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SEMI TAIL UPPER BOUNDS ON THE CLASS OF ADMISSIBLE ESTIMATORS IN DISCRETE EXPONENTIAL FAMILIES WITH APPLICATIONS TO POISSON AND NEGATIVE BINOMIAL DISTRIBUTIONS[1]

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Admissibility problems involving simultaneous estimation in discrete exponential families are studied by solving difference inequalities. It is shown that if an estimator is admissible under the loss function $L_{\mathbf{m}}(\boldsymbol{\theta}, \mathbf{a}) = \sum_{i=1}^{p} \theta_i^{m_i} (\theta_i - a_i)^2$, then in the tail (i.e., for large values of the observations), this estimator has to be less than certain bounds. Specific bounds, called Semi Tail Upper Bounds (STUB), are given here.

These STUBs are not only of theoretical interest, but also are sharp enough that they establish many new results. Two of the most interesting ones are: (i) the establishment of Brown's conjecture concerning inadmissibility of some of the estimators proposed by Clevenson and Zidek (1975), and (ii) the establishment of inadmissibility of Hudson's (1978) estimator which improves upon the uniformly minimum variance unbiased estimator in Negative Binomial families.

1. Introduction. Let $\mathbf{X} = (X_1, \cdots, X_p)$ be a random observation having a distribution characterized by $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)$. Consider the problem of estimating $\boldsymbol{\theta}$ under the loss function

$$L_{\mathbf{m}}(\boldsymbol{\theta}, \boldsymbol{\delta}) = \sum_{i=1}^{p} \theta_i^{m_i} (\delta_i - \theta_i)^2 \tag{1.1}$$

where $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_p)$ is an estimator of $\boldsymbol{\theta}$. In this paper, the notion of a Semi Tail Upper Bound (STUB) on the class of admissible estimators is introduced.

To illustrate the idea, we focus on the sum of squared error loss, i.e., $L_{\mathbf{m}}$ with $\mathbf{m} = (0, \cdots, 0)$. A statistic or an estimator $\mathbf{S}(\mathbf{X}) = (S_1(\mathbf{X}), \cdots, S_p(\mathbf{X}))$ is called a STUB in the direction $\mathbf{d}(\mathbf{x}) = (d_1(\mathbf{x}), \cdots, d_p(\mathbf{x}))$ if any estimator $\boldsymbol{\delta}(\mathbf{X})$, satisfying $\boldsymbol{\delta}(\mathbf{x}) \cdot \mathbf{d}(\mathbf{x}) \geq \mathbf{S}(\mathbf{x}) \cdot \mathbf{d}(\mathbf{x})$ for all $\mathbf{x}$ outside a compact set in $\mathbb{R}^p$, is inadmissible. The existence of a STUB provides a very efficient way to check whether an estimator is inadmissible. STUBs were found by Brown (1971) for the case in which the observations X_i, $i = 1, \cdots, p$, are independent normal random variables with mean θ_i and variance one. By associating the estimation problem with diffusion processes, Brown proved that the estimator of the James-Stein type

$$\boldsymbol{\delta}_{\ell}^{JS}(\mathbf{X}) = \left(1 - \frac{\ell}{\sum_i X_i^2}\right)\mathbf{X}$$

is a STUB in the direction $\mathbf{x}$ if $\ell < p - 2$. His arguments, although interesting from both probabilistic and statistical viewpoints, are difficult to generalize to other distributions.

In this paper, by using a different argument outlined at the end of Section 1, a general type of estimator is shown to give rise to STUBs for the case when the observations X_i, $i = 1, \cdots, p$, are independent random variables from discrete exponential families with

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density

$$f_i(x_i \mid \theta_i) = \rho_i(\theta_i)t_i(x_i)\theta_i^{x_i},\ x_i = 0, 1, 2, \cdots \tag{1.2}$$

under the loss function $L_{\mathbf{m}}$. (The definition of a STUB for $L_{\mathbf{m}}$ is similar to the one above, but with some modification). Special cases of (1.2) are the Poisson distribution (denoted by $Po(\theta_i)$), for which $\rho_i(\theta_i) = e^{-\theta_i}$ and $t_i(x_i) = 1/x_i!$, and the Negative Binomial distribution (denoted by NB (r_i, θ_i), r_i known), for which $\rho_i(\theta_i) = (1 - \theta_i)^{r_i}$ and $t_i(x_i) = C(r_i + x_i - 1, r_i - 1)$ where $C(k_1, k_2)$ denotes $k_1!/\{k_2!(k_1 - k_2)!\}$.

In particular, for the Poisson case, some of the improved estimators in Clevenson and Zidek (1975), Peng (1975), and Tsui and Press (1978) are STUBs under certain loss functions. For the Negative Binomial distributions, some of Hwang's (1982a) improved estimators are also STUBs under the sum of squared error loss.

All the directions $\mathbf{d}$ corresponding to the STUBs developed here have positive components and therefore the existence of a STUB indicates that, in order to be admissible, an estimator cannot be too large in the tail region. Many new results can be established easily using the STUBs. In this paper two particularly interesting facts are proved: (i) that Brown's conjecture holds concerning inadmissibility of some of the estimators proposed by Clevenson and Zidek (1975), and (ii) that Hudson's (1978) estimator, which dominates the uniformly minimum variance unbiased estimator (UMVUE) in Negative Binomial families, is inadmissible.

The procedure of proving an estimator $\mathbf{S}(\mathbf{X})$ to be a STUB is very similar to the procedure of improving upon $\mathbf{S}(\mathbf{X})$. If, for some $\mathbf{\Phi}(\mathbf{x})$, $\mathbf{S}(\mathbf{X}) + \mathbf{\Phi}(\mathbf{X})$ dominates $\mathbf{S}(\mathbf{X})$, and $\mathbf{\Phi}(\mathbf{x})$ satisfies the conditions in Theorem 2.1, then $\mathbf{S}(\mathbf{X})$ is a STUB in some direction $\mathbf{d}(\mathbf{x})$; $\mathbf{d}(\mathbf{x}) = -\mathbf{\Phi}(\mathbf{x})$ for the sum of squared error loss. To find $\mathbf{\Phi}(\mathbf{x})$ satisfying the conditions in Theorem 2.1, we solve an inequality using the discrete analog of Stein's (1973, 1981) identity which appears in Hudson (1978).

2. Definition and sufficient conditions of a STUB. Let $\mathbf{X} = (X_1, \cdots, X_p)$ where the X_i are p independent observations having discrete density (1.2). Under the loss $L_{\mathbf{m}}$, we give, in this section, the definition of a STUB and develop a set of conditions sufficient for a statistic (an inadmissible estimator itself) to be a STUB.

