Semi-Symmetric Metric T-Connection
in an Almost Contact Metric Manifold

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Abstract

In present paper, we studied the properties of semi-symmetric metric $T-$connection in almost contact metric manifolds. It has been shown that a generalised co-symplectic manifold with semi-symmetric metric $T-$connection is a generalised quasi-Sasakian manifold. Further, an almost contact metric manifold equipped with semi-symmetric metric $T-$connection is either projectively or con-circularly flat if and only if it is locally isometric to the hyperbolic space $H^n(-1)$.

1 Introduction

The idea of semi-symmetric metric connection on a Riemannian manifold was initiated by Yano [1]. He proved that a Riemannian metric admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian metric be conformally flat. Various properties of such connection have been studied by Sharfuddin and Husain [2], Imai [3], Pathak and De [5], Barua and Ray [7], Ray [9], Pandey and Dubey

[11] and many others. Ojha and Prasad [4] studied the properties of semi-symmetric non-metric connection in almost Grayan manifold and they proved that a generalised co-symplectic manifold is completely integrable with respect to the semi-symmetric non-metric connection. Ojha [8] further studied almost contact manifolds with specified affine connection D and obtained its applications, especially to curvature tensors and Nijenhuis tensor. Chaubey and Ojha [10] defined and studied semi-symmetric non-metric connections in almost contact metric manifold and quarter-symmetric metric connection in almost contact metric and Einstein manifolds. Mishra and Pandey [6] defined semi-symmetric metric $T-$connections and studied some properties of almost Grayan and Sasakian manifolds. In this paper we studied the properties of semi-symmetric metric T-connection in almost contact metric manifolds. It has been also proved that an almost contact metric manifold M_n equipped with semi-symmetric metric $T-$connection is locally isometric to the hyperbolic space $H^n(-1)$ if and only if it is either projectively or con-circularly flat.

2   Preliminaries

If on an odd dimensional differentiable manifold M_n, $n = 2m + 1$, of differentiability class C^{r+1}, there exist a vector valued real linear function ϕ, a $1-$form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{1}$$

$$\eta(\phi X) = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for arbitrary vector fields X and Y, then (M_n, g) is said to be an almost contact metric manifold and the structure $\{\phi, \eta, \xi, g\}$ is called an almost contact metric structure to M_n [13].

In view of (1), (2) and (3), we find

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \phi(\xi) = 0 \tag{4}$$

An almost contact metric manifold satisfying

$$(D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) = 0, \tag{5}$$

$$(D_X{}'F)(Y, Z) = \eta(Y)(D_X\eta)(\phi Z) - \eta(Z)(D_X\eta)(\phi Y), \tag{6}$$

$$
\begin{aligned}
(D_X{}'F)(Y, Z) \ + \ & (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) + \eta(X)[(D_Y\eta)(\phi Z) \\
- \ & (D_Z\eta)(\phi Y)] + \eta(Y)[(D_Z\eta)(\phi X) - (D_X\eta)(\phi Z)] \\
+ \ & \eta(Z)[(D_X\eta)(\phi Y) - (D_Y\eta)(\phi X)] = 0,
\end{aligned} \tag{7}
$$

$$(D_{\phi X}{}'F)(Y, Z) \;+\; (D_X{}'F)(\phi Y, Z) - \eta(Y)(D_{\phi X}\eta)(\phi Z)$$
$$+ \;\; \eta(Z)[(D_{\phi X}\eta)(\phi Y) - (D_X\eta)(Y)] = 0, \tag{8}$$

$$(D_{\phi X}{}'F)(\phi Y, Z) \;+\; (D_{\phi Y}{}'F)(\phi Z, X) + (D_{\phi Z}{}'F)(\phi X, Y)$$
$$- \;\; (D_X{}'F)(Y, Z) - (D_Y{}'F)(Z, X) - (D_Z{}'F)(X, Y)$$
$$+ \;\; \eta(X)[(D_Z\eta)(\phi Y) - (D_{\phi Y}\eta)(Z) - (D_Y\eta)(\phi Z)]$$
$$+ \;\; \eta(Y)[(D_X\eta)(\phi Z) - (D_{\phi Z}\eta)(X) - (D_Z\eta)(\phi X)]$$
$$+ \;\; \eta(Z)[(D_Y\eta)(\phi X) - (D_{\phi X}\eta)(Y) - (D_X\eta)(\phi Y)] = 0, \tag{9}$$

