Semiclassical equations for weakly inhomogeneous cosmologies

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The in-in effective action formalism is used to derive the semiclassical correction to Einstein's equations due to a massless scalar quantum field conformally coupled to small gravitational perturbations in spatially flat cosmological models. The vacuum expectation value of the stress tensor of the quantum field is directly derived from the renormalized in-in effective action. The usual in-out effective action is also discussed and it is used to compute the probability of particle creation. As one application, the stress tensor of a scalar field around a static cosmic string is derived and the back-reaction effect on the gravitational field of the string is discussed.

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I. INTRODUCTION

Our picture of the evolution of the early Universe relies in the so-called semiclassical theory of gravity which describes the interaction of quantum fields with the classical gravitational field. Order of magnitude arguments and Heisenberg's uncertainty principle tell us that there must be a period in the Universe evolution, well after the Planck time, when the quantization of the gravitational field may be ignored but still the scale of its time variations is short enough to create elementary particles, so that matter quantization cannot be ignored. Since we lack a theory of quantum gravity, it is still not known to what extent and in what sense this theory may be considered as a true semiclassical limit of quantum gravity interacting with matter fields. Plausibility arguments have been advanced by Hartle and Horowitz [1] who show that the quantum corrections to the classical action of gravity interacting with N identical non-self-interacting matter fields reduce in the leading-order $1/N$ approximation to such semiclassical theory. Some authors [2] have also used arguments based on quantum cosmology.

The semiclassical approach provides the framework for some realistic scenarios which may explain some of the features of the present Universe. One of these scenarios is inflation [3], which may explain the homogeneity and flatness problems of the standard big-bang cosmology. In the inflationary model the quantum fluctuations of the inflaton field may be the source of the small gravitational inhomogeneities which seed galaxies or gravitational waves. This may explain the Universe large-scale structure [4] and the presence of a hypothetical background of gravitational radiation [5]. Another scenario is the possible formation of topological defects [6] as the Universe undergoes some phase transitions. Topological defects, in particular cosmic strings, may seed structure [7] and may be an alternative to inflation for the generation of structure in the Universe.

In both scenarios the picture of the gravitational field that emerges is that of a conformally flat Friedmann-Robertson-Walker (FRW) background in which small gravitational perturbations are present. Large anisotropies and inhomogeneities might be present only if the Universe had emerged highly inhomogeneous from the Planck era into the classical regime [8]. But since the quantum consequences of a highly inhomogeneous model are difficult to estimate one assumes that by studying small perturbations on a FRW background, a qualitative picture of the evolution of the more extreme case may result.

Here we are interested in the quantum effects produced by the presence of small perturbations in conformally flat backgrounds. Quantum effects due to small anisotropies were first considered by Zeldovich [9], Starobinsky and Zeldovich [10], Hu and Parker [11], Hartle and Hu [12], and Birrell and Davies [13], who computed the creation of conformally coupled particles interacting with the anisotropies. Conformally coupled particles are not created in conformally flat backgrounds (FRW) [14] but the anisotropies break the conformal symmetry. Different techniques were used for such computations, these techniques go from a perturbative evaluation of the Bogoliubov transformations relating two vacua of the quantum field, to the evaluation of the in-out effective action of this field in the given gravitational background. These results were extended to the presence of arbitrary perturbations, including inhomogeneities, [15] by a technique based in the perturbative evaluation of the scattering matrix which had been used in flat backgrounds by Sexl and Urbantke [16] and Starobinsky and Zeldovich [17].

Quantum effects on the geometry, the so-called back-reaction effect, are more difficult to evaluate because this requires, on the one hand, the computation of the renormalized stress tensor of the quantum field in order to modify the classical Einstein equations, and on the other hand, it requires the solution to these semiclassical equations. It was argued by Zeldovich [9] that the back reaction would tend to dissipate the inhomogeneities as in a sort of gravitational Lenz's law effect. This is a mechanism to homogenize the Universe, but it is usually not advocated because in the standard scenario one assumes cosmological models which cannot explain the present large scale homogeneity by any causal mechanism after

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the Planck era; the inflationary scenario, on the other hand, seems to solve the homogeneity problem quite naturally. It is nevertheless a mechanism for entropy production. Early work on the back-reaction effect on the geometry due to anisotropies was done by Lukash and Starobinsky [18] and Lukash *et al.* [19], who assumed very special conditions near the Planck time, and by Hu and Parker [11] who considered a Bianchi type I anisotropic model, evaluated the stress tensor in the low-frequency approximation, and computed the resulting modified Einstein's equations numerically. The results of such work indicate that the dynamical mechanism of particle production achieves a rapid damping of the anisotropy if the calculations are extrapolated to the Planck era.

The computation of the quantum stress tensor is generally difficult in practice. However, for small perturbations on a conformally flat background one may use perturbative methods to get explicit expressions. One of the most powerful and efficient methods and, one that is very well adapted to a perturbative scheme, is based on the one-loop order computation of the so-called in-in effective action for quantum fields interacting with the gravitational perturbations. This technique is an effective action technique adapted to compute expectation values of quantum operators. It was first proposed by Schwinger [20] and Keldysh [21] and developed by Chou *et al.* [22]. Jordan [23] and Calzetta and Hu [24] developed the technique on a curved background, and it was then applied to derive the stress tensor of a quantum scalar field coupled to small anisotropies on a cosmological background [24].

The use of effective action methods in the back-reaction context was first considered by Hartle [25], and Fischetti, Hartle, and Hu [26] and Hartle and Hu [12] studied the effect of anisotropies. But in their formalism the basic element is the usual effective action which is related to the generating functional of the in-out vacuum persistence amplitude. This in-out formalism leads to matrix elements rather than expectation values for the quantum operators. Thus, one does not get directly from the in-out effective action the vacuum expectation value of the stress tensor of the quantum field, and one still needs to compute the Bogoliubov transformation between the in and out vacua. This method is, however, very useful for the computation of the particles created, since the probability amplitude for particle creation is directly related to the vacuum persistence amplitude.

In this paper we compute the in-in effective action to the one-loop order for a massless scalar field conformally coupled to small gravitational perturbations on a spatially flat FRW background. The in-in effective action is used to derive the quantum stress tensor and the corresponding semiclassical Einstein's equations. Our results generalize the Calzetta and Hu [24] results to the case of arbitrary small perturbations including inhomogeneities, and the stress tensor we derive coincides with that obtained by Horowitz and Wald [27], who used an axiomatic approach to derive it, and by Starobinsky [28], who used a modified Pauli-Villars regularization method [10]. One should stress, however, that the stress tensor computed does not include the energy of the particles created, which is a second-order correction to the computed terms. One might wonder that although the energy of the particles created is small it might have a long-term cumulative effect.

We should mention that the axiomatic approach to derive the stress tensor has been quite successful in several situations. Thus Horowitz [29] obtained the stress tensor due to a scalar field minimally coupled to arbitrary gravitational linear perturbations on a flat spacetime background applying the axiomatic arguments outlined by Wald [30]. This tensor was rederived by Jordan [31] using the in-in effective action method; note that the case of conformally coupled fields may also be obtained from our cosmological model when the conformal factor is taken constant. Another approach to the quantum stress tensor based on an iteratively evaluated mode decomposition was developed by Davies and Unruh [32].

In this paper we shall not consider the solutions to the semiclassical equations, except in a simple example involving a cosmic string. The correct approach to this problem is still controversial. In fact, the semiclassical equations are known to admit runaway solutions as a consequence of the fact that they are dynamical equations with higher order derivatives. Horowitz [29] and Jordan [31] found from these solutions that flat space is unstable against quantum effects. Whether these solutions are physical and thus signal a true instability, or unphysical and thus spurious, has been the subject of some discussion in recent years. Simon [33] has argued that the semiclassical correction to Einstein's equations must be seen as analytic perturbations, in terms of the Planck constant $\hbar$, to the classical Einstein's equations and that, as such, only solutions which are also analytic in $\hbar$ are physical. A consistent perturbative approach to find reduced equations, i.e., dynamical equations which are second order at each order of perturbation is known [34]. When this is applied it is found that flat space is perturbatively stable to first order in $\hbar$ [33]. Reduced semiclassical equations have been obtained also in some cosmologies [35]. Suen [36], on the other hand, has argued, on the basis of how the stress tensor is renormalized, that this tensor cannot be considered the first term of an expansion and therefore the previous reduction methods should not be applied.

In order to make this paper reasonably self-contained the in-in effective action formulation is summarized in Sec. II with a view to practical applications. In Sec. III the in-in effective action to the one-loop order is derived for a scalar field conformally coupled to a nearly conformally flat metric. Since along this computation one also derives all the terms needed for the in-out effective action, this action is also discussed, and it is used to derive the probability for pair creation; the results agree with those obtained by other methods. In Sec. IV the stress tensor for the quantum field is derived from the in-in effective action and the semiclassical correction to Einstein's equations is written down. The stress tensor is seen to agree with that obtained by Horowitz and Wald [27] and Starobinsky [28]. As an exercise the semiclassical equations in two-dimensional spacetime are also derived using the same formalism. In Sec. V we apply the previous formula to compute the stress tensor of a quantum field around a static cosmic string and we discuss the back-reaction effect on the gravitational field of the string. Note that since the gravitational field of a cosmic string can be considered a small perturbation on a flat background the above perturbative technique (in the sense of metric perturbations) can be applied. The results are in agreement with those found by other nonperturbative methods [37,38] but this perturbative method opens the possibility of computing the quantum stress tensor even in time-dependent situations. Work along these lines is in progress.

II. IN-IN FUNCTIONAL FORMALISM

In this section we summarize the in-in functional formalism for the evaluation of the in-in effective action with a view to the applications of this paper. We follow, essentially, the presentations by Jordan [23], Calzetta and Hu [24], and Paz [39].

Quantum corrections to a classical field theory can be studied with the help of the effective action. For simplicity, we consider the quantization of a scalar field $\phi(x)$. The usual in-out effective action is based in the generating functional $W[J]$ which is related to the vacuum persistence amplitude in the presence of some classical source $J(x)$ by

$$e^{iW[J]} \equiv \langle 0,\text{out}|\text{in},0\rangle_J \ . \tag{2.1}$$

This functional carries all the quantum information of the connected graphs of the theory.

When one couples an external field $J(x)$ it is convenient to use the interaction picture in which the states $|\psi\rangle$ evolve in time according to the Schrödinger equation $H_I|\psi\rangle = i\partial_t|\psi\rangle$, where H_I is the interaction Hamiltonian operator

$$H_I = \int d^{n-1}x \, J(x)\phi(x) \ ,$$

$\phi(x)$ is now the field operator in the Heisenberg representation, and n the number of spacetime dimensions. The solution of this equation may be formally written as

$$|\psi\rangle_{t_2} = T^{(t)}\exp\left[i\int_{t_1}^{t_2} dt\, H_I\right]|\psi\rangle_{t_1} \ , \tag{2.2}$$

where $T^{(t)}$ is the usual time-ordering operator, and (2.1) can be written as

$$e^{iW[J]} = \langle 0,\text{out}|T^{(t)}\exp\left[i\int_{-\infty}^{\infty} dt\, H_I\right]|\text{in},0\rangle \ . \tag{2.3}$$

It is easy to see from the classical field equations for $\phi(x)$ in the presence of $J(x)$ that $\exp(iW[J])$ satisfies the integro-differential Schwinger-Dyson equation, and that one may give a path integral representation for its solution as

$$e^{iW[J]} = \int \mathcal{D}[\phi]e^{i(S[\phi]+J\phi)} \ , \tag{2.4}$$

where $S[\phi]$ is the classical action of the field theory and the common shorthand notation $J\phi$ for the integral $\int d^n x\, J(x)\phi(x)$ has been used. The functional integral is taken with the following boundary conditions: $\phi \to e^{\mp i\omega t}$, where $\omega > 0$, when the time $t \to \pm\infty$, i.e., the scalar field has only negative and positive frequency modes in the in and out regions, respectively; the interaction is assumed to be switched off at these asymptotic regions. By differentiating with respect to the source one generates matrix elements from $W[J]$:

$$\frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0,\text{out}|\phi(x)|\text{in},0\rangle_J}{\langle 0,\text{out}|\text{in},0\rangle_J} \equiv \bar{\phi}[J] \ . \tag{2.5}$$

If we assume that the above expression can be reversed, the effective action is defined as the Legendre transformation of the generating functional:

$$\Gamma[\bar{\phi}] = W[J] - J\bar{\phi} \ . \tag{2.6}$$

This functional of $\bar{\phi}$ is the generator of the one-particle-irreducible graphs (graphs that remain connected when any internal line is cut) and contains all the quantum corrections to the classical action. From (2.6) one may derive the dynamical equation for the effective mean field $\bar{\phi}[0]$, i.e., the matrix element of the field ϕ in the absence of the source $J(x)$, as

$$\left.\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}}\right|_{\bar{\phi}=\bar{\phi}[0]} = 0 \ , \tag{2.7}$$

which expresses the quantum corrections to the classical equation as a variational problem of the effective action.

