Semicontinuity Problems in the Calculus of Variations

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Introduction

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function, and for $u \in W^{1,1}(a, b)$ set

$$F(u; a, b) = \int_a^b f(x, u(x), u'(x))\, dx.$$

In a paper of TONELLI [17] it is proved that the functional F is lower semicontinuous (lsc) in the topology of $L^\infty(a, b)$ if and only if the function f is convex in the last variable. Later, several authors generalized this result: among the many theorems obtained, in which x is allowed to belong to $\mathbb{R}^n$ and considerably less regularity on f is required, we recall particularly Theorem 12 of SERRIN [15], in which for the first time differentiability conditions on f are dropped, and the following result due to MARCELLINI & SBORDONE [11]:

If $f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *satisfies*:

(i) f *is measurable in* x, *and continuous in* (s, ξ), *and*

(ii) $0 \leqq f(x, s, \xi) \leqq g(x, |s|, |\xi|)$,

where g *is increasing in* $|s|$ *and* $|\xi|$, *and is locally summable in* x, *then the functional*

$$F(u, \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx \tag{0.1}$$

is sequentially weakly lower semicontinuous on* $W^{1,p}(\Omega)$, *with* $1 \leqq p \leqq +\infty$, *if and only if* f *is convex in* ξ.

(See also EKELAND and TEMAM [8] for the case in which f does not depend on u).

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* That is, $F(u, \Omega) \leqq \liminf F(u_n, \Omega)$ whenever $u_n \to u$ in the weak topology of $W^{1,p}(\Omega)$. When $p = \infty$, weak convergence should be replaced by weak* convergence. In what follows we shall use the abbreviation *sw l sc* for "sequential weak lower semicontinuity", or *sw* l sc* when "weak" is replaced by "weak*".

On the other hand, if we allow the function u to be vector-valued, i.e. $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, then the convexity hypothesis turns out to be sufficient, but too strong to be necessary, for F to be lsc: MORREY proved in [13] that, under strong regularity assumptions, F is *sw* l sc* on $W^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if f is quasi-convex, that is for every $s \in \mathbb{R}^m$ and for almost every $x \in \mathbb{R}^n$ the function $\xi \mapsto f(x, s, \xi)$ satisfies the condition

$$f(x, s, \xi) \cdot \operatorname{meas}(\Omega) \leqq \int_\Omega f(x, s, \xi + Dw(y))\, dy \tag{0.2}$$

for every $\xi \in \mathbb{R}^{nm}$, for every bounded open set $\Omega \subset \mathbb{R}^n$, and for every $w \in C_0^\infty(\Omega; \mathbb{R}^m)$. Although it is technically not easy to handle, this condition arises in a natural way in many problems (especially in elastostatics); moreover, it is equivalent to convexity in ξ in the case $m = 1$.

The theorem of MORREY was extended by MEYERS [12] to the semicontinuity (on $W^{k,p}(\Omega; \mathbb{R}^m)$) of functionals of the type

$$\int_\Omega f(x, u(x), \ldots, D^k u(x))\, dx, \tag{0.3}$$

always under strong continuity hypotheses.

In section II, by means of techniques basically relying on a recent theorem of LIU [10], which allows us to deduce semicontinuity on $W^{1,p}(\Omega; \mathbb{R}^m)$ from semicontinuity on $W^{1,\infty}(\Omega; \mathbb{R}^m)$, we prove the following main result (theorem [II.4]):

If $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ *satisfies*

(i) f *is measurable in* x, *continuous in* (s, ξ), *and*

(ii) $0 \leqq f(x, s, \xi) \leqq a(x) + C(|s|^p + |\xi|^p)$,
where $p \geqq 1$, *a is a non-negative locally summable function, and C is a non-negative constant,*

then the functional (0.1) *is swlsc on* $W^{1,p}(\Omega; \mathbb{R}^m)$ *if and only if* $\xi \mapsto f(x, s, \xi)$ *is quasi-convex, for every s and almost every x.*

Counterexamples valid even in the convex case show that these hypotheses are almost the minimal ones necessary to obtain a theorem of this kind.

As a particular case of our result, we deduce the weak semicontinuity on $W^{1,n}(\Omega; \mathbb{R}^n)$ of the functional

$$\int_\Omega b(x)\, |\det Du(x)|\, dx$$

(where b is non-negative, and bounded on bounded sets of $\mathbb{R}^n$). The integrand satisfies a stronger hypothesis than quasi-convexity (namely *polyconvexity*, a condition introduced and studied by BALL in [2], [3], [4], [5]), but the result does not seem to be previously known.

In the last section we prove a representation theorem for the greatest swlsc functional which is less than or equal to $\int_\Omega f(x, u(x), Du(x))\, dx$, where f is not necessarily quasi-convex. We show that, under reasonable continuity assumptions on f, this functional has the form $\int_\Omega \phi(x, u(x)\, Du(x))\, dx$, where ϕ is the greatest quasi-convex function which is less than or equal to f. A similar result has been proved by DACOROGNA [6], if f is a polyconvex function.

We remark that, using more complicated notations as in [12], [5], our results can be extended to the case of functionals of the type (0.3). For other results and additional bibliography on quasi-convexity, see the many important papers by BALL, MEYERS, and MORREY.

I. Notation and Preliminary Lemmas

If $a \in \mathbb{R}^n$, then $|a|$ is its euclidean norm; if ξ is an $m \times n$ matrix, $|\xi|$ is the norm of ξ when regarded as a vector in $\mathbb{R}^{mn}$. The Lebesgue measure of a measurable subset S of $\mathbb{R}^n$ will be denoted by meas (S).

Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leqq p \leqq +\infty$, $m \geqq 1$; we define $L^p(\Omega; \mathbb{R}^m)$ to be the collection of all m-tuples $(f^{(1)}, \ldots, f^{(m)})$ of real functions in $L^p(\Omega)$. Analogously, we say that $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ if u belongs to $L^p(\Omega; \mathbb{R}^m)$ together with its distribution derivatives $\frac{\partial u^{(i)}}{\partial x_j}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. The $m \times n$ matrix of these derivatives will be denoted by the symbol Du; $W^{1,p}(\Omega; \mathbb{R}^m)$ becomes a Banach space if it is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} = \||u|\|_{L^p(\Omega)}\, p + \||Du|\|_{L^p(\Omega)},$$

where

$$|u|\,(x) = |u(x)|, \qquad |Du|\,(x) = |Du(x)|.$$

Finally, $u \in C_0^1(\Omega; \mathbb{R}^m)$ if each $u^{(i)}$ is a C^1 function on Ω with compact support, while $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is the closure of $C_0^1(\Omega; \mathbb{R}^m)$ in the topology of $W^{1,p}(\Omega; \mathbb{R}^m)$.

Definition [I.1]. $f: \mathbb{R}^{nm} \to \mathbb{R}$ *is weakly quasi-convex if for every* $\tilde{\xi} \in \mathbb{R}^{nm}$, $\tilde{\eta} \in \mathbb{R}^m$ *and* $\tilde{\lambda} \in \mathbb{R}^n$ *the functions*

$$\lambda \mapsto f(\tilde{\xi} + \lambda \otimes \tilde{\eta}), \qquad \eta \mapsto f(\tilde{\xi} + \tilde{\lambda} \otimes \eta)$$

are convex, where $(\lambda \otimes \eta)_{ij} = \lambda_i \eta_j$.

