BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **28***(2) (2005), 205–213*

Semicompactness in Fuzzy Topological Spaces

R.P. CHAKRABORTY, ANJANA BHATTACHARYYA AND M.N. MUKHERJEE
Department of Pure Mathematics, University of Calcutta
35, Ballygunge Circular Road, Kolkata – 700019, India

Abstract. The paper deals with the concept of semicompactness in the generalized setting of a fuzzy topological space. We achieve a number of characterizations of a fuzzy semicompact space. The notion of semicompactness is further extended to arbitrary fuzzy sets. Such fuzzy sets are formulated in different ways and a few pertinent properties are discussed. Finally we compare semicompact fuzzy sets with some of the existing types of compact-like fuzzy sets. We ultimately show that so far as the mutual relationships among different existing allied classes of fuzzy sets are concerned, the class of semicompact fuzzy sets occupies a natural position in the hierarchy.

2000 Mathematics Subject Classification: Primary 54A40; Secondary 54D99

Key words and phrases: Fuzzy semicompact space, semi-q-nbd, fuzzy semicluster point.

1. Introduction

Barring paracompactness, there exists in the literature, a number of allied forms of compactness studied in a classical topological space. Among these, the most widely studied compact-like covering properties are almost compactness or quasi H-closedness of Porter and Thomas [18], near compactness of Singal and Mathur [20], S-closedness of Thompson [22], s-closedness of Maio and Noiri [8] and semicompactness of Dorsett [3]. The thorough investigations and the applicational aspects of these covering properties have prompted topologists to generalize these concepts (with the exception of semicompactness) to fuzzy setting. Malakar [9], in course of his study of certain functions, incidentally suggested the definition of a fuzzy semicompact space. In [12] some of interesting properties of fuzzy semicompactness are investigated. Our intention here is to go into some details towards characterizations of semicompactness for a fts. These characterizations are effected with the help of fuzzy nets, prefilterbases and similar other concepts, which comprise the deliberation in the next section.

It is seen from the literature that the process of generalization of different covering properties, akin to compactness, to fuzzy perspective was continued further ahead in the form of extension of such concepts to arbitrary fuzzy sets. This gave rise to the introduction and study of fuzzy compact, nearly compact, s-closed, S-closed, θ-rigid,

Received: April 27, 2004; *Revised:* March 9, 2005.

θ*-rigid, β-compact and almost compact sets respectively abbreviated as FC-sets, FNC-sets, FsC-sets, FSC-sets, FθR-sets, Fθ*R-sets, FβC-sets and FAC-sets (see [4], [13], [21], [14],[13], [10], [6], [16] for details). The interrelations among all these types of fuzzy sets are found to be as displayed by the following diagram:

$$\begin{array}{ccccccc} & & \text{FsC-set} & \Rightarrow & \text{F}\theta^*\text{R-set} & \Rightarrow & \text{FSC-set} \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{FC-set} & \Rightarrow & \text{FNC-set} & \Rightarrow & \text{F}\theta\text{R-set} & \Rightarrow & \text{FAC-set} \end{array}$$

where the concepts of fuzzy θ-rigidity (θ*-rigidity) and fuzzy almost compactness (S-closedness) coincide if these are considered for the whole fuzzy topological space, and no other implications than those described above, is true in general.

Our aspiration, in Section 3, would be to generalize the idea of fuzzy semicompactness to arbitrary fuzzy sets. Calling such fuzzy sets Fs*C-sets, we shall find some characterizations of such sets along with a few pertinent properties. Our ultimate purpose of initiating such fuzzy sets is fulfilled by the way of establishing the following implication diagram, more balanced than the above one.

$$\begin{array}{ccccccccc} (\text{F}\beta\text{C-set} & \Rightarrow) & \text{Fs}^*\text{C-set} & \Rightarrow & \text{FsC-set} & \Rightarrow & \text{F}\theta^*\text{R-set} & \Rightarrow & \text{FSC-set} \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ & & \text{FC-set} & \Rightarrow & \text{FNC-set} & \Rightarrow & \text{F}\theta\text{R-set} & \Rightarrow & \text{FAC-set} \end{array}$$

We construct examples, to this end, to show that a FsC-set or a FC-set need not be a Fs*C-set.