Since our observations take only integer values, all the X_i and x_i below are assumed to be integer valued. Let $\mathbf{e}_i$ be the ith coordinate vector of p components, i.e., the ith component of $\mathbf{e}_i$ is one and the rest are zero. We will say that a statement holds for sufficiently large $\mathbf{x}$, if there exists an M such that the statement is true for all $\mathbf{x}$ of which all the components $x_i \geq M$, $i = 1, \cdots, p$. The definition of a STUB is now stated below.

DEFINITION 2.1. Under the loss function $L_{\mathbf{m}}$, a statistic $\boldsymbol{\delta}^0$ (or an estimator of $\boldsymbol{\theta}$) is a STUB in the direction $\mathbf{d}(\mathbf{x}) = (d_1(\mathbf{x}), \cdots, d_p(\mathbf{x}))$, if every estimator $\boldsymbol{\delta}$ satisfying

$$\textstyle\sum_{i=1}^p \delta_i^0(\mathbf{x} - m_i\mathbf{e}_i)d_i(\mathbf{x}) \leq \sum_{i=1}^p \delta_i(\mathbf{x} - m_i\mathbf{e}_i)d_i(\mathbf{x}), \tag{2.1}$$

for sufficiently large $\mathbf{x}$, is inadmissible.

By definition, $\boldsymbol{\delta}^0$ itself is clearly an inadmissible estimator. The statistic $\boldsymbol{\delta}^0$ is called a STUB because for any admissible estimator $\boldsymbol{\delta}^A = (\delta_1^A, \cdots, \delta_p^A)$, it must be true that

$$\textstyle\sum_i \delta_i^A(\mathbf{x} - m_i\mathbf{e}_i)d_i(\mathbf{x}) \leq \sum_i \delta_i^0(\mathbf{x} - m_i\mathbf{e}_i)d_i(\mathbf{x}) \tag{2.2}$$

for a certain sequence of $\mathbf{x}$'s approaching $(\infty, \cdots, \infty)$. However this inequality does not necessarily hold for any sufficiently large $\mathbf{x}$. This accounts for the prefix "Semi".

The definition seems most natural for the sum of squared error loss. In this case, condition (2.1) reduces to the comparison of the projections of $\boldsymbol{\delta}^0(\mathbf{x})$ and of $\boldsymbol{\delta}(\mathbf{x})$ in the direction $\mathbf{d}(\mathbf{x})$. For general $L_{\mathbf{m}}$, we can establish STUBs only in the sense of (2.1). The following lemma does present one important special case, however, in which $\mathbf{m}$ can be ignored.

LEMMA 2.1. *Under $L_{\mathbf{m}}$, assume that $\boldsymbol{\delta}^0$ is a* STUB *in the direction* $\mathbf{d}$, *where* $d_i(\mathbf{x}) \geq 0$, $i = 1, \cdots, p$. *If* $\delta_i(\mathbf{x}) \geq \delta_i^0(\mathbf{x})$, $i = 1, \cdots, p$, *for sufficiently large* $\mathbf{x}$, *then* $\boldsymbol{\delta}$ *is inadmissible.* □

Before developing sufficient conditions, we describe Stein's technique (1973, 1981) in our context. As usual, the risk function of an estimator $\boldsymbol{\delta}$, with respect to $L_{\mathbf{m}}$, is denoted by $R(\boldsymbol{\theta}, \boldsymbol{\delta}) = E_\theta\{L_{\mathbf{m}}(\boldsymbol{\theta}, \boldsymbol{\delta}(\mathbf{X}))\}$, where E_θ denotes the expectation when $\mathbf{X}$ is distributed according to $\boldsymbol{\theta}$. Often E_θ will be denoted simply by E. Let $\mathscr{Z}$ be the set of all nonnegative integers and $\mathscr{Z}^p$ be the p-fold Cartesian product space of $\mathscr{Z}$.

To attempt to improve on $\boldsymbol{\delta}^0$, write the competitor $\boldsymbol{\delta}^*(\mathbf{X}) = \boldsymbol{\delta}^0(\mathbf{X}) + \boldsymbol{\Phi}(\mathbf{X})$ where $\boldsymbol{\Phi}(\mathbf{X}) = (\Phi_1(\mathbf{X}), \cdots, \Phi_p(\mathbf{X}))$. The following lemma gives the unbiased estimate of the difference of the risks of $\boldsymbol{\delta}^*$ and $\boldsymbol{\delta}^0$.

LEMMA 2.2. *Under the loss function $L_{\mathbf{m}}$, assume that $R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) < \infty$ and $E_\theta \Phi_i^2(\mathbf{X}) < \infty$ for all $i = 1, 2, \cdots, p$ and all $\boldsymbol{\theta}$. If $\Phi_i(\mathbf{x}) = 0$ whenever $x_i < -m_i$, then*

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^*) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) = E_\theta \mathscr{D}(\boldsymbol{\delta}^0(\mathbf{X}), \boldsymbol{\Phi}(X)), \text{ where} \tag{2.3}$$

$$\begin{aligned} \mathscr{D}(\boldsymbol{\delta}^0(\mathbf{x}), \boldsymbol{\Phi}(\mathbf{x})) = \textstyle\sum_{i=1}^p \{&2\, \delta_i^0(\mathbf{x} - m_i\mathbf{e}_i)\Phi_i(\mathbf{x} - m_i\mathbf{e}_i)t_i(x_i - m_i) \\ &- 2\, \Phi_i(\mathbf{x} - (m_i + 1)\mathbf{e}_i)t_i(x_i - m_i - 1) \\ &+ \Phi_i^2(\mathbf{x} - m_i\mathbf{e}_i)t_i(x_i - m_i)\}/t_i(x_i). \end{aligned} \tag{2.4}$$

PROOF. It is proved in Hwang (1979) that for any function $g(\mathbf{x})$ defined on $\mathscr{Z}^p$ for which $E_\theta|g(\mathbf{X})| < \infty$, the equation

$$E\theta_i^m g(\mathbf{X}) = E[g(\mathbf{X} - m_i\mathbf{e}_i)t_i(X_i - m_i)/t_i(X_i)] \tag{2.5}$$

holds if $g(x_1, \cdots, x_i, \cdots, x_p) = 0$ whenever $x_i < -m$; special cases can also be found in Hudson (1975) and Tsui and Press (1982). Expanding the quadratic parts in (1.1) gives

$$R(\boldsymbol{\theta}, \boldsymbol{\delta}^*) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) = E \textstyle\sum_{i=1}^p \{2\, \theta_i^{m_i}(\delta_i^0(\mathbf{X}) - \theta_i)\Phi_i(\mathbf{X}) + \theta_i^{m_i}\Phi_i^2(\mathbf{X})\} \tag{2.6}$$

which, together with (2.5), implies this lemma. □

Stein's idea is to find $\boldsymbol{\Phi}(\mathbf{x})$ such that $\mathscr{D}(\boldsymbol{\delta}^0(\mathbf{x}), \boldsymbol{\Phi}(\mathbf{x}))$ is nonpositive for all $\mathbf{x}$ and is negative for some $\mathbf{x} \in Z^p$. Under the assumptions of Lemma 2.2, this clearly implies that $\boldsymbol{\delta}^*$ dominates $\boldsymbol{\delta}^0$. For convenience, $\mathscr{D}(\boldsymbol{\delta}^0(\mathbf{x}), \boldsymbol{\Phi}(\mathbf{x}))$ will be denoted by $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi})$.