Definition [I.2]. *A continuous function* $f: \mathbb{R}^{nm} \to \mathbb{R}$ *is quasi-convex if for every* $\tilde{\xi} \in \mathbb{R}^{nm}$, *for every open subset* Ω *of* $\mathbb{R}^n$, *and every function* $z \in C_0^1(\Omega; \mathbb{R}^m)$ *we have*

$$\text{(I.1)} \qquad \text{meas}\,(\Omega) \cdot f(\tilde{\xi}) \leqq \int_\Omega f(\tilde{\xi} + Dz(x))\, dx.$$

In what follows, if f is a real function defined in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$, we will say that f is quasi convex in ξ if there exists a set $I \subset \mathbb{R}^n$, with meas $(I) = 0$, such that for every $\tilde{x} \in \mathbb{R}^n \setminus I$ and $\tilde{s} \in \mathbb{R}^m$ the function $\xi \mapsto f(\tilde{x}, \tilde{s}, \xi)$ is quasi-convex.

To prove that a function f is quasi-convex, note that it is enough to verify (I.1) for one open set Ω, and for every $z \in C_0^\infty(\Omega; \mathbb{R}^m)$; moreover if f is quasi-convex then (I.1) holds for every $z \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$.

Quasi-convexity implies weak quasi-convexity, which in turn implies that the function locally satisfies a Lipschitz condition. If $m = 1$ or $n = 1$, then quasi-convexity is equivalent to convexity; in general, a function $f \in C^2(\mathbb{R}^{nm})$ is weakly quasi-convex if and only if for every $\xi \in \mathbb{R}^{mn}$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$ there holds

$$\sum_{i,j=1}^{n} \sum_{h,k=1}^{m} \frac{\partial^2 f}{\partial \xi_{ih}\, \partial \xi_{jk}} (\xi)\, \lambda_i \lambda_j \eta_h \eta_k \geqq 0$$

(Legendre-Hadamard condition).

The proofs of the previous remarks may be found in [2], [12], [14]. The following result ([12], Lemma 1) will be useful to disengage Definition [I.1] from the boundary condition on z.

Lemma [I.3]. *Let $f: \mathbb{R}^{nm} \to \mathbb{R}$ be quasi-convex. For every bounded open set $\Omega \subset \mathbb{R}^n$ and every sequence $(z_k) \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ which is weakly* convergent to zero, we have*

$$\operatorname{meas}(\Omega) \cdot f(\xi) \leqq \liminf_{k\to\infty} \int_\Omega f(\xi + Dz(x))\, dx$$

for every $\xi \in \mathbb{R}^{nm}$.

Definition [I.4]. *$f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ is a Carathéodory function if the following conditions are satisfied:*

for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{nm}$, $x \mapsto f(x, s, \xi)$ is measurable;

for almost all $x \in \mathbb{R}^n$, $(s, \xi) \mapsto f(x, s, \xi)$ is continuous.

The following result of SCORZA-DRAGONI ([8], page 235) characterizes the class of Carathéodory functions.

Lemma [I.5]. *A mapping $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ is a Carathéodory function if and only if for every compact set $K \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset K$, with $\operatorname{meas}(K \setminus K_\varepsilon) < \varepsilon$, such that the restriction of f to $K_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^{nm}$ is continuous.*

The next lemma may be found in [7].

Lemma [I.6]. *Let $G \subset \mathbb{R}^n$ be measurable, with $\operatorname{meas}(G) < \infty$. Assume (M_k) is a sequence of measurable subsets of G such that, for some $\varepsilon > 0$, the following estimate holds:*

$$\operatorname{meas}(M_k) \geqq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Then a subsequence (M_{k_h}) can be selected such that $\bigcap_{h \in N} M_{k_h} \neq \emptyset$.

Lemma [I.7]. *Let (ϕ_k) be a bounded sequence in $L^1(\mathbb{R}^n)$. Then to each $\varepsilon > 0$ there exists a triple $(A_\varepsilon, \delta, S)$, where A_ε is measurable and $\operatorname{meas}(A_\varepsilon) < \varepsilon$, $\delta > 0$,* *and S is an infinite subset of* $\mathbb{N}$, *such that for all* $k \in S$

$$\int_B |\phi_k(x)|\, dx < \varepsilon$$

whenever B and A_ε are disjoint and meas $(B) < \delta$.

Proof. We reason by contradiction. Hence we suppose that there exists a $\varepsilon > 0$ such that, for every $(A_\varepsilon, \delta, S)$ as in the statement of the theorem, we may choose a measurable set B, with $B \cap A_\varepsilon = \emptyset$ and meas $(B) < \delta$, and an index $k \in S$ such that

$$\int_B |\phi_k(x)|\, dx \geqq \varepsilon.$$

This implies that for every set A, with meas $(A) < \varepsilon$, and every infinite set $S \subset \mathbb{N}$, there exists both a set C, with $C \cap A = \emptyset$ and meas $(A \cup C) < \varepsilon$, and an infinite subset T of S such that

$$\int_C |\phi_k(x)|\, dx \geqq \varepsilon \quad \text{for} \quad \text{all } k \in T.$$

This will be proved later; now we show that we are led to a contradiction.

Set $A = \emptyset$, $S = \mathbb{N}$, and let C_1 and T_1 be as above. Starting from $A = C_1$ and $S = T_1$, we pass to C_2 and T_2, where $C_1 \cap C_2 = \emptyset$ and

$$\int_{C_1 \cup C_2} |\phi_k(x)|\, dx = \int_{C_1} |\phi_k(x)|\, dx + \int_{C_2} |\phi_k(x)|\, dx \geqq 2\varepsilon$$

for all $k \in T_2$. Since meas $(C_1 \cup C_2) < \varepsilon$, we may set $A = C_1 \cup C_2$, $S = T_2$, and continue with the same argument. If

$$N > \varepsilon^{-1} \sup_{k \in \mathbb{N}} \|\phi_k\|_{L^1(\mathbb{R}^n)},$$

then after N iterations we obtain the contradiction.

We return to the interrupted proof: let A and S be as stated, set $S_1 = S$ and take $\delta_1 < (\varepsilon - \text{meas}\,(A))/2$. There exist a set B_1 disjoint from A, with meas $(B_1) < \delta_1$, and an index $k_1 \in S_1$, such that

$$\int_{B_1} |\phi_{k_1}(x)|\, dx \geqq \varepsilon.$$

Applying induction, put

$$\delta_n = \frac{1}{2}\delta_{n-1},\ S_n = \{k \in S_{n-1} : k > k_{n-1}\},$$

and set

$$C = \bigcup_{h \in \mathbb{N}} B_h,\ T = \{k_h : h \in \mathbb{N}\},$$

then C and T satisfy our requirements: $C \cap A = \emptyset$, meas (C) + meas $(A) < \varepsilon$, $T \subset S$ is infinite, and

$$\int_C |\phi_k(x)|\, dx \geqq \varepsilon$$

for all $k \in T$. This completes the proof.

If $r > 0$ and $x \in \mathbb{R}^n$, set $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and $\text{meas}(B_r(x)) = \omega_n r^n$.

Definition [I.8]. *Let* $u \in C_0^\infty(\mathbb{R}^n)$. *We define*

$$(M^*u)(x) = (Mu)(x) + \sum_{i=1}^{n} (MD_i u)(x),$$

where we set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B_r(x)} |f(y)|\, dy$$

for every locally summable f.

Lemma [I.9]. *If* $u \in C_0^\infty(\mathbb{R}^n)$ *then* $M^*u \in C^0(\mathbb{R}^n)$ *and*

$$|u(x)| + \sum_{i=1}^{n} |D_i u(x)| \leqq (M^*u)(x)$$

for all $x \in \mathbb{R}^n$. *Moreover (see* [16]) *if* $p > 1$ *then*

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \leqq c(n, p)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

and if $p = 1$ *then*

$$\text{meas}\,\{x \in \mathbb{R}^n : (M^*u) \geqq \lambda\} \leqq \frac{c(n)}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}^n)}$$

for all $\lambda > 0$.