In what follows, by a fts (X, τ) or simply by a fts X we shall mean a fuzzy topological space as defined by Chang [2]. The notations $\operatorname{cl} A$, $\operatorname{int} A$ and $1 - A$ will stand respectively for the fuzzy closure [2], interior [2] and complement [21] of a fuzzy set A in a fts X. The support of a fuzzy set A in X will be denoted by $\operatorname{supp} A$ (i.e., $\operatorname{supp} A = \{x \in X : A(x) \neq 0\}$). A fuzzy point [19] in X with the singleton $\operatorname{supp}\{x\} \subset X$ and the value $\alpha (0 < \alpha \leq 1)$ will be denoted by x_α. The fuzzy sets in X taking on respectively the constant values 0 and 1 are denoted by 0_X and 1_X respectively. For two fuzzy sets A and B in X, we write $A \leq B$ if $A(x) \leq B(x)$, for each $x \in X$ [23], while we write $A \,\mathrm{q}\, B$ if A is quasi-coincident (q-coincident, for short) with B [19], i.e., if $A(x) + B(x) > 1$, for some $x \in X$. The negation of $A \,\mathrm{q}\, B$ is written as $A \bar{\mathrm{q}} B$. A fuzzy set A in X is said to be fuzzy regular open (semiopen) if $\operatorname{int} \operatorname{cl} A = A$ (resp. $U \leq A \leq \operatorname{cl} U$, for some fuzzy open set U) [1]. The complement $1 - A$ of a fuzzy semiopen set A is called semiclosed. The semiclosure of a fuzzy set A in X, to be denoted by $\operatorname{scl} A$, is the union of all those fuzzy points x_t such that for any fuzzy semiopen set U with $U(x)+t > 1$, there exists $y \in X$ with $U(y)+A(y) > 1$ [5]. A fuzzy semiopen set U is called a semi-q-nbd of a fuzzy point x_α in a fts X if $x_\alpha \,\mathrm{q}\, U$. A collection $\mathcal{F}$ of fuzzy sets in a fts X is said to form a prefilterbase [7] in X if $0_X \notin \mathcal{F}$ and for any F_1, $F_2 \in \mathcal{F}$, there exists $F_3 \in \mathcal{F}$ such that $F_3 \leq F_1 \cap F_2$. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of X if $\sup\{U(x) : U \in \mathcal{U}\} = 1$ for all $x \in X$.

2. Fuzzy semicompact spaces

We start with the definition of fuzzy semicompact spaces as suggested in [9].

Definition 2.1. [9] *A fts X is said to be a fuzzy semicompact space if every fuzzy cover of X by fuzzy semiopen sets (such a cover will be called a fuzzy semiopen cover of X) has a finite subcover.*

A straightforward consequence of the above definition yields the following alternative formulation of a fuzzy semicompact space.

Theorem 2.1. *A fts X is fuzzy semicompact iff each family $\mathcal{U}$ of fuzzy semiclosed sets in X with finite intersection property (i.e., for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, $\cap\,\mathcal{U}_0 \neq 0_X$) has a non-null intersection.*

In order to characterize fuzzy semicompact spaces by fuzzy nets and prefilterbases we need the following two definitions.

Definition 2.2. *A fuzzy point x_α in a fts X is said to be a fuzzy semi-cluster point of a prefilterbase $\mathcal{B}$ on X if $x_\alpha \leq \operatorname{scl} B$, for all $B \in \mathcal{B}$.*

Definition 2.3. [11] *A fuzzy point x_α in a fts X is said to be a fuzzy semi-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ [19] if for every semi-q-nbd W of x_α and for each $n \in D$, there exists $m \in D$ with $m \geq n$ such that $S_m \,\mathrm{q}\, W$.*

We now go on to find some characterizations of fuzzy semicompact spaces.

Theorem 2.2. *A fts X is fuzzy semicompact iff every prefilterbase on X has a fuzzy semi-cluster point.*

Proof. Let X be fuzzy semicompact and let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a prefilterbase on X having no fuzzy semi-cluster point. Let $x \in X$. Corresponding to each $n \in \mathbb{N}$ (here and hereafter $\mathbb{N}$ denotes the set of natural numbers), there exists a semi-q-nbd U_x^n of the fuzzy point $x_{1/n}$ and an $F_x^n \in \mathcal{F}$ such that $U_x^n \overline{\mathrm{q}} F_x^n$. Since $U_x^n(x) > 1-1/n$, we have $U_x(x) = 1$, where $U_x = \cup\{U_x^n : n \in \mathbb{N}\}$. Thus $\mathcal{U} = \{U_x^n : n \in \mathbb{N}, x \in X\}$ is a fuzzy semiopen cover of X. Since X is fuzzy semicompact, there exist finitely many members $U_{x_1}^{n_1}$, $U_{x_2}^{n_1}$, $\cdots$, $U_{x_k}^{n_k}$ of $\mathcal{U}$ such that $\cup_{i=1}^{k} U_{x_i}^{n_i} = 1_X$. If $F \in \mathcal{F}$ such that $F \leq F_{x_1}^{n_1} \cap F_{x_2}^{n_2} \cap \cdots \cap F_{x_k}^{n_k}$, then $F \overline{\mathrm{q}} 1_X$. Consequently, $F = 0_X$ and this contradicts the definition of a prefilterbase.