Sufficient conditions for $\boldsymbol{\delta}^0$ to be a STUB can now be established. Let $I_M(\mathbf{x})$ be one if $x_i \geq M$ for all i and zero otherwise.

THEOREM 2.1. *Under the assumptions of Lemma 2.2, if $\Phi_i(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ and $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}) < 0$ for sufficiently large* $\mathbf{x}$, *then $\boldsymbol{\delta}^0$ is a* STUB *in the direction* $\mathbf{d}(\mathbf{x})$ *with* ith *component*

$$d_i(\mathbf{x}) = -\,\Phi_i(\mathbf{x} - m_i\mathbf{e}_i)t_i(x_i - m_i)/t_i(x_i).$$

PROOF. Let $\boldsymbol{\delta}$ be as in Definition 2.1. If $R(\boldsymbol{\theta}, \boldsymbol{\delta}) = \infty$ for all $\boldsymbol{\theta}$, then $\boldsymbol{\delta}$ is dominated by a constant estimator and is inadmissible. Thus suppose $R(\boldsymbol{\theta}, \boldsymbol{\delta}) < \infty$ for some $\boldsymbol{\theta}$, and consider this $\boldsymbol{\theta}$ in the remainder of the proof. Let M_0 be a number such that (2.1) is satisfied for all $\mathbf{x} = (x_1, \cdots, x_p)$ with $x_i > M_0$, $i = 1, \cdots, p$. To complete the proof, it suffices to prove that $\boldsymbol{\delta}(\mathbf{X})$ is dominated by $\boldsymbol{\delta}^*(\mathbf{X}) = \boldsymbol{\delta}^0(\mathbf{X}) + I_M(\mathbf{X})\boldsymbol{\Phi}(\mathbf{X})$ for some $M > M_0$.

From (2.3) and (2.4), $R(\boldsymbol{\theta}, \boldsymbol{\delta}^*) - R(\boldsymbol{\theta}, \boldsymbol{\delta}) = E\, \mathscr{D}(\boldsymbol{\delta}, I_M\boldsymbol{\Phi})$. By the monotonicity of $I_M(x)$ and the fact that

$$\Phi_i(\mathbf{x}) \leq 0,\ \mathscr{D}(\boldsymbol{\delta}, I_M\boldsymbol{\Phi}) \leq I_M(\mathbf{x})\mathscr{D}(\boldsymbol{\delta}, \boldsymbol{\Phi}) \leq I_M(\mathbf{x})\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}).$$

By assumption $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi})$ is negative for sufficiently large $\mathbf{x}$, hence an appropriate choice of M guarantees that $\mathscr{D}(\boldsymbol{\delta}, I_M\boldsymbol{\Phi})$ is nonnegative for all $\mathbf{x}$, and negative for sufficiently large $\mathbf{x}$. This implies that $\boldsymbol{\delta}$ is dominated by $\boldsymbol{\delta}^*$ and is inadmissible. □

The above proof also establishes the following Corollary.

COROLLARY 2.2. *Under the conditions of Theorem* 2.1, $\boldsymbol{\delta}^0(\mathbf{X})$ *is inadmissible and is dominated by* $\boldsymbol{\delta}^0(\mathbf{X}) + I_M(\mathbf{X})\boldsymbol{\Phi}(\mathbf{X})$ *for some large enough* M.

PROOF. Let $\boldsymbol{\delta} = \boldsymbol{\delta}^0$ in the proof of Theorem 2.1.

3. Development of STUBs. By Theorem 2.1, in order to prove an estimator $\boldsymbol{\delta}^0$ to be a STUB, it is sufficient to find $\Phi_i \le 0$ so that $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}) < 0$ for sufficiently large $\mathbf{x}$. The general theory we develop for doing this is, unfortunately, very cumbersome notationally. The examples following the theory (i.e., after Theorem 3.2) can be read independently, however.

Let η_{ij} be the Kronecker constant, i.e., $\eta_{ij} = 1$ or 0 depending on whether $i = j$ or $i \neq j$. Let a^+ denote $\max(a, 0)$ for any number a. In this section, we will consider an estimator $\boldsymbol{\delta}^0$ with ith component

$$\delta_i^0(\mathbf{X}) = \frac{t_i(X_i - 1)}{t_i(X_i)} S_i(X_i + m_i) \left\{ 1 - \frac{\ell_0 B_i H_i^{B_i - 1}(X_i + m_i)\Delta H_i(X_i + m_i)}{\sum_{j=1}^p H_j^{B_j}(X_j + m_i\eta_{ij})} \right\}^+, \tag{3.1}$$

where we use the notation $\Delta G(x_i) = G(x_i) - G(x_i - 1)$ for any function $G(x_i)$. The constants, $B_i \ge 0$ and ℓ_0, $-\infty < \ell_0 < \infty$ and the functions S_i and H_i are arbitrary, and are assumed to satisfy the following conditions:

(3.2) $H_i(x_i)$ is nonnegative and strictly increasing;

(3.3) $H_i(x_i - 1)/H_i(x_i) \to 1$ as $x_i \to \infty$;

(3.4) $S_i(x_i)$ is nonnegative for all x_i and is positive for $x_i \ge 0$.

It can be seen, as in the following Theorems and Corollaries (3.3 through 3.8), that many improved estimators are of the form (3.1) and satisfy (3.2), (3.3), and (3.4). Although $S_i(x_i) \equiv 1$ in these theorems, it is probably useful for future study to consider the more general $\boldsymbol{\delta}^0$.