In order to work with expectation values rather than matrix elements one can define a new generating functional whose dynamics is determined by two different external classical sources J_+ and J_-, by letting the in vacuum evolve independently under these sources:

$$e^{iW[J_+,J_-]} = \sum_\alpha \langle 0,\text{in}|\alpha,T\rangle_{J_-}\langle\alpha,T|\text{in},0\rangle_{J_+} \ . \tag{2.8}$$

Here we have assumed that $\{|\alpha,T\rangle\}$ is a complete basis of eigenstates of the field operator $\phi(x)$ at some future time T, i.e., $\phi(T,\mathbf{x})|\alpha,T\rangle = \alpha(\mathbf{x})|\alpha,T\rangle$. Then (2.8) may be written according to (2.2) as

$$e^{iW[J_+,J_-]} \equiv \int d\alpha \langle 0,\text{in}|T^{(a)}\exp\left[-i\int_{-\infty}^{T} dt\int d^{n-1}x J_-(x)\phi(x)\right]|\alpha,T\rangle$$

$$\times\langle\alpha,T|T^{(t)}\exp\left[i\int_{-\infty}^{T} dt\int d^{n-1}x J_+(x)\phi(x)\right]|\text{in},0\rangle \ , \tag{2.9}$$

where $T^{(t)}$ and $T^{(a)}$ mean, respectively, time and anti-time ordered-operators and $d\alpha$ means $d\alpha=\prod_{\mathbf{x}} d\alpha(\mathbf{x})$ where $\mathbf{x}$ are the points of the hypersurface Σ defined by $t=T$. The generating functional has also a path integral representation

$$e^{iW[J_+,J_-]}=\int d\alpha \int \mathcal{D}[\phi_-]e^{-i(S[\phi_-]+J_-\phi_-)} \times \int \mathcal{D}[\phi_+]e^{i(S[\phi_+]+J_+\phi_+)} , \quad (2.10)$$

with the boundary conditions that $\phi_+=\phi_-=\alpha$ on Σ and that the fields ϕ_+ and ϕ_- are pure negative and pure positive modes, respectively, in the in region, i.e., $\phi_\pm \to e^{\pm i\omega t}$ at $t\to -\infty$ (vacuum boundary conditions in the remote past). In a more compact form one may write

$$e^{iW[J_+,J_-]}=\int \mathcal{D}[\phi_+]\mathcal{D}[\phi_-] \times e^{i(S[\phi_+]+J_+\phi_+-S[\phi_-]-J_-\phi_-)} , \quad (2.11)$$

where it is understood that the sum is over all fields ϕ_+ and ϕ_- with negative and positive frequency modes, respectively, in the remote past but which coincide at time $t=T$. These boundary conditions can be made explicit by substituting m^2 by $m^2-i\epsilon$, where m is the field mass, in $S[\phi_+]$ and by substituting m^2 by $m^2+i\epsilon$ in $S[\phi_-]$; the latter is also sometimes indicated by writing $S^*[\phi_-]$ instead of $S[\phi_-]$ [24]. This integral can be thought of as the path sum of two different fields evolving in two different time branches [40]: one going forward in time in the presence of J_+ from the in vacuum to a time $t=T$, and the other backward in time in the presence of J_- from the time $t=T$ to the in vacuum, with the constraint $\phi_+=\phi_-$ on Σ. Because of such a path-integral representation, this formalism is often called closed-time path formalism.

The functional $W[J_+,J_-]$ generates expectation values of the field rather than matrix elements. For instance, we have

$$\left.\frac{\delta W[J_+,J_-]}{\delta J_+}\right|_{J_\pm=J}=\langle 0,\mathrm{in}|\phi(x)|\mathrm{in},0\rangle_J \equiv \bar{\phi}[J] \quad (2.12)$$

instead of Eq. (2.5). This functional generates not only the desired expectation values of time-ordered field operators but also the anti-time-ordered ones in the same footing:

$$\left.\frac{\delta e^{iW[J_+,J_-]}}{i\delta J_+(x_1)\cdots(-i)\delta J_-(y_1)\cdots}\right|_{J_\pm=J=0} =\langle 0,\mathrm{in}|\,T^{(a)}[\phi(y_1)\cdots]T^{(t)}[\phi(x_1)\cdots]|\mathrm{in},0\rangle . \quad (2.13)$$

In analogy with the in-out formalism the in-in effective action is defined as the Legendre transform of the new generating functional as

$$\Gamma[\bar{\phi}_+,\bar{\phi}_-]=W[J_+,J_-]-J_+\bar{\phi}_+ +J_-\bar{\phi}_- , \quad (2.14)$$

where the external sources are functionals of the fields $\bar{\phi}_+$ and $\bar{\phi}_-$, through the definitions

$$\frac{\delta W[J_+,J_-]}{\delta J_\pm}\equiv \pm\bar{\phi}_\pm[J_+,J_-] , \quad (2.15)$$

which we assume can be reversed.

From the definitions (2.14) and (2.15) we get the equation for the expectation values $\bar{\phi}_\pm[J_+,J_-]_J$, i.e.,

$$\frac{\delta\Gamma[\bar{\phi}_+,\bar{\phi}_-]}{\delta\bar{\phi}_\pm}=\mp J_\pm , \quad (2.16)$$

and by taking $J_\pm=0$ in (2.15) we recover the equation for the vacuum expectation value of the field

$$\bar{\phi}[0]\equiv\bar{\phi}_\pm[0,0]=\langle 0,\mathrm{in}|\phi(x)|\mathrm{in},0\rangle ,$$
$$\left.\frac{\delta\Gamma[\bar{\phi}_+,\bar{\phi}_-]}{\delta\bar{\phi}_+}\right|_{\bar{\phi}_\pm=\bar{\phi}_\pm[0,0]\equiv\bar{\phi}[0]}=0 . \quad (2.17)$$

This equation does not follow from a simple variational principle in terms of a single field $\bar{\phi}$: in the in-in action we have two fields $\bar{\phi}_+$ and $\bar{\phi}_-$ that are treated independently and, only when the sources have been eliminated, they become the vacuum expectation value. Note also that $\Gamma[\bar{\phi},\bar{\phi}]=0$ as a consequence of (2.14) and of the fact that $W[J,J]=0$, which follows from (2.8) and the usual normalization for the states. Equation (2.17) is a dynamical field equation which admits an initial value formulation: the solution $\bar{\phi}[0]=\bar{\phi}_\pm[0,0]$ is real and causal; i.e., the solution at one spacetime point depends only on data on the past of that point [23].

For a free field theory, i.e., a theory with a quadratic action, we can compute $W_0[J_+,J_-]$ from (2.11), which becomes now a Gaussian integration for the two independent fields ϕ_+ and ϕ_-. The corresponding propagators will be determined by the very particular boundary conditions of this problem. In fact, let us assume that the free action for ϕ_+ is

$$S_0[\phi_+]=-\int d^nx\,\tfrac{1}{2}[\partial_\mu\phi_+\partial^\mu\phi_+ +(m^2-i\epsilon)\phi_+^2]$$

and that we have an analogous action $S_0^*[\phi_-]$ for ϕ_-, then the classical field equations are

$$(\Box-m^2\pm i\epsilon)\phi_\pm^0(x)=-J_\pm(x) . \quad (2.18)$$

At this stage we can introduce the compact notation

$$\mathcal{S}[\phi_a]=S[\phi_+]-S^*[\phi_-] ,$$
$$\phi_a(x)=\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix},\quad J_a(x)=\begin{pmatrix}J_+\\ -J_-\end{pmatrix} , \quad (2.19)$$

where a and b take the two values $+$ and $-$, to simplify the mathematical expressions. The solutions of the classical field equations, which satisfy the boundary conditions, $\phi_\pm^0\to e^{\pm i\omega t}$, when $t\to-\infty$ with $\omega\geq 0$, and $\phi_+^0(T,\mathbf{x})=\phi_-^0(T,\mathbf{x})$ in the hypersurface Σ, which we take here at $t=T\to\infty$, can be written as

$$\phi_a^0(x)=-\int d^ny\,G_{ab}^0(x,y)J_b(y) , \quad (2.20)$$

where $G_{ab}^0(x,y)$ is the matrix

$$G^0_{ab}=\begin{bmatrix}\Delta_F & -\Delta^+\\ \Delta^- & -\Delta_D\end{bmatrix}, \tag{2.21}$$

defined with the Feynman Δ_F, Dyson Δ_D, and the positive Δ^+, and negative Δ^- Wightman functions:

$$\Delta_F(x-y)=-\int\frac{d^np}{(2\pi)^n}\frac{e^{ip\cdot(x-y)}}{p^2+m^2-i\epsilon},$$
$$\Delta_D(x-y)=-\int\frac{d^np}{(2\pi)^n}\frac{e^{ip\cdot(x-y)}}{p^2+m^2+i\epsilon}, \tag{2.22}$$
$$\Delta^{\pm}(x-y)=(\pm 2\pi i)\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-y)}\delta(p^2+m^2)\theta(\mp p^0).$$

These propagators are solutions of the equations

$$A^0_{ac}G^0_{cb}(x,y)=\delta^n(x-y)\delta_{ab}, \tag{2.23}$$

where the operator A^0_{ab} is the diagonal matrix defined by

$$A^0_{ab}=\text{diag}[(\Box-m^2+i\epsilon),-(\Box-m^2-i\epsilon)]. \tag{2.24}$$

Furthermore, the Feynman and Dyson Green's functions have a mode decomposition:

$$\Delta_F(x-y)=\theta(x^0-y^0)\Delta^-(x-y)-\theta(y^0-x^0)\Delta^+(x-y),$$
$$\Delta_D(x-y)=\theta(x^0-y^0)\Delta^+(x-y)-\theta(y^0-x^0)\Delta^-(x-y), \tag{2.25}$$

which reflect the boundary conditions imposed over each classical field solution $\phi^0_a(x)$ because the Green's functions

$$\Delta^{\pm}(x-y)\sim e^{\pm i\omega\cdot(x^0-y^0)}$$

correspond to negative and positive frequency modes, respectively (here $\omega=\sqrt{\mathbf{p}^2+m^2}$). Note that it is also satisfied that

$$\partial_t\phi^0_+(T,\mathbf{x})=\partial_t\phi^0_-(T,\mathbf{x})$$

at Σ. With these propagators to guarantee the boundary conditions the Gaussian integration of (2.11) for a free field is

$$W_0[J_a]=-\frac{1}{2}\int d^nx\,d^ny J_a(x)G^0_{ab}(x-y)J_b(y), \tag{2.26}$$

where a term independent of J_a has been discarded to satisfy $W_0[J,J]=0$, and one can use now (2.13) to generate time-ordered and anti-time-ordered expectation values of field operators. In particular, we have that

$$\langle 0,\text{in}|T^{(t)}\phi(x)\phi(y)|\text{in},0\rangle=i\Delta_F(x-y),$$
$$\langle 0,\text{in}|T^{(a)}\phi(x)\phi(y)|\text{in},0\rangle=-i\Delta_D(x-y), \tag{2.27}$$
$$\langle 0,\text{in}|\phi(y)\phi(x)|\text{in},0\rangle=-i\Delta^+(y-x)=i\Delta^-(x-y).$$

For interacting fields one can proceed as usual by writing

$$e^{iW[J_a]}=\exp\left[i\int d^nx\,\mathcal{L}_{\text{int}}\left[\frac{\delta}{i\delta J_a}\right]\right]e^{iW_0[J_a]}, \tag{2.28}$$

where we have separated the Lagrangian into a free and an interacting part, $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}}$. Then one may continue in the usual perturbative fashion; however, we are not going to consider self-interacting theories in this paper; the only interaction will be with the gravitational field.

Let us now proceed to the main objective, namely, the evaluation of the effective action $\Gamma[\bar{\phi}_a]$ up to the one loop order, which corresponds to the first order expansion of $W[J_a]$ in powers of $\hbar$. As usual [41], if we assume that the action is bounded from above then we can go to Euclidean space and solve (2.11) by the steepest descent method; we keep, however, the Minkowskian notation. Let us denote by $\phi^{(0)}_+(x)$ and $\phi^{(0)}_-(x)$ the solutions of the classical field equations which may, or may not, include self-interactions,

$$\frac{\delta\mathcal{S}[\phi^{(0)}_b]}{\delta\phi^{(0)}_a(x)}=-J_a(x), \tag{2.29}$$

and let us expand the exponent in (2.11) about these background fields:

$$\mathcal{S}[\phi_a]+\int d^nx\,J_a(x)\phi_a(x)=\mathcal{S}[\phi^{(0)}_a]+\int d^nx\,J_a(x)\phi^{(0)}_a(x)+\frac{1}{2}\int d^nx\,d^ny[\phi_a(x)-\phi^{(0)}_a(x)]A_{ab}(x,y)[\phi_b(y)-\phi^{(0)}_b(y)]+\cdots, \tag{2.30}$$

where

$$A_{++}(x,y)\equiv\left[\frac{\delta^2S[\phi_+]}{\delta\phi_+(x)\delta\phi_+(y)}\right]_{\phi_+=\phi^{(0)}_+},$$
$$A_{--}(x,y)\equiv-\left[\frac{\delta^2S^*[\phi_-]}{\delta\phi_-(x)\delta\phi_-(y)}\right]_{\phi_-=\phi^{(0)}_-}, \tag{2.31}$$

and, of course,

$$A_{+-}(x,y)=A_{-+}(x,y)\equiv 0.$$

Substituting this into (2.11) the integration is now Gaussian and we can write, to this one-loop order,

$$e^{iW[J_a]}\simeq e^{iW^{(0)}[J_a]}[\det A_{ab}(x,y)]^{-1/2}, \tag{2.32}$$

where

$$W^{(0)}[J_a]=\mathcal{S}[\phi^{(0)}_a]+\int d^nx\,J_a\phi^{(0)}_a.$$

In terms of the propagator G, which is a functional of the background fields $\phi_a^{(0)}(x)$ and takes a 2×2 matrix form, i.e., $G(x,y)=A^{-1}(x,y)$, we can write (2.32) as

$$W[J_a]\simeq W^{(0)}[J_a]-\frac{i}{2}\mathrm{Tr}(\ln G)\ . \qquad (2.33)$$

The effective action, which is a functional of $\bar\phi_a$, can now be explicitly found to the same order. Using (2.14), (2.15), and the fact that $\bar\phi_a$ differs from $\phi_a^{(0)}$ by a term of order $\hbar$ we can show that

$$W^{(0)}[J_a]\simeq \mathcal{S}[\bar\phi_a]+\int d^n x\, J_a\bar\phi_a\ ,$$

so that finally we have

$$\Gamma[\bar\phi_a]\simeq \mathcal{S}[\bar\phi_a]-\frac{i}{2}\mathrm{Tr}(\ln G)\ . \qquad (2.34)$$

Now the equations for $\bar\phi_a$ can be deduced from (2.16) using the explicit functional dependence on the fields given by (2.34). However, we should note from (2.17) that in order to get the field equations for the expectation value of $\phi(x)$ we only need the explicit dependence of the effective action on one of the fields $\bar\phi_+$ or $\bar\phi_-$. Therefore, we are only interested in the dependence of (2.34) on $\bar\phi_+$, for example. Following Paz [39] we can write

$$\begin{aligned}\frac{\delta}{\delta\bar\phi_+(x)}&[\mathrm{Tr}(\ln G)]\\ &=-\int dy\, dz\, G_{ab}(z,y)\frac{\delta}{\delta\bar\phi_+(x)}G^{-1}_{ba}(z,y)\\ &=-\int dy\, dz\, G_{++}(z,y)\frac{\delta}{\delta\bar\phi_+(x)}G^{-1}_{++}(z,y)\\ &=\frac{\delta}{\delta\bar\phi_+(x)}[\mathrm{Tr}(\ln G_{++})]\ ,\end{aligned} \qquad (2.35)$$

where we have used that $G^{-1}_{ab}\equiv A_{ab}$ is diagonal, see (2.31). Thus we have

$$\Gamma[\bar\phi_+,\bar\phi_-]\simeq S[\bar\phi_+]-\frac{i}{2}\mathrm{Tr}(\ln G_{++})+F\ , \qquad (2.36)$$

where F includes all the terms which do not contribute to the variation of the field $\bar\phi_+$, i.e., $\delta F/\delta\bar\phi_+(x)|_{\bar\phi_\pm=\bar\phi}=0$. This expression is very similar to the one-loop in-out effective action, which is given by the above equation (2.36), where G_{++} is substituted by the Feynman propagator Δ_F, the main difference is in the boundary conditions: the propagator G_{++} is defined as an expectation value and not as an in-out matrix element.