Conversely, let every prefilterbase have a fuzzy semi-cluster point. We have to show that X is fuzzy semicompact. Let $\mathcal{B} = \{F_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy semiclosed sets having finite intersection property. The set of finite intersections of members of $\mathcal{B}$ then forms a prefilterbase $\mathcal{F}$ on X. So by the given condition $\mathcal{F}$ has a fuzzy semi-cluster point. Let x_α be a fuzzy semi-cluster point of $\mathcal{F}$. So, $x_\alpha \leq \cap_{\alpha\in\Lambda} \operatorname{scl} F_\alpha = \cap_{\alpha\in\Lambda} F_\alpha$. Thus $\cap\{F : F \in \mathcal{F}\} \neq 0_X$. Hence by Theorem 2.1, X is fuzzy semicompact. □

Theorem 2.3. *A fts X is fuzzy semicompact iff every fuzzy net in X has a fuzzy semi-cluster point.*

Proof. Let X be a fuzzy semicompact space. If possible, let $\{S_n : n \in (D, \geq)\}$, where $(D, \geq)$ is a directed set, be a fuzzy net in X which has no fuzzy semi-cluster point. For each fuzzy point x_α, there is a semi-q-nbd U_{x_α} of x_α and an $n_{U_{x_\alpha}} \in D$ such that $\mathrm{S}_m \overline{\mathrm{q}} U_{x_\alpha}$ for all $m \in D$ with $m \geq n_{U_{x_\alpha}}$. Let $\mathcal{U}$ denote the collection of all such U_{x_α}, where x_α runs over all fuzzy points in X. Now the collection $\mathcal{V} = \{1-U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$

is a family of fuzzy semiclosed sets in X possessing finite intersection property. In fact, let $\mathcal{V}_0 = \{1 - \overline{q}U_{x^i_{\alpha_i}} : i = 1, 2, \ldots, m\}$ be a finite subfamily of $\mathcal{V}$. Then there exists $k \in D$ such that $k \geq n_{U_{x^1_{\alpha_1}}}, \ldots, n_{U_{x^m_{\alpha_m}}}$ and so $S_p\ \overline{q}U_{x^i_{\alpha_i}}$ for $i = 1, 2, \ldots, m$ and for all $p \geq k$ $(p \in D)$, i.e., $S_p \leq 1 - \cup^m_{i=}U_{x^i_{x_{\alpha_i}}} = \cap^m_{i=}(1 - U_{x^i_{x_{\alpha_i}}})$ for all $p \geq k$. Hence $\cap\ \mathcal{V}_0 \neq 0_X$. Since X is fuzzy semicompact, by Theorem 2.1 there exists a fuzzy point y_β in X such that

$$y_\beta \leq \cap\{1 - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\} = 1 - \cup\{U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}.$$

Thus $y_\beta \leq 1 - U_{x_\alpha}$, for all $U_{x_\alpha} \in \mathcal{U}$, and hence in particular, $y_\beta \leq 1 - U_{y_\beta}$ i.e., $y_\beta \overline{q} U_{y_\beta}$. But by construction, for each fuzzy point x_α there exists a $U_{x_\alpha} \in \mathcal{U}$ such that $x_\alpha\, q\, U_{x_\alpha}$, and we arrive at a contradiction.

To prove the converse, it suffices to prove, in view of Theorem 2.2, that every prefilterbase on X has a fuzzy semi-cluster point. Let $\mathcal{F}$ be a prefilterbase in X. As each $F \in \mathcal{F}$ is non-null, we choose a fuzzy point $x(F) \leq F$. Let $S = \{x(F) : F \in \mathcal{F}\}$. Let a relation "$\succeq$" be defined in $\mathcal{F}$ as follows:

$$F_\alpha \succeq F_\beta \quad \text{iff} \quad F_\alpha \leq F_\beta \text{ in } X, \text{ for } F_\alpha, F_\beta \in \mathcal{F}.$$