In the remainder of this paper, for any function g, $\prod_{u=a}^x g(u)$ and $\sum_{u=a}^x g(u)$ are interpreted, respectively, as 1 and 0 if $x < a$. Define

$$h_i(x_i) = \sum_{k=(m_i+1)^+}^{x_i} \frac{t_i(k)}{t_i(k - m_i - 1)} \prod_{u=0}^{k-1} S_i^{-1}(u), \tag{3.5}$$

$D_0(\mathbf{x}) = \sum_{j=1}^p H_j^{B_j}(x_j)$ and $D(\mathbf{x}) = \sum_{j=1}^p h_j^{\beta_j}(x_j)$ for some positive numbers $\beta_1, \cdots, \beta_p$. Often $D_0(\mathbf{x})$ and $D(\mathbf{x})$ will be denoted by D_0 and D. Furthermore, define for some constants $\ell_1 > \ell_0$, $\lambda \ge 0$ and $c > 0$,

$$\Phi_i(\mathbf{x}) = q_i(\mathbf{x})\phi_i(\mathbf{x}), \quad q_i(\mathbf{x}) = D_0^{\ell_1}(\mathbf{x} + m_i\mathbf{e}_i) \prod_{u=0}^{x_i+m_i} S_i^{-1}(u), \tag{3.6}$$

$$\phi_i(\mathbf{x}) = -ch_i(x_i + m_i)/D^{\lambda}(\mathbf{x} + m_i\mathbf{e}_i).$$

Now the general inadmissibility result is stated below:

THEOREM 3.1. *Suppose that Conditions* (3.2), (3.3), *and* (3.4) *hold. If, for* $i = 1, \cdots, p$, (i) $R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) < \infty$ *and* $E\Phi_i^2(\mathbf{x}) < \infty$ *for all* $\boldsymbol{\theta}$, (ii) $h_i(x_i)/h_i(x_i - 1) \to 1$ *as* $x_i \to \infty$, (iii) *for some finite constant* K,

$$\sum_{i=1}^p h_i^2(x_i) \frac{t_i(x_i - m_i)}{t_i(x_i)} \{\prod_{u=0}^{x_i} S_i^{-2}(u)\} D_0^{\ell_1}/D^{\lambda} < K,$$

(iv) $p > \lambda \max_{1 \le j \le p} \beta_j$, *then* $\boldsymbol{\delta}^0$ *is inadmissible. Indeed* $\boldsymbol{\delta}^0(\mathbf{X})$ *is dominated by* $\boldsymbol{\delta}^0(\mathbf{X}) + I_M(\mathbf{X})\boldsymbol{\Phi}(\mathbf{X})$ *for some* $M > 0$, *and* $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}) < 0$ (*as in* (2.4)) *for sufficiently large* $\mathbf{x}$.

PROOF. See the Appendix. □

The main theorem of this paper, below, is now a consequence of Theorems 3.1 and 2.1.

THEOREM 3.2. *Assume that all the conditions of Theorem* 3.1 *hold. Under the loss function* $L_{\mathbf{m}}$, $\boldsymbol{\delta}^0$ *as in* (3.1) *is a* STUB *in the direction* $\mathbf{d}$ *with* ith *component*

$$d_i(x_i) = h_i(x_i)t_i(x_i - m_i)/\{t_i(x_i) \textstyle\prod_{u=0}^{x_i} S_i(u)\}.$$

PROOF. From Theorems 3.1 and 2.1, $\boldsymbol{\delta}^0$ is a STUB in the direction $\mathbf{d}^*$ with

$$d_i^*(x_i) = cD_0^{\ell}D^{-\lambda}h_i(x_i)t_i(x_i - m_i)/\{t_i(x_i) \textstyle\prod_{u=0}^{x_i} S_i(u)\}.$$

Since $cD_0^{\ell}D^{-\lambda}$ is positive for sufficiently large $\mathbf{x}$ and is independent of i, a STUB in the direction $\mathbf{d}^*$ is clearly a STUB in the direction $\mathbf{d}$.

We specialize the above general results to some particular distributions below. In Theorems 3.3, 3.4 and 3.5, $X_1, \cdots, X_p$ are assumed to be independent Poisson random variables with means $\theta_1, \cdots, \theta_p$. Each of the estimators considered in these theorems was shown by previous authors to dominate the standard estimator $\mathbf{X}$ under a certain loss function and yet is shown here to be an inadmissible STUB under the same loss function. Let $\mathscr{L}_m$ denote the loss function $L_{\mathbf{m}}$ as in (1.1) with $\mathbf{m} = (m, m, \cdots, m)$, i.e.,

$$\mathscr{L}_m(\boldsymbol{\theta}, \mathbf{a}) = \textstyle\sum_i \theta_i^m(\theta_i - a_i)^2.$$

The proofs of Theorems 3.3 through 3.6 are direct applications of Theorem 3.2 and therefore are omitted. Note that these theorems hold for any dimension p.

THEOREM 3.3. *Let* $h(x_i) = \sum_{k=1}^{x_i} k^{-1}$. *Under the loss function* $\mathscr{L}_0$, *the estimator* $\boldsymbol{\delta}^P$ *given componentwise by*

$$\delta_i^P(\mathbf{X}) = \left\{X_i - \frac{\ell h(X_i)}{\sum h^2(X_j)}\right\}^+, \quad \ell < p - 2, \tag{3.7}$$

is inadmissible and is a STUB *in the direction* $\mathbf{d}(\mathbf{x}) = (h(x_1), \cdots, h(x_n))$. □

Note that $\boldsymbol{\delta}^P$ is similar to Peng's (1975) estimator.

THEOREM 3.4. *Consider the loss function* $\mathscr{L}_m$, *with m a negative integer and let* $h(x_i) = \prod_{k=1}^{-m} (x_i + k)$. *The estimator* $\boldsymbol{\delta}$ *of Tsui and Press* (1982), *given componentwise by*

$$\delta_i^T(\mathbf{X}) = \left\{X_i - \frac{\ell h(X_i + m)}{\sum_j h(X_j + mn\eta_{ij})}\right\}^+, \quad \ell < -m(p - 1), \tag{3.8}$$

is inadmissible and is a STUB *in the direction* $\mathbf{d}(\mathbf{x})$, *with* $d_i(x_i) = 1$.

It was proven in Clevenson and Zidek (1975) that

$$\boldsymbol{\delta}^{CZ}(\mathbf{X}) = \left(1 - \frac{\ell}{p - 1 + \sum X_j}\right)^+ \mathbf{X} \tag{3.9}$$

dominates the standard estimator $\mathbf{X}$ under $\mathscr{L}_m$, with m being any negative integer, if $p \ge 2$ and $0 < \ell \le 2(p - 1)$. However, $\boldsymbol{\delta}^C$ is inadmissible for the values of ℓ given in the following theorem.

THEOREM 3.5. *Under $\mathscr{L}_m$, m being a negative integer, the estimator δ^{CZ}, with $\ell < (-m)(p-1)$, is inadmissible and is a* STUB *in the direction* **d** *given componentwise by*

$$d_i(x_i) = (x_i + 1)!/(x_i - m)! \quad \square$$

Results similar to Theorem 3.5 can also be established for the estimators in Ghosh and Parsian (1981). Assume in the next theorem that X_i, $i = 1, \cdots, p$, are independent random variables having $NB(r_i, \theta_i)$ distributions. The parameters r_i are assumed to be known.