$$(D_X{}'F)(Y, Z) = \eta(Y)(D_Z\eta)(\phi X) + \eta(Z)(D_{\phi X}\eta)(Y), \tag{10}$$

$$(D_{\phi X}{}'F)(\phi Y, Z) \;+\; (D_{\phi Y}{}'F)(\phi Z, X) + (D_{\phi Z}{}'F)(\phi X, Y)$$
$$= \;\; (D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X)$$
$$+ \;\; (D_Z{}'F)(X, Y) - \eta(X)(D_Z\eta)(\phi Y)$$
$$- \;\; \eta(Y)(D_X\eta)(\phi Z) - \eta(Z)(D_Y\eta)(\phi X), \tag{11}$$

$$(D_{\phi X}{}'F)(\phi Y, Z) = (D_X{}'F)(Y, Z) - \eta(Y)(D_X\eta)(\phi Z) \tag{12}$$

$$(D_{\phi X}{}'F)(\phi Y, Z) - (D_X{}'F)(Y, Z) + \eta(Y)(D_X\eta)(\phi Z)$$
$$= (D_{\phi Y}{}'F)(\phi Z, X) - (D_Y{}'F)(Z, X) + \eta(Z)(D_Y\eta)(\phi X) \tag{13}$$

$$(D_{\phi X}{}'F)(\phi Y, Z) \;-\; (D_X{}'F)(Y, Z) + \eta(Y)(D_X\eta)(\phi Z)$$
$$- \;\; \eta(Z)[(D_{\phi X}\eta)(Y) + (D_X\eta)(\phi Y)]$$
$$= \;\; (D_{\phi Y}{}'F)(\phi Z, X) - (D_Y{}'F)(Z, X) + \eta(Z)(D_Y\eta)(\phi X)$$
$$- \;\; \eta(X)[(D_{\phi Y}\eta)(Z) + (D_Y\eta)(\phi Z)] \tag{14}$$

for arbitrary vector fields X, Y and Z, are respectively called quasi-Sasakian manifold, generalised co-symplectic manifold, generalised quasi-Sasakian manifold, generalised almost contact normal metric manifold, generalised quasi-normal manifold, normal quasi-Sasakian manifold, quasi-normal manifold, almost contact normal metric manifold, almost contact nearly normal metric manifold and generalised almost contact nearly normal metric manifold [14]. Here D denotes the Riemannian connection with respect to the Riemannian metric g.

3   Semi-symmetric metric $T-$connection

Let D be a Riemannian connection, then a linear connection ∇ defined as

$$\nabla_X Y = D_X Y + \pi(Y)X - g(X, Y)\rho \tag{15}$$

for arbitrary vector fields X and Y, where π is any $1-$form associated with the vector field ρ, i.e.,

$$\pi(Y) \overset{\text{def}}{=} g(Y, \rho), \tag{16}$$

is called a semi-symmetric metric connection [1]. The torsion tensor S of the connection ∇ and metric tensor g are given by

$$S(X, Y) = \pi(Y)X - \pi(X)Y \tag{17}$$

and

$$\nabla_X g = 0. \tag{18}$$

Agreement The manifold (M_n, g) is considered to be an almost contact metric manifold. The equations (15), (16) and (17) become

$$\nabla_X Y = D_X Y + \eta(Y)X - g(X, Y)\xi, \tag{19}$$

$$\eta(Y) = g(Y, \xi) \tag{20}$$

$$S(X, Y) = \eta(Y)X - \eta(X)Y \tag{21}$$

If in addition,

$$(a) \quad \nabla_X \xi = 0 \quad or \quad (b) \quad (\nabla_X \eta)(Y) = 0 \tag{22}$$

hold for arbitrary vector fields X and Y, then the connection ∇ is said to be a semi-symmetric metric $T-$connection [6]. Also, from (19) and (22), we have

$$D_X \xi + X - \eta(X)\xi = 0 \iff (D_X \eta)(Y) + g(\phi X, \phi Y) = 0. \tag{23}$$