Then $(\mathcal{F}, \succeq)$ is a directed set. Now S is a fuzzy net with the directed set $(\mathcal{F}, \succeq)$ as domain. By hypothesis the fuzzy net S has a fuzzy semi-cluster point x_t $(0 < t \leq 1)$. Then for every semi-q-nbd W of x_t and for each $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \succeq \mathcal{F}$ such that $x(G)\, q\, W$. As $x(G) \leq G \leq F$, it then follows that $F\, q\, W$ for each $F \in \mathcal{F}$. Hence x_t is a fuzzy semi-cluster point of $\mathcal{F}$. □

Definition 2.4. *A fuzzy net $\{S_n : n \in (D, \succeq)\}$, where $(D, \succeq)$ is a directed set, is said to be semi convergent to a fuzzy point x_α, if for every semi-q-nbd W of x_α, there exists $m \in D$ such that $S_n\, q\, W$, for all $n \succeq m$.*

Lemma 2.1. *A fuzzy point x_α is a fuzzy semi-cluster point of a fuzzy net $\{S_n : n \in (D, \succeq)\}$, where $(D, \succeq)$ is a directed set, in a fts X iff it has a fuzzy subnet which fuzzy semi converges to x_α.*

Proof. Let x_α be a semi-cluster point of the fuzzy net $\{S_n : n \in (D, \succeq)\}$, with the directed set $(D, \succeq)$ as the domain. Let $\mathcal{W}$ denote the collection of all semi-q-nbds of x_α. Now x_α being a fuzzy semi-cluster point of the net $\{S_n : n \in (D, \succeq)\}$, for each $W \in \mathcal{W}$ there exists S_n such that $S_n\, q\, W$. Let $\mathcal{C}$ denote the set of all ordered pairs (n, W) with the above property, i.e., $n \in D$, $W \in \mathcal{W}$ and $S_n\, q\, W$. Let us define a relation "$\Box$" on $\mathcal{C}$ given by $(m, U)\Box(n, V)$ iff $m \succeq n$ in D and $U \leq V$. Then $(\mathcal{C}, \Box)$ is a directed set and it is easy to see that $T : (\mathcal{C}, \Box) \to (X, \tau)$ given by $T(m, U) = S_m$ is a fuzzy subnet of the given fuzzy net. Let W be any semi-q-nbd of x_α. Then there is an $n \in D$ such that $(n, W) \in \mathcal{C}$ and hence $S_n\, q\, W$. Now, $(m, U) \in \mathcal{C}$ and $(m, U)\Box(n, W) \Rightarrow T(m, U) = S_m\, q\, U$ and $U \leq W \Rightarrow T(m, U) qW$. Hence T semi converges to x_α. The converse is clear. □

It now follows from Theorem 2.3 and Lemma 2.1 that

Lemma 2.2. *A fts X is fuzzy semicompact iff each fuzzy net in X has a fuzzy semi convergent subnet.*

Definition 2.5. *A fuzzy point x_α in a fts X is called a complete semi accumulation point of a fuzzy set A in X iff for each semi-q-nbd U of x_α, $|\operatorname{supp} A| = |\{y \in X : A(y) + U(y) > 1\}|$, where for a subset B of X, by $|B|$ we mean, as usual, the cardinality of B.*

Theorem 2.4. *A necessary condition for a fts X to be fuzzy semicompact is that every fuzzy set A in X with $|\operatorname{supp} A| \geq N_0$ (where N_0 denotes the cardinal number of the set of integers) has a complete semi accumulation point.*

Proof. Let A be a fuzzy set in a fuzzy semicompact space X such that $|\operatorname{supp} A| \geq N_0$, and if possible, suppose A has no complete semi accumulation point in X. Then for each $x \in X$ and each $n \in \mathbb{N}$, there is a semi-q-nbd U_x^n of the fuzzy point $x_{1/n}$ (with support x and value $1/n$) such that

$$|\{x \in X : A(x) + U_x^n(x) > 1\}| < |\operatorname{supp} A|. \tag{2.1}$$

Now, since $U_x^n(x) + 1/n > 1$, it follows that $\{U_x^n : x \in X, n \in \mathbb{N}\}$ is a fuzzy cover of X by fuzzy semiopen sets. As X is fuzzy semicompact, there exists a finite subset $\{x_1, x_2, \cdots, x_m\}$ of X and finitely many positive integers $n_1,\ n_2,\ \cdots,\ n_m$ such that $\cup_{i=1}^m U_{x_i}^{n_i} = 1_X$. Now, $x \in \operatorname{supp} A \Rightarrow U_{x_k}^{n_k} = 1$, for some k $(1 \leq k \leq m)$ $\Rightarrow U_{x_k}^{n_k}(x) + A(x) > 1 \Rightarrow x \in \{y \in X : A(y) + U_{x_k}^{n_k}(y) > 1\} = A_{U_{n_k}}$ (say). As $A_{U_{n_k}} \subseteq \cup_{i=1}^m A_{U_{n_i}}$, We have

$$\operatorname{supp} A \subseteq \cup_{i=1}^m A_{U_{n_i}}. \tag{2.2}$$

But $|A_{U_{n_k}}| < |\operatorname{supp} A|$ by (2.1) for $i = 1, 2, \cdots, m$. Thus

$$\left|\cup_{i=1}^m A_{U_{n_i}}\right| = \max_{1 \leq\ i \leq m} |A_{U_{n_i}}| < |\operatorname{supp} A|.$$

