution follows, and hence it suffices to show the relative compactness of $\{Y_n\}$ in $\mathcal{P}(\mathcal{D}_K)$. It follows by Proposition 3.1, Fubini's theorem, and Formula (2) that

$$\begin{aligned} P(\|X(t)\| \geq q^l) &= 1 - \int_K dy \hat{\mu}_t(y) \int_{B_{l-1}} \chi_x(-y) dx \\ &= q^{l-1} \int_{B_{-l+1}} (1 - \hat{\mu}_t(y)) dy \,. \end{aligned}$$

Using the canonical representation (3) : $\hat{\mu}_t(y) = \exp(t\varphi(y))$, $\varphi(y) = \int_K (\chi_y(x) - 1)\nu(dx)$, the dominated convergence theorem, and (2), we proceed to

$$\begin{aligned} t^{-1} P(\|X(t)\| \geq q^l) &= q^{l-1} \int_{B_{-l+1}} t^{-1}(1 - \exp(t\varphi(y))) dy \\ &\overset{t\to 0}{\to} q^{l-1} \int_{B_{-l+1}} (-\varphi(y)) dy \\ &= q^{l-1} \int_K \nu(dx) \int_{B_{-l+1}} (1 - \chi_y(x)) dy \\ &= q^{l-1} \int_K q^{-l+1} (1 - \mathbf{1}_{B_{l-1}}(x)) \nu(dx) \\ &= \nu(B_{l-1}^c) \,. \end{aligned} \tag{11}$$

Let $\|b\| = q^{-k}$ and put $s := [l/k]$. Then, by the self-similarity of the Lévy measure $\nu$, we obtain that

$$\nu(B_{l-1}^c) \leq \nu(b^{-s} B_{-1}^c) = a^s \nu(B_{-1}^c) \leq a^{l/k} a^{-1} \nu(B_{-1}^c) \,. \tag{12}$$

Here note that

$$P(\|\xi_i\| \geq \|b^n\|^{-1} q^l) = P(\|b^n X(1)\| \geq q^l) = P(\|X(a^n)\| \geq q^l) \,.$$

Then we have

$$a^{-l/k} \sup_n a^{-n} P(\|\xi_i\| \geq \|b^n\|^{-1} q^l) \leq a^{-l/k} \sup_{t \leq 1} t^{-1} P(\|X(t)\| \geq q^l) \,,$$

and by (11) and (12), this is finite. Therefore (6) is fulfilled, and hence $\{Y_n\}$ is relatively compact. □

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FACULTY OF BUSINESS AND COMMERCE
KEIO UNIVERSITY
KANAGAWA 223–8521
JAPAN

*E-mail address*: yasuda@hc.cc.keio.ac.jp