Proceedings of the 34th
Conference on Decision & Control
New Orleans, LA - December 1995

Semialgebraic sets, Stabilization, and Computability*

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Abstract

We show that the simultaneous stabilization question: *When are three linear systems stabilizable by the same controller?* cannot be solved by a semialgebraic set description nor be answered by computational machines.

1 Introduction

Let $\beta_1, \beta_2, \beta_3$ be real numbers. The three first order systems

$$(1) \qquad \frac{s+1}{s-1} + \beta_1 \quad \frac{s+1}{s-1} + \beta_2 \quad \frac{s+1}{s-1} + \beta_3$$

are said to be *simultaneously stabilizable* if there exists a controller that stabilizes each system.

It is shown in [6] that, when $\beta_1 = 0$ and $\beta_2 = -\beta_3 = \beta$, the three systems are simultaneously stabilizable if and only if $|\beta| < \Gamma^4(\frac{1}{4})/(4\pi^2) = 4.377\cdots$, where $\Gamma(.)$ is the usual factorial function.

In this paper we complete this analysis by providing a completely general necessary and sufficient condition for the three systems (1) to be simultaneously stabilizable. The condition involves the parameters β_i and a transcendental function known as the elliptic modular function (see below for a definition).

From our analysis of these three systems we then draw general conclusions on the structure of the set of systems that are simultaneously stabilizable. Our conclusions are twofold. First, the set of triplets of systems that are simultaneously stabilizable is not semialgebraic; second, simultaneous stabilizability of more than two systems is not decidable by standard computation machines.

The paper is organized as follows.

In Section 2 we show that the stabilizability of the systems (1) depends upon the existence of a rational function that satisfies nonclassical interpolation constraints. We then use a theorem from geometric function theory – Landau's theorem – to derive necessary and sufficient conditions for the solvability of this interpolation problem.

In Section 3 we use the condition obtained to show that the set of coefficients β_i for which the systems are simultaneously stabilizable is not a semialgebraic set. This result extends an earlier result of Blondel and Gevers [6] and answers a question raised by Ghosh in several of his papers (e.g., in [10] and [11]). Semialgebraic sets are particular subsets of $\mathbb{R}^n$ that can be used to describe a large variety of decision problems (see later for some examples).

In a last section we interpret the results of Section 3 in terms of computability. We prove that the problem of determining whether our three systems are simultaneously stabilizable cannot be decided by certain computational machines.

The systems that we consider are linear, time-invariant, single-input, single-output and are given by their (real rational) transfer functions. A controller c stabilizes a system p if the four closed loop transfer functions $cp/(1+cp)$, $p/(1+cp)$, $c/(1+cp)$, $1/(1+cp)$ are stable (i.e., are proper and have no poles in the closed right half plane). The k systems p_i $(i = 1, \cdots, k)$ are simultaneously stabilizable if there exist a controller c that stabilizes each system p_i. We use the following notation: $\mathbb{C}$ and $\mathbb{R}$ are the sets of complex and real numbers, $\Re(z)$ and $\Im(z)$ are

*Modified version of "Transcendence in Simultaneous stabilization" to appear in J. Math. Systems, Estimation, and Control. This work was completed while Blondel was Göran Gustafsson Research Fellow at the Royal Institute of Technology, Stockholm, Sweden

0-7803-2685-7/95 $4.00

the real and imaginary parts of z. $\mathbb{R}(z)$ is the set of real rational functions. $D(R) = \{z \in \mathbb{C} \mid |z| < R\}$ is the open disc with center 0 and radius R. $D = D(1)$ is the open unit disc, and $\overline{D}$ is its closure. $\Pi^+ = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ is the open upper half plane.

2 Three special systems

Our condition for the systems (1) to be simultaneously stabilizable involves a function known as the *elliptic modular function.* There are several related functions associated with this name. The one that we are considering here is a conformal mapping λ of the upper half plane Π^+ onto $\mathbb{C}\setminus\{0,1\}$. For the construction and properties of λ, see Segal [16, pp. 68-76] and Rudin [14, sec. 16.17-16.20]. Local inverses of λ will be denoted by ν.