Theorem 3.1 *A generalised co-symplectic manifold with semi-symmetric metric $T-$connection ∇ is a generalised quasi-Sasakian manifold.*

Proof In consequence of (1), (4) and (23), (6) becomes

$$(D_X{}'F)(Y, Z) = \eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y) \tag{24}$$

Taking cyclic sum of (24) in X, Y and Z and then using (3) and (4), we get

$$\begin{aligned}(D_X{}'F)(Y, Z) \quad &+ \quad (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) \\ &= \quad 2[\eta(X)g(Y, \phi Z) + \eta(Y)g(Z, \phi X) + \eta(Z)g(X, \phi Y)]\end{aligned} \tag{25}$$

In view of (7) and (23), we obtain (25) and hence the statement of the theorem.

Theorem 3.2 *A normal quasi-Sasakian manifold with semi-symmetric metric $T-$connection ∇ is a quasi-Sasakian.*

Proof Using (23) in (10) and then using (1), (3) and (4), we have

$$(D_X{}'F)(Y, Z) = \eta(Y)g(X, \phi Z) + \eta(Z)g(X, \phi Y) \tag{26}$$

Taking cyclic sum of (26) in X, Y, Z and using (3) and (4), we obtain

$$(D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) = 0. \tag{27}$$

Hence the theorem.

We also proved the following theorems by straight forward calculation as done above.

Theorem 3.3 *A generalised almost contact normal metric manifold admitting a semi-symmetric metric $T-$connection ∇ is an almost contact normal metric manifold.*

Theorem 3.4 *A generalised quasi-normal manifold equipped with semi-symmetric metric $T-$connection ∇ is a quasi-normal manifold.*

Theorem 3.5 *A generalised almost contact nearly normal metric manifold admitting a semi-symmetric metric $T-$connection ∇ is an almost contact nearly normal metric manifold.*

4 Curvature tensor with respect to the semi-symmetric metric $T-$connection

The curvature tensor of the semi-symmetric metric $T-$connection ∇

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and that of the Riemannian connection D

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

are related as [6]

$$R(X, Y, Z) = K(X, Y, Z) + g(Y, Z)X - g(X, Z)Y \tag{28}$$

Contracting above equation with respect to X, we have

$$\tilde{Ric}(Y, Z) = Ric(Y, Z) + (n-1)g(Y, Z), \tag{29}$$

where $\tilde{Ric}$ and Ric are the Ricci-tensors of the connections ∇ and D respectively.

Theorem 4.1 *If an almost contact metric manifold admitting a semi-symmetric metric $T-$connection ∇, then the Ricci-tensor with respect to the semi-symmetric metric $T-$connection ∇ is symmetric.*

Proof Interchanging Y and Z in (29), we get

$$\tilde{Ric}(Z,Y) = Ric(Z,Y) + (n-1)g(Y,Z), \qquad (30)$$

Subtracting (30) from (29) and then using $Ric(Y,Z) = Ric(Z,Y)$, we obtain the statement of the theorem.

Theorem 4.2 *The necessary and sufficient condition for an almost contact metric manifold to be flat with respect to semi-symmetric metric $T-$connection ∇ if and only if it is locally isometric to the hyperbolic space $H^n(-1)$.*

Proof If $R(X,Y,Z) = 0$, then (28) becomes

$$K(X,Y,Z) = -[g(Y,Z)X - g(X,Z)Y] \qquad (31)$$

A space form is said to be hyperbolic, elliptic or euclidean if and only if the scalar curvature tensor is negative, positive or zero [12]. Thus, (31) gives the necessary part of the theorem. Sufficient part is obvious from (28) and (31).