This formalism can be extended to curved spacetimes without difficulties assuming that the spacetime is globally hyperbolic [23]. The hypersurfaces of constant time are now Cauchy hypersurfaces and the in and out states are defined in the Cauchy hypersurfaces corresponding to the far past and far future, respectively. Now the spacetime integrals must be performed with the volume element $d^n x\sqrt{-g}$ where $g_{\mu\nu}$ is the spacetime metric. The above expressions (2.20), (2.23), and (2.36) are still valid except that now the Feyman, Dyson, and Wightman functions have a different representation to that of (2.22). If the spacetime is asymptotically flat in the in and out regions the previous boundary conditions for the fields $\phi^0_\pm(x)$ will also apply; if not, in order to be able to define physically meaningful in and out vacua we must assume that we are still able to define positive and negative frequency solutions in the asymptotic regions. This is always possible, for instance, if the asymptotic regions admit approximate timelike Killing fields. But, generally, in a curved spacetime the in and out vacua are not equivalent. Jordan [23] has shown that for quantum scalar fields in a curved spacetime the field equations are real and causal up to the two-loop order and he has also checked the unitarity of the formalism restricted to vacuum states.

Before ending this section let us rewrite (2.36) in a more convenient form for us. In general, the propagator G_{++} cannot be found exactly and has to be evaluated perturbatively. For instance, in the next section we will take perturbations $h_{\mu\nu}(x)$ to a given background metric and only the exact propagator corresponding to the background is known. Thus we write

$$A_{ab}=A^0_{ab}+(V^{(1)}_{ab}+V^{(2)}_{ab}+\cdots)\ , \qquad (2.37)$$

where A^0_{ab} is the unperturbed (diagonal) operator whose propagator G^0_{ab} is known,

$$A^0_{ac}G^0_{cb}=\delta_{ab}\ , \qquad (2.38)$$

and the diagonal operators $V^{(1)}_{ab}+V^{(2)}_{ab}+\cdots$ contain the perturbative terms (in the next section they will correspond to perturbations of order $|h_{\mu\nu}|$ and $|h_{\mu\nu}|^2$, respectively). We can write

$$\begin{aligned}G_{ab}&=G^0_{ab}-G^0_{ac}(V^{(1)}_{cd}+V^{(2)}_{cd}+\cdots)G_{db}\\ &=G^0_{ab}-G^0_{ac}V^{(1)}_{cd}G^0_{db}-G^0_{ac}V^{(2)}_{cd}G^0_{db}\\ &\quad+G^0_{ac}V^{(1)}_{cd}G^0_{de}V^{(1)}_{ef}G^0_{fb}+\cdots\ ,\end{aligned} \qquad (2.39)$$

where the products are operator products. In particular,

$$\begin{aligned}G_{++}&=G^0_{++}-G^0_{+a}V^{(1)}_{ab}G^0_{b+}-G^0_{+a}V^{(2)}_{ab}G^0_{b+}\\ &\quad+G^0_{+a}V^{(1)}_{ab}G^0_{bc}V^{(1)}_{cd}G^0_{d+}+\cdots\ ,\end{aligned} \qquad (2.40)$$

expanding the logarithmic term in (2.36) and using that $A^0_{++}G^0_{+-}=0$, we finally get

$$\begin{aligned}\Gamma[\bar\phi_+,\bar\phi_-]\simeq{}& S[\bar\phi_+]-\frac{i}{2}\mathrm{Tr}(\ln G^0_{++})+F\\ &+\frac{i}{2}\mathrm{Tr}(V^{(1)}_+G^0_{++}+V^{(2)}_+G^0_{++}-\tfrac{1}{2}V^{(1)}_+G^0_{++}V^{(1)}_+G^0_{++}+V^{(1)}_+G^0_{+-}V^{(1)}_-G^0_{-+}+\cdots)\ .\end{aligned} \qquad (2.41)$$

where we have defined $V_{+}^{(i)} \equiv V_{++}^{(i)}$ and $V_{-}^{(i)} \equiv -V_{--}^{(i)}$ following (2.31). Note that if it were not for the last term which involves the propagator G_{+-}^{0} this expression for the in-in effective action would agree with the in-out effective action which involves only one field $\phi(x)$; see, for instance, Hartle and Hu [12]. Therefore, the term containing the propagator G_{+-}^{0} is the only new term that contributes to the field equation for $\bar{\phi}_{+}(x)$. It can be seen [39] that the effect of the last term in (2.41) is to make the field equation for $\bar{\phi}(x)$ causal: if one takes the derivative of the in-in effective action with respect to $\bar{\phi}_{+}(x)$ and puts $\bar{\phi}_{+}=\bar{\phi}_{-}=\bar{\phi}$, the resulting field equation is causal.

Notice that in the case of a free scalar field on a nearly flat background the propagator G_{ab}^{0} which solves (2.38), corresponding to the flat background, and which provides the boundary conditions of the in-in problem is simply given by $G_{++}^{0}=\Delta_F, G_{--}^{0}=-\Delta_D$, $G_{-+}^{0}=\Delta^{-}$, and $G_{+-}^{0}=-\Delta^{+}$, as can be seen from (2.23). In fact, from (2.31) A^0 is, in this case, the operator defined in (2.24).

III. IN-IN EFFECTIVE ACTION

In this section we compute the in-in effective action (2.41) for a conformal field in a nearly conformally flat spacetime. The cosmological background consists of a spatially flat homogeneous FRW with small perturbations as

$$\tilde{g}_{\mu\nu}(x) \equiv a^2(\eta)[\eta_{\mu\nu}+h_{\mu\nu}(x)] \ . \tag{3.1}$$

where $a(\eta)=\exp[\omega(\eta)]$ is the conformal factor, η is the conformal time $d\eta=dt/a$, t is the cosmological time, and $h_{\mu\nu}(x)$ is a symmetric tensor representing arbitrary small perturbations; we take the metric signature $(-+\cdots+)$. The classical action for a free (i.e., with no self-interactions) massless conformally coupled scalar field $\Phi(x)$ is given by

$$S_m[\tilde{g}_{\mu\nu},\phi]=-\frac{1}{2}\int d^nx\sqrt{-\tilde{g}}\,[\tilde{g}^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi+\xi(n)\tilde{R}\,\Phi^2] \ , \tag{3.2}$$

where $\xi(n)=n-2/4(n-1)$, $\tilde{R}$ is the Ricci scalar for the metric $\tilde{g}_{\mu\nu}$, and we take the spacetime dimensions n arbitrary for the moment in view of dimensional regularization. Because of the conformal coupling one may simplify the problem by defining a new matter field $\phi(x)$ and a new metric $g_{\mu\nu}(x)$ as

$$\phi(x) \equiv \exp\left[\frac{n-2}{2}\omega(\eta)\right]\Phi(x) \ , \qquad g_{\mu\nu}(x) \equiv \eta_{\mu\nu}+h_{\mu\nu}(x) \ . \tag{3.3}$$

Then the action (3.2), after integration by parts and assuming no contributions of the surface integrals, is equivalent to the action for the field $\phi(x)$ in the nearly flat metric $g_{\mu\nu}$:

$$S_m[g_{\mu\nu},\phi]=-\frac{1}{2}\int d^nx\sqrt{-g}\,[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\xi(n)R\,\phi^2] \ , \tag{3.4}$$

where R is the Ricci scalar for the metric $g_{\mu\nu}(x)$. Therefore, the problem has been reduced to that of a scalar field $\phi(x)$ in a nearly flat background. We must keep in mind that the physical field is $\Phi(x)$, but the fact that the two fields differ by just a power of the conformal factor, which is a function of time only, considerably simplifies the connection between the vacua of the two fields. For instance, a positive frequency mode in flat space will correspond to a positive frequency mode in the conformally related space. Since for a free field in flat space the in and out vacua are equivalent (there is no particle creation) the same is true for the vacua of the conformal field in the conformally flat background. As a consequence nontrivial quantum effects can be produced only by the breaking of conformal flatness which in this case is due to the coupling of the quantum field with the gravitational perturbations. The above action can be expanded in terms of these perturbations as

$$S_m[h_{\mu\nu},\phi]=S_m^{(0)}[\phi]+\sum_{n=1}^{\infty}S_m^{(n)}[h_{\mu\nu},\phi] \ , \tag{3.5}$$

where the first term is simply the action for the field ϕ in flat spacetime, and the higher perturbative terms carry all the information on the interaction with the perturbations.

Since we are interested in deriving the semiclassical correction to Einstein's equations due to the quantum effect of the scalar field but keeping the gravitational field as classical, we have to add to the effective action Γ_m (m stands for matter fields), the classical action of the gravitational field $S_g[\tilde{g}_{\mu\nu}]$. We should also add the action of any other classical source but we shall ignore this for simplicity; note that its effects on the semiclassical equations may be taken into account by simply adding the corresponding classical stress tensor to the quantum stress tensor. Furthermore, in order to renormalize the effective action it is sufficient to add to the usual Einstein's action. terms quadratic in the Riemann tensor:

$$S_g[\tilde{g}_{\mu\nu}] \equiv \int d^nx[-\tilde{g}(x)]^{1/2}\left[\frac{1}{16\pi G_N}\tilde{R}(x)+\frac{\mu^{n-4}}{2880\pi^2(n-4)}[\tilde{R}_{\mu\nu\alpha\beta}(x)\tilde{R}^{\mu\nu\alpha\beta}(x)-\tilde{R}_{\mu\nu}(x)\tilde{R}^{\mu\nu}(x)]\right] \ , \tag{3.6}$$

where μ is an arbitrary mass scale which will be useful in dimensional regularization. The quadratic terms with poles at $n=4$ are those which are necessary to cancel the divergencies of Γ_m; notice that with this election one obtains the correct trace anomaly.

Following the previous section we introduce now two fields $\phi_+(x)$ and $\phi_-(x)$ that, at some future hypersurface Σ, coincide: $\phi_+(x)=\phi_-(x)$. Since the scalar field is proved by the gravitational field we must assume that the two fields evolve in two different geometries $g^+_{\mu\nu}(x)$ and $g^-_{\mu\nu}(x)$, respectively, where $g^\pm_{\mu\nu}(x)\equiv\eta_{\mu\nu}+h^\pm_{\mu\nu}(x)$, and since we assume that the fields have no interaction other than the gravitational we need not introduce the classical external currents $J_\pm(x)$. Thus we shall write the total in-in effective action for the gravitational and the matter fields as

$$\Gamma_{(ii)}[\omega,h^\pm_{\mu\nu}]=S_g[\omega,h^+_{\mu\nu}]-S_g[\omega,h^-_{\mu\nu}]+\Gamma_m[h^+_{\mu\nu},\phi_+;h^-_{\mu\nu},\phi_-]\ , \tag{3.7}$$

where $\Gamma_m[h^+_{\mu\nu},\phi_+;h^-_{\mu\nu},\phi_-]$ contains the quantum effects of the scalar field.

Following Eq. (2.41), we write Γ_m in a perturbative expansion in $h_{\mu\nu}$ as $\Gamma_m=\Gamma^{(0)}_m+\Gamma^{(1)}_m+\Gamma^{(2)}_m+\cdots$ and write only the terms which contribute to the variation of $h^+_{\mu\nu}$, which are also those which contribute to the variation of $\phi_+(x)$. Thus we can write, $\Gamma^{(0)}_{(ii)}=S^{(0)}_g+\Gamma^{(0)}_m$, $\Gamma^{(1)}_{(ii)}=S^{(1)}_g+\Gamma^{(1)}_m$, and $\Gamma^{(2)}_{(ii)}=S^{(2)}_g+\Gamma^{(2)}_m$, where

$$\begin{aligned}
\Gamma^{(0)}_m[\bar\phi_+]&=S^{(0)}_m[\bar\phi_+]-\frac{i}{2}\mathrm{Tr}(\ln G^0_{++})\ ,\\
\Gamma^{(1)}_m[h^+_{\mu\nu},\bar\phi_+]&=S^{(1)}_m[\bar\phi_+]+\frac{i}{2}\mathrm{Tr}(V^{(1)}_+G^{(0)}_{++})\ ,\\
\Gamma^{(2)}_m[h^+_{\mu\nu},\bar\phi_+]&=S^{(2)}_m[\bar\phi_+]+\frac{i}{2}\mathrm{Tr}(V^{(2)}_+G^0_{++})\\
&\quad-\frac{i}{4}\mathrm{Tr}(V^{(1)}_+G^0_{++}V^{(1)}_+G^0_{++})\\
&\quad+\frac{i}{2}\mathrm{Tr}(V^{(1)}_+G^0_{+-}V^{(1)}_-G^0_{-+})\ .
\end{aligned} \tag{3.8}$$

To compute $V^{(1)}$ and $V^{(2)}$ we use (2.31) and (2.37) and expand $S_m[\phi]$ as in (3.5):

$$S^{(0)}_m[\phi]=-\frac{1}{2}\int d^nx\,[\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi]\ , \tag{3.9}$$

$$S^{(1)}_m[h_{\mu\nu},\phi]=\frac{1}{2}\int d^nx\,[\bar h^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-\xi(n)R^{(1)}\phi^2]\ , \tag{3.10}$$

$$S^{(2)}_m[h_{\mu\nu},\phi]=-\frac{1}{2}\int d^nx[\,\hat h^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\xi(n)(R^{(2)}+\tfrac{1}{2}hR^{(1)})\phi^2]\ , \tag{3.11}$$

where $\bar h_{\mu\nu}\equiv h_{\mu\nu}-\frac{1}{2}h\eta_{\mu\nu}$,

$$\hat h_{\mu\nu}\equiv h_\mu{}^\alpha h_{\alpha\nu}-\tfrac{1}{2}hh_{\mu\nu}+\tfrac{1}{8}h^2\eta_{\mu\nu}-\tfrac{1}{4}h_{\alpha\beta}h^{\alpha\beta}\eta_{\mu\nu}\ ,$$

and $R^{(1)}$ and $R^{(2)}$ are the first- and second-order terms, respectively, of the scalar curvature [see Eq. (B9) from the Appendix]. From these expressions one gets the operators A^0, $V^{(1)}$, and $V^{(2)}$ by differentiation with respect to the field $\phi(x)$:

$$A^0\phi(x)=\Box\phi(x)\ , \tag{3.12}$$

$$V^{(1)}(x)\phi(x)=-\{[\partial_\mu\bar h^{\mu\nu}(x)]\partial_\nu+\bar h^{\mu\nu}(x)\partial_\mu\partial_\nu+\xi(n)R^{(1)}\}\phi(x)\ , \tag{3.13}$$

$$V^{(2)}(x)\phi(x)=\{[\partial_\mu\hat h^{\mu\nu}(x)]\partial_\nu+\hat h^{\mu\nu}(x)\partial_\mu\partial_\nu-\xi(n)(R^{(2)}+\tfrac{1}{2}hR^{(1)})\}\phi(x)\ . \tag{3.14}$$

The explicit form of the operator $V^{(2)}$ will not be needed, however. Now we can write the propagator, from (3.12), as

$$G^0_{++}(x,x')=\Delta_F(x,x')=-\int\frac{d^nk}{(2\pi)^n}\frac{e^{ik\cdot(x-x')}}{k^2-i\epsilon}\ ; \tag{3.15}$$

recall that this is the propagator for the field $\phi(x)$; the propagator for the physical field $\Phi(x)$ is related to this by the conformal factor [13], but we do not need it here. All we need to know is that the boundary conditions for the physical field are determined by the boundary conditions of $\phi(x)$ in flat space.

