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# SEMIDISCRETE FINITE ELEMENT APPROXIMATIONS OF A LINEAR FLUID-STRUCTURE INTERACTION PROBLEM*

Q. DU†, M. D. GUNZBURGER‡, L. S. HOU§, AND J. LEE¶

**Abstract.** Semidiscrete finite element approximations of a linear fluid-structure interaction problem are studied. First, results concerning a divergence-free weak formulation of the interaction problem are reviewed. Next, semidiscrete finite element approximations are defined, and the existence of finite element solutions is proved with the help of an auxiliary, discretely divergence-free formulation. A discrete inf-sup condition is verified, and the existence of a finite element pressure is established. Strong a priori estimates for the finite element solutions are also derived. Then, by passing to the limit in the finite element approximations, the existence of a strong solution is demonstrated and semidiscrete error estimates are obtained.

**Key words.** fluid-structure interactions, finite element methods, error estimates

**AMS subject classifications.** 65M60, 76M10, 76D07, 73V05, 73C02

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## 1. Introduction.

Fluid-structure interaction problems have been extensively studied in recent years both analytically and computationally. The book [28] and the special issue [30] give accounts of the state of the art from the engineering points of view. In addition, a short discussion of the literature can be found in [10]. The references in [10] include [4, 18, 29] for fluid-structure interactions involving elementary fluids, [2, 3, 32] for fluid-structure interactions involving inviscid fluids, and [6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 20, 21, 22, 26, 27, 33, 34] for interactions between viscous, incompressible fluids and elastic solids.

In [10], we analyzed a model for the interactions between Stokesian fluids and linear elastic solids. This paper is devoted to the finite element analysis of that model. As in [10], we assume that the fluid and solid occupy two adjacent open Lipschitz domains, $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^d$, respectively, where $d = 2$ or 3 is the space dimension. We denote by $\Omega$ the entire fluid-solid region under consideration; i.e., $\Omega$ is the interior of $\overline{\Omega}_1 \cup \overline{\Omega}_2$. Let $\Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$ denote the interface between the fluid and solid, and let $\Gamma_1 = \partial\Omega_1 \setminus \Gamma_0$ and $\Gamma_2 = \partial\Omega_2 \setminus \Gamma_0$ denote the parts of the fluid and solid boundaries, respectively, excluding the interface $\Gamma_0$. For obvious reasons we assume that $\text{meas}(\Gamma_1 \cup \Gamma_2) \neq 0$.

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In the fluid region $\Omega_1$, we apply the Stokes system

$$(1.1)\qquad \begin{cases} \rho_1 \mathbf{v}_t + \nabla p - \mu_1 \nabla \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T) = \rho_1 \mathbf{f}_1 & \text{in } \Omega_1, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega_1, \\ \mathbf{v} = 0 & \text{on } \Gamma_1, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_1, \end{cases}$$

where $\mathbf{v}$ denotes the fluid velocity, $p$ the fluid pressure, $\mathbf{f}_1$ the given body force per unit mass, $\rho_1$ and $\mu_1$ the constant fluid density and viscosity, and $\mathbf{v}_0$ the given initial velocity.

In the solid region, we apply the equations of linear elasticity

$$(1.2)\qquad \begin{cases} \rho_2 \mathbf{u}_{tt} - \mu_2 \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \lambda_2 \nabla(\nabla \cdot \mathbf{u}) = \rho_2 \mathbf{f}_2 & \text{in } \Omega_2, \\ \mathbf{u} = 0 & \text{on } \Gamma_2, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{and} \quad \mathbf{u}_t|_{t=0} = \mathbf{u}_1 & \text{in } \Omega_2, \end{cases}$$

where $\mathbf{u}$ denotes the displacement of the solid, $\mathbf{f}_2$ the given loading force per unit mass, $\mu_2$ and $\lambda_2$ the Lamé constants, $\rho_2$ the constant solid density, and $\mathbf{u}_0$ and $\mathbf{u}_1$ the given initial data.

Across the *fixed* interface $\Gamma_0$ between the fluid and solid, the velocity and stress vector are continuous. Thus, we have

$$(1.3)\qquad \mathbf{u}_t = \mathbf{v} \qquad \text{on } \Gamma_0$$

and

$$(1.4)\qquad \mu_2(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \mathbf{n}_2 + \lambda_2(\nabla \cdot \mathbf{u})\mathbf{n}_2 = p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \mathbf{n}_1 \qquad \text{on } \Gamma_0,$$

where $\mathbf{n}_i$ is the outward-pointing unit normal vector along $\partial\Omega_i$, $i = 1, 2$.