Theorem 4.3 *If an almost contact metric manifold M_n admitting a semi-symmetric metric $T-$connection ∇ whose scalar curvature tensor vanishes, then the curvature tensor of ∇ coincides with the con-circular curvature tensor of the manifold.*

Proof In view of (4), (29) becomes

$$\tilde{R}Y = RY + (n-1)Y \qquad (32)$$

and

$$\tilde{r} = r + n(n-1), \qquad (33)$$

where the Ricci operators $\tilde{R}$ and R of the connections ∇ and D are defined by

$$\tilde{Ric}(Y,Z) \stackrel{\text{def}}{=} g(\tilde{R}Y,Z) \; ; \; Ric(Y,Z) \stackrel{\text{def}}{=} g(RY,Z)$$

and the scalar curvature tensors $\tilde{r}$ and r of ∇ and D are

$$\tilde{r} \stackrel{\text{def}}{=} trace(\tilde{R}) \; ; \; r \stackrel{\text{def}}{=} trace(R)$$

respectively. If we take $\tilde{r} = 0$, then (33) gives

$$r = -n(n-1) \qquad (34)$$

The con-circular curvature tensor of the Riemannian connection D [13] is

$$C(X, Y, Z) = K(X, Y, Z) - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y] \qquad (35)$$

In consequence of (28) and (34), (35) becomes

$$C(X, Y, Z) = R(X, Y, Z) \qquad (36)$$

Corollary 4.4 *If an almost contact metric manifold M_n admitting a semi-symmetric metric $T-$connection ∇ whose curvature tensor vanishes, then the manifold will be con-circularly flat.*

Theorem 4.5 *Let M_n be an almost contact metric manifold admitting a semi-symmetric metric $T-$connection ∇, then it is locally isometric to the hyperbolic space $H^n(-1)$ if and only if M_n is con-circularly flat.*

Proof of the theorem is obvious from theorems (4.2) and (4.3).

Theorem 4.6 *An almost contact metric manifold M_n equipped with a semi-symmetric metric $T-$connection ∇ whose Ricci-tensor vanishes, then the curvature tensor of the connection ∇ is equal to the Weyl projective curvature tensor of the manifold.*

Proof If $\tilde{Ric}(Y, Z) = 0$, then (29) gives

$$Ric(Y, Z) = -(n-1)g(Y, Z) \qquad (37)$$

The Weyl projective curvature tensor of the Riemannian connection D [13] is

$$W(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-1}[Ric(Y, Z)X - Ric(X, Z)Y] \qquad (38)$$

Using (28) and (37) in (38), we have

$$W(X, Y, Z) = R(X, Y, Z). \qquad (39)$$

Corollary 4.7 *An almost contact metric manifold M_n equipped with a semi-symmetric metric $T-$connection ∇ whose curvature tensor vanishes, then the manifold is projectively flat.*

Remark- The theorem (2.2) proved by Prof. R. S. Mishra and S. N. Pandey [6] is a particular case of the theorem (4.6).

In consequence of theorems (4.2) and (4.6), we state

Theorem 4.8 *If an almost contact metric manifold admitting a semi-symmetric metric $T-$connection ∇, then the manifold is projectively flat if and only if it is locally isometric to the hyperbolic space $H^n(-1)$.*

Theorem 4.9 *Let M_n be an almost contact metric manifold admitting a semi-symmetric metric $T-$connection ∇ whose Ricci-tensor vanishes, then the curvature tensor with respect to the semi-symmetric metric $T-$connection is equal to the conformal curvature tensor of the manifold.*

Proof The conformal curvature tensor V of the Riemannian connection D is [13]

$$
\begin{aligned}
V(X,Y,Z) = {}& K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X \\
& - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY] \\
& + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]
\end{aligned}
\tag{40}
$$

In consequence of (30), (34), (36), and (39), (40) gives

$$
V(X,Y,Z) = R(X,Y,Z) \tag{41}
$$

Corollary If in an almost contact metric manifold M_n, the curvature tensor of a semi-symmetric metric $T-$connection ∇ vanishes, then it is conformally flat.

Theorems (4.2) and (4.9) state

Theorem 4.10 *If an almost contact metric manifold admitting a semi-symmetric metric $T-$connection ∇, then the manifold is conformally flat if and only if it is locally isometric to the hyperbolic space $H^n(-1)$.*

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