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Semi-Symmetric Non-Metric Connection

A. K. Dubey[1] and R. H. Ojha[2]

Department of Mathematics, Faculty of Science
Banaras Hindu University, Varanasi-221005, India
arun22_math@rediffmail.com[1]
rh_ojha@rediffmail.com[2]

S. K. Chaubey[3]

P. K. Institute of Technology and Management
Birhana (Raya), Mathura-281206, India
sk22_math@yahoo.co.in[3]

Abstract

Recently S. K. Chaubey and R. H. Ojha [1] introduced a semi-symmetric non-metric connection in almost contact metric manifold. The purpose of the present paper is to study this connection in a Sasakian manifold. We have also studied curvature tensors of a semi-symmetric non-metric T−connection in an almost contact metric manifold.

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1 Introduction

Let there exist an odd dimensional differentiable manifold M_n, $n = 2m+1$, of differentiability class C^∞, a C^∞ vector valued linear function F, a C^∞ vector field T and a C^∞ $1-$form A satisfying

$$\overline{\overline{X}} + X = A(X)T, \tag{1}$$

$$A(\overline{X}) = 0, \tag{2}$$

where $\overline{X} \overset{def}{=} FX$, for arbitrary vector field X, then M_n is called an almost contact metric manifold and the structure (F, T, A) is called an almost contact structure. In view of (1) and (2), we find

$$(a) \quad \overline{T} = 0, \quad (b) \quad A(T) = 1, \quad (c) \quad rank(F) = n - 1. \tag{3}$$

If a non-singular metric tensor g of type $(0, 2)$ satisfies

$$g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y), \tag{4}$$

for arbitrary vector fields X and Y, then an almost contact manifold M_n endowed with g is called an almost contact metric structure to M_n [2]. Putting T for X in (4) and using (3) (a) and (3) (b), we get

$$g(T, Y) = A(Y), \tag{5}$$

Put

$$'F(X, Y) \stackrel{def}{=} g(\overline{X}, Y), \tag{6}$$

then we have

$$'F(X, Y) +' F(Y, X) = 0, \tag{7}$$

An almost contact metric manifold on which

$$D_X T = \overline{X}, \tag{8}$$

holds for arbitrary vector field X is called a $K-$contact Riemannian manifold. For a $K-$contact Riemannian manifold

$$(D_X A)(Y) = g(\overline{X}, Y), \tag{9}$$

If on a $K-$contact Riemannian manifold

$$(D_X{}'F)(Y, Z) = A(Y)g(X, Z) - A(Z)g(X, Y) \tag{10}$$

holds, then the manifold is known as a Sasakian manifold [3]. Where D denotes the Riemannian connection.

2 Semi-symmetric non-metric connection

A linear connection $\tilde{B}$ on (M_n, g) defined as

$$\tilde{B}_X Y = D_X Y - A(Y)X - g(X, Y)T \tag{11}$$

for arbitrary vector fields X and Y, is said to be a semi-symmetric non-metric connection [1].The torsion tensor $\tilde{S}$ of the connection $\tilde{B}$ and the metric tensor g are given by

$$\tilde{S}(X, Y) = A(X)Y - A(Y)X, \tag{12}$$

and

$$(\tilde{B}_X g)(Y,Z) = 2[A(Y)g(X,Z) + A(Z)g(X,Y)] \tag{13}$$

If in addition to (12) and (13)

$$(a) \quad \tilde{B}_X T = 0, \quad or \quad (b) \quad (\tilde{B}_X A)(Y) = 0 \tag{14}$$

hold for arbitrary vector fields X and Y, then the connection $\tilde{B}$ is said to be a semi-symmetric non-metric $T-$connection. Also, from (11) and (14), we have

$$D_X T - X - A(X)T = 0 \Leftrightarrow (D_X A)(Y) + g(X,Y) + A(X)A(Y) = 0 \tag{15}$$

Now, we put (11) as

$$\tilde{B}_X Y = D_X Y + H(X,Y), \tag{16}$$

where

$$H(X,Y) = -A(Y)X - g(X,Y)T \tag{17}$$

Let us define

$$(a) \quad \tilde{S}(X,Y,Z) \overset{\text{def}}{=} g(\tilde{S}(X,Y),Z) \quad (b) \quad 'H(X,Y,Z) \overset{\text{def}}{=} g(\tilde{H}(X,Y),Z) \tag{18}$$

then we can write

$$(a) \quad \tilde{S}(X,Y,Z) = A(X)g(Y,Z) - A(Y)g(X,Z) \tag{19}$$

$$(b) \quad 'H(X,Y,Z) = -A(Y)g(X,Z) - A(Z)g(X,Y)$$

Theorem 2.1 *Let $\tilde{B}$ be a semi-symmetric non-metric connection in a Sasakian manifold with a Riemannian connection D, then we have*

$$(a) \quad (\tilde{B}_X {'F})(Y,\overline{Z}) = A(Y)[(D_X A)(\overline{Z}) - (D_X A)(Z)] \tag{20}$$

$$(b) \quad (\tilde{B}_X {'F})(\overline{Y},Z) = A(Z)[(D_X A)(Y) - (D_X A)(\overline{Y})]$$