THEOREM 3.6. *Let $h(x_i) = \sum_{k=1}^{x_i} (r_i - 1 + k)/k$. Let δ^{NB} be the estimator of $\boldsymbol{\theta}$ given componentwise by*

$$\delta_i^{NB}(\mathbf{X}) = \left\{\frac{X_i}{r_i - 1 + X_i} - \frac{\ell h(X_i)}{\Sigma_j h^2(X_j)}\right\}^+, \quad \ell < p - 2. \tag{3.10}$$

Under $\mathscr{L}_0$, δ^{NB} is inadmissible and is a STUB *in the direction* $\mathbf{d}(\mathbf{x})$ *with* $d_i(x_i) = h(x_i)$. $\square$

Many inadmissibility results of previous authors are immediate consequences of Theorems 3.2 through 3.6 and Lemma 2.1 For the Poisson families, Theorem 3.3 implies that the UMVUE, **X**, is inadmissible under $\mathscr{L}_0$ for $p \ge 3$, a result first obtained in Peng (1975). It is also a corollary of Theorem 3.4 that the same estimator **X** is inadmissible under $\mathscr{L}_m$ if $m < 0$ and $p \ge 2$, a result of Clevenson and Zidek (1975). From Theorem 3.6 it follows, for Negative Binomial Families, that the UMVUE, $(X_1/(r_1 + X_1 - 1), \cdots, X_p/(r_p + X_p - 1))$, is inadmissible under $\mathscr{L}_0$ if $p \ge 3$, a result by Hwang (1982a); Hudson (1978) proved a special case of this result.

More elaborate applications of Theorems 3.2 through 3.6 yield strong new results. Only the most interesting ones are given below.

In Corollaries 3.7 and 3.8, the observations $X_1, \cdots, X_p$ are assumed to be independent Poisson variables with means $\theta_1, \cdots, \theta_p$.

COROLLARY 3.7. *Under $\mathscr{L}_{-1}$, the estimator δ^*, given componentwise by*

$$\delta_i^*(\mathbf{X}) = (\textstyle\sum_{j=1}^p X_j)X_i/(\beta + p - 1 + \sum_{j=1}^p X_j) \tag{3.11}$$

is inadmissible if $\beta < 0$.

PROOF. Compare δ^* with the STUB δ^{CZ} given in (3.9), where $\mathscr{L}_m = \mathscr{L}_{-1}$ and ℓ is chosen to be such that $\beta + p - 1 < \ell < p - 1$. $\square$

Brown (1979) conjectured that δ^* is admissible if and only if $\beta \ge 0$. Here, for the inadmissibility part, we have shown this conjecture to be true. For $\beta \ge 0$, Brown and Hwang (1981) established the admissibility of δ^*. Clevenson and Zidek (1975) proved the same result for $\beta > 1$. Therefore, the condition $\beta < 0$ for the inadmissibility of δ^* is as weak as possible. Thus in this case, Theorem 3.5 provides very sharp STUBs.

Even though δ^* is admissible under $\mathscr{L}_{-1}$ if $\beta \ge 0$, it is inadmissible under some other loss functions, as the following corollary shows. Therefore, the admissibility property seems to be fairly sensitive to the loss function used.

COROLLARY 3.8. *Under the loss function $\mathscr{L}_m$, m a negative integer, the estimator δ^* is inadmissible if $\beta < (-m - 1)(p - 1)$.*

PROOF. This follows from Theorem 3.5 and Lemma 2.1. $\square$

The corollary below concerns the case in which the independent observations have $NB(r, \theta_i)$ distributions where r is assumed to be known. Hudson (1978) developed the estimator δ^H, given componentwise by

$$(3.12)\qquad \delta_i^H(\mathbf{X}) = \frac{X_i}{r-1+X_i} - \frac{[N(\mathbf{X})-3]^+ h(X_i)}{\Sigma_j h^2(X_j)},\ h(x_i) = \sum_{k=1}^{x_i} (r-1+k)/k,$$

where $N(\mathbf{X}) = \#\{i: X_i > 0\}$. He proved that $\boldsymbol{\delta}^H$ dominates the UMVUE of $\boldsymbol{\theta}$ if the loss function is $\mathscr{L}_0$ and $p \geq 4$. However, it is again inadmissible.

COROLLARY 3.9. *If $p \geq 3$, then $\boldsymbol{\delta}^H$ and its positive part $(\boldsymbol{\delta}^H)^+$ are inadmissible under $\mathscr{L}_0$.*

PROOF. Compare $\boldsymbol{\delta}^H$ and $(\boldsymbol{\delta}^H)^+$ to the STUB $\boldsymbol{\delta}^{NB}$ in (3.10) with $\ell = p - 2.5$ and $r_i = r$ for all i. □

4. Discussion. Of interest is the statistical significance of the direction $\mathbf{d}(\mathbf{x})$ associated with a particular STUB, $\boldsymbol{\delta}^0$. For $\mathscr{L}_0$, it can be seen that, by moving $\boldsymbol{\delta}^0$ (which satisfies all the conditions in Theorem 3.1) toward the direction $-\mathbf{d}(\mathbf{x})$, one can improve upon $\boldsymbol{\delta}^0$. Some STUBs can be improved in several directions; the directions lie in a certain limited range.

In discrete exponential families, the estimator $\boldsymbol{\delta}(\mathbf{X}) = (0, \cdots, 0)$ is clearly admissible. Therefore, nontrivial "lower bounds" on the class of admissible estimators do not exist. However, in the continuous case (e.g., gamma families) under certain loss functions, one must pull the standard estimator away from $(0, 0, \cdots, 0)$ to improve upon it. (See Berger, 1980). In such a situation, nontrivial "lower bounds" do exist; for details, see Hwang (1982b).

The technique developed here can be used to provide a simpler proof of some of Brown's (1973) inadmissiblity results by showing that $\boldsymbol{\delta}_\ell^{JS}$ is in fact a STUB in the direction $\mathbf{x} = (x_1, \cdots, x_p)$. These results are again established by proving that $\boldsymbol{\delta}_\ell^{JS}$ can be improved by shrinking it toward the direction $(x_1, \cdots, x_p)$. See Hwang (1982b).

Note that Brown (1971) also established, for the case described in Section 1, that admissible rules have to be generalized Bayes and that any generalized Bayes rule $\boldsymbol{\delta}^B$, for which $|\boldsymbol{\delta}^B(\mathbf{X}) - \mathbf{X}|$ is uniformly bounded, is admissible, if $\mathbf{x}\cdot\boldsymbol{\delta}^B(\mathbf{x}) \leq \mathbf{x}\cdot\boldsymbol{\delta}_{p-2}^{JS}(\mathbf{x})$ for sufficiently large $\mathbf{x}$. These results indicate that $\{\boldsymbol{\delta}_\ell^{JS}, \ell < p-2\}$ are not only STUBs, but also approach a "dividing line" $\boldsymbol{\delta}_{p-2}^{JS}$ (a term borrowed from Strawderman and Cohen, 1971) between admissible and inadmissible rules. In each problem considered here, it is possible that the STUBs also approach a dividing line between admissibility and inadmissibility. Of course, all the results developed here apply not only to simultaneous estimation problems, but also to the one dimensional case ($p = 1$) as well.