Hence, by (2.2) we get

$$|\operatorname{supp} A| \leq \left|\cup_{i=1}^m A_{U_{n_i}}\right| < |\operatorname{supp} A|$$

which is a contradiction. This proves the result. □

Remark 2.1. Notice that the converse of the theorem is false which follows from the following example.

Example 2.1. Consider a fuzzy set X with the fuzzy topology $\tau = [0,1]^X$. Then the condition of the theorem is vacuously satisfied; but the fuzzy semiopen cover $\tau \backslash \{1_X\}$ of X has no finite subcover proving that the fts (X, τ) is not semicompact.

3. Fuzzy semicompact sets

Before we introduce fuzzy semicompact sets, let us recall, to make the exposition clear, the definitions of certain existing allied classes of fuzzy sets as follows.

Definition 3.1. *Let A be a fuzzy set in a fts X. A collection $\mathcal{U}$ of fuzzy sets in X is said to be a fuzzy cover of A [4] if $\sup\{U(x) : U \in \mathcal{U}\} = 1$, for all $x \in \operatorname{supp} A$. If the members of $\mathcal{U}$ are fuzzy open (resp. regular open, semiopen) in X, then $\mathcal{U}$ is called a fuzzy open (resp. regular open, semiopen) cover of A. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to have a finite (proximate, semi-proximate) subcover $\mathcal{U}_0$ for A if $\mathcal{U}_0$ is a finite subfamily of $\mathcal{U}$ and $\cup \mathcal{U}_0 \geq A$ (resp. $\cup\{\operatorname{cl} U : U \in \mathcal{U}_0\} \geq A, \{\operatorname{scl} U : U \in \mathcal{U}_0\} \geq A$).*

Definition 3.2. *A fuzzy set A in a fts X is said to be*

(a) *a fuzzy compact set or simply a FC-set [4] if every fuzzy open cover of A has a finite subcover for A,*

(b) *a fuzzy nearly compact set, or a FNC-set [13] if every fuzzy regular open cover of A has a finite subcover for A,*

(c) *a fuzzy s-closed set (FsC-set [21]) if every fuzzy semiopen cover of A has a semi-proximate subcover for A,*

(d) *a fuzzy almost compact set or a FAC-set [16] if every fuzzy open cover of A has a finite proximate subcover for A,*

(e) *a fuzzy θ-rigid set or simply a FθR-set [13] if for every fuzzy open cover $\mathcal{U}$ of A, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \leq \operatorname{int}\operatorname{cl}(\cup\mathcal{U}_0)$.*

(f) *A fuzzy θ^*-rigid or simply a Fθ^*R-set [10] if for every semiopen cover $\mathcal{U}$ of A, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \leq \operatorname{scl}(\cup\{\operatorname{scl} U : U \in \mathcal{U}_0\})$.*

We now set the following definition:

Definition 3.3. *A fuzzy set A in a fts X is said to be a fuzzy semicompact set (Fs*C-set, for short) if every fuzzy cover of A by fuzzy semiopen sets of X has a finite subcover for A.*

We would now proceed to obtain some characterizations of the above type of fuzzy sets.

Theorem 3.1. *For a fuzzy set A in a fts X, the following are equivalent:*

(a) *A is a Fs*C-set.*

(b) *For every family $\mathcal{F}$ of fuzzy semiclosed sets in X with $\cap\{F : F \in \mathcal{F}\} \cap A = 0_X$, there exists a finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$ such that $\cap\mathcal{F}_0 \overline{q} A$.*

(c) *If $\mathcal{B}$ is a prefilterbase of fuzzy semiclosed sets in X such that each element of $\mathcal{B}$ is q-coincident with A, then $(\cap\mathcal{B}) \cap A \neq 0_X$.*