Lemma [I.10]. *Let* $u \in C_0^\infty(\mathbb{R}^n)$, *and put*

$$U(x, y) = \frac{\left|u(y) - u(x) - \sum_{i=1}^{n} D_i u(x)\,(y_i - x_i)\right|}{|y - x|}.$$

Then for every $x \in \mathbb{R}^n$ *and* $r > 0$

$$\int_{B_r(x)} U(x, y)\, dy \leqq 2\omega_n r^n (M^*u)(x).$$

The proof is contained in [10], Lemma 2. By modifying another demonstration of [10], we are also able to prove

Lemma [I.11]. *Let* $u \in C_0^\infty(\mathbb{R}^n)$ *and* $\lambda > 0$, *and set*

$$H^\lambda = \{x \in \mathbb{R}^n : (M^*u)(x) < \lambda\}.$$

Then for every $x, y \in H^\lambda$ *we have*

$$\frac{|u(y) - u(x)|}{|y - x|} \leqq c(n)\,\lambda.$$

Proof. Let $c'(n)$ be such that for every $x, y \in \mathbb{R}^n$ with $|x-y|=r$, we have

$$\text{meas}\,(B_r(x) \cap B_r(y)) > \frac{2}{c'(n)}\,\omega_n r^n. \tag{I.2}$$

For $\delta > 0$ set

$$W_r(x, \delta) = \{y \in B_r(x) \colon U(x, y) < \delta\},$$

whence by Lemma [I.10]

$$\text{meas}\,(B_r(x) \setminus W_r(x, \delta)) \leqq \frac{2}{\delta}\,\omega_n r^n (M^* u)(x).$$

If $z \in H^\lambda$, then

$$\text{meas}\,(B_r(z) \setminus W_r(z, 2c'(n)\,\lambda)) \leqq \frac{2\omega_n r^n}{2c'(n)\,\lambda}(M^* u)(z) < \frac{\omega_n r^n}{c'(n)}. \tag{I.3}$$

Let $x, y \in H^\lambda$ with $r = |x-y|$. By (I.2) and (I.3)

$$W_r(x, 2c'(n)\,\lambda) \cap W_r(y, 2c'(n)\,\lambda) \neq \emptyset.$$

Choose $\tilde{z}$ in this intersection, so that $|\tilde{z} - x| < r$, $|\tilde{z} - y| < r$. Then

$$\begin{aligned}\frac{|u(y)-u(x)|}{|y-x|} &\leqq U(y, \tilde{z}) + \sum_{i=1}^{n} [|D_i u(x)| + |D_i u(y)|] \\ &\leqq (4c'(n) + 2)\,\lambda,\end{aligned}$$

as required.

Lemma [I.12]. *Let X be a metric space, E a subspace of X, and k a positive real number. Then any k-Lipschitz mapping from E into $\mathbb{R}$ can be extended by a k-Lipschitz mapping from X into $\mathbb{R}$.*

For the proof see [8], page 298. We conclude this preliminary section by defining

$$G^\nu = \{2^{-\nu}(x+Y) \colon x \in \mathbb{Z}^n\}, \qquad \nu \in \mathbb{N},$$

where $Y = (0, 1)^n = \{y \in \mathbb{R}^n \colon 0 < y_i < 1 \text{ for } 1 \leqq i \leqq n\}$.

II. Semicontinuity Theorems

If f is a real function defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$, and if the left hand side of (II.1) makes sense, then we define (for every measurable set $S \subset \mathbb{R}^n$)

$$\int_S f(x, u(x), Du(x))\,dx = F(u, S). \tag{II.1}$$

Theorem [II.1]. *Let* $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ *satisfy:*

(II.2) *f is a Carathéodory function;*

(II.3) *f is quasi-convex in ξ;*

(II.4) $0 \leqq f(x, s, \xi) \leqq a(x) + b(s, \xi)$ *for every* $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, *and* $\xi \in \mathbb{R}^{nm}$, *where a is a non-negative locally summable function on $\mathbb{R}^n$, and $b \geqq 0$ is locally bounded on* $\mathbb{R}^m \times \mathbb{R}^{nm}$.

*Then for every open set Ω in $\mathbb{R}^n$ the functional $u \mapsto F(u, \Omega)$ is sw*lsc on* $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proof. Let us suppose first that $\Omega = (0, 1)^n$. Fix $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and $(z_k) \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $z^k \rightharpoonup 0$ (weak* convergence) in $W^{1,\infty}(\Omega; \mathbb{R}^m)$; we must prove that

$$F(u, \Omega) \leqq \liminf_{k \to \infty} F(u + z_k, \Omega).$$

Without loss of generality we may suppose $a(x) < +\infty$ for every x. Put

$$\lambda = \|u\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)} + \sup_{k \in \mathbb{N}} \|z_k\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)}$$

$$M = \sup \{b(s, \xi): |s| \leqq \lambda, |\xi| \leqq \lambda\}.$$

Now take $\varepsilon > 0$, and let $\alpha \geqq 1$ be so large that if

$$E = \{x \in \Omega: a(x) \leqq \alpha\} \setminus I$$

then

$$\operatorname{meas}(\Omega \setminus E) < \frac{\varepsilon}{M}, \qquad \int_{\Omega \setminus E} a(x)\, dx < \varepsilon.$$

By Lemma [I.5] there exists a compact set $K \subset \Omega$ such that f is continuous on $K \times \mathbb{R}^m \times \mathbb{R}^{nm}$ and

$$\operatorname{meas}(\Omega \setminus K) < \frac{\varepsilon}{\alpha + M}.$$

If we neglect sets of measure zero, then for all $\nu \in \mathbb{N}$ we can write

$$\Omega = \bigcup_{h=1}^{2^{n\nu}} Q_h^\nu$$

with $Q_h^\nu \in G^\nu$. The range $1 \leqq h \leqq 2^{n\nu}$ will be assumed henceforth, and we shall also write $\sum_h$ and $\bigcup_h$ when h ranges from 1 to $2^{n\nu}$. Define

$$(u)_h^\nu = 2^{-n\nu} \int_{Q_h^\nu} u(y)\, dy, \qquad (u)^\nu(x) = \sum_h (u)_h^\nu \chi_{Q_h^\nu}(x)$$

$$(Du)_h^\nu = 2^{-n\nu} \int_{Q_h^\nu} Du(y)\, dy, \qquad (Du)^\nu(x) = \sum_h (Du)_h^\nu \chi_{Q_h^\nu}(x).$$

Note that

$$\|(u)^\nu\|_{L^\infty(\Omega;\mathbb{R}^m)} + \|(Du)^\nu\|_{L^\infty(\Omega;\mathbb{R}^{nm})} \leqq \|u\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)},$$

and that the sequences $((u)^\nu)$ and $((Du)^\nu)$ converge pointwise a.e. to u and Du respectively.