The effective action depends on the fields $\bar\phi_\pm$ and the metric perturbations $h^\pm_{\mu\nu}$. We may obtain the equation for the field $\bar\phi(x)$ (field equation) by functional derivation with respect to $\bar\phi_+$, i.e., Eq. (2.17), and the equations for the metric perturbations (back-reaction equations) by functional derivation with respect to $h^+_{\mu\nu}$. Our primary interest is to obtain the back-reaction equations. From Eq. (2.17) one can see that the vacuum expectation value

$$\bar\phi\equiv\langle 0,\mathrm{in}|\phi|\mathrm{in},0\rangle=O(h_{\mu\nu})$$

(it would be zero in flat space), and thus the contribution to the back-reaction equations coming from the matter action term will be at least of second order in the perturbations when the field equations are substituted. Formally, one can compute the effective action as a functional of $h^\pm_{\mu\nu}$ only, and thus, the terms involving the field $\bar\phi_+(S^{(i)}_m[\bar\phi_+])$ are not necessary.

We can now compute each of the (divergent) terms (3.8). As it is well known the first term $\Gamma^{(0)}_m$ is easily renormalized by adding a suitable counterterm that cancels the divergencies which lead to the conformal anomaly [13,26], but, this term does not play any role in the stress tensor of the field because it is independent of $h_{\mu\nu}$.

We can now go to the next term in (3.8), $\Gamma^{(1)}_m$, but this formally divergent (tadpole) term has no contribution, since $\mathrm{Tr}(V^{(1)}_+G^0_{++})$ involves n-dimensional integrals of the form $1/k^2$, k_α/k^2, and $k_\alpha k_\beta/k^2$ (where k_α is the integration momentum variable), which are identically zero in dimensional regularization [42]. Therefore, there is no term linear in $h_{\mu\nu}$ in the effective action; i.e., we have $\Gamma^{(1)}_m=0$.

The first nontrivial quantum contributions to the stress tensor coming from Γ_m are quadratic in $h_{\mu\nu}$ and we thus need to compute $\Gamma^{(2)}_m$. Here again the second (tadpole) term in (3.8), $\mathrm{Tr}(V^{(2)}_+G^0_{++})$, gives no contribution in dimensional regularization, since the typical integrals are of the same type that those of $\mathrm{Tr}(V_+^{(1)}G^0_{++})$. For this reason we do not need the explicit form of the operator $V^{(2)}$.

The problem is thus reduced to the evaluation of the third and fourth terms in (3.8). As we have stressed in Sec. II the third term also appears in the evaluation of the in-out effective action and the fourth term is typical of the in-in contribution to this order. Let us evaluate the third term, i.e.,

$$T_1 \equiv -\frac{i}{4}\mathrm{Tr}(V_+^{(1)}G^0_{++}V_+^{(1)}G^0_{++})$$

(recall that $G^0_{++}=\Delta_F$, with $m=0$):

$$T_1=-\frac{i}{4}\int d^nx\, d^nx'\, V_+^{(1)}(x)\Delta_F(x,x')V_+^{(1)}(x')\Delta_F(x',x)$$
$$=-\frac{i}{4}\int d^nx\, d^nx' \int \frac{d^np}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\left[\{[\partial_\mu \bar{h}_+^{\mu\nu}(x)]\partial_\nu+\bar{h}_+^{\mu\nu}(x)\partial_\mu\partial_\nu+\xi(n)R_+^{(1)}(x)\}\frac{e^{iq\cdot(x-x')}}{q^2-i\epsilon}\right]$$
$$\times\left[\{[\partial'_\alpha \bar{h}_+^{\alpha\beta}(x')]\partial'_\beta+\bar{h}_+^{\alpha\beta}(x')\partial'_\alpha\partial'_\beta+\xi(n)R_+^{(1)}(x')\}\frac{e^{ip\cdot(x'-x)}}{p^2-i\epsilon}\right]. \tag{3.16}$$

We now introduce the projector $P^{\mu\nu}=\eta^{\mu\nu}-p^\mu p^\nu/p^2$, the symbol $\eta^{\mu\nu\alpha\beta}$, $(\bar{h}^{\mu\nu}\equiv\eta^{\mu\alpha\nu\beta}h_{\alpha\beta})$, change the p integration by $p'\equiv q-p$, rename p' as p again, and write T_1 as

$$T_1=-i\int d^nx\, d^ny\, h^+_{\mu\nu}(x)h^+_{\alpha\beta}(y)\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-y)}\hat{K}^{\mu\nu\alpha\beta}(p)\ ,$$
$$\hat{K}^{\mu\nu\alpha\beta}(p)=\frac{1}{4}\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-i\epsilon)[(p-q)^2-i\epsilon]}[\eta^{\rho\mu\tau\nu}(q-p)_\rho q_\tau-\xi(n)p^2P^{\mu\nu}][\eta^{\lambda\alpha\sigma\beta}(q-p)_\lambda q_\sigma-\xi(n)p^2P^{\alpha\beta}]\ . \tag{3.17}$$

The momentum integrals can be computed in the standard way (see Appendix D), and expanding around $n=4$ we get, after a rather long calculation,

$$\hat{K}^{\mu\nu\alpha\beta}(p)=\frac{p^4I_1(p)}{1440}\left[(3P^{\mu\beta}P^{\nu\alpha}-P^{\mu\nu}P^{\alpha\beta})+\frac{(n-4)}{15}[8(P^{\mu\nu}P^{\alpha\beta}-3P^{\mu\beta}P^{\nu\alpha})+5P^{\mu\nu}P^{\alpha\beta}]+O(n-4)^2\right], \tag{3.18}$$

where

$$I_1(p)=-\frac{i}{8\pi^2}\left[\frac{1}{n-4}+\frac{1}{2}\ln\left[\frac{p^2-i\epsilon}{\mu_0^2}\right]+O(n-4)\right], \tag{3.19}$$

We now introduce the projector $P^{\mu\nu}=\eta^{\mu\nu}-p^\mu p^\nu/p^2$, the symbol $\eta^{\mu\nu\alpha\beta}$ $(\bar{h}^{\mu\nu}\equiv\eta^{\mu\alpha\nu\beta}h_{\alpha\beta})$, change the p integration by $p'\equiv q-p$, rename p' as p again, and write T_1 as

$$T_1=-\frac{\alpha}{4}\left[\frac{1}{n-4}\int d^4x[3R^+_{\mu\nu\alpha\beta}(x)R^{+\mu\nu\alpha\beta}(x)-R^{+2}(x)]+\frac{1}{3}\int d^4x\, R^{+2}(x)\right.$$
$$\left.-\int d^4x\, d^4y[3R^+_{\mu\nu\alpha\beta}(x)R^{+\mu\nu\alpha\beta}(y)-R^+(x)R^+(y)]K_1(x-y;\mu_0)+O(n-4)\right], \tag{3.20}$$

where

$$K_1(x-y;\mu_0)\equiv-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\ln\left[\frac{p^2-i\epsilon}{\mu_0^2}\right], \tag{3.21}$$

and $\alpha\equiv(2880\pi^2)^{-1}$. Note that the divergent terms with a pole at $n=4$ are local and quadratic in the curvature. They may be compensated by counterterms in the gravitational part of the action $S_g^{(2)}$ coming from (3.6). Recall that the curvature terms here depend on the metric $g_{\mu\nu}$ rather than the physical metric $\tilde{g}_{\mu\nu}$.

Let us compute now the fourth term of $\Gamma_m^{(2)}$, i.e.,

$$T_2\equiv\frac{i}{2}\mathrm{Tr}(V_+^{(1)}G^0_{+-}V_-^{(1)}G^0_{-+})\ ,$$

which depends on the propagators $G^0_{+-}=-\Delta^+$ and $G^0_{-+}=\Delta^-$, represented in (2.22) with $m=0$. We can write

$$T_2=-\frac{i}{2}\int d^nx\,d^nx' V_+^{(1)}(x)\Delta^+(x,x')V_-^{(1)}(x')\Delta^-(x',x)$$

$$=(-2i\pi^2)\int d^nx\,d^nx'\int \frac{d^np}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\delta(q^2)\theta(-q^0)\delta(p^2)\theta(p^0)\{[\partial_\mu \bar{h}_+^{\mu\nu}(x)]\partial_\nu+\bar{h}_+^{\mu\nu}(x)\partial_\mu\partial_\nu+\xi(n)R_+^{(1)}(x)\}e^{iq\cdot(x-x')}$$
$$\times\{[\partial'_\alpha \bar{h}_-^{\alpha\beta}(x')]\partial'_\beta+\bar{h}_-^{\alpha\beta}(x')\partial'_\alpha\partial'_\beta+\xi(n)R_-^{(1)}(x')\}e^{ip\cdot(x'-x)}\ , \tag{3.22}$$

changing the integration variable from p to $q-p$ as in the previous case we can write

$$T_2=-8i\pi^2\int d^nx\,d^ny h_{\mu\nu}^+(x)h_{\alpha\beta}^-(y)\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-y)}\hat{L}^{\mu\nu\alpha\beta}(p)\ ,$$

$$\hat{L}^{\mu\nu\alpha\beta}(p)=\frac{1}{4}\int\frac{d^nq}{(2\pi)^n}\delta((p-q)^2)\theta(q^0-p^0)\delta(q^2)\theta(-q^0)[\eta^{\rho\mu\tau\nu}(q-p)_\rho q_\tau-\xi(n)p^2P^{\mu\nu}][\eta^{\lambda\alpha\sigma\beta}(q-p)_\lambda q_\sigma-\xi(n)p^2P^{\alpha\beta}]\ . \tag{3.23}$$

After performing the phase-space integrations and expanding around $n=4$ (see Appendix D), we obtain

$$\hat{L}^{\mu\nu\alpha\beta}(p)=\frac{p^4I_2(p)}{1440}[(3P^{\mu\beta}P^{\nu\alpha}-P^{\mu\nu}P^{\alpha\beta})+O(n-4)]\ , \tag{3.24}$$

where

$$I_2(p)=\frac{1}{8\pi^2}\left[\frac{1}{4\pi}\theta(-p^2)\theta(-p^0)+O(n-4)\right]\ , \tag{3.25}$$

which has no poles at $n=4$, therefore the term T_2 requires no counterterms in the action to be renormalized. Using the expressions of the Riemann components in terms of $P_{\mu\nu}$ (see Appendix B 2) we get

$$T_2=\frac{\alpha}{2}\int d^4x\,d^4y[3R_{\mu\nu\alpha\beta}^+(x)R^{-\mu\nu\alpha\beta}(y)-R^+(x)R^-(y)]\times K_2(x-y)+O(n-4)\ , \tag{3.26}$$

where

$$K_2(x-y)\equiv-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}(2\pi i)\theta(-p^2)\theta(-p^0)\ . \tag{3.27}$$

Here again the Riemann components refer to the metric $g_{\mu\nu}$. We now have $\Gamma_m^{(2)}=T_1+T_2$ which must be renormalized by adding the gravitational action up to the second order in $h_{\mu\nu}$. The explicit expansion of S_g, in (3.6), up to this order in terms of the curvature components of the metric $g_{\mu\nu}$ is

$$S_g[\tilde{g}_{\mu\nu}]\equiv S_g^{(0)}+S_g^{(1)}+S_g^{(2)}+\cdots$$
$$=\frac{1}{16\pi G_N}\int d^4x[-g(x)]^{1/2}e^{2\omega}[R(x)+6\omega_{;\mu}\omega^{;\mu}]+\frac{\alpha}{4(n-4)}\int d^4x[3R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(x)-R^2(x)]$$
$$+\alpha\int d^4x[R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(x)-R_{\mu\nu}(x)R^{\mu\nu}(x)]\ln(\mu e^{\omega})$$
$$+\alpha\int d^4x[-g(x)]^{1/2}[2R^{\mu\nu}\omega_{;\mu}\omega_{;\nu}+R\Box_g\omega-4(\Box_g\omega)\omega_{;\nu}\omega^{;\nu}-3(\Box_g\omega)^2-2(\omega_{;\nu}\omega^{;\nu})^2]+O(n-4)\ , \tag{3.28}$$

where we have dropped the plus sign on the fields for simplicity. Finally adding these terms to Γ_m and including only the terms which contribute to the variation of $h_{\mu\nu}^+$ we get the renormalized effective action

$$\Gamma_{(ii)}^{(2)}[\omega,h_{\mu\nu}^{\pm}]=\int d^4x[-\tilde{g}^+(x)]^{1/2}\left[\frac{\tilde{R}^+(x)}{16\pi G_N}-\frac{\alpha}{12}\tilde{R}^+(x)\tilde{R}^+(x)\right]$$
$$+2\alpha\int d^4x[-g^+(x)]^{1/2}[G^{+\mu\nu}(x)\omega_{;\mu}\omega_{;\nu}+\Box_g\omega(\omega_{;\nu}\omega^{;\nu})+\tfrac{1}{2}(\omega_{;\mu}\omega^{;\mu})^2]$$
$$+\alpha\int d^4x[-g^+(x)]^{1/2}\{[R_{\mu\nu\alpha\beta}^+(x)R^{+\mu\nu\alpha\beta}(x)-R_{\mu\nu}^+(x)R^{+\mu\nu}(x)]\}\omega(x)$$
$$+\frac{\alpha}{4}\int d^4x\,d^4y[-g^+(x)]^{1/2}[-g^+(y)]^{1/2}[3R_{\mu\nu\alpha\beta}^+(x)R^{+\mu\nu\alpha\beta}(y)-R^+(x)R^+(y)]K_1(x-y;\bar{\mu})$$
$$+\frac{\alpha}{2}\int d^4x\,d^4y[-g^+(x)]^{1/2}[-g^-(y)]^{1/2}[3R_{\mu\nu\alpha\beta}^+(x)R^{-\mu\nu\alpha\beta}(y)-R^+(x)R^-(y)]K_2(x-y)+O(h_{\mu\nu}^3)\ , \tag{3.29}$$

where $\bar{\mu}\equiv\mu\mu_0$ and we have substituted 1 by the volume densities $\sqrt{-g}$ in all the integrals involving quadratic curvature terms of the metric $g_{\mu\nu}$ in order to facilitate the identification of the exact variational formulas of the Appendix E needed in the computations of the next section.