The physical validity of the model (1.1)–(1.4) was explained in [10]. Previous work concerning this model include, as cited in [10], eigenmode analysis [34], homogenization [8], the one-dimensional case [11], and a numerical algorithm [13]. In [10], weak formulations for (1.1)–(1.4) were defined, and the existence of weak solutions was established. The proof for the existence result was based on Galerkin approximations using divergence-free basis functions, and the pressure term was absent in the Galerkin approximations.

The objective of this paper is to define semidiscrete finite element approximations, prove the convergence of finite element solutions, and derive error estimates for the finite element approximations. We point out that finite element basis functions in general are not divergence-free, and finite element formulations must be studied with the pressure term. The proof for the convergence of finite element solutions provides an alternative proof to that found in [10] for the existence of a weak solution; the results of this paper do not rely on those of [10] concerning the existence of a divergence-free weak solution. Moreover, the regularity and compatibility assumptions made on the data in this paper lead to a stronger solution. The details for the divergence-free Galerkin approximations of [10] and the discretely divergence-free finite element approximations are sufficiently different so that separate treatments are warranted.

A few technical aspects contained in this paper are particularly noteworthy: the finite element initial conditions are defined asymmetrically about the two subdomains $\Omega_1$ and $\Omega_2$; two inf-sup conditions are verified that facilitate the analysis of certain steady-state saddle point problems (these inf-sup conditions are also useful in dealing with approximations of mixed boundary value problems for the Stokes equations); and error estimates for a weighted $L^2$ projection onto discretely divergence-free spaces are derived.

The plan of the paper is as follows. In section 2, we recall relevant results of [10], in particular the weak formulations and the existence theorems. In section 3, we define semidiscrete finite element approximations and establish the existence of and a priori estimates for the finite element solutions. In section 4, we show the convergence of finite element solutions and derive error estimates.

**2. Notations and results concerning divergence-free weak formulations.** In this section we will recall the notation, weak formulations, and existence results of [10].

Throughout this paper, $C$ denotes a positive constant, depending on the domains $\Omega$, $\Omega_1$, and $\Omega_2$, whose meaning and value changes with context. $H^s(\mathcal{D})$, $s \in \mathbb{R}$, denotes the standard Sobolev space of order $s$ with respect to the set $\mathcal{D}$ equipped with the standard norm $\|\cdot\|_{s,\mathcal{D}}$. Vector-valued Sobolev spaces are denoted by $\mathbf{H}^s(\mathcal{D})$, with norms still denoted by $\|\cdot\|_{s,\mathcal{D}}$. $H^1_0(\mathcal{D})$ denotes the space of functions belonging to $H^1(\mathcal{D})$ that vanish on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$; $\mathbf{H}^1_0(\mathcal{D})$ denotes the vector-valued counterpart.

We will use the following $L^2$ inner product notations on scalar and vector-valued $L^2$ spaces:

$$[p,q]_{\mathcal{D}} = \int_{\mathcal{D}} pq\, d\mathcal{D} \quad \forall\, p,q \in L^2(\mathcal{D}), \qquad [\mathbf{u},\mathbf{v}]_{\mathcal{D}} = \int_{\mathcal{D}} \mathbf{u}\cdot\mathbf{v}\, d\mathcal{D} \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\mathcal{D}),$$

where the spatial set $\mathcal{D}$ is $\Omega$ or $\Gamma_0$ or $\Omega_i$, for $i = 1, 2$.

We introduce the function spaces

$$X_i = [\mathbf{H}^1_0(\Omega)]|_{\Omega_i} \quad \text{with the norm } \|\cdot\|_{X_i} = \|\cdot\|_{1,\Omega_i}, \quad i = 1,2,$$

and

$$\boldsymbol{\Psi} = \{\boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega) : \operatorname{div}\boldsymbol{\eta} = 0 \text{ in } \Omega_1\} \quad \text{with the norm } \|\cdot\|_{1,\Omega}.$$

We define the weighted $\mathbf{L}^2(\Omega)$ inner product $[[\cdot,\cdot]]$ by

$$[[\boldsymbol{\xi},\boldsymbol{\eta}]] = [\rho_1\boldsymbol{\xi},\boldsymbol{\eta}]_{\Omega_1} + [\rho_2\boldsymbol{\xi},\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\, \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbf{L}^2(\Omega). \tag{2.1}$$