Theorem 1: The systems (1) are simultaneously stabilizable if and and only if

$$|a_1| < \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}, \tag{2}$$

where
a_0 and a_1 are defined by $a_0 = (\beta_2-\beta_3)/(\beta_2-\beta_1)$, $a_1 = (\beta_2-\beta_3)(\beta_1-\beta_3)/(\beta_2-\beta_1)$, and ν is a local inverse of the elliptic modular function λ. The right hand side in (2) does not depend on the particular choice of ν.

The proof of this result will follow from an adaptation of Landau's Theorem.

Landau's theorem: Suppose that $f : D(R) \to \mathbb{C}\setminus\{0,1\}$ is analytic. Let $a_0 = f(0)$, $a_1 = f'(0)$ and let ν be a local inverse of λ in a neighbourhood of a_0. Then

$$R|a_1| \le \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}. \tag{3}$$

Equality holds in (3) if and only if $f = \lambda\circ\psi$, where

$$\psi(z) = \frac{\overline{\nu(a_0)}\alpha z + R\nu(a_0)}{\alpha z + R}, \tag{4}$$

and α is a complex number of modulus 1.
Conversely, suppose that a_0 and a_1 are complex numbers such that $a_0 \ne 0,1$ and inequality (3) holds. Then there exists an analytic function $f : D(R) \to \mathbb{C}\setminus\{0,1\}$ such that $f(0) = a_0$ and $f'(0) = a_1$. If a_0 and a_1 are real numbers, then f can be chosen so that $f(\overline{z}) = \overline{f(z)}$. ■

We need a rational version of Landau's Theorem.

Landau's Theorem [rational version]: Suppose that a_0 and a_1 are real numbers such that $a_0 \ne 0,1$. Then there exists a real rational function q such that $q(0) = a_0$, $q'(0) = a_1$ and $q(z) \ne 0,1,\infty$ for all z in the closed unit disc $\overline{D}$ if and only if

$$|a_1| < \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}, \tag{5}$$

where ν is a local inverse of the elliptic modular function λ. The right hand side in (5) does not depend on the particular choice of ν.

Proof: *"Only if" part:* Assume that q is such a function. By continuity, there exists an $R > 1$ such that $q(z) \ne 0,1,\infty$ for all z in the open disc $D(R)$. Landau's theorem now gives us the inequality (3), from which the strict inequality (5) follows since $R > 1$.
"If" part: Assume that (5) holds. In the trivial case $a_1 = 0$ we may choose q to be constant. Otherwise, define

$$R = \frac{2\Im(\nu(a_0))}{|a_1||\nu'(a_0)|} \tag{6}$$

The strict inequality (5) shows that $R > 1$. By Landau's theorem and equality (6), there exists an analytic function $f : D(R) \to \mathbb{C}\setminus\{0,1\}$ such that $f(0) = a_0, f'(0) = a_1$ and $f(\overline{z}) = \overline{f(z)}$. We now approximate f with a real polynomial q which does not assume the values 0 and 1 in $\overline{D}$ and satisfies $q(0) = a_0$ and $q'(0) = a_1$. This will complete the "if" part. For this purpose, let

$$\eta = \min\{\inf_{z\in\overline{D}}|f(z)|, \inf_{z\in\overline{D}}|f(z)-1|\}.$$

Since $f(z) \ne 0,1$ for all $z \in D(R)$, and since $\overline{D}$ is compact, $\eta > 0$. The function defined by $h(z) \triangleq (f(z) - a_0 - a_1 z)/z^2$ is analytic and such that $h(\overline{z}) = \overline{h(z)}$ in $D(R)$. By Runge's theorem (see Rudin [14, Theorem 13.7]) there is a polynomial p such that

$$|h(z) - p(z)| < \eta \qquad (\forall z \in \overline{D}). \tag{7}$$

Define a real polynomial by $p_1(z) \triangleq (p(z) + \overline{p(\overline{z})})/2$. Since $h(\overline{z}) = \overline{h(z)}$, it is easy to see that p_1 satisfies (7) also. Now the real polynomial $q(z) \triangleq a_0 + a_1 z + z^2 p_1(z)$ satisfies

$$|f(z) - q(z)| = |z^2 h(z) - z^2 p_1(z)| < \eta \qquad (\forall z \in \overline{D}).$$