For every ν and h fix $x_h^\nu \in Q_h^\nu \cap K \cap E$, if this set is not empty. Then

$$F(u + z_k, \Omega) \geqq F(u + z_k, K \cap E) = a_k + b_k^\nu + c_k^\nu + d^\nu + e,$$

where we put

$$a_k = \int_{K\cap E} [f(x, (u + z_k)(x), (Du + Dz_k)(x)) - f(x, u(x), (Du + Dz_k)(x))]\,dx;$$

$$b_k^\nu = \sum_h \int_{Q_h^\nu \cap K \cap E} [f(x, u(x), (Du + Dz_k)(x)) - f(x_h^\nu, (u)_h^\nu, (Du)_h^\nu + Dz_k(x))]\,dx;$$

$$c_k^\nu = \sum_h \int_{Q_h^\nu \cap K \cap E} [f(x_h^\nu, (u)_h^\nu, (Du)_h^\nu + Dz_k(x)) - f(x_h^\nu, (u)_h^\nu, (Du)_h^\nu)]\,dx;$$

$$d^\nu = \sum_h \int_{Q_h^\nu \cap K \cap E} [f(x_h^\nu, (u)_h^\nu, (Du)_h^\nu) - f(x, u(x), Du(x))]\,dx.$$

By the uniform continuity of f on the bounded sets of $K \times \mathbb{R}^n \times \mathbb{R}^{nm}$ we have $\lim_{k\to\infty} a_k = 0$. Similarly the uniform continuity of f and the pointwise convergence of $((u)^\nu)$ and $((Du)^\nu)$ imply that

$$\lim_{\nu\to\infty} d^\nu = 0, \qquad \lim_{\nu\to\infty} b_k^\nu = 0 \text{ uniformly with respect to } k.$$

Hence we may suppose that ν is large enough to ensure that $|b_k^\nu| + |d^\nu| < \varepsilon$ for all k.

Now note that

$$\begin{aligned}
&\left|\sum_h \int_{Q_h^\nu \setminus (K\cap E)} [f(x_h^\nu, (u)_h^\nu, (Du)_h^\nu + Dz_k(x)) - f(x_h^\nu, (u)_h^\nu, (Du)_h^\nu)]\,dx\right| \\
&\qquad \leqq 2 \sum_h \int_{Q_h^\nu \setminus (K\cap E)} [a(x_h^\nu) + M]\,dx \\
&\qquad \leqq 2\left[(\alpha + M)\,\mathrm{meas}\,(\Omega \setminus K) + M\,\mathrm{meas}\,(\Omega \setminus E) + \int_{\Omega\setminus E} \alpha\,dx\right] \\
&\qquad \leqq 4\varepsilon + 2 \int_{\Omega\setminus E} a(x)\,dx \leqq 6\varepsilon.
\end{aligned}$$

Applying Lemma [I.3] to each Q_h^ν, we find that

$$\liminf_{k\to\infty} c_k^\nu \geqq -6\varepsilon.$$

Finally,

$$e = F(u, K \cap E) \geqq F(u, \Omega) - 3\varepsilon.$$

As $k \to \infty$, the foregoing estimates yield

$$\liminf_{k\to\infty} F(u + z_k, \Omega) \geqq F(u, \Omega) - 10\varepsilon,$$

Since ε was arbitrary, this proves our result for the special choice of Ω noted at the beginning.

It is easy to see that the same argument applies to every hypercube Ω with edges parallel to the coordinate axes; the assertion for a generic Ω follows from the fact that the supremum of a family of lsc functions is lsc. This completes the proof.

A slight modification of the proof yields the same theorem even if f satisfies (II.2) and (II.3), and $|f|$ satisfies (II.4) (see [12], [9]). Note that if f is defined on $\Omega\times\mathbb{R}^m\times\{\xi\in\mathbb{R}^{nm}: |\xi|<r\}$ for some $r>0$ and the hypotheses of theorem [II.1] hold, then the functional $u\mapsto F(u,\Omega)$ is sw*lsc on the space of functions $u\in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})}<r$.

The inverse to theorem [II.1] is given by

Theorem [II.2]. *Let* $f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{nm}\to\mathbb{R}$ *satisfy* (II.2) *and* (II.4). *Assume the functional* $u\mapsto F(u,\Omega)$ *to be sw*lsc on* $W^{1,\infty}(\Omega;\mathbb{R}^m)$ *for every open set* $\Omega\subset\mathbb{R}^n$. *Then* f *is quasi-convex in* ξ.

Proof. We have to show that, if we fix an open set $\Omega\subset\mathbb{R}^n$, then there exists a set $I\subset\Omega$, with meas $(I)=0$, such that $\xi\mapsto f(x,s,\xi)$ is quasi-convex for every $x\in\Omega\setminus I$ and $s\in\mathbb{R}^m$. To this end, we will use only the fact that $u\mapsto F(u,\Omega)$ is lsc for that particular Ω.

By Lemma [I.5] we can choose a nondecreasing sequence (K_i) of compact sets, with meas $(\Omega\setminus K_i)<\frac{1}{i}$, such that f is continuous on each $K_i\times\mathbb{R}^m\times\mathbb{R}^{nm}$.

Define I in the following way: $x\in\Omega\setminus I$ if the following conditions are satisfied:

(II.5)
$$x\in\bigcup_{i\in\mathbb{N}}K_i;$$
$$a(x)<+\infty;$$
$$x \text{ is a Lebesgue point for } \chi_{K_i}, \text{ for every } i;^*$$
$$x \text{ is a Lebesgue point for } a\cdot\chi_{\Omega\setminus K_i}, \text{ for every } i.$$

Fix $\tilde{x}\in\Omega\setminus I$, $\tilde{s}\in\mathbb{R}^m$, $\tilde{\xi}\in\mathbb{R}^{nm}$, where clearly we may suppose $\tilde{x}=0$, and also set $u(x)=\tilde{s}+\tilde{\xi}\cdot x$, where $\tilde{\xi}$ is regarded as an $m\times n$ matrix. Let $z\in C_0^\infty(Y;\mathbb{R}^m)$, and put

$$\lambda=\|u\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)}+\|z\|_{W^{1,\infty}(Y;\mathbb{R}^m)}.$$

Define z periodically on $\mathbb{R}^n$, setting $z(x)=z(x+y)$ for every $y\in\mathbb{Z}^n$. Let $\tilde{k}$ be so large that $2^{-\tilde{k}}Y\subset\Omega$; for $k\geqq\tilde{k}$ and $\nu\in\mathbb{N}$ define

$$z_k^\nu(x)=\begin{cases}2^{-k\nu}z(2^{k\nu}x) & \text{if } x\in 2^{-k}Y\\ 0 & \text{otherwise,}\end{cases}$$

* This means that $\lim\limits_{r\to 0}\,[\text{meas}\,(B_r(x))]^{-1}\int_{B_r(x)}\chi_{K_i}(y)\,dy=1$.

so that $\|z_k^\nu\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)} \leqq \lambda$. For every k, $z_k^\nu \rightharpoonup 0$ (weak*) in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ as $\nu \to +\infty$, hence $z_k^\nu \to 0$ strongly in $L^\infty(\Omega;\mathbb{R}^m)$. Also for fixed k if we neglect sets of measure 0, then

$$2^{-k}Y = \bigcup_h Q_h^{k\nu}$$

with $Q_h^{k\nu} \in G^{k\nu}$ for $1 \leqq h \leqq 2^{n\nu}$. We denote by x_h^ν the corner of $Q_h^{k\nu}$ nearest to the origin, so that $Q_h^{k\nu} = x_h^\nu + 2^{-k\nu}Y$.