In applying a STUB to prove inadmissibility of $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_p)$, there is an easy generalization of our result. Clearly, inadmissibility of the "sub-estimator" (e.g., $(\delta_1, \delta_2, \cdots, \delta_q)$ where $q < p$) under the corresponding "sub-loss function" (e.g., $\sum_{i=1}^q \theta_i^{m_i}(\delta_i - \theta_1)^2$) implies the inadmissibility of $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_p)$ under $L_{\mathbf{m}}$. Therefore, instead of trying to compare the whole estimator $\boldsymbol{\delta}$ to a STUB, we can deal with a sub-estimator and compare it with a STUB of the same dimension.

Another generalization is that any estimator which is proved to be inadmissible under $L_{\mathbf{m}}$ in this paper is also inadmissible under the loss function $L_{\mathbf{m}}^{\mathbf{k}}(\boldsymbol{\theta}, \mathbf{a}) = \sum_{j=1}^p k_i \theta_i^{m_i}(\theta_1 - a_i)^2$ where $k_i > 0$. This is true since in the proof of Theorem 3.1 the existence of a nontrivial solution of (A.4) implies the same for the difference inequality

$$\sum_{i=1}^p k_i\{v_i(x_i)F_i(\mathbf{x})\Delta_i\phi_i(\mathbf{x}) + w_i(\mathbf{x})\phi_i^2(\mathbf{x})\} \leq 0.$$

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APPENDIX

To use Lemma 2.2 to prove Theorem 3.1 (or in general to use the same lemma to improve upon any inadmissible estimator $\boldsymbol{\delta}^0$), one need obtain solutions $\boldsymbol{\Phi}$ to $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}) \leq$

0. In order to find nontrivial solutions to this inequality, we derive Theorems A.1 and A.2. The general scheme is to write $\Phi_i = q_i\phi_i$ and then decompose $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi})$ into the sum of two manageable terms as in Theorem A.1 below.

THEOREM A.1. *Let $\mathscr{D}_i$ be the* i*th term on the right hand side of* (2.4). *Then*

$$\mathscr{D}_i = \mathscr{D}_i' + 2\mathscr{E}_i'\phi_i(\mathbf{x} - m_i\mathbf{e}_i)/t_i(x_i) \tag{A.1}$$

where

$$\mathscr{D}_i' = 2\{t_i(x_i - m_i - 1)/t_i(x_i)\}q_i(\mathbf{x} - (m_i + 1)\mathbf{e}_i)\Delta_i\phi_i(\mathbf{x} - m_i\mathbf{e}_i) + \{t_i(x_i - m_i)/t_i(x_i)\}q_i^2(\mathbf{x} - m_i\mathbf{e}_i)\phi_i^2(\mathbf{x} - m_ie_i) \tag{A.2}$$

and

$$\mathscr{E}_i' = \delta_i^0(\mathbf{x} - m_i\mathbf{e}_i)t_i(x_i - m_i)q_i(\mathbf{x} - m_i\mathbf{e}_i) - t_i(x_i - m_i - 1)q_i(\mathbf{x} - (m_i + 1)\mathbf{e}_i). \tag{A.3}$$

PROOF. Direct calculation. □

Assuming $\phi_i(\mathbf{x}) \le 0$, it is not difficult to choose $q(\mathbf{x})$ independent of ϕ so that $\mathscr{E}_i'\phi_i(\mathbf{x} - m_i\mathbf{e}_i) \le 0$. Therefore to solve $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}) \le 0$, it suffices to find ϕ, $\phi_i \le 0$ so that $\sum \mathscr{D}_i' \le 0$ for such a choice of $\mathbf{q}$. The analogous decomposition for the continuous case was first demonstrated in Berger (1980).

To solve $\sum \mathscr{D}_i' \le 0$, we consider a general difference inequality which has the form

$$\mathscr{D}^*(\phi) = \textstyle\sum_{i=1}^{p} v_i(x_i)F_i(\mathbf{x})\Delta_i\phi_i(\mathbf{x}) + w_i(\mathbf{x})\phi_i^2(\mathbf{x}) \le 0, \tag{A.4}$$

where $v_i(x_i)$, $F_i(\mathbf{x})$, and $w_i(\mathbf{x})$ are nonnegative functions and there exists a nonnegative integer α such that $v_i(x_i) > 0$ whenever $x_i \ge \alpha$. It is assumed that there exists an $M > 0$, such that

$$|F_i(\mathbf{x}) - 1| < \varepsilon < 1 \tag{A.5}$$

for all $i = 1, \cdots, p$ and all $\mathbf{x} \in A_M$, where

$$A_M = \{\mathbf{x}: x_i \ge M, i = 1, \cdots, p\}. \tag{A.6}$$

We will interpret 0/0 to be 0. Let $h_i(x_i) = \sum_{k=\alpha}^{x_i} v_i^{-1}(k)$. For the definition of $I_M(\mathbf{x})$, M an arbitrary number, see the paragraph before Theorem 2.1.

THEOREM A.2. Suppose that there exist some positive constants β_i, $\alpha' > \max(M, \alpha + 1)$, U_i, K, *and a nonnegative constant* λ *such that for* $\mathbf{x} \in A_{\alpha'}$ *and* $i = 1, \cdots, p$, (i) $h_i(x_i)/h_i(x_i - 1) \le U_i$ *and* (ii) $\sum_{i=1}^{p} w_i(\mathbf{x})h_i^2(x_i)/D^\lambda \le K$, *where* $D = \sum_{i=1}^{p} h_i^{\beta_i}(x_i)$. *Let*

$$\beta' = \lambda\{\max_{1\le i\le p}\beta_i U_i^{(\beta_i - 1)^+}\}\{\max_{1\le i\le p} U_i^{(\beta_i(\lambda-1)^+)}\}$$

and $\phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \cdots, \phi_p(\mathbf{x}))$, *where*

$$\phi_i(\mathbf{x}) = -ch_i(x_i)/D^\lambda. \tag{A.7}$$

If $p > \beta'(1 + \varepsilon)/(1 - \varepsilon)$, *then for any* c, $0 < c < \{p(1 - \varepsilon) - \beta'(1 + \varepsilon)\}/K$, $I_{\alpha'}(\mathbf{x})\phi(\mathbf{x})$ *is a solution to* (A.4), *and indeed*

$$\mathscr{D}^*(I_{\alpha'}\phi) \le -c\{p(1 - \varepsilon) - \beta'(1 + \varepsilon) - cK\}I_{\alpha'}(\mathbf{x})/D^\lambda. \tag{A.8}$$

Further $\mathscr{D}^*(\phi) < 0$ *for sufficiently large* $\mathbf{x}$.