We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between $\boldsymbol{\Psi}^*$ and $\boldsymbol{\Psi}$ that is generated from the weighted $\mathbf{L}^2(\Omega)$ inner product $[[\cdot,\cdot]]$. The norm on the dual space $\boldsymbol{\Psi}^*$ is defined in the conventional manner:

$$\|\mathbf{g}\|_{\boldsymbol{\Psi}^*} = \sup_{\boldsymbol{\eta}\in\boldsymbol{\Psi},\, \|\boldsymbol{\eta}\|_{1,\Omega}\le 1} |\langle\langle \mathbf{g},\boldsymbol{\eta}\rangle\rangle| \qquad \forall\, \mathbf{g} \in \boldsymbol{\Psi}^*.$$

We define the bilinear forms

$$a_1[\mathbf{u},\mathbf{v}] = 2\int_{\Omega_1} \mu_1(\nabla\mathbf{u}+\nabla\mathbf{u}^T):(\nabla\mathbf{v}+\nabla\mathbf{v}^T)\,d\Omega \qquad \forall\,\mathbf{u},\mathbf{v}\in X_1,$$

$$a_2[\mathbf{u},\mathbf{v}] = \int_{\Omega_2} \Big\{2\mu_2(\nabla\mathbf{u}+\nabla\mathbf{u}^T):(\nabla\mathbf{v}+\nabla\mathbf{v}^T)+\lambda_2(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\Big\}\,d\Omega \;\; \forall\mathbf{u},\mathbf{v}\in X_2,$$

$$b[\mathbf{v},q] = -\int_{\Omega_1} q\nabla\cdot\mathbf{v}\,d\Omega \qquad \forall\,\mathbf{v}\in X_1,\ \forall\,q\in L^2(\Omega_1).$$

It can be verified with the help of Korn's inequalities [31, pp. 31, 120] that for $i=1,2$,

$$a_i[\boldsymbol{\eta},\boldsymbol{\eta}] \ge k_i\|\boldsymbol{\eta}\|^2_{1,\Omega_i} \quad \forall\,\boldsymbol{\eta}\in X_i \qquad \text{if meas }(\Gamma_i)\neq 0 \tag{2.2}$$

and

$$[\boldsymbol{\eta},\boldsymbol{\eta}]_{\Omega_i} + a_i[\boldsymbol{\eta},\boldsymbol{\eta}] \ge k_i\|\boldsymbol{\eta}\|^2_{1,\Omega_i} \quad \forall\,\boldsymbol{\eta}\in X_i \qquad \text{if meas }(\Gamma_i)= 0. \tag{2.3}$$

The bounded bilinear form $b[\cdot,\cdot]$ was shown in [10] to satisfy the inf-sup conditions

$$\inf_{q\in L^2(\Omega_1)} \sup_{\boldsymbol{\eta}\in\mathbf{H}^1_0(\Omega)} \frac{b[\boldsymbol{\eta},q]}{\|\boldsymbol{\eta}\|_{1,\Omega}\|q\|_{0,\Omega_1}} \ge k_b \tag{2.4}$$

and

$$\inf_{q\in L^2(\Omega_1)} \sup_{\mathbf{v}\in X_1} \frac{b[\mathbf{v},q]}{\|\mathbf{v}\|_{1,\Omega_1}\|q\|_{0,\Omega_1}} \ge k_b, \tag{2.5}$$

where $k_b>0$ is a constant.

For functions that also depend on time, we introduce the space $L^2(0,T;X)$ that consists of $L^2$-integrable functions from $[0,T]$ into the space $X$ and which is equipped with the norm

$$\left(\int_0^t \|f\|_X^2\,dt\right)^{1/2}.$$

Similarly, we introduce the space $C(0,T;X)$ that consists of continuous functions from $[0,T]$ into the space $X$ and which is equipped with the norm

$$\sup_{t\in[0,T]} \|f\|_X.$$

The divergence-free weak formulation for (1.1)–(1.4) was defined in [10] as follows. Given

$$\begin{cases} \mathbf{f}_1\in C([0,T];\mathbf{L}^2(\Omega_1)), \quad \mathbf{f}_2\in C([0,T];\mathbf{L}^2(\Omega_2)), \quad \mathbf{u}_0\in X_2, \\ \mathbf{v}_0\in X_1, \quad \operatorname{div}\mathbf{v}_0=0 \text{ in } \Omega_1, \quad \mathbf{u}_1\in X_2, \quad \mathbf{v}_0|_{\Gamma_0}=\mathbf{u}_1|_{\Gamma_0}, \end{cases} \tag{2.6}$$

seek a pair $(\mathbf{v},\mathbf{u})$ such that

$$(\mathbf{v},\mathbf{u})\in L^2(0,T;X_1)\times L^2(0,T;X_2), \qquad \operatorname{div}\mathbf{v}=0, \tag{2.7}$$