By (II.5), we may suppose that $0 \in K_i$ for all i. Choose $\varepsilon > 0$. Then there exists $\tilde{i}$ such that for $i \geqq \tilde{i}$ we have

$$\int_{\Omega \setminus K_i} [a(x) + M]\, dx < \varepsilon,$$

where

$$M = \sup\{b(s,\xi): |s| + |\xi| \leqq 2\lambda\}.$$

Let $\tilde{f}_i$ be a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$, coinciding with f on $K_i \times \{(s,\xi): |s| + |\xi| \leqq 2\lambda\} = K_i \times B_{2\lambda}$. We may also suppose that $\tilde{f}_i$ satisfies $0 \leqq \tilde{f}_i \leqq \max\limits_{K_i \times B_{2\lambda}} f$. Choose a function $\psi \in C_0^0(\Omega)$ so that

$$0 \leqq \psi(x) \leqq 1 \text{ for all } x \in \Omega,$$

$$\psi(x) = 1 \text{ for all } x \in K_i,$$

$$\int_{\Omega \setminus K_i} \psi(x)\, dx < \varepsilon / \max_{K_i \times B_{2\lambda}} f.$$

The function $f_i = \psi \tilde{f}_i$ is another continuous extension of f outside $K_i \times B_{2\lambda}$. We can split the functional $F(u + z_k^\nu, 2^{-k}Y)$ as follows:

$$F(u + z_k^\nu, 2^{-k}Y) = a^\nu + b^\nu + c^\nu,$$

where we set

$$a^\nu = \int_{2^{-k}Y} [f(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x)) - f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))]\, dx;$$

$$b^\nu = \sum_h \int_{Q_h^{k\nu}} [f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x)) - f_i(x_h^\nu, u(x_h^\nu), Du(x_h^\nu) + Dz_k^\nu(x))]\, dx;$$

$$c^\nu = \sum_h \int_{Q_h^{k\nu}} f_i(x_h^\nu, u(x_h^\nu), Du(x_h^\nu) + Dz(2^{k\nu}x))\, dx = \sum_h 2^{-nk\nu} \int_Y f_i(x_h^\nu, u(x_h^\nu), Du(x_h^\nu) + Dz(y))\, dy.$$

Our choice of f_i yields $|a^\nu| < 2\varepsilon$ for every ν and $i \geqq \tilde{i}$. Moreover since $u \in C^1(\bar{\Omega};\mathbb{R}^m)$ and f_i is uniformly continuous, we have $\lim\limits_{\nu\to\infty} b^\nu = 0$. Finally c^ν has the form of a Cauchy sum, over the cube $2^{-k}Y$, of the continuous function

$$x \mapsto \int_Y f_i(x, u(x), Du(x) + Dz(y))\, dy.$$

Hence it is convergent as $\nu \to \infty$, with

$$\lim_{\nu\to\infty} c^\nu = \int_{2^{-k}Y} \left[\int_Y f_i(x, u(x), Du(x) + Dz(y))\, dy\right] dx.$$

Combining the above three lines we have

$$\limsup_{\nu\to\infty} F(u + z_k^\nu, 2^{-k}Y) \leqq 2\varepsilon + \int_{2^{-k}Y} \left[\int_Y f_i(x, u(x), Du(x) + Dz(y))\, dy\right] dx.$$

Let ψ tend to χ_{K_i}. Since $f = f_i$ on $K_i \times B_{2\lambda}$, it follows from the dominated convergence theorem that

$$\limsup_{\nu\to\infty} F(u + z_k^\nu, 2^{-k}Y) \leqq 2\varepsilon + \int_{K_i \cap 2^{-k}Y} \left[\int_Y f(x, u(x), Du(x) + Dz(y))\, dy\right] dx.$$

By the semicontinuity of $u \mapsto F(u, \Omega)$ and the fact that $z_k^\nu \equiv 0$ on $\Omega \setminus 2^{-k}Y$

$$\begin{aligned} F(u, \Omega) &= F(u, 2^{-k}Y) + F(u, \Omega \setminus 2^{-k}Y) \\ &\leqq \liminf_{\nu\to\infty} [F(u + z_k^\nu, 2^{-k}Y) + F(u, \Omega \setminus 2^{-k}Y)]. \end{aligned}$$

Hence for $i \geqq \tilde{i}$

$$F(u, 2^{-k}Y) \leqq 2\varepsilon + \iint_{2^{-k}Y \times Y} \chi_{K_i}(x) f(x, u(x), Du(x) + Dz(y))\, dx\, dy.$$

Letting $i \to +\infty$, and using the fact that ε is arbitrary, we get

$$F(u, 2^{-k}Y) \leqq \int_{2^{-k}Y} \left[\int_Y f(x, u(x), Du(x) + Dz(y))\, dy\right] dx,$$

so that

$$2^{nk} \int_{2^{-k}Y} \left[f(x, u(x), Du(x)) - \int_Y f(x, x, u(x), Du(x) + Dz(y))\, dy\right] dx \leqq 0.$$

Call $\mu(x; u, z)$ the integrand in the square brackets; our hypotheses on the set I, and the continuity of f on $K_{\tilde{i}} \times B_{2\lambda}$, then yield

$$\begin{aligned} &\lim_{k\to\infty} 2^{nk} \int_{2^{-k}Y \cap K_{\tilde{i}}} \mu(x; u, z)\, dx \\ &= \lim_{k\to\infty} \left(2^{nk} \int_{2^{-k}Y} \chi_{K_{\tilde{i}}}(x)\, dx\right) \left([\text{meas}\,(2^{-k}Y \cap K_{\tilde{i}})]^{-1} \int_{2^{-k}Y \cap K_{\tilde{i}}} \mu(x; u, z)\, dx\right) \\ &= f(0, \tilde{s}, \tilde{\xi}) - \int_Y f(0, \tilde{s}, \tilde{\xi} + Dz(y))\, dy. \end{aligned}$$

On the other hand the integral of μ on $2^{-k}Y \setminus K_{\tilde{i}}$ is small because

$$\left| 2^{nk} \int_{2^{-k}Y \setminus K_{\tilde{i}}} \mu(x; u, z)\, dx \right| \leqq 2^{nk} \int_{2^{-k}Y \setminus K_{\tilde{i}}} [a(x) + M]\, dx$$
$$= 2^{nk} \int_{2^{-k}Y} [a(x) + M]\, \chi_{\Omega \setminus K_{\tilde{i}}}(x)\, dx,$$

which tends to zero as $k \to \infty$. These estimates show that (I.1) is satisfied on the open set Y, hence on every open set $\Omega \subset \mathbb{R}^n$. □

Note that the proof remains almost unchanged if we suppose that (II.2) holds, that $|f|$ satisfies (II.4), and that the functional $u \mapsto F(u, \Omega)$ is sw*lsc on each Dirichlet class $\tilde{u} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$, with $\tilde{u}$ a polynomial of degree one.

Remark [II.3]. *Let f satisfy* (II.2) *and* (II.4). *Assume that the functional* $u \mapsto F(u, \Omega)$ *is sw*lsc on the space of functions u in* $W^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that* $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$ (*where* $r > 0$). *Then there exists a set* $I \subset \Omega$, *with* $\text{meas}(I) = 0$, *such that for every* $\tilde{x} \in \Omega \setminus I$, $\tilde{s} \in \mathbb{R}^m$, *and* $\tilde{\xi} \in B_r(0) \subset \mathbb{R}^{nm}$, *and for every* $z \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ *such that* $\|\tilde{\xi} + Dz\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$, *we have*

$$\text{meas}(\Omega) \cdot f(\tilde{x}, \tilde{s}, \tilde{\xi}) \leqq \int_\Omega f(\tilde{x}, \tilde{s}, \tilde{\xi} + Dz(x))\, dx.$$

Theorems [II.1] and [II.2] generalize results contained in [12], [9]. Our next theorem deals with semicontinuity in $W^{1,p}$, $p \geqq 1$.

Theorem [II.4]. *Let* $1 \leqq p < +\infty$, *and assume that* $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ *satisfies* (II.2), (II.3) *and*

(II.6) $0 \leqq f(x, s, \xi) \leqq a(x) + C(|s|^p + |\xi|^p)$ *for every* $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{nm}$, *where C is a non-negative constant and a is a non-negative locally summable function on* $\mathbb{R}^n$.