If one is just interested in the production of particles, the in-in effective action is not the most direct approach because the probability of particle creation is related to the transition amplitude from the in to the out vacua, and this amplitude is directly related to the in-out effective action [25]. Let us write the vacuum persistence amplitude in the presence of an external source, i.e.,

$$\langle 0,\mathrm{out}|\mathrm{in},0\rangle_J=\exp(iW[J]) .$$

The probability of pair creation is proportional to the imaginary part of W,

$$P=2\,\mathrm{Im}W ,$$

but if we take $J=0$ and consider the quantum fields propagating in the gravitational background, then W is just the in-out effective action: $\Gamma_{(io)}$. The calculations leading to such an action are similar to those for the in-in case, although they are simpler because we do not need to introduce two fields. The main work has already been done: consider just a single field in (3.8) and ignore the term T_2. The renormalized action is obtained again by adding the gravitational action (3.28), the final result can be read directly from (3.29): ignore the plus indices and the term involving $K_2(x-y)$. The nonlocal term now includes $K_1(x-y;\bar{\mu})$ only, this term is complex and it is responsible for the particle creation effect. It turns out that the pair creation probability is given, however, by a local term as is well known [15,25]. In fact, the imaginary part of the kernel $K_1(x-y;\bar{\mu})$ is

$$\mathrm{Im}K_1(x-y;\bar{\mu})=\frac{\pi}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\theta(-p^2) . \qquad (3.30)$$

From the expression for $\Gamma_{(io)}$ and (3.30), after performing the x and y integrations which lead to the Fourier transform of the curvature tensor, $R_{\mu\nu\alpha\beta}(p)$, we get

$$P=2\,\mathrm{Im}\Gamma_{(io)}$$
$$=\frac{\alpha\pi}{4}\int\frac{d^4p}{(2\pi)^4}[3R_{\mu\nu\alpha\beta}(p)R^{\mu\nu\alpha\beta}(-p)-R(p)R(-p)]\times\theta(-p^2) . \qquad (3.31)$$

Finally, using that the Gauss-Bonnet topological invariant is zero, the above relation can be written in terms of the Fourier transform of the Weyl tensor of the physical metric $\tilde{g}_{\mu\nu}$ as

$$P=\frac{1}{960\pi}\int\frac{d^4p}{(2\pi)^4}|\tilde{C}_{\mu\nu\alpha\beta}(p)|^2\theta(-p^2)\theta(-p^0) , \qquad (3.32)$$

in agreement with the expressions computed by other means [15].

IV. SEMICLASSICAL EQUATIONS

In this section we obtain the quantum-mechanically corrected Einstein's equations due to the presence of a massless conformal scalar quantum field. The semiclassical equations for the metric perturbation can be found by functional differentiation of the in-in effective action (3.29) with respect to $h^+_{\mu\nu}(x)$ and then restricting $h^+_{\mu\nu}(x)=h^-_{\mu\nu}(x)=h_{\mu\nu}(x)$, as

$$\left.\frac{\delta\Gamma^{(2)}_{(ii)}[\omega,h^\pm_{\mu\nu}]}{\delta h^+_{\mu\nu}(x)}\right|_{h^\pm_{\mu\nu}=h_{\mu\nu}}=0 . \qquad (4.1)$$

From Eq. (4.1) it is easy to derive the equations of motion to first order; we use that, for an arbitrary functional $A[\tilde{g}_{\mu\nu}]$,

$$\frac{\delta A[\omega,g_{\mu\nu}]}{\sqrt{-g}\,\delta g_{\mu\nu}}=e^{6\omega}\frac{\delta A[\tilde{g}_{\mu\nu}]}{\sqrt{-\tilde{g}}\,\delta\tilde{g}_{\mu\nu}} , \qquad (4.2)$$

to find the variation of the first two terms of (3.29). Notice that we assume that $\omega(x)$ is a scalar function independent of the metric [in general, we will assume that $\omega(x)$ depends on the spacetime point only, in particular, in the flat FRW case it will be a function of the cosmological time t only].

Using the expressions listed in the Appendix E one can show that the semiclassical equations can be written as

$$e^{6\omega}\left[-\frac{1}{16\pi G_N}(\tilde{G}^{\mu\nu}_{(0)}+\tilde{G}^{\mu\nu}_{(1)})-\frac{\alpha}{12}(\tilde{B}^{\mu\nu}_{(0)}+\tilde{B}^{\mu\nu}_{(1)})+\frac{\alpha}{2}(\tilde{H}^{\mu\nu}_{(0)}+\tilde{H}^{\mu\nu}_{(1)})\right]$$
$$-\alpha\tilde{R}^{(0)}_{\alpha\beta}C^{\mu\alpha\nu\beta}_{(1)}+\frac{3\alpha}{2}\left[-4(C^{\mu\alpha\nu\beta}_{(1)}\omega)_{,\alpha\beta}+\int d^4y\;A^{\mu\nu}_{(1)}(y)H(x-y;\bar{\mu})\right]+O(h^2_{\mu\nu})=0 , \qquad (4.3)$$

where $G^{\mu\nu}(x)$ is the Einstein's tensor, $C^{\mu\alpha\nu\beta}(x)$ Weyl's tensor, $B^{\mu\nu}(x)$ and $A^{\mu\nu}(x)$ are the exact spacetime tensors given by the variation of $\int d^4x\,R^2(x)$ and $\int d^4x\,C_{\mu\alpha\nu\beta}C^{\mu\alpha\nu\beta}$, respectively (see Appendix E 2 and use that $C_{\mu\alpha\nu\beta}C^{\mu\alpha\nu\beta}=R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2$), with respect to an arbitrary metric $g_{\mu\nu}(x)$,

$$B^{\mu\nu}(x)\equiv\tfrac{1}{2}g^{\mu\nu}R^2-2RR^{\mu\nu}+2R^{;\mu\nu}-2g^{\mu\nu}\Box_gR ,$$
$$A^{\mu\nu}(x)\equiv\tfrac{1}{2}g^{\mu\nu}C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma}-R^{\mu\alpha\beta\rho}R^{\nu}{}_{\alpha\beta\rho}+4R^{\mu\alpha}R_{\alpha}{}^{\nu}-\tfrac{2}{3}RR^{\mu\nu}-2\Box_gR^{\mu\nu}+\tfrac{2}{3}R^{;\mu\nu}+\tfrac{1}{3}g^{\mu\nu}\Box_gR ,$$

and $H^{\mu\nu}(x)$ is the spacetime tensor defined by

$$H^{\mu\nu}(x)\equiv -R^{\mu\alpha}R_{\alpha}{}^{\nu}+\tfrac{2}{3}RR^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}-\tfrac{1}{4}g^{\mu\nu}R^{2} .$$

In Eq. (4.3) we are only interested in the expressions of these tensors up to first order in the perturbations, as indicated by the bracketed subindices and we recall that an over tilde on the tensors refer to the physical metric. The nonlocal part $H(x-y;\bar{\mu})$ is the sum of the integrals (3.21) and (3.27):

$$H(x-y;\bar{\mu})\equiv K_1(x-y;\bar{\mu})+K_2(x-y)$$
$$=-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\left[\ln\left[\frac{p^2-i\epsilon}{\bar{\mu}^2}\right]+(2\pi i)\theta(-p^2)\theta(-p^0)\right] , \quad (4.4)$$

which can be simplified to

$$H(x-y;\bar{\mu})=-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\left[\ln\left[\frac{|p^2|}{\bar{\mu}^2}\right]+i\pi\theta(-p^2)\mathrm{sgn}(-p^0)\right] , \quad (4.5)$$

by using

$$\ln(\pm i)=\pm i\frac{\pi}{2}$$

and

$$\lim_{\epsilon\to 0}\ln(\epsilon+ix)=\ln|x|+\mathrm{sgn}(x)i\frac{\pi}{2} .$$

Note that this equation is real, in spite of appearances, because the imaginary part of the integrand is an odd term with respect to the integration variable p^μ. One can also notice that $H(x-y;\bar{\mu})$ differs from that defined by Horowitz [29] by a factor $(1/4\pi)$.

Equations (4.3) are dynamical equations with higher-order derivative terms. When the background is flat they reduce to the field equations studied by Horowitz [29] and Jordan [31]. As we have noted in the previous section one can add a classical stress matter source to these field equations.

To compare the functional method used in this paper with other techniques and, in particular, to compare Eqs. (3.29) and (4.3) with previous results one can give, for example, the energy-momentum tensor of the quantum field and the expression of the semiclassical equations in two dimensions.

A. Stress tensor to first order

From (4.3) one can read the zero and first order vacuum expectation value of the energy-momentum tensor of the scalar field:

$$\langle T^{\mu\nu}_{(0)}\rangle=\alpha(\tilde{H}^{\mu\nu}_{(0)}-\tfrac{1}{6}\tilde{B}^{\mu\nu}_{(0)}) , \quad (4.6)$$

$$\langle T^{\mu\nu}_{(1)}\rangle=\alpha\left[(\tilde{H}^{\mu\nu}_{(1)}-2\tilde{R}^{(0)}_{\alpha\beta}\tilde{C}^{\mu\alpha\nu\beta}_{(1)})-\tfrac{1}{6}\tilde{B}^{\mu\nu}_{(1)}+3e^{-6\omega}\left[-4(C^{\mu\alpha\nu\beta}_{(1)}\omega)_{,\alpha\beta}+\int d^4y\, A^{\mu\nu}_{(1)}(y)H(x-y;\bar{\mu})\right]\right] . \quad (4.7)$$

The stress tensor to first order in $h_{\mu\nu}$, $\langle T^{\mu\nu}_{(1)}\rangle$, is in agreement with that obtained by Horowitz and Wald [27] and Starobinsky [28]. On the other hand, the zeroth-order tensor $\langle T^{\mu\nu}_{(0)}\rangle$, which gives the exact stress tensor for a conformal scalar field in a conformally flat spacetime, agrees with that found by other techniques [13,43]. Note also that we recover the trace anomaly result to this order in $h_{\mu\nu}$:

$$\langle T^{\mu}_{\mu}\rangle=\langle T^{\mu}_{(0)\mu}\rangle+\langle T^{\mu}_{(1)\mu}\rangle+O(h^2_{\mu\nu})$$
$$=\alpha[\Box_{g}\tilde{R}+(\tilde{R}^{\mu\nu}\tilde{R}_{\mu\nu}-\tfrac{1}{3}\tilde{R}^2)]+O(h^2_{\mu\nu}) . \quad (4.8)$$

A comment on the computed stress tensor is now in order. As we have emphasized in the introduction the stress tensor to first order in $h_{\mu\nu}$ does not include the energy of the particles created [27]. The reason is that to the lowest order the energy would appear in the expectation value of the in vacuum $\langle T_{\mu\nu}\rangle$ in terms such as

$$\langle 0,\mathrm{in}|\mathrm{out},2\rangle\langle 2,\mathrm{out}|T_{\mu\nu}|\mathrm{out},2\rangle\langle 2,\mathrm{out}|\mathrm{in},0\rangle ,$$

but the scattering matrix that gives the transition from the in vacuum to the two out particle states is first order in $h_{\mu\nu}$. As a consequence, if we want to include the energy of the particles created consistently in the semiclassical correction we should compute $\langle T_{\mu\nu}\rangle$ to second order in $h_{\mu\nu}$, of course such computation would include also vacuum polarization effects to second order in $h_{\mu\nu}$.

This does not mean that the stress tensor (4.7) does not include particle creation effects. This should be clear from the fact that the in-in effective action, as we have seen, includes the in-out effective action which leads directly to the probability of pair creation. More important, Calzetta and Hu [24,44] have argued that such effects are in the nonlocal terms of (4.7), by showing that the energy dissipated by such nonlocal terms is the energy carried away by the created particles.