$$\begin{aligned} &\frac{d}{dt}\Big(\rho_1[\mathbf{v},\boldsymbol{\eta}]_{\Omega_1}+\rho_2[\partial_t\mathbf{u},\boldsymbol{\eta}]_{\Omega_2}\Big)+a_1[\mathbf{v},\boldsymbol{\eta}]+a_2[\mathbf{u},\boldsymbol{\eta}] \\ &\qquad = \rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1}+\rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\,\boldsymbol{\eta}\in\boldsymbol{\Psi}, \end{aligned} \tag{2.8}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \qquad \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}_t|_{t=0} = \mathbf{u}_1, \tag{2.9}$$

and

$$\int_0^t \mathbf{v}(s)|_{\Gamma_0}\, ds = \mathbf{u}(t)|_{\Gamma_0} - \mathbf{u}_0|_{\Gamma_0} \quad \text{a.e. } t. \tag{2.10}$$

The "natural" interface condition (1.4) is built into (2.8), and the "essential" interface condition (1.3) is enforced weakly in the sense of (2.10).

By defining

$$\boldsymbol{\xi} = \begin{cases} \mathbf{v} & \text{in } \Omega_1, \\ \mathbf{u}_t & \text{in } \Omega_2, \end{cases} \qquad \boldsymbol{\xi}_0 = \begin{cases} \mathbf{v}_0 & \text{in } \Omega_1, \\ \mathbf{u}_1 & \text{in } \Omega_2, \end{cases} \qquad \text{and} \quad \mathbf{f} = \begin{cases} \mathbf{f}_1 & \text{in } \Omega_1, \\ \mathbf{f}_2 & \text{in } \Omega_2, \end{cases} \tag{2.11}$$

(2.7)–(2.10) was conveniently recast in [10] into the following equivalent, auxiliary divergence-free weak formulation: seek a $\boldsymbol{\xi}$ such that

$$\begin{aligned} &\boldsymbol{\xi} \in L^2(0,T;\mathbf{L}^2(\Omega)), \qquad \partial_t\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{\Psi}^*), \\ &\boldsymbol{\xi}|_{\Omega_1} \in L^2(0,T;X_1), \qquad \operatorname{div}\boldsymbol{\xi}|_{\Omega_1} = 0, \qquad \textstyle\int_0^t \boldsymbol{\xi}(s)|_{\Omega_2}\, ds \in L^2(0,T;X_2), \end{aligned} \tag{2.12}$$

(2.13)
$$\langle\langle \boldsymbol{\xi}_t, \boldsymbol{\eta}\rangle\rangle + a_1[\boldsymbol{\xi},\boldsymbol{\eta}] + a_2\left[\int_0^t \boldsymbol{\xi}(s)\, ds, \boldsymbol{\eta}\right] = [[\mathbf{f},\boldsymbol{\eta}]] - a_2[\mathbf{u}_0,\boldsymbol{\eta}] \quad \forall\, \boldsymbol{\eta} \in \boldsymbol{\Psi}, \text{ a.e. } t,$$

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \tag{2.14}$$

and

$$\int_0^t (\boldsymbol{\xi}(s)|_{\Omega_1})\Big|_{\Gamma_0} ds = \int_0^t (\boldsymbol{\xi}(s)|_{\Omega_2})\Big|_{\Gamma_0} ds \qquad \text{a.e. } t. \tag{2.15}$$

The existence and uniqueness of a solution for the auxiliary problem (2.12)–(2.15) was proved in [10].

THEOREM 2.1. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2$, *and* $\mathbf{u}_0$ *satisfy* (2.6). *Then, there exists a unique solution* $\boldsymbol{\xi}$ *for* (2.12)–(2.15). *Moreover,* $\boldsymbol{\xi}$ *satisfies the estimates*

$$\begin{aligned} &\|\boldsymbol{\xi}(t)\|_{0,\Omega}^2 + \|\boldsymbol{\xi}\|_{L^2(0,T;\mathbf{H}^1(\Omega_1))}^2 + \left\|\int_0^t \boldsymbol{\xi}(s)\, ds\right\|_{\mathbf{H}^1(\Omega_2)}^2 \\ &\quad \le Ce^{CT}(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2) \quad \forall\, t \in [0,T] \end{aligned} \tag{2.16}$$

*and*

$$\begin{aligned} &\|\partial_t\boldsymbol{\xi}\|_{L^2(0,T;\boldsymbol{\Psi}^*)}^2 \\ &\quad \le Ce^{CT}(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{u}_0\|_{1,\Omega_2}^2 + \|\mathbf{v}_0\|_{1,\Omega_1}^2 + \|\mathbf{u}_1\|_{1,\Omega_2}^2). \end{aligned} \tag{2.17}$$

Using relation (2.11) reversely, i.e., setting $\mathbf{v} = \boldsymbol{\xi}|_{\Omega_1}$ and $\mathbf{u} = \mathbf{u}_0 + \int_0^t \boldsymbol{\xi}(s)|_{\Omega_2}\, ds$, Theorem 2.1 immediately yields the following existence result for (2.7)–(2.10).