Then for every open set $\Omega \subset \mathbb{R}^n$ *the functional* $u \mapsto F(u, \Omega)$ *is swlsc on* $W^{1,p}(\Omega; \mathbb{R}^m)$.

Proof. As in theorem [II.1] we may confine ourselves to a particular set Ω, say a ball. Take $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $(z_k) \subset W^{1,p}(\Omega; \mathbb{R}^m)$ with $z_k \rightharpoonup 0$ (weakly) in $W^{1,p}(\Omega; \mathbb{R}^m)$. We may suppose

$$\liminf_{k \to \infty} F(u + z_k, \Omega) = \lim_{k \to \infty} F(u + z_k, \Omega).$$

This will allow us to select subsequences without altering $\liminf_{k \to \infty} F(u + z_k, \Omega)$; hence we need not indicate subsequences, denoting all of them with the same index k.

By an extension theorem ([1], Theorem 4.26) we may assume each z_k to be defined on $\mathbb{R}^n$, with $\|z_k\|_{W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)}$ bounded uniformly with respect to k. Since $C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$ is dense in $W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$ and $u \mapsto F(u, \Omega)$ is continuous in the strong topology of $W^{1,p}(\Omega; \mathbb{R}^m)$, there exists a sequence $(w_k) \subset C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\|w_k - z_k\|_{W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)} < \frac{1}{k}, \quad |F(u + w_k, \Omega) - F(u + z_k, \Omega)| < \frac{1}{k}.$$

Hence we may assume the sequence (z_k) to be in $C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$, and to be bounded in $W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$.

Let $\eta:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous increasing function, with $\eta(0)=0$, such that for every measurable set $B\subset\Omega$

$$\int_B [a(x)+C(|u(x)|^p+|Du(x)|^p)]\,dx<\eta(\text{meas}\,(B)).$$

Fix $\varepsilon>0$, and apply Lemma [I.7] to each of the m sequences $((M^*z_k^{(i)})^p)$, $1\leqq i\leqq m$. This gives a subsequence (z_k), a set $A_\varepsilon\subset\Omega$, with $\text{meas}\,(A_\varepsilon)<\varepsilon$, and a real number $\delta>0$ such that

$$\int_B [(M^*z_k^{(i)})(x)]^p\,dx<\varepsilon$$

for all k, for $1\leqq i\leqq m$, and for every $B\subset\Omega\setminus A_\varepsilon$ with $\text{meas}\,(B)<\delta$. By Lemma [I.9] we may take $\lambda>0$ so large that for all i,k

$$\text{(II.7)}\qquad \text{meas}\,\{x\in\mathbb{R}^n:(M^*z_k^{(i)})(x)\geqq\lambda\}<\min(\varepsilon,\delta).$$

For all i,k set

$$H_{i,k}^\lambda=\{x\in\mathbb{R}^n:(M^*z_k^{(i)})(x)<\lambda\},\qquad H_k^\lambda=\bigcap_{i=1}^m H_{i,k}^\lambda.$$

Lemma [I.11] ensures that, for all $x,y\in H_k^\lambda$ and $1\leqq i\leqq m$,

$$\frac{|z_k^{(i)}(y)-z_k^{(i)}(x)|}{|y-x|}\leqq c(n)\,\lambda.$$

Let $g_k^{(i)}$ be a Lipschitz function extending $z_k^{(i)}$ outside H_k^λ, with Lipschitz constant not greater than $c(n)\,\lambda$ (Lemma [I.12]). Since H_k^λ is an open set we have

$$g_k^{(i)}(x)=z_k^{(i)}(x),\qquad Dg_k^{(i)}(x)=Dz_k^{(i)}(x)$$

for all $x\in H_k^\lambda$, and

$$\|Dg_k^{(i)}\|_{L^\infty(\mathbb{R}^n)}\leqq c(n)\,\lambda.$$

We may also assume

$$\|g_k^{(i)}\|_{L^\infty(\mathbb{R}^n)}\leqq\|z_k^{(i)}\|_{L^\infty(H_k^\lambda)}\leqq\lambda.$$

We may suppose that, at least for a subsequence,

$$g_k^{(i)}\rightharpoonup v^{(i)}\ (\text{weak*})\ \text{in}\ W^{1,\infty}(\Omega)$$

for $1\leqq i\leqq m$. Put $(g_k^{(1)},\dots,g_k^{(m)})=g_k$, $(v^{(1)},\dots,v^{(m)})=v$; we have

$$\begin{aligned}F(u+z_k,\Omega)&\geqq F(u+g_k,(\Omega\setminus A_\varepsilon)\cap H_k^\lambda)\\&=F(u+g_k,\Omega\setminus A_\varepsilon)-F(u+g_k,(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda).\end{aligned}$$

Since

$$\text{meas}\,[(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda]\leqq\sum_{i=1}^m\text{meas}\,[(\Omega\setminus A_\varepsilon)\setminus H_{i,k}^\lambda]<m\min(\varepsilon,\delta)$$

by (II.6) and by our choice of A_ε we obtain

$$F(u+g_k,(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda)\leqq 2^{p-1}\{\eta(m\varepsilon)+c(n,\Omega)\,\lambda^p\operatorname{meas}[(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda]\}$$

$$\leqq 2^{p-1}\{\eta(m\varepsilon)+c(n,\Omega)\sum_{i=1}^{m}\int_{(\Omega\setminus A_\varepsilon)\setminus H_{i,k}^\lambda}[(M^*z_k^{(i)})(x)]^p\,dx\}$$

$$\leqq 2^{p-1}\{\eta(m\varepsilon)+mc(n,\Omega)\,\varepsilon\}=O(\varepsilon).$$

Thus

$$F(u+z_k,\Omega)\geqq F(u+g_k,\Omega\setminus A_\varepsilon)-O(\varepsilon).$$

Choose an open set $\Omega'\subset\Omega$ containing $\Omega\setminus A_\varepsilon$ and such that

$$|F(u+g_k,\Omega')-F(u+g_k,\Omega\setminus A_\varepsilon)|<\varepsilon$$

(this is possible since the functions g_k are uniformly bounded in $W^{1,\infty}(\Omega;\mathbb{R}^m)$).

Applying Theorem [II.1] to the functional

$$\Gamma(w,S)=\int_S\gamma(x,w(x),Dw(x)\,dx),$$

where

$$\gamma(x,s,\xi)=f(x,u(x)+s,Du(x)+\xi),$$

we are led to

$$\lim_{k\to\infty}F(u+z_k,\Omega)\geqq\liminf_{k\to\infty}F(u+g_k,\Omega')-\varepsilon-O(\varepsilon)$$

$$\geqq F(u+v,\Omega')-\varepsilon-O(\varepsilon).$$

At least for a subsequence we may suppose that $z_k(x)\to 0$ for almost all $x\in\Omega$. Set

$$G=\{x\in\Omega: v(x)\neq 0\}$$

and

$$\tilde G=G\cap\{x\in\Omega: z_k(x)\to 0\},$$

so that $\operatorname{meas}(G)=\operatorname{meas}(\tilde G)$. Since the functions g_k are continuous and converge to v in L^∞, we have

$$g_k(x)\to v(x)$$

for all $x\in\Omega$, hence for all $x\in G$. If we now suppose

$$\operatorname{meas}(G)>(m+1)\,\varepsilon$$

we obtain a contradiction. Indeed by (II.7)

$$\operatorname{meas}(\tilde G\cap H_k^\lambda)>\varepsilon \text{ for all } k,$$

and by Lemma [I.6], for a subsequence,

$$\Big(\bigcap_{h\in\mathbb{N}}H_{k_h}^\lambda\Big)\cap\tilde G\neq\emptyset.$$

If $\tilde{x}$ belongs to this set, then

$$v(\tilde{x}) = \lim_{h\to\infty} g_{k_h}(\tilde{x}) = \lim_{h\to\infty} z_{k_h}(\tilde{x}) = 0,$$

contrary to the definition of G.