Proof. (a) $\Rightarrow$ (b): Let A be a Fs*C-set in X and let $\mathcal{F}$ be a family of fuzzy semiclosed sets in X such that $\cap\{F : F \in \mathcal{F}\} \cap A = 0_X$. Then, for every $x \in \operatorname{supp} A$, $\inf\{F(x) : F \in \mathcal{F}\} = 0$, so that $\{1-F : F \in \mathcal{F}\}$ is a fuzzy semiopen cover of A. Hence there exists a finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$ such that $A \leq \cup\{1 - F : F \in \mathcal{F}_0\} \leq 1 - A$ and consequently $(\cap\mathcal{F}_0)\overline{q}A$.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): If A is not a Fs*C-set in X, then there exists a fuzzy semiopen cover $\mathcal{U}$ of A which has no finite subcover for A. So for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ there exists $x \in \operatorname{supp} A$ such that $\sup\{U(x) : U \in \mathcal{U}_0\} < A(x)$, i.e., $\inf\{1 - U(x) : U \in \mathcal{U}_0\} > 1 - A(x) \geq 0$. Thus if $\mathcal{B} = \{1 - U : U \in \mathcal{U}\}$, then finite intersections of members of $\mathcal{B}$ form a prefilterbase $\mathcal{F}$ (say) of fuzzy semiclosed sets in X for which there is no member F of $\mathcal{F}$ such that $F\overline{q}A$. In fact otherwise there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \leq 1 - \cap\{1 - U : U \in \mathcal{U}_0\} = \cup\{U : U \in \mathcal{U}_0\}$, contradicting our hypothesis. By (c) we then have $\cap\{1 - U : U \in \mathcal{U}\} \cap A \neq 0_X$ and hence there exists $x \in \operatorname{supp} A$ such that $\inf\{1 - U(x) : U \in \mathcal{U}\} > 0$, i.e., $\sup\{U(x) : U \in \mathcal{U}\} < 1$, which contradicts that U is a fuzzy cover of A. □

Theorem 3.2. *A fuzzy set A in a fts X is a Fs*C-set iff whenever $\mathcal{B}$ is a prefilterbase on X with the property that for any $F \in \mathcal{B}$ and for any fuzzy semiopen set U with $A \leq U$, $F \operatorname{q} U$ holds, then $\mathcal{B}$ has a fuzzy semi-cluster point in A.*

Proof. Let A be a Fs*C-set. If possible, let $\mathcal{B}$ be a prefilterbase with the given property, which has no fuzzy semi-cluster point in A. For each $x \in \operatorname{supp} A$, there exists a positive integer m_x such that $1/m_x < A(x)$. For any positive integer $n \geq m_x$, since $x_{1/n} \leq A$, $x_{1/n}$ is not a fuzzy semi-cluster point of $\mathcal{B}$. Hence there is a semi-q-nbd V_x^n of $x_{1/n}$ and a $B_x^n \in \mathcal{B}$ such that $V_x^n \overline{\mathrm{q}} B_x^n$. Since $V_x^n(x) + 1/n > 1$, we obtain $\sup_{n \geq m_x} V_x^n(x) = 1$.

The collection $\mathcal{U}$ of all such V_x^n for $x \in \operatorname{supp} A$ and $n \geq m_x (> 1/A(x))$, forms a fuzzy semiopen cover of A such that for each $V_x^n \in \mathcal{U}$, there exists $B_x^n \in \mathbb{B}$ with $V_x^n \overline{\mathrm{q}} B_x^n$. Since A is a fuzzy semicompact set in X, there exist a finite number of members $V_{x_1}^{n_1}, \cdots, V_{x_k}^{n_k}$ of $\mathcal{U}$ such that $A \leq \cup_{i=1}^k V_{x_i}^{n_i} = V$ (say). Let $B \in \mathcal{B}$ such that $B \leq \cup_{i=1}^k B_{x_i}$. Then V is a fuzzy semiopen set containing A such that $V \overline{q} B$.

Conversely, let $\mathcal{B}$ be a prefilterbase on X consisting of fuzzy semiclosed sets such that $\cap\{F : F \in \mathcal{B}\} \cap A = 0_X$.

It then follows that $\mathcal{B}$ has no fuzzy semi-cluster point in A. By hypothesis, there exists $F \in \mathcal{B}$ and there exists a fuzzy semiopen set U with $A \leq U$ such that $F \overline{\mathrm{q}} U$. Then $A \overline{\mathrm{q}} F$. It then follows by Theorem 3.1(c) that A is a Fs*C-set. □

The following theorem is a generalization of the sufficiency part of Theorem 2.2 for arbitrary fuzzy sets.