THEOREM 2.2. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, *and* $\mathbf{u}_1$ *satisfy* (2.6). *Then, there exists a unique solution* $(\mathbf{v}, \mathbf{u}) \in L^2(0,T;X_1) \times L^2(0,T;X_2)$ *for* (2.7)–(2.10), *where* (2.8) *holds in the sense of distributions on* $(0,T)$. *Moreover,*

$$\|\mathbf{v}(t)\|^2_{0,\Omega_1} + \|\mathbf{u}_t(t)\|^2_{0,\Omega_2} + \|\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|\mathbf{u}(t)\|^2_{\mathbf{H}^1(\Omega_2)} \le Ce^{CT}(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{v}_0\|^2_{0,\Omega_1} + \|\mathbf{u}_1\|^2_{0,\Omega_2}) \quad \forall\, t \in [0,T]. \tag{2.18}$$

The existence of a stronger solution and an $L^2$-integrable pressure was also established in [10].

THEOREM 2.3. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, *and* $\mathbf{u}_1$ *satisfy* (2.6) *and*

$$\partial_t \mathbf{f}_i \in L^2(0,T;\mathbf{L}^2(\Omega_i)),\ i=1,2, \quad \mathbf{v}_0 \in \mathbf{H}^2(\Omega_1), \quad \mathbf{u}_1 \in \mathbf{H}^1(\Omega_2), \quad \mathbf{u}_0 \in \mathbf{H}^2(\Omega_2).$$

*Assume further that there exists a* $p_0 \in H^1(\Omega_1)$ *such that*

$$(p_0\mathbf{n}_1 - \mu_1(\nabla \mathbf{v}_0 + \nabla \mathbf{v}_0^T)\cdot \mathbf{n}_1)|_{\Gamma_0} = (\mu_2(\nabla \mathbf{u}_0 + \nabla \mathbf{u}_0^T)\cdot \mathbf{n}_2 + (\lambda_2+\mu_2)(\operatorname{div} \mathbf{u}_0)\mathbf{n}_2)|_{\Gamma_0},$$

*where* $\mathbf{n}_i$ *denotes the outward-pointing normal along* $\partial\Omega_i$. *Then, the solution* $(\mathbf{v}, \mathbf{u})$ *to* (2.7)–(2.10) *satisfies*

$$\mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\Omega_1)) \cap L^2(0,T;X_1), \quad \mathbf{u} \in L^\infty(0,T;X_2),$$

$$\mathbf{v}_t \in L^\infty(0,T;\mathbf{L}^2(\Omega_1)) \cap L^2(0,T;X_1), \quad \mathbf{u}_t \in L^\infty(0,T;X_2), \quad \mathbf{u}_{tt} \in L^\infty(0,T;\mathbf{L}^2(\Omega_2)),$$

*and*

$$\|\partial_t \mathbf{v}(t)\|^2_{0,\Omega_1} + \|\partial_{tt}\mathbf{u}(t)\|^2_{0,\Omega_2} + \|\partial_t \mathbf{v}\|^2_{L^2(0,T;X_1)} + \|\partial_t \mathbf{u}(t)\|^2_{1,\Omega_2)} \le Ce^{CT}(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{1,\Omega_2}) \ \forall t \in [0,T].$$

*Furthermore, there exists a unique* $p \in L^2(0,T;L^2(\Omega_1))$ *such that*

$$\rho_1[\mathbf{v}_t, \boldsymbol{\eta}]_{\Omega_1} + b[\boldsymbol{\eta}, p] + a_1[\mathbf{v}, \boldsymbol{\eta}] + \rho_2[\mathbf{u}_{tt}, \boldsymbol{\eta}]_{\Omega_2} + a_2[\mathbf{u}, \boldsymbol{\eta}] = \rho_1[\mathbf{f}_1, \boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}]_{\Omega_2} \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega),\ a.e.\ t \tag{2.19}$$

*and*

$$\|p\|_{L^2(0,T;L^2(\Omega_1))} \le Ce^{CT}(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{1,\Omega_2}).$$

**3. Semidiscrete finite element approximations.** In this section we will define semidiscrete finite element approximations, prove the existence of finite element solutions on discretely divergence-free spaces and derive energy estimates, and establish the existence of a discrete pressure by verifying inf-sup conditions for finite element space pairs.