We may thus write, by the positivity of f,

$$\begin{aligned}\lim_{k\to\infty} F(u + z_k, \Omega) &\geqq F(u, \Omega' \setminus G) - O(\varepsilon) - \varepsilon \\ &\geqq F(u, \Omega) - O(\varepsilon) - \varepsilon - \eta[(m+2)\,\varepsilon],\end{aligned}$$

which concludes the proof since ε is arbitrary.

In this proof the role played by the hypothesis $f \geqq 0$ is fundamental. Indeed if (II.6) is changed to

$$|f(x, s, \xi)| \leqq a(x) + C(|s|^p + |\xi|^p),$$

and (II.2), (II.3) are satisfied, then Theorem [II.4] is false, at least for $n > 2$, but one can prove that for all $\varepsilon > 0$ the funtional $u \mapsto F(u, \Omega)$ is swlsc on $W^{1,p+\varepsilon}(\Omega; \mathbb{R}^m)$ (see [9]).

Since semicontinuity on $W^{1,p}$ implies semicontinuity on $W^{1,\infty}$, we may summarize the results of this section as follows:

Statement [II.5]. *Let* $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ *be a Carathéodory function which satisfies* (II.6) *for some* $p \geqq 1$ *[or alternately satisfies* (II.4)*]. Then the functional* $u \mapsto F(u, \Omega)$ *is swlsc on* $W^{1,p}(\Omega; \mathbb{R}^m)$ *[or is sw*lsc on* $W^{1,\infty}(\Omega; \mathbb{R}^m)$*] if and only if f is quasi-convex in* ξ.

III. A Representation Theorem

In this section, given a functional of the type (II.1) with f not necessarily quasi-convex in ξ, we deal with the problem of finding its lsc envelope on $W^{1,p}(\Omega;\mathbb{R})$, i.e. the greatest functional less than or equal to F which is swlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$. As a consequence of statement [II.5], it will suffice to treat the case $p = +\infty$.

Let $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ be a Carathéodory function satisfying (II.4). For every $r > 0$ and for every Ω bounded open set of $\mathbb{R}^n$, if $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})} \leqq r$, we define

$$\begin{aligned} F(r, u, \Omega) = \inf\{&\liminf_{k\to\infty} F(u_k, \Omega): u_k \rightharpoonup u \text{ (weak*) in } W^{1,\infty}(\Omega; \mathbb{R}^m) \\ &\text{and } \|Du_k\|_{L^\infty(\Omega;\mathbb{R}^{nm})} \leqq r\} \end{aligned}$$

$$\begin{aligned} F_0(r, u, \Omega) = \inf\{&\liminf_{k\to\infty} F(u_k, \Omega): (u_k - u) \rightharpoonup 0 \text{ (weak*) in } W_0^{1,\infty}(\Omega; \mathbb{R}^m) \\ &\text{and } \|Du_k\|_{L^\infty(\Omega;\mathbb{R}^{nm})} \leqq r\}, \end{aligned}$$

where $F(u, \Omega)$ is defined by (II.1). The argument employed in [11], Lemmas 3.3 and 4.5, leads us to the following results.

Lemma [III.1]. *If f satisfies the foregoing hypotheses, then for every $r > 0$ and every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})} < r$ there exists a function $h_u \in L^1(\Omega)$ such that*

$$F(r, u, \Omega') = F_0(r, u, \Omega') = \int_{\Omega'} h_u(x)\, dx$$

for every open set $\Omega' \subset \Omega$.

Lemma [III.2]. *Let $u_1, u_2 \in W^{1,\infty}(\Omega; \mathbb{R}^m)$, with $\|Du_i\|_{L^\infty(\Omega;\mathbb{R}^{nm})} < r$ and $\|u_i\|_{L^\infty(\Omega;\mathbb{R}^m)} < d$, $i = 1, 2$. Then for every open set $\Omega' \subset \Omega$ we have*

$$|F(r, u_1, \Omega') - F(r, u_2, \Omega')| \leqq \int_{\Omega'} \omega(x, d, 3r, \|u_1 - u_2\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)})\, dx,$$

where

$$\omega(x, d, r, \delta) = \sup\{|f(x, s_1, \xi_1) - f(x, s_2, \xi_2)| : |s_i| < d, |\xi_i| < r \text{ for } i = 1, 2, \text{ and } |s_1 - s_2| + |\xi_1 - \xi_2| < \delta\}.$$

We now use these results to prove

Lemma [III.3]. *To each $r > 0$ there exists a Carathéodory function ϕ_r defined on $\Omega \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$ such that for every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})} < r$ we have*

$$\phi_r(x, u(x), Du(x)) = h_u(x) \text{ for almost every } x \in \Omega.$$

Proof. Let $\mathscr{A}_r$ be the class of all affine functions on $\mathbb{R}^n$ with rational coefficients and with gradient less than r in norm. Also let L be the set of the points in $\mathbb{R}^n$ which are Lebesgue points for every function h_u, with $u \in \mathscr{A}_r$. For $x \in L$, $s \in \mathbb{Q}^m$, $\xi \in \mathbb{Q}^{nm}$, with $|\xi| < r$, put

$$\phi_r(x, s, \xi) = h_u(x),$$

where $u \in \mathscr{A}_r$, $u(x) = s$, $Du = \xi$. Lemma [III.2] implies that ϕ_r is continuous in (s, ξ) for almost every $x \in L$. Since $L \times \mathbb{Q}^m \times \mathbb{Q}^{nm}$ is dense in $L \times \mathbb{R}^m \times \mathbb{R}^{nm}$, we may therefore extend the definition of ϕ_r to $\Omega \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$, obtaining

$$|\phi_r(x, s_1, \xi_1) - \phi_r(x, s_2, \xi_2)| \leqq \omega(x, d, 3r, \delta)$$

or almost every $x \in \Omega$ and for $|s_i| < d$, $|\xi_i| < r$ $(i = 1, 2)$, whenever $s_1 - s_2| + |\xi_1 - \xi_2| < \delta$. This inequality yields

$$h_u(x) = \phi_r(x, u(x), Du(x))$$

for every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})} < r$ and for almost every $x \in \Omega$.

It remains to be proved that for all (s, ξ) the function $x \mapsto \phi_r(x, s, \xi)$ is measurable. Let $s_1 \in \mathbb{R}$ and let u be affine with $u(0) = s_1$, $Du = \xi$. For almost every $x \in \Omega$ we have

$$\phi_r(x, s_1 + \xi \cdot x, \xi) = h_u(x),$$

hence this function is measurable. If ψ is a simple function, i.e. $\psi(x) = \sum_{i=1}^{k} s_i \chi_{E_i}(x)$, with each E_i measurable and $E_i \cap E_j = \emptyset$ if $i \neq j$, then

$$\phi_r(x, \psi(x) + \xi \cdot x, \xi) = \sum_{i=1}^{k} \phi_r(x, s_i + \xi \cdot x, \xi)\, \chi_{E_i}(x).$$

Therefore by an approximation argument we can prove that $x \mapsto \phi_r(x, \theta(x) + \xi \cdot x, \xi)$ is measurable, for $\theta \in L^1(\Omega)$. This happens in particular if $\theta(x) = s - \xi \cdot x$, and the proof is complete.