Proof We have

$$X('F(Y,Z)) = (D_X {'F})(Y,Z) +' F(D_X Y,Z) +' F(Y,D_X Z)$$

$$= (\tilde{B}_X {'F})(Y,Z) +' F(\tilde{B}_X Y,Z) +' F(Y,\tilde{B}_X Z)$$

With the help of (11), this equation becomes

$$(\tilde{B}_X {'F})(Y,Z) = (D_X {'F})(Y,Z) +' H(X,Y,\overline{Z}) -' H(X,Z,\overline{Y}), \tag{21}$$

Using (19) (b) in this equation, we obtain

$$(\tilde{B}_X {'F})(Y,Z) = (D_X {'F})(Y,Z) - A(Y)g(X,\overline{Z}) + A(Z)g(X,\overline{Y}), \tag{22}$$

From (9), (22) becomes

$$(\tilde{B}_X {'F})(Y,Z) = (D_X {'F})(Y,Z) + A(Y)(D_X A)(Z) - A(Z)(D_X A)(Y), \tag{23}$$

Barring Z on both sides and using (10) in (23), we have (20) (a). Also from (10) and (23), we find (20) (b).

Theorem 2.2 *Let $\tilde{B}$ be a semi-symmetric non-metric connection in a Sasakian manifold with a Riemannian connection D, then we have*

$$(\tilde{B}_{\overline{X}}{}'F)(Y,Z)+(\tilde{B}_X{}'F)(\overline{Y},Z)+(\tilde{B}_X{}'F)(Y,\overline{Z})=0 \tag{24}$$

Proof Barring X, Y, Z in (23) respectively, we get

$$(a)\quad (\tilde{B}_{\overline{X}}{}'F)(Y,Z)=(D_{\overline{X}}{}'F)(Y,Z)+A(Y)(D_{\overline{X}}A)(Z)-A(Z)(D_{\overline{X}}A)(Y) \tag{25}$$

$$(b)\quad (\tilde{B}_X{}'F)(\overline{Y},Z)=(D_X{}'F)(\overline{Y},Z)-A(Z)(D_XA)(\overline{Y})$$

$$(c)\quad (\tilde{B}_X{}'F)(Y,\overline{Z})=(D_X{}'F)(Y,\overline{Z})+A(Y)(D_XA)(\overline{Z})$$

Adding these equations, we obtain

$$(\tilde{B}_{\overline{X}}{}'F)(Y,Z)+(\tilde{B}_X{}'F)(\overline{Y},Z)+(\tilde{B}_X{}'F)(Y,\overline{Z})=(D_{\overline{X}}{}'F)(Y,Z)$$
$$+(D_X{}'F)(\overline{Y},Z)+(D_X{}'F)(Y,\overline{Z})+A(Y)(D_{\overline{X}}A)(Z)$$
$$-A(Z)(D_{\overline{X}}A)(Y)-A(Z)(D_XA)(\overline{Y})+A(Y)(D_XA)(\overline{Z})$$

Using (9) and (10) in the above equation, we get the result.

Theorem 2.3 *Let $\tilde{B}$ be a semi-symmetric non-metric connection in a Sasakian manifold with a Riemannian connection D, then we have*

$$(\tilde{B}_X{}'F)(Y,\overline{Z})+(\tilde{B}_Y{}'F)(\overline{Z},X)+(\tilde{B}_{\overline{Z}}{}'F)(X,Y)+2'\tilde{S}(X,Y,Z)=0$$

Proof From (23), we have

$$(a)\quad (\tilde{B}_X{}'F)(Y,\overline{Z})=(D_X{}'F)(Y,\overline{Z})+A(Y)(D_XA)(\overline{Z}) \tag{26}$$

Similarly,

$$(b)\quad (\tilde{B}_Y{}'F)(\overline{Z},X)=(D_Y{}'F)(\overline{Z},X)-A(X)(D_YA)(\overline{Z})$$

$$(c)\quad (\tilde{B}_{\overline{Z}}{}'F)(X,Y)=(D_{\overline{Z}}{}'F)(X,Y)+A(X)(D_{\overline{Z}}A)(Y)-A(Y)(D_{\overline{Z}}A)(X)$$

Adding these equations, we obtain

$$(\tilde{B}_X{}'F)(Y,\overline{Z})+(\tilde{B}_Y{}'F)(\overline{Z},X)+(\tilde{B}_{\overline{Z}}{}'F)(X,Y)=(D_X{}'F)(Y,\overline{Z})$$
$$+(D_Y{}'F)(\overline{Z},X)+(D_{\overline{Z}}{}'F)(X,Y)+A(Y)(D_XA)(\overline{Z})$$
$$-A(X)(D_YA)(\overline{Z})+A(X)(D_{\overline{Z}}A)(Y)-A(Y)(D_{\overline{Z}}A)(X)$$

Using (9), (10) and (19) (a) in the above equation, we get the result.