PROOF OF THEOREM A.2. Clearly $\mathscr{D}^*(I_{\alpha'}(\mathbf{x})\phi(\mathbf{x})) \le I_{\alpha'}(\mathbf{x})\mathscr{D}^*(\phi(\mathbf{x}))$. To complete the proof, we need only show that, for $\mathbf{x} \in A_{\alpha'}$,

$$\mathscr{D}^*(\phi(\mathbf{x})) \le -c\{p(1 - \varepsilon) - \beta'(1 + \varepsilon) - cK\}/D^\lambda. \tag{A.9}$$

In the following arguments, consider only $\mathbf{x} \in A_{\alpha'}$. Let $D_i = h_i^{\beta_i - 1}(x_i - 1) + \sum_{j \neq i} h_j^{\beta_j}(x_j)$ and $D' = \sum_{j=1}^{p} h_j^{\beta_j}(x_j - 1)$. Direct calculation gives

$$\sum_{i=1}^{p} v_i(x_i) F_i(\mathbf{x}) \Delta_i \phi_i(\mathbf{x}) = -c \sum_{i=1}^{p} v_i(x_i) F_i(\mathbf{x}) \left\{ \frac{\Delta_i h_i(x_i)}{D^\lambda} - \frac{h_i(x_i - 1)\Delta_i D^\lambda}{D^\lambda D_i^\lambda} \right\} \tag{A.10}$$
$$\leq \frac{-cp(1-\varepsilon)}{D^\lambda} + \frac{c(1+\varepsilon)}{D^\lambda (D')^\lambda} \Sigma v_i(x_i) h_i(x_i - 1) \Delta_i D^\lambda,$$

where the last inequality follows from (A.5) and the fact that $D_i \geq D'$. If we can prove that,

$$\sum_{i=1}^{p} v_i(x_i) h_i(x_i - 1) \Delta_i D^\lambda / (D')^\lambda \leq \beta', \tag{A.11}$$

then, together with (A.10) and assumption (ii) of this theorem, (A.9) will be established.

Now for $\lambda \geq 1$, applying the mean value theorem gives

$$v_i(x_i) h_i(x_i - 1) \Delta_i D^\lambda \leq \lambda D^{\lambda - 1} v_i(x_i) h_i(x_i - 1) \Delta_i h_i^{\beta_i}(x_i). \tag{A.12}$$

From assumption (i), it follows that

$$\Delta_i h_i^{\beta_i}(x_i) \leq \beta_i U_i^{(\beta_i - 1)^+} h_i^{\beta_i - 1}(x_i - 1)/v_i(x_i) \tag{A.13}$$

which, in addition to (A.12), implies that

$$v_i(x_i) h_i(x_i - 1) \Delta_i D^\lambda \leq \lambda \beta_i U_i^{(\beta_i - 1)^+} h_i^{\beta_i}(x_i - 1) D^{\lambda - 1}. \tag{A.14}$$

For $\lambda < 1$, (A.12) again holds if, on the right hand side, $D^{\lambda-1}$ is replaced by $(D')^{\lambda-1}$. Together with (A.14), this shows that, for any λ,

$$v_i(x_i) h_i(x_i - 1) \Delta_i D^\lambda \leq \lambda \beta_i U_i^{(\beta_i - 1)^+} h_i^{\beta_i}(x_i - 1)(D')^{\lambda - 1}(D/D')^{(\lambda - 1)^+}. \tag{A.15}$$

By assumption (i), $D/D' \leq \max U_i^{\beta_i}$. This and (A.15) therefore establish (A.11). □

Although the above theorem appears complicated, the following immediate corollary is simpler and will be what is actually used to prove Theorem 3.1.

COROLLARY A.3. *Assume that $h_i(x_i)/h_i(x_i - 1) \to 1$ as $x_i \to \infty$ and for every $\varepsilon > 0$, there exists $M > 0$ such that* (A.5) *is satisfied. Suppose that condition* (ii) *of Theorem* A.2 *is satisfied for some positive finite constants K, β_i and λ. If $p > \lambda \max \beta_j$, then there exists ϕ, of the form* (A.7), *with $c > 0$, such that $\mathscr{D}^*(\phi) < 0$ for sufficiently large* $\mathbf{x}$. □

To apply Corollary A.3, we choose β_i and λ such that D^λ has the same order as $\sum w_i(\mathbf{x}) h_i^2(x_i)$, so that assumption (ii) of Theorem A.2 is satisfied. This is the procedure that is used in establishing Theorem 3.1.

PROOF OF THEOREM 3.1. In this proof let $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi})$ be as in (2.4) and $\boldsymbol{\delta}^0$, $\boldsymbol{\Phi}$, q_i, and ϕ_i be as defined in the paragraphs between (3.1) and (3.6). By Corollary 2.2, one need only show that $\mathscr{D}(\boldsymbol{\delta}^0, \boldsymbol{\Phi}) < 0$ for sufficiently large $\mathbf{x}$. From Theorem A.1, the inequality follows if one shows that for sufficiently large $\mathbf{x}$,

$$\mathscr{E}_i' \geq 0, \quad i = 1, \cdots, p, \text{ and} \tag{A.16}$$

$$\mathscr{D}' = \Sigma \mathscr{D}_j' < 0 \tag{A.17}$$

where $\mathscr{E}_i'$ and $\mathscr{D}_i'$ are given in (A.2) and (A.3).