The above lemma, together with the semicontinuity of $F(r, u, \Omega)$ and Remark [II.3], implies

Remark [III.4]. *For every* $(x, s, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm}$ *set*

$$\phi(x, s, \xi) = \lim_{\substack{r \to \infty \\ r > |\xi|}} \phi_r(x, s, \xi) = \inf_{r > |\xi|} \phi_r(x, s, \xi).$$

The function ϕ is measurable in x, upper semi-continuous in s, continuous in ξ, and quasi-convex in ξ.

Let $\tilde{x} \in \Omega$, $\tilde{s} \in \mathbb{R}^m$. Lemma [III.3] implies that for all $r > 0$ there exists a function $g_r^{(\tilde{x},\tilde{s})}$ such that

$$\int_\Omega g_r^{(\tilde{x},\tilde{s})}(Du(x))\, dx = \inf \{\liminf_{k \to \infty} \int_\Omega f(\tilde{x}, \tilde{s}, Du_k(x))\, dx : u_k \rightharpoonup u$$

$$\text{(weak*) in } W^{1,\infty}(\Omega; \mathbb{R}^m) \text{ and } \|Du_k\|_{L^\infty(\Omega;\mathbb{R}^{nm})} \leqq r\}$$

$$\text{for all } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \text{ with } \|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})} < r.$$

Put

$$g^{(\tilde{x},\tilde{s})}(\xi) = \lim_{\substack{r \to \infty \\ r > |\xi|}} g_r^{(\tilde{x},\tilde{s})}(\xi).$$

Theorem [III.5]. *For almost every $x \in \Omega$ and every $s \in \mathbb{R}^m$ the function $\xi \mapsto \phi(x, s, \xi)$ is the greatest quasi-convex function less than or equal to $\xi \mapsto f(x, s, \xi)$.*

Proof. Let $K \subset \Omega$ be a compact set such that f is continuous on $K \times \mathbb{R}^m \times \mathbb{R}^{nm}$. For $x \in K$, $s \in \mathbb{R}^m$, set

$$g_r(x, s, \xi) = g_r^{(x,s)}(\xi).$$

By the uniform continuity of f on bounded subsets of $K \times \mathbb{R}^m \times \mathbb{R}^{nm}$, g_r is continuous on $K \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$. Since K is arbitrary, g_r is defined for almost every $x \in \Omega$, and every $s \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{nm}$, with $|\xi| < r$. Moreover, Lemma [I.5] implies that g_r is a Carathéodory function, and it is quasi-convex in ξ since the same holds for all $g_r^{(x,s)}$. As we remarked after Theorem [II.1], the functional

$$G_r(u, \Omega) = \int_\Omega g_r(x, u(x), Du(x))\, dx$$

is sw*lsc on $\{u \in W^{1,\infty}(\Omega; \mathbb{R}^m) : \|Du\|_{L^\infty(\Omega;\mathbb{R}^{nm})} < r\}$.

If we set, for all x, s, ξ,

$$g(x, s, \xi) = g^{(x,s)}(\xi),$$

then the functional

$$u \mapsto \int_\Omega g(\tilde{x}, \tilde{s}, Du(x))\, dx$$

is the lsc envelope on $W^{1,\infty}(\Omega; \mathbb{R}^m)$ of the functional

$$u \mapsto \int_\Omega f(\tilde{x}, \tilde{s}, Du(x))\, dx.$$

Hence $\xi \mapsto g(\tilde{x}, \tilde{s}, \xi)$ is the greatest quasi-convex function not greater than $\xi \mapsto f(\tilde{x}, \tilde{s}, \xi)$. This implies $g \geqq \phi$. For every $r > 0$, G_r is semicontinuous, hence $G_r(u, \Omega) \leqq F(r, u, \Omega)$, and $g_r \leqq \phi_r$, whence $g \leqq \phi$. □

Note that the function ϕ does not necessarily represent the lsc envelope of $u \mapsto F(u, \Omega)$. Indeed, if ϕ is not a Carathéodory function, there is a counterexample even if f is convex in ξ (example 3.11 in [11]).

We give here some conditions which ensure that ϕ is a Carathéodory function.

Theorem [III.6]. *If either of the conditions*

(III.1) $f = f(x, \xi)$, *or*

(III.2) $|f(x, s_1, \xi) - f(x, s_2, \xi)| < \omega(x, |s_1 - s_2|)\, \beta(|\xi|)$,
where $\omega: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$ *is a Carathéodory function,* $\omega(x, 0) = 0$, *and* β *is increasing and non-negative,*

is satisfied, then ϕ *is a Carathéodory function.*

Proof. If (III.1) holds, the result follows from Remark [III.4]. Next assume that (III.2) holds. We note (see [8], Corollary 2.4) that if $\psi: \mathbb{R}^q \to \mathbb{R}^+$ is a convex function and if we set $M = \max_{|y| \leqq R} \psi(y)$, then for all $r < R$ and $y_1, y_2 \in B_r(0)$ there holds

$$|\psi(y_1) - \psi(y_2)| \leqq \frac{M}{R - r} |y_1 - y_2|.$$

Since quasi-convexity implies weak quasi-convexity, this estimate shows that if $\psi: \mathbb{R}^{nm} \to \mathbb{R}^+$ is quasi-convex then for all $r < R$ and $\xi_1, \xi_2 \in B_r(0)$ we have

$$|\psi(\xi_1) - \psi(\xi_2)| \leqq \frac{M \sqrt{\min(m, n)}}{R - r} |\xi_1 - \xi_2|,$$

where we have put $M = \max_{|\xi \leqq R|} \psi(\xi)$.

Note that ϕ almost everywhere satisfies the inequality

$$|\phi(x, s_1, \xi) - \phi(x, s_2, \xi)| < \omega(x, |s_1 - s_2|)\, \beta(|\xi|),$$

as one can see by proving the same estimate for the function g_r and then using the equality $g = \phi$.

Choose $R > 0$, and for all $x \in \Omega$ put

$$M(x) = a(x) + \sup\{b(s, \xi): |s| \leqq R, |\xi| \leqq R\}$$
$$\geqq \sup\{\phi(x, s, \xi): |s| \leqq R, |\xi| \leqq R\}.$$

For almost all $x \in \Omega$ one has, for all $r < R$, $s_1, s_2 \in B_r(0) \subset \mathbb{R}^m$, and $\xi_1, \xi_2 \in B_r(0) \subset \mathbb{R}^{nm}$,

$$|\phi(x, s_1, \xi_1) - \phi(x, s_2, \xi_2)| \leqq \frac{M(x)\sqrt{\min(m, n)}}{R - r}|\xi_1 - \xi_2| +$$
$$+ \omega(x, |s_1 - s_2|)\,\beta(R). \quad \square$$

We summarize the results of section III as follows.

Statement [III.7]. *Let* $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ *be a Carathéodory function which satisfies* (II.6) *for some* $p \geqq 1$ [*or alternately satisfies* (II.4)], *and let either one of the conditions* (III.1), (III.2) *hold. Then the lsc envelope on* $W^{1,p}(\Omega; \mathbb{R}^m)$ [*on* $W^{1,\infty}(\Omega; \mathbb{R}^m)$] *of* $u \mapsto F(u, \Omega)$ *is the functional*

$$u \mapsto \int_\Omega \phi(x, u(x), Du(x))\, dx,$$

where for almost all $x \in \Omega$ *and for all* $s \in \mathbb{R}^m$ *the function*

$$\xi \mapsto \phi(x, s, \xi)$$

is the greatest quasi-convex function which is less than or equal to $\xi \mapsto f(x, s, \xi)$.

This theorem provides an extension of the results of [6] to the case in which f depends on u as well as on x and ξ.

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