Theorem 3.3. *A fuzzy set A in a fts X is a Fs*C-set if every prefilterbase in A has a fuzzy semi-cluster point in A.*

Proof. If A is not a Fs*C-set, then there exists a fuzzy semiopen cover $\mathcal{U}$ of A such that for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$

$$A \not\leq \cup\{U : U \in \mathcal{U}_0\}.$$

Corresponding to each $U \in \mathcal{U}$, we define a fuzzy set F_U as follows :

$$F_U(x) = \begin{cases} \min\{1 - U(x), A(x), |A(x) - U(x)|\} & (x \in \operatorname{supp} A) \\ 0 & (x \notin \operatorname{supp} A). \end{cases}$$

For every finite collection $\{F_{U_1}, F_{U_2}, \cdots, F_{U_n}\}$ of members of $\mathcal{F} = \{F_U : U \in \mathcal{U}\}$, we have

$$\sup_{1 \leq i \leq n} U_i(x) < A(x) \leq 1, \text{ for some } x \in \operatorname{supp} A$$

so that

$$\min[A(x) - U_1(x), \cdots, A(x) - U_n(x)] > 0.$$

Thus $\cap_{i=1}^n F_{U_i} \neq 0_X$ and consequently the family $\mathcal{B}$ of finite intersections of members of $\mathcal{F}$ is a prefilterbase in A. Now for each fuzzy point $x_\alpha \leq A$, obviously there exists $U \in \mathcal{U}$ such that $x_\alpha \, \mathrm{q} \, U$. Since $F_U \overline{\mathrm{q}} U$, we have that $\mathcal{B}$ has no fuzzy semi-cluster point in A. □

As to the converse of the last theorem, we have the following result, the proof of which is somewhat parallel to the necessity part of Theorem 2.2.

Theorem 3.4. *If A is a Fs*C-set in a fts X, then every prefilterbase $\mathcal{F}$ in A, each of whose members is q-coincident with A, has a fuzzy semi-cluster point in A.*

Proof. Let $\mathcal{F}$ be a prefilterbase in A with the given property, such that $\mathcal{F}$ has no fuzzy semi-cluster point in A. Consider any $a \in \operatorname{supp} A$. Then for each positive integer n with $n \geq 1/A(a)$, as the fuzzy point $a_{1/n}(\leq A)$ is not a fuzzy semi-cluster point of $\mathcal{F}$, there exist a semi-q-nbd U_a^n of $a_{1/n}$ and an $F_a^n \in \mathcal{F}$ such that $U_a^n \overline{q} F_a^n$. As $U_a^n(a) > 1 - 1/a$, if we put $U_a = \cup\{U_a^n : n$ is a natural number with $n \geq 1/A(a)\}$, then $U_a(a) = 1$. Hence $\mathcal{U} = \{U_a^n : a \in \operatorname{supp} A, n \geq 1/A(a)\}$ is a fuzzy semiopen cover of A. As A is a Fs*C-set, there exist finitely many members $U_{a_1}^{n_1}, \cdots, U_{a_k}^{n_k}$ of $\mathcal{U}$ such that $\cup_{i=1}^k U_{a_i}^{n_i} \geq A$. Now there is an $F \in \mathcal{F}$ with $F \leq F_{a_1}^{n_1} \cap \cdots \cap F_{a_k}^{n_k}$. Then $F \overline{q} \cup_{i=1}^k U_{a_i}^{n_i}$ so that $F \overline{q} A$, and this contradicts the stated condition on members of $\mathcal{F}$. □

From the last two theorems, we obtain:

Corollary 3.1. *A fuzzy set A in a fts X is a Fs*C-set iff every prefilterbase $\mathcal{F}$ in A, each of whose members is q-coincident with A, has a fuzzy semi-cluster point in A.*

In the rest of this section, we derive a few elementary properties concerning Fs*C-sets.

Theorem 3.5. *In a fts , union of finite number of Fs*C-sets is a Fs*C-set.*

Proof. Clear. □

In order to look for the type of functions under which fuzzy semicompactness remains invariant, we recall the following definition.