As alluded to previously, finite element solutions in general are not divergence-free, and finite element formulations should include the pressure term. Of course, the corresponding continuous weak formulation should also contain the pressure term. Such a weak formulation requires additional regularity on $\mathbf{v}_t$ and $\mathbf{u}_{tt}$. The continuous weak formulation we consider is as follows: given $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2$, and $\mathbf{u}_0$ satisfying (2.6), seek a triplet $(\mathbf{v}, p, \mathbf{u})$ such that

$$(\mathbf{v}, p, \mathbf{u}) \in L^2(0,T;X_1) \times L^2(0,T;L^2(\Omega_1)) \times L^2(0,T;X_2), \tag{3.1}$$

$$\mathbf{v}_t \in L^2(0,T;\mathbf{L}^2(\Omega_1)), \qquad \mathbf{u}_t \in L^2(0,T;X_2), \qquad \mathbf{u}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega_2)), \tag{3.2}$$

$$\begin{aligned}&\rho_1[\mathbf{v}_t,\boldsymbol{\eta}]_{\Omega_1} + b[\boldsymbol{\eta},p] + a_1[\mathbf{v},\boldsymbol{\eta}] + \rho_2[\mathbf{u}_{tt},\boldsymbol{\eta}]_{\Omega_2} + a_2[\mathbf{u},\boldsymbol{\eta}]\\ &\quad = \rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\,\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega),\ \text{a.e. } t \in [0,T],\end{aligned} \tag{3.3}$$

$$b[\mathbf{v},q] = 0 \qquad \forall\, q \in L^2(\Omega_1),\ \text{a.e. } t\in[0,T], \tag{3.4}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \qquad \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}_t|_{t=0} = \mathbf{u}_1, \tag{3.5}$$

$$\mathbf{v}|_{\Gamma_0} = \mathbf{u}_t|_{\Gamma_0} \quad \text{a.e. } t. \tag{3.6}$$

We will define finite element approximations to (3.3)–(3.6). By showing the convergence of finite element solutions, we establish the existence of a solution for (3.1)–(3.6). For reasons connected with the derivation of the regularity results (3.2), we will define finite element initial conditions in a nonstandard manner.

**3.1. Finite element discretization.** In what follows we assume that $\Omega_1$ and $\Omega_2$ are two-dimensional polygons or three-dimensional polyhedra. Let $h$ denote a discretization parameter associated with the triangulation $\mathcal{T}^h(\Omega)$ of $\Omega$. We assume that elements of $\mathcal{T}^h$ do not cross the interface $\Gamma_0$. We assume that the triangulation $\mathcal{T}^h$ consists of triangular elements in two dimensions or tetrahedral elements in three dimensions, though our results can be extended to other types of triangulations. Furthermore, we assume that there exists a triangulation $\mathcal{T}^{h_0}(\Omega)$ such that, for each $h < h_0$, $\mathcal{T}^h(\Omega)$ is a refinement of $\mathcal{T}^{h_0}(\Omega)$.

For each $h$, we choose $X^h \subset \mathbf{C}(\overline{\Omega}) \cap \mathbf{H}_0^1(\Omega)$ and $Q_1^h \subset L^2(\Omega_1)$ as finite element subspaces over the triangulation $\mathcal{T}^h(\Omega)$. We assume that $X^h$ contains piecewise linear functions. We set

$$X_i^h = X^h|_{\Omega_i}, \quad i = 1,2,$$

and

$$\boldsymbol{\Psi}^h = \{\boldsymbol{\eta}_h \in X^h : b[\boldsymbol{\eta}_h, q_h] = 0\ \forall\, q_h \in Q_1^h\}.$$

We assume that the finite element spaces $X_1^h$, $X_2^h$, and $Q_1^h$ satisfy the standard approximation properties [5]; i.e., there exist an integer $k > 0$ and constant $C > 0$ such that

$$\inf_{\mathbf{v}^h \in X_i^h} \|\mathbf{v} - \mathbf{v}^h\|_{0,\Omega_i} \le Ch^{r+1}\|\mathbf{v}\|_{r+1,\Omega_i} \qquad \forall\,\mathbf{v} \in \mathbf{H}^{r+1}(\Omega_i) \cap X_i,\ r \in [0,k], \tag{3.7}$$

$$\inf_{\mathbf{v}^h \in X_i^h} \|\mathbf{v} - \mathbf{v}^h\|_{1,\Omega_i} \le Ch^{r}\|\mathbf{v}\|_{r+1,\Omega_i} \qquad \forall\,\mathbf{v} \in \mathbf{H}^{r+1}(\Omega_i) \cap X_i,\ r \in [0,k], \tag{3.8}$$

and

$$\inf_{q^h \in Q_1^h} \|q - q^h\|_{0,\Omega_1} \le Ch^{r}\|p\|_{r,\Omega_1} \qquad \forall\, q \in H^r(\Omega_1),\ r \in [0,k]. \tag{3.9}$$