B. Two-dimensional gravity

As an exercise we will derive here the semiclassical corrections to the stress tensor of the scalar field in two spacetime dimensions. At the classical level there is no dynamics for the gravitational field in two dimensions because the Einstein tensor is identically zero, but the semiclassical corrections due to the presence of a quantum field lead to nontrivial effects. Since we have worked in n dimensions in order to use dimensional regularization, we can use most of the work done in the previous section to analyze the two-dimensional (2D) case. From Eqs. (3.17) and Appendix D it is not difficult to show that

$$\hat{R}^{(2D)}_{\mu\nu\alpha\beta}(q)=\frac{I_1^{(2D)}(q)}{96}(n-2)q^4P_{\mu\nu}P_{\alpha\beta}+O((n-2)^2)\ , \tag{4.9}$$

where

$$I_1^{(2D)}(q)=\frac{1}{\pi(q^2-i\epsilon)}\left[\frac{1}{n-2}+\frac{\gamma}{2}+O(n-2)\right]. \tag{4.10}$$

Then, the in-out contribution to the effective action T_1 becomes

$$T_1=-\frac{1}{96\pi}\int d^2x\, d^2y\, R(x)\Delta_F^{(2D)}(x-y)R(y)\ , \tag{4.11}$$

where $\Delta_F^{(2D)}(x-y)$ is the Feynman propagator in two dimensions:

$$\Delta_F^{(2D)}(x-y)=-\int\frac{d^2q}{(2\pi)^2}\frac{e^{iq\cdot(x-y)}}{q^2-i\epsilon}\ . \tag{4.12}$$

From Eq. (3.23) it is straightforward to see that there is no contribution to the effective action due to the typical in-in term T_2. Quantum corrections appear to be nonlocal and quadratic in the scalar curvature, but contrary to the four-dimensional case there are no divergent terms in the regularization process and counteraction terms are not needed. Finally, one can express the effective action to one loop as

$$\Gamma^{(2D)}[\omega,h_{\mu\nu}]=\frac{1}{16\pi G_N}\int d^2x\,[-g(x)]^{1/2}(R-2\Box_g\omega)$$
$$-\frac{1}{96\pi}\int d^2x\, d^2y\,[-g(x)]^{1/2}[-g(y)]^{1/2}R(x)\Delta_F^{(2D)}(x-y)R(y)+O(h_{\mu\nu}^3)\ . \tag{4.13}$$

The stress tensor of the quantum field can be obtained from the second term of the above equation by differentiating with respect to the metric. To order zero in the perturbation the energy-momentum tensor vanishes, but to first order there is a nonlocal contribution,

$$\langle T^{\mu\nu}_{(1)}(x)\rangle^{(2D)}=\frac{1}{24\pi}\int d^2y\,[\eta^{\mu\nu}R_{,\alpha}{}^{\alpha}-R^{,\mu\nu}]\times\Delta_F^{(2D)}(x-y)+O(h_{\mu\nu}^2)\ , \tag{4.14}$$

and the trace is local

$$\langle T^{\mu}_{(1)\mu}(x)\rangle^{(2D)}=R/(24\pi)+O(h_{\mu\nu}^2)\ ,$$

as expected [13]. Note that in two dimensions there is no difference between the in-out and the in-in effective actions; thus, the use of the usual in-out effective action to derive the semiclassical equations is justified.

V. BACK REACTION ON THE FIELD OF A STATIC COSMIC STRING

In this section, as a simple example, we discuss the back reaction due to one loop quantum effects on the gravitational field of a static cosmic string. First, we must compute the vacuum expectation value of the stress tensor of a conformally coupled massless scalar quantum field outside the core of a straight and static cosmic string.

In the weak field approximation the metric of a cosmic string can be seen as a small perturbation about flat space:

$$g_{\mu\nu}(x)=\eta_{\mu\nu}+h_{\mu\nu}(x)\ . \tag{5.1}$$

Let $T_c^{\mu\nu}$ be the stress tensor of a cosmic string (or of any other classical source); the semiclassical equations up to first order in the perturbation can be written as

$$G^{\mu\nu}(x)=8\pi G_N[T_c^{\mu\nu}(x)+\langle T^{\mu\nu}_{(1)}(x)\rangle+O(h_{\mu\nu}^2)]\ . \tag{5.2}$$

There is no zero-order correction $\langle T^{\mu\nu}_{(0)} \rangle$ to the classical Einstein equations because the background is flat; see (4.6). The vacuum expectation value $\langle T^{\mu\nu}_{(1)} \rangle$ is obtained from Eq. (4.7) when the conformal function is $\omega=0$. We have

$$\langle T^{\mu\nu}_{(1)}(x) \rangle = -\frac{\alpha}{6} B^{\mu\nu}_{(1)}(x) + 3\alpha \int d^4y\, H(x-y;\bar{\mu}) A^{\mu\nu}_{(1)}(y) , \tag{5.3}$$

where

$$B^{\mu\nu}_{(1)}(x) = 2\eta^{\mu\nu} G_\alpha{}^\alpha{}_{,\beta}{}^\beta - 2G_\alpha{}^{\alpha,\mu\nu} ,$$
$$A^{\mu\nu}_{(1)}(x) = -2G^{\mu\nu}{}_{,\alpha}{}^\alpha - \tfrac{2}{3} G_\alpha{}^{\alpha,\mu\nu} + \tfrac{2}{3}\eta^{\mu\nu} G_\alpha{}^\alpha{}_{,\beta}{}^\beta . \tag{5.4}$$

Note that the quantum correction term $\langle T^{\mu\nu}_{(1)} \rangle$ depends on the Einstein tensor $G^{\mu\nu}$; thus, one may use Einstein's equations to the classical order, which is already first order in $h_{\mu\nu}$, to substitute $G^{\mu\nu}$ by $8\pi G_N T^{\mu\nu}_c$. This simplifies considerably the problem since the explicit gravitational field of the string (or the classical source) is not required to compute $\langle T^{\mu\nu}_{(1)} \rangle$.

The stress tensor of a static cosmic string along the z axis can be written in the thin line approximation as

$$T_{c\mu}{}^\nu(x,y) = -\mu\delta(x)\delta(y)\mathrm{diag}(1,0,0,1) , \tag{5.5}$$

where μ is the mass per unit length of the string. Since $G_N\mu \sim 10^{-6}$ for grand unified theory (GUT) strings we can assume that $G_N\mu \ll 1$ and the linear approximation (5.1) is justified. In fact, ignoring quantum effects the stress tensor (5.5) leads in the linear approximation to the conical metric with a deficit angle of $8\pi G_N\mu$ [45].

Equations (5.4) can be expressed in terms of the string stress tensor and its trace

$$T \equiv T_c = -2\mu\delta(x)\delta(y) ,$$

as

$$B_{(1)\mu}{}^\nu(x) = 16\pi G_N(\delta_\mu{}^\nu T_{,\alpha}{}^\alpha - T_{,\mu}{}^\nu) ,$$

$$A_{(1)\mu}{}^\nu(x) = 8\pi G_N(-2T_\mu{}^\nu{}_{,\alpha}{}^\alpha - \tfrac{2}{3}T_{,\mu}{}^\nu + \tfrac{2}{3}\delta_\mu{}^\nu T_{,\alpha}{}^\alpha) . \tag{5.6}$$

In this case, $B_{(1)\mu}{}^\nu(x)$ in (5.3) gives no contribution outside the core of the string because it is proportional to partial derivatives of a δ function with support on the z axis. The tensor components $A_{(1)\mu}{}^\nu(x)$, in the Minkowskian coordinates (t,x,y,z) in which Eq. (5.1) is given, are

$$A_{(1)\mu}{}^\nu = \frac{8\pi G_N}{3} \begin{bmatrix} -T_{,xx}-T_{,yy} & 0 & 0 & 0 \\ 0 & 2T_{,yy} & -2T_{,xy} & 0 \\ 0 & -2T_{,yx} & 2T_{,xx} & 0 \\ 0 & 0 & 0 & -T_{,xx}-T_{,yy} \end{bmatrix} .$$

The nonlocal term $H(x-y;\bar{\mu})$ in Eq. (5.3) can be expressed as a δ function; following Jordan [31] we can write

$$H(x-y;\bar{\mu}) = -\frac{1}{2\pi}\delta'[(x-y)^2] . \tag{5.7}$$

Introducing these expressions into (5.3), we may perform the space integrations; this is easy due to the presence of the δ functions. In fact, as an example, let us compute the xy component of the stress tensor, $\langle T_x{}^y \rangle = \langle T_y{}^x \rangle$, to first order in $G_N\mu$. The only contribution to this component comes from $A_{(1)x}{}^y = -16\pi G_N T_{,xy}/3 = 32\pi G_N \mu\delta'(x)\delta'(y)/3$:

$$\langle T_x{}^y \rangle = 3\alpha \int d^4x' H(x-x';\bar{\mu}) A_{(1)x}{}^y(x')$$
$$= -\frac{3\alpha}{2\pi}\int d^4x' \delta'[(x-x')^2] A_{(1)x}{}^y(x') . \tag{5.8}$$

This expression can be written as

$$\langle T_x{}^y \rangle = \frac{3\alpha}{2\pi} \lim_{\lambda\to 0^-} \frac{d}{d\lambda} \int d^4x' \delta(x'^2-\lambda) A_{(1)x}{}^y(x'^\mu + x^\mu) , \tag{5.9}$$

where $A_{(1)x}{}^y(x'^\mu + x^\mu)$ is now

$$A_{(1)x}{}^y(x'^\mu + x^\mu) = \left[\frac{32\pi}{3}\right] G_N\mu \frac{\partial}{\partial x}\frac{\partial}{\partial y} \times [\delta(x'+x)\delta(y'+y)] . \tag{5.10}$$

Following straightforward steps,

$$
\begin{aligned}
\langle T_x{}^y\rangle &= 16\alpha G_N\mu\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[\lim_{\lambda\to 0^-}\frac{d}{d\lambda}\int d^4x'\delta(x'^2-\lambda)\delta(x'+x)\delta(y'+y)\right]\\
&= 16\alpha G_N\mu\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[\lim_{\lambda\to 0^-}\frac{d}{d\lambda}\int dz'\frac{1}{\sqrt{z'^2+x^2+y^2-\lambda}}\right]\\
&= 16\alpha G_N\mu\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[\int_0^\infty\frac{dz'}{(z'^2+x^2+y^2)^{3/2}}\right]\\
&= 32\alpha G_N\mu\left[\frac{4xy}{(x^2+y^2)^3}\right] .
\end{aligned}
\tag{5.11}
$$

The remaining non-null components of the stress tensor in Minkowskian coordinates to the same order $G_N\mu$ are

$$
\begin{aligned}
\langle T_y{}^y\rangle &= 32\alpha G_N\mu\left[\frac{y^2-3x^2}{(x^2+y^2)^3}\right] ,\\
\langle T_x{}^x\rangle &= 32\alpha G_N\mu\left[\frac{x^2-3y^2}{(x^2+y^2)^3}\right] ,\\
\langle T_t{}^t\rangle = \langle T_z{}^z\rangle &= 32\alpha G_N\mu\left[\frac{1}{(x^2+y^2)^2}\right] .
\end{aligned}
\tag{5.12}
$$

Because of the cylindrical symmetry of the problem it is better to express this tensor components in polar coordinates as

$$\langle T_\mu{}^\nu\rangle = \frac{32\alpha G_N\mu}{r^4}\mathrm{diag}(1,1,-3,1) , \tag{5.13}$$

which is independent of the arbitrary renormalization scale $\bar{\mu}$ (as expected because it would contribute with δ functions with support on the core of the string which we do not consider). This tensor coincides with the first order development in $G_N\mu$ of previous exact results obtained by other techniques [38]. Note that in the exact case, i.e., when the classical solution is found explicitly, the one-loop quantum stress tensor is simply (5.13) where one changes $32G_N\mu$ by

$$2[(1-4G_N\mu)^{-4}-1] .$$

The back-reaction equations (5.2) have been solved by Hiscock [37] who found that the linear corrections to the metric outside the string are such that the spacetime is no longer flat space with a deficit angle: the two surface perpendicular to the string is an hyperboloid which asymptotically approaches the conical surface at large distances [the one-loop quantum corrections to $h_{\mu\nu}(x)$ are of the form $G_N\mu\hbar/r^2$]. Note that the semiclassical equations here have no higher-order derivatives because we have treated the quantum terms as a perturbative correction (as Hiscock does) in line with Simon's arguments [33]. Work on the back reaction on dynamic cosmic strings [46] is in progress.

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APPENDIX A: USEFUL RELATIONS

1. Bianchi identities

The Bianchi identities are

$$R^{\mu\alpha\nu\beta;\sigma}+R^{\mu\alpha\beta\sigma;\nu}+R^{\mu\alpha\sigma\nu;\beta}=0 , \tag{A1}$$

$$R^{\mu\alpha\nu\beta}{}_{;\alpha}=R^{\mu\nu;\beta}-R^{\mu\beta;\nu} , \tag{A2}$$

$$2R^{\alpha\beta}{}_{;\alpha}=R^{;\beta} , \tag{A3}$$

$$R^{\mu\alpha\nu\beta}{}_{;\alpha\beta}=\Box_g R^{\mu\nu}-R^{\mu\alpha;\nu}{}_{\alpha} , \tag{A4}$$

$$2R^{\alpha\beta}{}_{;\alpha\beta}=\Box_g R . \tag{A5}$$

2. Commutation of covariant derivatives

The commutation of the covariant derivatives proceeds as follows:

$$
\begin{aligned}
A^{a_1\cdots a_n}{}_{;\alpha\beta}-A^{a_1\cdots a_n}{}_{;\beta\alpha}
&=-\sum_{k=1}^{n}R^{a_k}{}_{b_k\alpha\beta}A^{a_1\cdots b_k\cdots a_n} ,
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
R^{\mu\alpha;\nu}{}_{\alpha}&=g^{\nu\alpha}R^{\mu\beta}{}_{;\beta\alpha}-R^{\mu\alpha\nu\beta}R_{\alpha\beta}+R^{\mu\alpha}R_{\alpha}{}^{\nu}\\
&=\tfrac{1}{2}R^{;\mu\nu}-R^{\mu\alpha\nu\beta}R_{\alpha\beta}+R^{\mu\alpha}R_{\alpha}{}^{\nu} ,
\end{aligned}
\tag{A7}
$$

$$
\begin{aligned}
R^{\mu\alpha\nu\beta}{}_{;\beta\alpha}&=R^{\mu\alpha\nu\beta}{}_{;\alpha\beta}=\Box_g R^{\mu\nu}-R^{\mu\alpha;\nu}{}_{\alpha}\\
&=\Box_g R^{\mu\nu}-\tfrac{1}{2}R^{;\mu\nu}+R^{\mu\alpha\nu\beta}R_{\alpha\beta}-R^{\mu\alpha}R_{\alpha}{}^{\nu} .
\end{aligned}
\tag{A8}
$$