Theorem 2.4 *In a Sasakian manifold with a semi-symmetric non-metric connection $\tilde{B}$, we have*

$$(\tilde{B}_XA)(\overline{Y})=(D_XA)(\overline{Y})-{}'F(X,Y) \tag{27}$$

Proof Covariant derivative of (5) with respect to $\tilde{B}$ gives

$$(\tilde{B}_XA)(Y)=(\tilde{B}_Xg)(Y,T)+g(Y,\tilde{B}_XT) \tag{28}$$

Using (11) and (13), (28) becomes

$$(\tilde{B}_XA)(Y)=(D_XA)(Y)+g(Y,Z)+A(X)A(Y) \tag{29}$$

Barring Y in (29) and using (6), we get the result.

3 Curvature tensor of a semi-symmetric non-metric $T-$connection $\tilde{B}$

The curvature tensor of the connection $\tilde{B}$

$$R(X, Y, Z) = \tilde{B}_X \tilde{B}_Y Z - \tilde{B}_Y \tilde{B}_X Z - \tilde{B}_{[X,Y]} Z$$

and that of the connection D

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

are related as [1]

$$R(X, Y, Z) = K(X, Y, Z) - \beta(X, Z)Y + \beta(Y, Z)X \tag{30}$$

$$-g(Y, Z)(D_X T - A(X)T) + g(X, Z)(D_Y T - A(Y)T),$$

where

$$\beta(X, Y) = (D_X A)(Y) + A(X)A(Y) + g(X, Y) \tag{31}$$

is a tensor field of type $(0, 2)$. In view of (14) (b) and (29), (31) gives

$$\beta(X, Y) = 0 \tag{32}$$

and hence in view of (31), (30) becomes

$$R(X, Y, Z) = K(X, Y, Z) - g(Y, Z)X + g(X, Z)Y \tag{33}$$

Contracting (33) with respect to X, we get

$$\tilde{R}ic(Y, Z) = Ric(Y, Z) - (n - 1)g(Y, Z), \tag{34}$$

where

$$\tilde{R}ic(Y, Z) \overset{\text{def}}{=} (C_1^1 R)(Y, Z) \quad \textit{and} \quad Ric(Y, Z) \overset{\text{def}}{=} (C_1^1 K)(Y, Z)$$

are the Ricci-tensors with respect to the connections $\tilde{B}$ and D respectively. Again from (34), we have

$$\tilde{R}Y = RY - (n - 1)Y \tag{35}$$

and

$$\tilde{r} = r - n(n - 1), \tag{36}$$

where the Ricci operators $\tilde{R}$ and R of the connections $\tilde{B}$ and D are defined by

$$\tilde{R}ic(Y, Z) \overset{\text{def}}{=} g(\tilde{R}Y, Z) \quad ; \quad Ric(Y, Z) \overset{\text{def}}{=} g(RY, Z)$$

and the scalar curvature tensor $\tilde{r}$ and r of $\tilde{B}$ and D are

$$\tilde{r} \overset{\text{def}}{=} trace(\tilde{R}) \quad ; \quad r \overset{\text{def}}{=} trace(R)$$

respectively.

Theorem 3.1 *If an almost contact metric manifold M_n equipped with a semi-symmetric non-metric $T-$connection $\tilde{B}$ whose Ricci-tensor vanishes, then the curvature tensor of $\tilde{B}$ coincides with the Weyl-projective curvature tensor of the manifold.*

Proof If $\tilde{R}ic(Y, Z) = 0$, then (34) gives

$$Ric(Y, Z) = (n - 1)g(Y, Z) \tag{37}$$

The Weyl projective curvature tensor of the Riemannian connection D [2] is

$$W(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-1}[Ric(Y, Z)X - Ric(X, Z)Y] \tag{38}$$

Using (33) and (37) in (38), we have

$$W(X, Y, Z) = R(X, Y, Z).$$

Corollary 3.1 If an almost contact metric manifold M_n equipped with a semi-symmetric non-metric $T-$connection $\tilde{B}$ whose curvature tensor vanishes, then the manifold is projectively flat.

Theorem 3.2 *Let M_n be an almost contact metric manifold admitting a semi-symmetric non-metric $T-$connection $\tilde{B}$ whose Ricci-tensor vanishes, then the curvature tensor with respect to the semi-symmetric non-metric $T-$connection coincides with the conformal curvature tensor of the manifold.*

Proof The conformal curvature tensor V of the Riemannian connection D is [2]

$$V(X, Y, Z) = K(X, Y, Z) - \frac{1}{(n-2)}[Ric(Y, Z)X - Ric(X, Z)Y + g(Y, Z)RX$$

$$-g(X, Z)RY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \tag{39}$$

In consequence of (33), (34), (35) and (36), (39) gives

$$V(X, Y, Z) = R(X, Y, Z)$$

Corollary 3.2 If an almost contact metric manifold M_n, the curvature tensor of a semi-symmetric non-metric $T-$connection $\tilde{B}$ vanishes, then it is conformally flat.

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