Also, $X^h$ satisfies the approximation properties

$$\inf_{\boldsymbol{\eta}^h \in X^h} \|\boldsymbol{\eta} - \boldsymbol{\eta}^h\|_{0,\Omega} \le Ch^{r+1} \|\boldsymbol{\eta}\|_{r+1,\Omega} \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^{r+1}(\Omega) \cap \mathbf{L}^2(\Omega),\ r \in [0,k], \tag{3.10}$$

and

$$\inf_{\boldsymbol{\eta}^h \in X^h} \|\boldsymbol{\eta} - \boldsymbol{\eta}^h\|_{1,\Omega_i} \le Ch^{r} \|\boldsymbol{\eta}\|_{r+1,\Omega} \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^{r+1}(\Omega) \cap \mathbf{H}^1_0(\Omega),\ r \in [0,k]. \tag{3.11}$$

We assume that the finite element pair $\{\widetilde{X}^h_1, M^h\} \equiv \{X^h_1 \cap \mathbf{H}^1_0(\Omega_1), Q^h_1 \cap L^2_0(\Omega_1)\}$ satisfies the discrete inf-sup condition

$$\inf_{q^h \in M^h(\Omega_1)} \sup_{\mathbf{v}^h \in \widetilde{X}^h(\Omega_1)} \frac{b[\mathbf{v}^h, q^h]}{\|\mathbf{v}^h\|_{1,\Omega_1} \|q^h\|_{0,\Omega_1}} \ge C. \tag{3.12}$$

Choices of finite element spaces satisfying (3.12) are well known [19]. Note that functions in $\widetilde{X}^h_1$ vanish on $\Gamma_0$.

We also assume that triangulations are uniformly regular so that the following inverse inequalities hold:

$$\begin{aligned} &\|\mathbf{v}^h\|_{1,\Omega} \le Ch^{-1} \|\mathbf{v}^h\|_{0,\Omega} \quad \forall \mathbf{v}^h \in X^h; \\ &\|\mathbf{v}^h\|_{1,\Omega_i} \le Ch^{-1} \|\mathbf{v}^h\|_{0,\Omega_i} \quad \forall \mathbf{v}^h \in X^h_i,\ i = 1, 2. \end{aligned} \tag{3.13}$$

Semidiscrete finite element approximations of the weak form (3.3)–(3.6) are defined as follows: seek $(\mathbf{v}^h, p^h, \mathbf{u}^h) \in C^1([0,T]; X^h_1) \times C([0,T]; Q^h_1) \times C^1([0,T]; X^h_2)$ such that

$$\begin{aligned} &\rho_1[\partial_t \mathbf{v}_h, \boldsymbol{\eta}_h]_{\Omega_1} + b[\boldsymbol{\eta}_h, p_h] + a_1[\mathbf{v}_h, \boldsymbol{\eta}_h] + \rho_2[\partial_{tt}\mathbf{u}_h, \boldsymbol{\eta}_h]_{\Omega_2} + a_2[\mathbf{u}_h, \boldsymbol{\eta}_h] \\ &\quad = \rho_1[\mathbf{f}_1, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}_h]_{\Omega_2} \qquad \forall \boldsymbol{\eta}_h \in X^h,\ \text{a.e. } t, \end{aligned} \tag{3.14}$$

$$b[\mathbf{v}_h, q_h] = 0 \qquad \forall q_h \in Q^h_1,\ \text{a.e. } t, \tag{3.15}$$

$$\mathbf{v}_h|_{\Gamma_0} = \partial_t \mathbf{u}_h|_{\Gamma_0} \qquad \text{a.e. } t \in [0,T], \tag{3.16}$$

$$\mathbf{v}_h|_{t=0} = \mathbf{v}_{0,h}, \qquad \mathbf{u}_h|_{t=0} = \mathbf{u}_{0,h}, \qquad \partial_t \mathbf{u}_h|_{t=0} = \mathbf{u}_{1,h}, \tag{3.17}$$

where $\mathbf{v}_{0,h} \in \boldsymbol{\Psi}^h|_{\Omega_1}$, $\mathbf{u}_{0,h} \in X^h_2$, and $\mathbf{u}_{1,h} \in X^h_2$ are finite element approximations of $\mathbf{v}_0$, $\mathbf{u}_0$, and $\mathbf{u}_1$, respectively. We assume that $(\mathbf{v}_{0,h}, \mathbf{u}_{1,h})$ satisfies