In conjunction with the definition of η, this shows that $q(z) \ne 0,1$ for all $z \in \overline{D}$. ■

We now have all what is needed to prove the theorem.

Proof of Theorem 1: Notice first that, using the standard bilinear transformation, the three systems (1) are simultaneously stabilizable if and only if the three systems

$$\frac{z}{1+\beta_1 z}, \quad \frac{z}{1+\beta_2 z}, \quad \frac{z}{1+\beta_3 z}$$

are simultaneously stabilizable in the closed unit disc. (We have used the fact that systems are simultaneously stabilizable if and only if their inverses are.) We now adopt the factorization approach given in Vidyasagar [17]. A controller n/d (where n and d are coprime stable rational functions) in closed loop with a system n_i/d_i (where n_i and d_i are coprime stable rational functions) leads to a stable closed-loop configuration if and only if $n_i n + d_i d$ has no zeros in the closed unit disc $\overline{D}$. The controller $n/d = -\beta_3/1$ stabilizes the third system $z/(1+\beta_3 z)$. Hence, by the Youla-Kucera parametrization, a factorization of all the controllers n/d that stabilize this system is given by

(8) $n(z) = -\beta_3 + r(z)(1+\beta_3 z), d(z) = 1 - r(z)z,$

where r is an arbitrary real rational function with no poles in $\overline{D}$. This controller n/d also stabilizes the first and second systems if and only if

(9) $$zn(z) + (1+\beta_i z)d(z) \neq 0$$

for $z \in \overline{D}$ and $i = 1, 2$. Putting (8) into (9) and simplifying, we get

(10) $$1 + (\beta_i - \beta_3)z + (\beta_3 - \beta_i)z^2 r(z) \neq 0$$

for $z \in \overline{D}$ and $i = 1, 2$. After division by $\beta_i - \beta_3$ and introduction of

$$a_0 = \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1},$$
$$a_1 = \frac{(\beta_2 - \beta_3)(\beta_1 - \beta_3)}{\beta_2 - \beta_1},$$

the condition (10) can be written as

(11) $a_0 + a_1 z - a_1 z^2 r(z) \neq 0, 1 \qquad$ for $z \in \overline{D}$.

We have thus shown that the systems (1) are simultaneously stabilizable if and only if there exists a real rational function r with no poles in $\overline{D}$ such that (11) holds. Denoting the left-hand side of (11) by $q(z)$ it is easy to see that the existence of a rational function r that has the required properties is equivalent to the existence of a real rational function q that is such that $q(0) = a_0$, $q'(0) = a_1$ and $q(z) \neq 0, 1, \infty$ for all z in $\overline{D}$. By the rational version of Landau's Theorem, this is equivalent to the strict inequality (2). ∎

3 Semialgebraic sets

Let S be the set of all triplets $(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ for which the systems (1) are simultaneously stabilizable. In this section we use the explicit description of S given in Theorem 1 to show that S is not a semialgebraic set.

Definition: A set $X \subseteq \mathbb{R}^n$ is semialgebraic if it is a finite union of sets of the type

$$\{x \in \mathbb{R}^n \mid P_1(x) = 0, ..., P_k(x) = 0, \\ P_{k+1}(x) > 0, ..., P_m(x) > 0\},$$

where $P_i(x) = P_i(x_1, ..., x_n)$ $(i = 1, 2, ..., m)$ are real polynomials in n variables (general references for semialgebraic sets are [2] and [9]).

As illustrated with the next four examples many control problems can be described by semialgebraic sets (see [1] for more examples).