Definition 3.4. [15] *A function $f : X \to Y$ is said to be a fuzzy irresolute function if $f^{-1}(V)$ is a fuzzy semiopen set in X for every fuzzy semiopen set V in Y.*

Theorem 3.6. *If A is a Fs*C-set in a fts X and $f : X \to Y$ is fuzzy irresolute then $f(A)$ is a Fs*C-set in the fts Y.*

Proof. For each fuzzy semiopen cover $\{V_\alpha : \alpha \in \Lambda\}$ of $f(A)$ in Y, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a fuzzy semiopen cover of A in X. Hence

$$A \leq \cup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha), \text{ for some finite subset } \Lambda_0 \text{ of } \Lambda.$$

Then

$$f(A) \leq f\left(\cup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha)\right) \leq f f^{-1}(\cup_{\alpha \in \Lambda_0} V_\alpha) \leq \cup_{\alpha \in \Lambda_0} V_\alpha.$$

Thus $f(A)$ is a Fs*C-set in Y. □

Corollary 3.2. *If $f : X \to Y$ is fuzzy irresolute and surjection then Y is fuzzy semicompact whenever X is fuzzy semicompact.*

Definition 3.5. [1] *A function $f : X \to Y$ is said to be fuzzy semicontinuous if $f^{-1}(V)$ is fuzzy semiopen in X for every fuzzy open set V in Y.*

Theorem 3.7. *If $f : X \to Y$ is fuzzy semicontinuous then for any fuzzy set A in X, $f(A)$ is a FC-set whenever A is a Fs*C-set.*

Proof. Clear. □

Corollary 3.3. *If $f : X \to Y$ is fuzzy semicontinuous and surjective then Y is fuzzy compact if X is fuzzy semicompact.*

Acknowledgment. The authors are thankful to the referee for certain comments towards some corrections needed in the earlier version of the paper. The second author is thankful to the CSIR, New Delhi for sponsoring the work.

References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.* **82**(1) (1981), 14–32.
[2] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* **24** (1968), 182–190.
[3] C. Dorsett, Semicompactness, semiseparation axioms, and product spaces, *Bull. Malaysian Math. Soc. (2)* **4**(1) (1981), 21–28.
[4] S. Ganguly and S. Saha, A note on compactness in fuzzy setting, *Fuzzy Sets and Systems* **34**(1) (1990), 117–124.
[5] B. Ghosh, Semi-continuous and semi-closed mappings and semi-connectedness in fuzzy setting, *Fuzzy Sets and Systems* **35**(3) (1990), 345–355.
[6] I. M. Hanafy, Fuzzy β-compactness and fuzzy β-closed spaces, *Turkish J. Math.* **28** (3) (2004), 281–293.
[7] R. Lowen, Convergence in fuzzy topological spaces, *General Topology Appl.* **10**(2) (1979), 147–160.
[8] G. Di Maio and T. Noiri, On s-closed spaces, *Indian J. Pure Appl. Math.* **18**(3) (1987), 226–233.
[9] S. Malakar, On fuzzy semi-irresolute and strongly irresolute functions, *Fuzzy Sets and Systems* **45**(2) (1992), 239–244.
[10] S. Malakar, R. P. Chakraborty and S. P. Sinha, On s-closedness and θ^*-rigidity of fuzzy sets, *J. Fuzzy Math.* **2**(1) (1994), 105–115.
[11] S. Malakar and M. N. Mukherjee, Some characterizations of S-closed fuzzy topological spaces, *J. Pure Math.* **10** (1993), 1–6.
[12] A. S. Mashhour, A. A. Allam and K. M. Abd El-Hakeim, On fuzzy semicompact spaces, *Bull. Fac. Sci. Assiut Univ. A* **16**(1) (1987), 277–285.
[13] M. N. Mukherjee and B. Ghosh, On nearly compact and θ-rigid fuzzy sets in fuzzy topological spaces, *Fuzzy Sets and Systems* **43**(1) (1991), 57–68.
[14] M. N. Mukherjee and B. Ghosh, On fuzzy S-closed spaces and FSC sets, *Bull. Malaysian Math. Soc. (2)* **12**(1) (1989), 1–14.
[15] M. N. Mukherjee and S. P. Sinha, Irresolute and almost open functions between fuzzy topological spaces, *Fuzzy Sets and Systems* **29**(3) (1989), 381–388.
[16] M. N. Mukherjee and S. P. Sinha, Almost compact fuzzy sets in fuzzy topological spaces, *Fuzzy Sets and Systems* **38**(3) (1990), 389–396.
[17] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.* **15** (1965), 961–970.
[18] J. Porter and J. Thomas, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.* **138** (1969), 159–170.
[19] P. M. Pu, and Y. M. Liu, Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.* **76**(2) (1980), 571–599.
[20] M. K. Singal and A. Mathur, On nearly compact spaces. II, *Boll. Un. Math. Ital. (4)* **9** (1974), 670–678.
[21] S. P. Sinha and S. Malakar, On s-closed fuzzy topological spaces, *J. Fuzzy Math.* **2**(1) (1994), 95–103.
[22] T. Thompson, S-closed spaces, *Proc. Amer. Math. Soc.* **60** (1976), 335–338.
[23] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.