3. 2D curvature tensors

The 2D curvature tensors are

$$R_{\mu\alpha\nu\beta}=\frac{R}{2}(g_{\mu\nu}g_{\alpha\beta}-g_{\mu\beta}g_{\nu\alpha}) , \tag{A9}$$

$$R_{\mu\nu}=\frac{R}{2}g_{\mu\nu} , \tag{A10}$$

$$G_{\mu\nu}=0 , \tag{A11}$$

$$R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}=R^2=2R_{\mu\nu}R^{\mu\nu} . \tag{A12}$$

APPENDIX B: EXPANSIONS AROUND FLAT SPACE

1. Curvature tensors

The curvature tensors are

$$g_{\mu\nu}(x)=\eta_{\mu\nu}+h_{\mu\nu}(x),\ (-,+,\dots,+) , \tag{B1}$$

$$g^{\mu\nu}(x)=\eta^{\mu\nu}-h^{\mu\nu}(x)+h^{\mu}{}_{\alpha}(x)h^{\alpha\nu}(x)+O(h^3_{\mu\nu}) , \tag{B2}$$

$$[-g(x)]^{1/2}=1+\tfrac{1}{2}h+\tfrac{1}{8}h^2-\tfrac{1}{4}h_{\mu\nu}h^{\mu\nu}+O(h^3_{\mu\nu}) , \tag{B3}$$

$$R^{\alpha}{}_{\beta\gamma\delta}=\partial_\gamma\Gamma^{\alpha}_{\beta\delta}-\partial_\delta\Gamma^{\alpha}_{\beta\gamma}+\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}-\Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma} , \tag{B4}$$

$$\Gamma^{\alpha}_{\beta\delta}=\tfrac{1}{2}\eta^{\alpha\lambda}S_{\lambda\delta,\beta}-\tfrac{1}{2}h^{\alpha\lambda}S_{\lambda\delta,\beta}+O(h^3_{\mu\nu}) , \tag{B5}$$

$$S_{\lambda\delta,\beta}\equiv h_{\lambda\delta,\beta}+h_{\beta\lambda,\delta}-h_{\beta\delta,\lambda} , \tag{B6}$$

$$R_{\sigma\beta\gamma\delta}=S_{\sigma[\delta,\beta\gamma]}-h^{\lambda}{}_{\sigma,[\gamma}S_{\lambda\delta],\beta}+\tfrac{1}{2}\eta^{\mu\rho}S_{\sigma[\gamma,\mu}S_{\rho\delta],\beta}+O(h^3_{\mu\nu}) , \tag{B7}$$

$$\begin{aligned}R_{\beta\delta}=&\ \tfrac{1}{2}\eta^{\mu\nu}(h_{\mu\delta,\beta\nu}-h_{\beta\delta,\mu\nu}-h_{\mu\nu,\beta\delta}+h_{\beta\nu,\mu\delta})-(h^{\mu\nu}{}_{,\mu}-\tfrac{1}{2}h^{,\nu})h_{\nu(\delta,\beta)}+\tfrac{1}{2}(h^{\mu\nu}h_{\beta\delta,\nu})_{,\mu}\\&+\tfrac{1}{4}h^{\mu\nu}{}_{,\delta}h_{\mu\nu,\beta}-\tfrac{1}{4}h^{,\mu}h_{\beta\delta,\mu}+h_{\mu\delta,\nu}h_{\beta}{}^{[\mu,\nu]}+\tfrac{1}{2}h^{\mu\nu}h_{\mu\nu,\beta\delta}-h^{\mu\nu}h_{\mu(\beta,\delta)\nu}+O(h^3_{\mu\nu}) ,\end{aligned} \tag{B8}$$

$$\begin{aligned}R=&\ h^{\alpha\beta}{}_{,\alpha\beta}-h_{,\alpha}{}^{\alpha}-h^{\mu\nu}{}_{,\mu}h_{\nu\alpha}{}^{,\alpha}+h^{\mu\nu}{}_{,\mu}h_{,\nu}+\tfrac{3}{4}h^{\mu\nu,\alpha}h_{\mu\nu,\alpha}-\tfrac{1}{4}h^{,\mu}h_{,\mu}\\&-\tfrac{1}{2}h^{\mu\nu,\alpha}h_{\alpha\mu,\nu}-2h^{\mu\nu}h_{\mu\alpha,\nu}{}^{\alpha}+h^{\mu\nu}h_{,\mu\nu}+h^{\mu\nu}h_{\mu\nu,\alpha}{}^{\alpha}+O(h^3_{\mu\nu}) ,\end{aligned} \tag{B9}$$

$$R_{\sigma\beta\gamma\delta}R^{\sigma\beta\gamma\delta}=h_{\sigma\delta,\beta\gamma}h^{\sigma\delta,\beta\gamma}-2h_{\sigma\delta,\beta\gamma}h^{\beta\delta,\sigma\gamma}+h_{\sigma\delta,\beta\gamma}h^{\beta\gamma,\sigma\delta}+O(h^3_{\mu\nu}) , \tag{B10}$$

$$\begin{aligned}R_{\beta\delta}R^{\beta\delta}=&\ \tfrac{1}{4}[\,2h^{\mu\nu}{}_{,\alpha\mu}h_{\beta\nu}{}^{,\alpha\beta}-4h^{\mu\nu}{}_{,\alpha\mu}h^{\alpha}{}_{\nu,\beta}{}^{\beta}-4h^{\mu\nu}{}_{,\alpha\mu}h^{,\alpha}{}_{\nu}+2h^{\mu\nu}{}_{,\alpha\mu}h^{\alpha\beta}{}_{,\beta\nu}\\&+h^{\mu\nu}{}_{,\alpha}{}^{\alpha}h_{\mu\nu,\beta}{}^{\beta}+2h^{\mu\nu}{}_{,\alpha}{}^{\alpha}h_{,\mu\nu}+h^{,\mu\nu}h_{,\mu\nu}+O(h^3_{\mu\nu})] ,\end{aligned} \tag{B11}$$

$$R^2=h^{\alpha\beta}{}_{,\alpha\beta}h^{\mu\nu}{}_{,\mu\nu}-2h^{\alpha\beta}{}_{,\alpha\beta}h_{,\mu}{}^{\mu}+h_{,\alpha}{}^{\alpha}h_{,\mu}{}^{\mu}+O(h^3_{\mu\nu}) . \tag{B12}$$

2. Curvature tensors in terms of the projector $P_{\mu\nu}$

In terms of the projector $P_{\mu\nu}$ the curvature tensors are

$$P_{\mu\nu}=\eta_{\mu\nu}-q^{\mu}q^{\nu}/q^2 , \tag{B13}$$

$$G\equiv\int d^nx\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2)=0+O(h^3_{\mu\nu}) , \tag{B14}$$

$$\int d^nx\,R^2(x)=\int d^nx\,d^ny\,h_{\mu\nu}(x)h_{\alpha\beta}(y)\int\frac{d^nq}{(2\pi)^n}e^{iq\cdot(x-y)}P^{\mu\nu}P^{\alpha\beta}q^4+O(h^3_{\mu\nu}) , \tag{B15}$$

$$\int d^nx\,R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(x)=\int d^nx\,d^ny\,h_{\mu\nu}(x)h_{\alpha\beta}(y)\int\frac{d^nq}{(2\pi)^n}e^{iq\cdot(x-y)}P^{\mu\beta}P^{\nu\alpha}q^4+O(h^3_{\mu\nu}) , \tag{B16}$$

$$\begin{aligned}&\int d^nx\,d^ny\,R(x)R(y)K_1(x-y;\mu)\\&\quad=-\frac{1}{2}\int d^nx\,d^ny\,h_{\mu\nu}(x)h_{\alpha\beta}(y)\int\frac{d^nq}{(2\pi)^n}e^{iq\cdot(x-y)}\ln\left|\frac{q^2-i\epsilon}{\mu^2}\right|P^{\mu\nu}P^{\alpha\beta}q^4+O(h^3_{\mu\nu}) ,\end{aligned} \tag{B17}$$

$$\begin{aligned}&\int d^nx\,d^ny\,R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(y)K_1(x-y;\mu)\\&\quad=-\frac{1}{2}\int d^nx\,d^ny\,h_{\mu\nu}(x)h_{\alpha\beta}(y)\int\frac{d^nq}{(2\pi)^n}e^{iq\cdot(x-y)}\ln\left|\frac{q^2-i\epsilon}{\mu^2}\right|P^{\mu\beta}P^{\nu\alpha}q^4+O(h^3_{\mu\nu}) ,\end{aligned} \tag{B18}$$

$$\int d^n x\, d^n y\, R(x)R(y)K_2(x-y)$$
$$=-\frac{1}{2}\int d^n x\, d^n y\, h_{\mu\nu}(x)h_{\alpha\beta}(y)\int \frac{d^n q}{(2\pi)^n}e^{iq\cdot(x-y)}(2\pi i)\theta(-q^2)\theta(-q^0)P^{\mu\nu}P^{\alpha\beta}q^4+O(h^3_{\mu\nu}) , \quad \text{(B19)}$$

$$\int d^n x\, d^n y\, R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(y)K_2(x-y)$$
$$=-\frac{1}{2}\int d^n x\, d^n y\, h_{\mu\nu}(x)h_{\alpha\beta}(y)\int \frac{d^n q}{(2\pi)^n}e^{iq\cdot(x-y)}(2\pi i)\theta(-q^2)\theta(-q^0)P^{\mu\beta}P^{\nu\alpha}q^4+O(h^3_{\mu\nu}) . \quad \text{(B20)}$$

APPENDIX C: CURVATURE TENSORS IN CONFORMALLY TRANSFORMED n-DIMENSIONAL SPACES

The curvature tensors are

$$\tilde{g}_{\mu\nu}(x)=e^{2\omega}g_{\mu\nu}(x) , \quad \text{(C1)}$$

$$\tilde{R}_{\mu\nu\alpha\beta}=e^{2\omega}[R_{\mu\nu\alpha\beta}+2g_{\beta[\mu}\delta_{\nu]}{}^{\rho}(\omega_{;\alpha\rho}-\omega_{;\alpha}\omega_{;\rho})-2g_{\alpha[\mu}\delta_{\nu]}{}^{\rho}(\omega_{;\beta\rho}-\omega_{;\beta}\omega_{;\rho})-2g_{\alpha[\mu}g_{\beta\nu]}\omega_{;\rho}\omega^{;\rho}] , \quad \text{(C2)}$$

$$\tilde{R}_{\mu\alpha}=R_{\mu\alpha}-(n-2)\omega_{;\mu\alpha}-g_{\mu\alpha}\Box_g\omega+(n-2)\omega_{;\mu}\omega_{;\alpha}-(n-2)g_{\mu\alpha}\omega_{;\rho}\omega^{;\rho} , \quad \text{(C3)}$$

$$\tilde{R}=e^{-2\omega}[R-2(n-1)\Box_g\omega-(n-1)(n-2)\omega_{;\rho}\omega^{;\rho}] , \quad \text{(C4)}$$

$$\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}=e^{-4\omega}[R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-8R^{\mu\nu}(\omega_{;\mu\nu}-\omega_{;\mu}\omega_{;\nu})-4R\,\omega_{;\mu}\omega^{;\mu}+4(n-2)\omega_{;\mu\nu}\omega^{;\mu\nu}$$
$$-8(n-2)\omega^{;\mu\nu}\omega_{;\mu}\omega_{;\nu}+4(\Box_g\omega)^2+8(n-2)\omega_{;\mu}\omega^{;\mu}\Box_g\omega+2(n-1)(n-2)(\omega_{;\mu}\omega^{;\mu})^2] , \quad \text{(C5)}$$

$$\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}=e^{-4\omega}[R_{\mu\nu}R^{\mu\nu}-2(n-2)R^{\mu\nu}(\omega_{;\mu\nu}-\omega_{;\mu}\omega_{;\nu})-2R\Box_g\omega-2(n-2)R\,\omega_{;\mu}\omega^{;\mu}+(n-2)^2\omega_{;\mu\nu}\omega^{;\mu\nu}$$
$$+(3n-4)(\Box_g\omega)^2-2(n-2)^2\omega^{;\mu\nu}\omega_{;\mu}\omega_{;\nu}+2(n-2)(2n-3)\omega_{;\mu}\omega^{;\mu}\Box_g\omega+(n-1)(n-2)^2(\omega_{;\mu}\omega^{;\mu})^2] , \quad \text{(C6)}$$

$$\tilde{R}^2=e^{-4\omega}[R^2-4(n-1)R\Box_g\omega-2(n-1)(n-2)R\,\omega_{;\mu}\omega^{;\mu}+4(n-1)^2(\Box_g\omega)^2$$
$$+4(n-1)^2(n-2)\omega_{;\mu}\omega^{;\mu}\Box_g\omega+(n-1)^2(n-2)^2(\omega_{;\mu}\omega^{;\mu})^2] , \quad \text{(C7)}$$

$$\tilde{C}_{\mu\nu\alpha\beta}\tilde{C}^{\mu\nu\alpha\beta}=\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}-\frac{4}{n-2}\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}+\frac{2}{(n-1)(n-2)}\tilde{R}^2$$
$$=e^{-4\omega}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} . \quad \text{(C8)}$$