Example 1: Polynomial stability. The set of coefficients $(a_0, a_1, ..., a_n) \in \mathbb{R}^{n+1}$ for which the polynomial $a_0 + a_1 s + ... + a_n s^n$ is Hurwitz stable (i.e., has no zeros in the closed right half plane) is semialgebraic.
Example 2: The space of systems. Let Σ_n be the set of vectors $(a_0, ..., a_n, b_0, ..., b_n) \in \mathbb{R}^{2n+2}$ for which the polynomial $b_0 + b_1 z + ... + b_n z^n$ has highest order coefficient equal to 1 and for which the polynomials $a_0 + a_1 z + ... + a_n z^n$ and $b_0 + b_1 z + ... + b_n z^n$ are coprime. Σ_n is semialgebraic. In the sequel we often identify a system with its coefficient vector in Σ_n, and we speak of Σ_n as the space of systems of order at most n. The cartesian product space Σ_n^k is the space of all k-tuples of systems of order at most n. A set of k-tuples of systems of order at most n is called semialgebraic if its corresponding subset of Σ_n^k is semialgebraic.
Example 3: Simultaneous stabilization of two systems. The set S_n^2 of pairs of systems of order at most n that are simultaneously stabilizable is semialgebraic.
Example 4: Simultaneous stabilization with a controller of a priori bounded order. The set $S_{n,N}^k$ of all k-tuples of systems that are of order at most n and that are are simultaneously stabilizable by a controller *of order at most* N is semialgebraic (for a more comprehensive treatment of this example, see Ghosh [11]).

We now show:

Theorem 2: The set S of triplets $(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ for which the systems (1) are simultaneously stabilizable is not semialgebraic. Furthermore, the set S is a countable union of semialgebraic sets but its complement in $\mathbb{R}^3$ is not.

We first need a lemma for proving this. From the discussion in Rudin [14] it is easy to see that λ has a local inverse $\nu_* : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ that is such that

$$\nu_*(x) = i|\nu_*(x)| \text{ and } \nu_*'(x) = -i|\nu_*'(x)|.$$

Lemma: The analytic function $F : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ defined by

$$F(z) = -2\frac{\nu_*(z)}{\nu_*'(z)}$$

is not algebraic. In addition to this we have

$$F(x) = \frac{2\Im(\nu_*(x))}{|\nu_*'(x)|}$$

for all $x < 0$.

Proof: Assume, to get a contradiction, that P is a nonzero polynomial such that

$$P(z, F(z)) = 0 \qquad \text{for all } z \in \mathbb{C} \setminus [0, \infty).$$

As proved in Theorem 16.20(a) of Rudin [14], the modular function λ has the property that $\lambda(z+2n) = \lambda(z)$ for all integers n and $z \in \Pi^+$. This implies that $\nu_n(z) \triangleq 2n + \nu_*(z)$ is a local inverse of λ for every integer n. Since ν_n and ν_* are both local inverses of the analytic function λ, they must be analytic continuations of each other (see Theorem 10.7.2 in Hille [13]). Hence $F_n(z) \triangleq -2\frac{\nu_n(z)}{\nu'(z)}$ is an analytic continuation of F, and likewise $P(z, F_n(z))$ is an analytic continuation of $P(z, F(z))$. But by assumption $P(z, F(z)) \equiv 0$, and so $P(z, F_n(z)) \equiv 0$. Now fix $z_0 \in \mathbb{C} \setminus [0, \infty)$. Since $F_n(z_0) = -2\frac{2n+\nu_*(z_0)}{\nu_*'(z_0)}$ has infinitely many values as n ranges over the integers, this shows that the polynomial $P_{z_0}(w) \triangleq P(z_0, w)$ has infinitely many zeros. Hence $P_{z_0} = 0$, so $P = 0$. This contradiction shows that the assumption that F is algebraic was false.
The statement about the values of $F(x)$ for $x < 0$ follows immediately from the fact that, for all $x < 0$, we have

$$\nu_*(x) = i|\nu_*(x)| \text{ and } \nu_*'(x) = -i|\nu_*'(x)|.$$

■

Proof of Theorem 2: If S was semialgebraic so would be its complement S^c. Hence, the first assertion follows from the second one.
We first prove the easy part of the second assertion. Namely, we prove that S is a countable union of semialgebraic sets.