To show (A.16), observe that

$$\mathscr{E}_i' = t_i(x_i - m_i - 1)\{S_i(x_i)(\mathrm{FSK})_i(\mathbf{x}) q_i(\mathbf{x} - m_i \mathbf{e}_i) - q_i(\mathbf{x} - (m_i + 1)\mathbf{e}_i)\}$$

where $(\mathrm{FSK})_i(\mathbf{x})$ denotes the factor of shrinkage in (3.1), i.e.,

$$(\mathrm{FSK})_i(\mathbf{x}) = \{1 - \ell_0 B_i H_i^{\beta_i - 1}(x_i) \Delta H_i(x_i)/D_0\}^+.$$

Note that from (3.6)

$$S_\iota(x_\iota)(\mathrm{FSK})_\iota(\mathbf{x})q_i(\mathbf{x}-m_\iota\mathbf{e}_\iota)-q_\iota(\mathbf{x}-(m_\iota+1)\mathbf{e}_i)$$
$$=\textstyle\prod_{u=0}^{x_\iota-1}S_i^{-1}(u)\{(\mathrm{FSK})_i(\mathbf{x})D_0^{\ell_1}(\mathbf{x})-D_0^{\ell_1}(\mathbf{x}-\mathbf{e}_i)\}.$$

To establish (A.16), what remains to be shown is that, for sufficiently large **x**,

$$D_0^{\ell_1}(\mathbf{x})/D_0^{\ell_1}(\mathbf{x}-\mathbf{e}_\iota)\geq 1/(\mathrm{FSK})_i(\mathbf{x}). \tag{A.18}$$

Note $(\mathrm{FSK})_i(\mathbf{x})>0$, for sufficiently large **x**, because

$$0\leq H_i^{B_\iota-1}(x_i)\Delta H_i(x_i)/D_0\leq \Delta H_i(x_i)/H_i(x_i), \tag{A.19}$$

and $\Delta H_\iota(x_\iota)/H_\iota(x_\iota)$ approaches zero as $x_\iota\to\infty$ by (3.3). Inequality (A.18) is thus equivalent to

$$\frac{\Delta_\iota D_0^{\ell_1}(\mathbf{x})}{D_0^{\ell_1}(\mathbf{x}-\mathbf{e}_\iota)}\geq\frac{\ell_0 B_\iota H_i^{B_\iota-1}(x_\iota)\Delta H_\iota(x_\iota)}{D_0-\ell_0 B_\iota H_i^{B_\iota-1}(x_i)\Delta H_\iota(x_\iota)}. \tag{A.20}$$

If $B_\iota=0$, (A.20) is satisfied for any ℓ_1 (ℓ_1 can be negative), since $\Delta_i D_0^{\ell_1}(\mathbf{x})=0$. Assume now that $B_\iota>0$. By the mean value theorem, let $D_i^*(\mathbf{x})$ be such that $D_0(\mathbf{x}-\mathbf{e}_i)<D_i^*(\mathbf{x})<D_0(\mathbf{x})$ and

$$\Delta_\iota D_0^{\ell_1}(\mathbf{x})=\ell_1(D_i^*(\mathbf{x}))^{\ell_1-1}\Delta H_i^{B_\iota}(x_\iota).$$

Clearly (A.20) is equivalent to $\ell_1\geq\ell_0/R_\iota(\mathbf{x})$ where

$$R_\iota(\mathbf{x})=\frac{\{\Delta H_i^{B_\iota}(x_\iota)\}\{D_0-\ell_0 B_i H_i^{B_\iota-1}(x_i)\Delta H_\iota(x_\iota)\}\{D_i^*(\mathbf{x})\}^{\ell_1-1}}{B_\iota H_i^{B_\iota-1}(x_\iota)\Delta H_\iota(x_i)D_0^{\ell_1}(\mathbf{x}-\mathbf{e}_\iota)}. \tag{A.21}$$

We now show that $R_\iota(\mathbf{x})\to 1$ as $x_\iota\to\infty$ uniformly in other variables, which, together with the fact that $\ell_1>\ell_0$, implies $\ell_1\geq\ell_0 R_\iota(\mathbf{x})$ for sufficiently large **x**. (The statement is true even for the case where either or both ℓ_0 and ℓ_1 are negative.) By the mean value theorem and (3.3), we have

$$\Delta H_i^{B_\iota}(x_i)/\{B_i H_i^{B_\iota-1}(x_\iota)\Delta H_\iota(x_i)\}\to 1$$

as $x_\iota\to\infty$. Hence, in order to establish $R_i(\mathbf{x})\to 1$, we need only show that

$$\{D_i^*(\mathbf{x})\}^{\ell-1}\{D_0-\ell_0 B_i H_i^{B_\iota-1}(x_\iota)\Delta H_i(x_i)\}/D_0^{\ell_1}(\mathbf{x}-\mathbf{e}_\iota)\to 1 \tag{A.22}$$

as $x_i\to\infty$ uniformly in all other variables. Now by condition (3.3),

$$1\leq\frac{D_i^*(\mathbf{x})}{D_0(\mathbf{x}-\mathbf{e}_\iota)}\leq\frac{D_0(\mathbf{x})}{D_0(\mathbf{x}-\mathbf{e}_\iota)}\leq\frac{H_i^{B_\iota}(x_\iota)}{H_i^{B_\iota}(x_\iota-1)}\to 1 \tag{A.23}$$

as $x_i\to\infty$ uniformly in the other variables. Therefore, it follows that

$$\{D_i^*(\mathbf{x})/D_0(\mathbf{x}-\mathbf{e}_\iota)\}\to 1 \tag{A.24}$$

in the same way. From (A.19), we have

$$\{H_i^{B_\iota-1}(x_i)\Delta H_i(x_i)/D_0(\mathbf{x})\}\to 0$$

as $x_i\to\infty$ uniformly in other variables, which together with (A.24) implies that

$$\{D_0-\ell_0 B_i H_i^{B_\iota-1}(x_i)\}/D_0(\mathbf{x}-\mathbf{e}_\iota)\to 1 \tag{A.25}$$

in the same manner. Equation (A.24) and (A.25) clearly establish (A.22) and consequently (A.16) for sufficiently large **x**.

To show (A.17), by plugging (3.6) into (A.2), one can observe that $\mathscr{D}'=2D_0^{\ell_1}(\mathbf{x})\mathscr{D}''$, where

(A.26)
$$\mathscr{D}'' = \sum_{i=1}^{p} \frac{t_i(x_i - m_i - 1)}{t_i(x_i)} \frac{D_0^{\ell_1}(\mathbf{x} - \mathbf{e}_i)}{D_0^{\ell_1}(\mathbf{x})} \prod_{u=0}^{x_i - 1} S_i^{-1}(u) \Delta_i \phi_i(\mathbf{x} - m_i \mathbf{e}_i) + \frac{1}{2} \frac{t_i(x_i - m_i)}{t_i(x_i)} D_0^{\ell_1}(\mathbf{x}) \{\prod_{u=0}^{x_i} 1/S_i^2(u)\} \phi_i^2(\mathbf{x} - m_i \mathbf{e}_i).$$

It is now sufficient to show that $\mathscr{D}'' < 0$ for sufficiently large $\mathbf{x}$. Now $\mathscr{D}''$ has the form (A.4) with $F_i(\mathbf{x}) = D_0^{\ell_1}/D_0^{\ell_1}\ (\mathbf{x} - \mathbf{e}_i)$ satisfying the assumptions of Corollary A.3. This together with assumptions (ii), (iii) and (iv) of this theorem, implies that, for some $c > 0$, ϕ as given componentwise in (3.6) is such that $\mathscr{D}'' < 0$ for sufficiently large $\mathbf{x}$. □

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