APPENDIX D: MOMENTUM INTEGRALS AND DIMENSIONAL REGULARIZATION

The momentum integrals are

$$I(q)\equiv\int \frac{d^n p}{(2\pi)^n}f(p,q) , \quad \text{(D1)}$$

$$I_\mu=\int \frac{d^n p}{(2\pi)^n}f(p,q)p_\mu=\frac{I(q)}{2}q_\mu , \quad \text{(D2)}$$

$$I_{\mu\nu}=\int \frac{d^n p}{(2\pi)^n}f(p,q)p_\mu p_\nu=\frac{I(q)}{4}\left[q_\mu q_\nu-\frac{q^2}{(n-1)}P_{\mu\nu}\right] , \quad \text{(D3)}$$

$$I_{\mu\nu\alpha}=\int \frac{d^n p}{(2\pi)^n}f(p,q)p_\mu p_\nu p_\alpha$$
$$=\frac{I(q)}{8}\left[q_\mu q_\nu q_\alpha-\frac{q^2}{n-1}(P_{\mu\nu}q_\alpha+P_{\mu\alpha}q_\nu+P_{\alpha\nu}q_\mu)\right] , \quad \text{(D4)}$$

$$I_{\mu\nu\alpha\beta}=\int \frac{d^n p}{(2\pi)^n}f(p,q)p_\mu p_\nu p_\alpha p_\beta$$
$$=\frac{I(q)}{16}\left[q_\mu q_\nu q_\alpha q_\beta-\frac{q^2}{n-1}(P_{\mu\nu}q_\alpha q_\beta+P_{\nu\alpha}q_\mu q_\beta+P_{\nu\beta}q_\mu q_\alpha+P_{\mu\alpha}q_\nu q_\beta+P_{\mu\beta}q_\nu q_\alpha+P_{\alpha\beta}q_\mu q_\nu)\right.$$
$$\left.+\frac{q^4}{n^2-1}(P_{\mu\nu}P_{\alpha\beta}+P_{\mu\beta}P_{\nu\alpha}+P_{\nu\beta}P_{\mu\alpha})\right] , \quad \text{(D5)}$$

$$I_1(p)=\int \frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-i\epsilon)[(p-q)^2-i\epsilon]}$$
$$=i(p^2-i\epsilon)^{n/2-2}\frac{\Gamma(2-n/2)[\Gamma(n/2)-1)]^2}{(4\pi)^{n/2}\Gamma(n-2)}\ , \tag{D6}$$

$$I_1^{(4D)}(p)=\left[\frac{-i}{8\pi^2}\right]\left[\frac{1}{n-4}+\frac{1}{2}\ln(p^2-i\epsilon)+\frac{1}{2}(\gamma-2-\ln 4\pi)+O(n-4)\right]\ , \tag{D7}$$

$$I_1^{(2D)}(p)=\frac{i}{\pi(p^2-i\epsilon)}\left[\frac{1}{n-2}+\frac{1}{2}\gamma+O(n-2)\right]\ , \tag{D8}$$

$$I_2(p)=\int \frac{d^nq}{(2\pi)^n}\delta(q^2)\theta(-q^0)\delta[(p-q)^2]\theta(q^0-p^0)$$
$$=\frac{\theta(-p^0)\theta(-p^2)(p^2)^{n-3}}{(n-3)2^{n-1}(2\pi)^3|\mathbf{p}|}\left[\frac{1}{(p^0+|\mathbf{p}|)^{n-3}}-\frac{1}{(p^0-|\mathbf{p}|)^{n-3}}\right]\ , \tag{D9}$$

$$I_2^{(4D)}(p)=\frac{\theta(-p^0)\theta(-p^2)}{4(2\pi)^3}\ , \tag{D10}$$

$$I_2^{(2D)}(p)=-\frac{\theta(-p^0)\theta(-p^2)}{(2\pi)^3(p^2-i\epsilon)}\ . \tag{D11}$$

APPENDIX E: VARIATIONAL CALCULUS

1. Variational equations

The variational equations are

$$g_{\mu\nu}(x)\to g_{\mu\nu}(x)+\delta g_{\mu\nu}(x)\ , \tag{E1}$$

$$\delta g^{\alpha\lambda}=-g^{\alpha\mu}g^{\lambda\nu}\delta g_{\mu\nu}\ , \tag{E2}$$

$$\delta\sqrt{-g}=\tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}\ , \tag{E3}$$

$$\delta\Gamma^{\alpha}_{\beta\delta}=\tfrac{1}{2}g^{\alpha\lambda}(\delta g_{\lambda\delta;\beta}+\delta g_{\beta\lambda;\delta}-\delta g_{\beta\delta;\lambda})\ , \tag{E4}$$

$$\delta R^{\alpha}{}_{\beta\gamma\delta}=\tfrac{1}{2}g^{\alpha\lambda}(\delta g_{\lambda\delta;\beta\gamma}+\delta g_{\beta\lambda;\delta\gamma}-\delta g_{\beta\delta;\lambda\gamma}-\delta g_{\lambda\gamma;\beta\delta}-\delta g_{\beta\lambda;\gamma\delta}+\delta g_{\beta\gamma;\lambda\delta})\ , \tag{E5}$$

$$\delta R_{\beta\delta}=\tfrac{1}{2}g^{\alpha\lambda}(\delta g_{\lambda\delta;\beta\alpha}+\delta g_{\beta\lambda;\delta\alpha}-\delta g_{\beta\delta;\lambda\alpha}-\delta g_{\lambda\alpha;\beta\delta})\ , \tag{E6}$$

$$\delta R=-R^{\beta\delta}\delta g_{\beta\delta}+g^{\beta\delta}g^{\alpha\lambda}(\delta g_{\lambda\delta;\beta\alpha}-\delta g_{\beta\delta;\lambda\alpha})\ , \tag{E7}$$

$$\delta(\Box_g\omega)=-\omega^{;\mu\nu}\delta g_{\mu\nu}-\tfrac{1}{2}\omega^{;\lambda}g^{\mu\nu}(\delta g_{\lambda\nu;\mu}+\delta g_{\mu\lambda;\nu}-\delta g_{\mu\nu;\lambda})\ . \tag{E8}$$

2. Functional differentiation

Through functional differentiation we get

$$\delta\int d^4x\sqrt{-g}\,R^2=\int d^4x\sqrt{-g}\,(\tfrac{1}{2}g^{\mu\nu}R^2-2RR^{\mu\nu}+2R^{;\mu\nu}-2g^{\mu\nu}\Box_g R)\delta g_{\mu\nu}\ , \tag{E9}$$

$$\delta\int d^4x\sqrt{-g}\,R^{\alpha\beta}R_{\alpha\beta}=\int d^4x\sqrt{-g}\,(\tfrac{1}{2}g^{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}-2R^{\mu\alpha}R_{\alpha}{}^{\nu}+2R^{\mu\alpha;\nu}{}_{\alpha}-\Box_g R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\Box_g R)\delta g_{\mu\nu}\ , \tag{E10}$$

$$\delta\int d^4x\sqrt{-g}\,R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}=\int d^4x\sqrt{-g}\,(\tfrac{1}{2}g^{\mu\nu}R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}-2R^{\mu\alpha\beta\rho}R^{\nu}{}_{\alpha\beta\rho}-4\Box_g R^{\mu\nu}+2R^{;\mu\nu}-4R^{\mu\alpha\nu\beta}R_{\alpha\beta}+4R^{\mu\alpha}R_{\alpha}{}^{\nu})\delta g_{\mu\nu}\ , \tag{E11}$$

$$\delta\int d^4x\sqrt{-g}\,R^{\alpha\beta}\omega_{;\alpha}\omega_{;\beta}=\int d^4x\sqrt{-g}\,\{g^{\mu\nu}R^{\alpha\beta}\omega_{;\alpha}\omega_{;\beta}-2R^{(\mu\alpha}\omega^{;\nu)}\omega_{;\alpha}+(\Box_g\omega)\omega^{;\mu\nu}-\tfrac{1}{2}g^{\mu\nu}[(\Box_g\omega)^2+\omega^{;\alpha\beta}\omega_{;\alpha\beta}]+[\omega^{;\mu\nu\alpha}-g^{\mu\nu}\Box_g(\omega^{;\alpha})]\omega_{;\alpha}\}\delta g_{\mu\nu}\ , \tag{E12}$$

$$\delta\int d^4x\sqrt{-g}\,R(\Box_g\omega)=\int d^4x\sqrt{-g}\,[-R^{\mu\nu}(\Box_g\omega)+R^{;(\mu}\omega^{;\nu)}-\tfrac{1}{2}g^{\mu\nu}R^{;\alpha}\omega_{;\alpha}+(\Box_g\omega)^{;\mu\nu}-g^{\mu\nu}\Box_g(\Box_g\omega)]\delta g_{\mu\nu}\ , \tag{E13}$$

$$\delta\int d^4x\sqrt{-g}\,R(\omega_{;\alpha}\omega^{;\alpha})=\int d^4x\sqrt{-g}\,\{\tfrac{1}{2}g^{\mu\nu}R(\omega_{;\alpha}\omega^{;\alpha})-R^{\mu\nu}(\omega_{;\alpha}\omega^{;\alpha})-R(\omega^{;\mu}\omega^{;\nu})+2R^{\mu\alpha\nu\beta}\omega_{;\alpha}\omega_{;\beta}+2(\omega^{;\mu\alpha}\omega^{;\nu}{}_{\alpha}-g^{\mu\nu}\omega_{;\alpha\beta}\omega^{;\alpha\beta})+2[\omega^{;\mu\nu\alpha}-g^{\mu\nu}\Box_g(\omega^{;\alpha})]\omega_{;\alpha}\}\delta g_{\mu\nu}\ , \tag{E14}$$

$$\delta\int d^4x\sqrt{-g}\,\Box_g\omega(\omega_{;\alpha}\omega^{;\alpha})=\int d^4x\sqrt{-g}\,[(2\omega^{;\alpha(\mu}\omega^{;\nu)}-g^{\mu\nu}\omega^{;\alpha\beta}\omega_{;\beta})\omega_{;\alpha}-(\Box_g\omega)\omega^{;\mu}\omega^{;\nu}]\delta g_{\mu\nu}\ , \tag{E15}$$

$$\delta\int d^4x\sqrt{-g}\,(\Box_g\omega)^2=\int d^4x\sqrt{-g}\,\{-g^{\mu\nu}[\tfrac{1}{2}(\Box_g\omega)^2+(\Box_g\omega)_{;\alpha}\omega^{;\alpha}]+2\omega^{;(\mu}(\Box_g\omega)^{;\nu)}\}\delta g_{\mu\nu}\ , \tag{E16}$$

$$\delta\int d^4x\sqrt{-g}\,(\omega_{;\alpha}\omega^{;\alpha})^2=\int d^4x\sqrt{-g}\,[\tfrac{1}{2}g^{\mu\nu}(\omega_{;\alpha}\omega^{;\alpha})^2-2\omega^{;\mu}\omega^{;\nu}(\omega_{;\alpha}\omega^{;\alpha})]\delta g_{\mu\nu}\ , \tag{E17}$$

$$\begin{aligned}\delta\int & d^4x\sqrt{-g}\,R^{\mu\nu}R_{\mu\nu}\omega(x)\\ &=\int d^4x\sqrt{-g}\,[\,(\tfrac{1}{2}g^{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}-2R^{\mu\alpha\nu\beta}R_{\alpha\beta}-R^{;\mu\nu}-\Box_g R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\Box_g R)\omega(x)+R^{;(\mu}\omega^{;\nu)}\\ &\qquad+(2R^{\alpha(\mu;\nu)}-2R^{\mu\nu;\alpha}-g^{\mu\nu}R^{;\alpha})\omega_{;\alpha}+2R^{\alpha(\mu}\omega^{;\nu)}{}_{\alpha}-R^{\mu\nu}(\Box_g\omega)-g^{\mu\nu}R^{\alpha\beta}\omega_{;\alpha\beta}]\delta g_{\mu\nu}\ ,\end{aligned} \tag{E18}$$

$$\begin{aligned}\delta\int d^4x\sqrt{-g}\,R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}\omega(x)=\int d^4x\sqrt{-g}\,[\,&(\tfrac{1}{2}g^{\mu\nu}R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}-2R^{\mu\beta\rho\sigma}R^{\nu}{}_{\beta\rho\sigma}\\ &-4R^{\mu\alpha\nu\beta}R_{\alpha\beta}+4R^{\mu\alpha}R_{\alpha}{}^{\nu}-4\Box_g R^{\mu\nu}+2R^{;\mu\nu})\omega(x)\\ &-8(R^{\mu\nu;\alpha}-R^{\alpha(\mu;\nu)})\omega_{;\alpha}-4R^{\mu\alpha\nu\beta}\omega_{;\alpha\beta}]\delta g_{\mu\nu}\ ,\end{aligned} \tag{E19}$$

$$\begin{aligned}\delta\int & d^4x\,d^4y\sqrt{-g^+(x)}\sqrt{-g^+(y)}R^+(x)R^+(y)K_1(x-y;\bar{\mu})\\ &=\int d^4x\sqrt{-g^+(x)}\left[2\int d^4y\sqrt{-g^+(y)}R^+(y)[-G^{+\mu\nu}(x)+\nabla^{\mu}_{(x)}\nabla^{\nu}_{(x)}-g^{+\mu\nu}(x)\Box_{(x)}]K_1(x-y;\bar{\mu})\right]\delta g^+_{\mu\nu}(x)\ ,\end{aligned} \tag{E20}$$

$$\begin{aligned}\delta\int & d^4x\,d^4y\sqrt{-g^+(x)}\sqrt{-g^+(y)}R^{+\alpha\beta\rho\sigma}(x)R^+_{\alpha\beta\rho\sigma}(y)K_1(x-y;\bar{\mu})\\ &=\int d^4x\sqrt{-g^+(x)}\left[\int d^4y\sqrt{-g^+(y)}[g^{+\mu\nu}(x)R^+_{\alpha\beta\rho\sigma}(x)R^{+\alpha\beta\rho\sigma}(y)\right.\\ &\qquad\left.-4R^{+\mu\alpha\nu\beta}(y)\nabla^{(x)}_{\alpha}\nabla^{(x)}_{\beta}]K_1(x-y;\bar{\mu})\right]\delta g^+_{\mu\nu}(x)\ ,\end{aligned} \tag{E21}$$

$$\begin{aligned}\delta\int & d^4x\,d^4y\sqrt{-g^+(x)}\sqrt{-g^-(y)}R^+(x)R^-(y)K_2(x-y)\\ &=\int d^4x\sqrt{-g^+(x)}\left[\int d^4y\sqrt{-g^-(y)}R^-(y)[-G^{+\mu\nu}(x)+\nabla^{\mu}_{(x)}\nabla^{\nu}_{(x)}-g^{+\mu\nu}(x)\Box_{(x)}]K_2(x-y)\right]\delta g^+_{\mu\nu}(x)\ ,\end{aligned} \tag{E22}$$

$$\begin{aligned}\delta\int & d^4x\,d^4y\sqrt{-g^+(x)}\sqrt{-g^-(y)}R^+_{\alpha\beta\rho\sigma}(x)R^{-\alpha\beta\rho\sigma}(y)K_2(x-y)\\ &=\int d^4x\sqrt{-g^+(x)}\left[\int d^4y\sqrt{-g^-(y)}[\tfrac{1}{2}g^{+\mu\nu}(x)R^+_{\alpha\beta\rho\sigma}(x)R^{-\alpha\beta\rho\sigma}(y)\right.\\ &\qquad\left.-2R^{-\mu\alpha\nu\beta}(y)\nabla^{(x)}_{\alpha}\nabla^{(x)}_{\beta}]K_2(x-y)\right]\delta g^+_{\mu\nu}(x)\ .\end{aligned} \tag{E23}$$

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