For that purpose, define S_n by

$$S_n = \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \,|\, \frac{s+1}{s-1} + \beta_i$$

$$\text{are sim. stab. by a controller of order } n\}.$$

Then $S = \bigcup_{n=0}^{\infty} S_n$. By Example 4, the sets S_n are semialgebraic and thus the first part is proved.
We now prove the second part of the assertion. Assume, to get a contradiction, that the complement S^c is a countable union of semialgebraic sets. Since the set $\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid \beta_1 < \beta_2 < \beta_3\}$ is semialgebraic, the set

$$B \triangleq S^c \cap \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid \beta_1 < \beta_2 < \beta_3\}$$

is a countable union of semialgebraic sets. Theorem 1 implies that

$$B = \{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \mid \beta_1 < \beta_2 < \beta_3$$

$$\text{and } |a_1| \geq \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}\},$$

where
a_0 and a_1 are defined by $a_0 = (\beta_2 - \beta_3)/(\beta_2 - \beta_1)$, $a_1 = (\beta_2 - \beta_3)(\beta_1 - \beta_3)/(\beta_2 - \beta_1)$. The function which maps $(\beta_1, \beta_2, \beta_3)$ to (a_0, a_1) maps B onto $A \triangleq \{(a_0, a_1) \in \mathbb{R}^2 \mid a_0 < 0 \text{ and } a_1 \geq \frac{2\Im(\nu(a_0))}{|\nu'(a_0)|}\}$. By using the Tarski-Seidenberg theorem, it is easy to prove that A is a countable union of semialgebraic sets, so we can write $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{(x, y) \in \mathbb{R}^2 \mid P_{n,1}(x, y) = 0, \ldots, P_{n,k_n}(x, y) = 0, P_{n,k_n+1}(x, y) > 0, \ldots, P_{n,m_n}(x, y) >$$

$0 \leq k_n < \infty$, $k_n \leq m_n < \infty$ and $P_{n,i}(x, y)$ are nonzero real polynomials. With the help of the nonalgebraic function F in the lemma we can write

$$A = \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ and } y \geq F(x)\}.$$

Now fix $x_0 < 0$. Then $(x_0, F(x_0)) \in A$, so $(x_0, F(x_0)) \in A_{n_0}$ for some n_0. If $k_{n_0} = 0$, then A_{n_0} would be open, so $(x_0, F(x_0))$ would be an interior point of A. But $(x_0, y) \notin A$ if $0 < y < F(x_0)$. Hence $k_{n_0} > 0$, so $P_{n_0,1}(x_0, F(x_0)) = 0$.
Therefore the sets $Z_n \triangleq \{x < 0 \mid P_{n,1}(x, F(x)) = 0\}$ $(n = 1, 2, \ldots)$ have union $(-\infty, 0)$. One of these sets, say Z_{n_1}, must be uncountable, and must thus have a limit point in $(-\infty, 0)$. This means that the set of zeros of the analytic function $P_{n_1,1}(z, F(z))$ has a limit point in its domain of definition, so it must be identically zero. Since $P_{n_1,1}$ is nonzero, this shows that F is an algebraic function, a contradiction. ■

4 Computability

In this section we show that the question of deciding whether the three systems (1) are simultaneously stabilizable cannot be decided by certain *computational machines*.

We first define what we mean by a computational machine over real numbers. Our model is an extension of the classical Blum-Shub-Smale or BSS model [8].

Definition: A machine consists of a possibly infinite set of nodes N. Associated to each node n there is:

1) A set X_n, the *input space*.
2) A function $t_n : X_n \to N$, the *transition function*.
3) For each node m in the range of t_n, a function $f_{n \to m} : t_n^{-1}(m) \to X_m$, the *data transformation*.

One node n_s is singled out as the *start node*, and another one is the *end node* n_e. The machine works in discrete time steps $0, 1, 2, \cdots$. At each time k the machine is at a certain node n_k and has a certain value $x_k \in X_{n_k}$ of its stored data. The machine starts at the start node ($n_0 = n_s$) and its data is initialized with an input x_0 belonging to the input space X_{n_s} of the start node. At time k two things can happen: If the machine is at the end node ($n_k = n_e$), then it stops and outputs the value x_k. Otherwise, the machine goes to node $n_{k+1} = t_{n_k}(x_k)$ and transforms the data according to $x_{k+1} = f_{n_k \to n_{k+1}}(x_k)$.

We impose the following restrictions on our machines. For each node n

a) The input space X_n is a semialgebraic subset of some euclidean space $\mathbb{R}^{d_n}$.

b) The transition function t_n has a finite range. This means that at each node there is only a finite number of nodes to which the machine may make a transition.

c) The inverse images $X_{n \to m} \triangleq t_n^{-1}(m)$ are semialgebraic subsets of $\mathbb{R}^{d_n}$. Note that the set $X_{n \to m}$ is the set of data in $\mathbb{R}^{d_n}$ that make the machine transit from node n to node m.

d) The data transformations $f_{n \to m}$ are semialgebraic functions (Let X be a semialgebraic subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}^k$ is called semialgebraic if its graph $\{(x, y) \in X \times \mathbb{R}^k \mid f(x) = y\}$ is a semialgebraic subset of $\mathbb{R}^{n+k}$).

Behind these abstract definitions lies a very natural idea of machine; basically one that uses an algorithm that involve only semialgebraic functions. Rational operations (addition, substraction, multiplication, division) are examples of operations that lead to algebraic functions. A less trivial example is polynomial root extraction.

Decidable sets for our machine are defined in the following way:

Definition: Let E and I be sets such that $E \subseteq I$. We say that E is *decidable* in I, if there is a machine such that:

1) The input space of the start node is I. The input space of the end node is $\{0, 1\}$.

2) For every input $x_0 \in E$, the machine eventually stops and outputs 1.

3) For every input $x_0 \in I \setminus E$, the machine eventually stops and outputs 0.

Theorem 3: The set of triplets $(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ for which the systems (1) are simultaneously stabilizable is not decidable. ■

The proof of this result follows at once from the second part of Theorem 2 and from the following characterisation of decidable sets.

Lemma: Let I be a semialgebraic set and let E be a subset of I. Then E is decidable in I if and only if both E and $I \setminus E$ are countable unions of semialgebraic sets.

Proof of Lemma: See the full version of the paper. ■

The results obtained in this paper are for triplets of systems belonging to the set $\{\frac{s+1}{s-1} + \beta \mid \beta \in \mathbb{R}\}$. However it is clear that our conclusions also hold for larger subsets of $\mathbb{R}(s)$ and for any number of systems greater or equal to three.

General Corollary: Assume that $k \geq 3$, P is a subset of $\mathbb{R}(s)$, and P contains $\{\frac{s+1}{s-1} + \beta \mid \beta \in \mathbb{R}\}$ (for example P is the set of first order systems). Then the set of k-uples of systems in P that are simultaneously stabilizable does not form a semialgebraic set, and simultaneous stabilizability of k systems in P is not decidable by the machines introduced in the fourth section. ■

5 Conclusion

We have shown that simultaneous stabilizability of three systems is not a "semialgebraic problem" and that it cannot be decided by our machines, which are allowed to evaluate semialgebraic functions.
Thus, every solution of the simultaneous stabilization problem for three or more first order systems must necessarily include some transcendental function. We have given one example of such triplets of systems whose stabilization condition can be expressed in terms of an inequality involving the elliptic modular function.
Can the general simultaneous stabilization problem be solved in terms of the elliptic modular function only? We believe not, but this remains an open problem.

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