$$b[\mathbf{v}_{0,h}, q_h] = 0 \quad \forall q_h \in Q^h_1, \qquad \mathbf{v}_{0,h}|_{\Gamma_0} = \mathbf{u}_{1,h}|_{\Gamma_0} \tag{3.18}$$

and that $\mathbf{u}_{0,h}$ is defined by

$$a_2[\mathbf{u}_{0,h}, \mathbf{w}_h] = a_2[\mathbf{u}_0, \mathbf{w}_h] \qquad \forall \mathbf{w}_h \in X^h_2. \tag{3.19}$$

**3.2. The existence of discretely divergence-free finite element solutions.** The existence of finite element solutions $\{(\mathbf{v}^h, \mathbf{u}^h)\}$ can be established in a manner analogous to the analysis of the Galerkin approximations $\{(\mathbf{v}_m, \mathbf{u}_m)\}$ in [10]. However, it should be noted that finite element approximations are not special cases of the Galerkin approximations due to the fact that the basis functions used in the Galerkin approximations are divergence-free in $\Omega_1$, whereas the finite element solutions are only discretely divergence-free in $\Omega_1$ in the sense of (3.15), i.e., they belong to the space of discretely divergence-free functions $\mathbf{\Psi}^h$.

We first formulate auxiliary semidiscrete finite element approximations on the discretely divergence-free space $\mathbf{\Psi}^h$. Through the relation

$$\boldsymbol{\xi}_h = \begin{cases} \mathbf{v}_h & \text{in } \Omega_1, \\ \partial_t \mathbf{u}_h & \text{in } \Omega_2, \end{cases} \tag{3.20}$$

we see that (3.14)–(3.19) can be recast into the system

$$\begin{aligned} &\rho_1[\partial_t \boldsymbol{\xi}_h, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\partial_t \boldsymbol{\xi}_h, \boldsymbol{\eta}_h]_{\Omega_2} + a_1[\boldsymbol{\xi}_h, \boldsymbol{\eta}_h] + a_2\left[\int_0^t \boldsymbol{\xi}_h(s)\,ds, \boldsymbol{\eta}_h\right] \\ &\qquad = \rho_1[\mathbf{f}_1, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}_h]_{\Omega_2} - a_2[\mathbf{u}_0, \boldsymbol{\eta}_h] \qquad \forall\, \boldsymbol{\eta}_h \in \mathbf{\Psi}^h,\ t \in [0,T] \end{aligned} \tag{3.21}$$

and

$$\boldsymbol{\xi}_h(0) = \boldsymbol{\xi}_{0,h} \equiv \begin{cases} \mathbf{v}_{0,h} & \text{in } \Omega_1, \\ \mathbf{u}_{1,h} & \text{in } \Omega_2. \end{cases} \tag{3.22}$$

Let $\{\boldsymbol{\psi}_j^h\}_{j=1}^{J_h}$ be a finite element basis for $\mathbf{\Psi}^h$. Assumption (3.18) implies that $\boldsymbol{\xi}_{0,h} \in \mathbf{\Psi}^h$, so that we can write

$$\boldsymbol{\xi}_{0,h} = \sum_{j=1}^{J_h} d_j \boldsymbol{\psi}_j^h.$$

The solution $\boldsymbol{\xi}^h \in C([0,T]; \mathbf{\Psi}^h)$ for (3.21)–(3.22) can be expressed in the form

$$\boldsymbol{\xi}_h = \sum_{j=1}^{J_h} g_j^h(t) \boldsymbol{\psi}_j^h(\mathbf{x}) \tag{3.23}$$

so that system (3.21)–(3.22) is equivalent to the following linear system of ordinary differential equations for $\{g_j^h\}_{j=1}^{J_h}$:

$$\begin{cases} \displaystyle\sum_{j=1}^{J_h} [[\boldsymbol{\psi}_j^h, \boldsymbol{\psi}_i^h]] \frac{d}{dt} g_j^h(t) + \sum_{j=1}^{J_h} a_1[\boldsymbol{\psi}_j^h, \boldsymbol{\psi}_i^h]\, g_j^h(t) + \sum_{j=1}^{J_h} a_2[\boldsymbol{\psi}_j^h, \boldsymbol{\psi}_i^h] \int_0^t g_j^h(s)\,ds \\ \qquad = [\rho_1 \mathbf{f}_1(t), \boldsymbol{\psi}_i^h]_{\Omega_1} + [\rho_2 \mathbf{f}_2(t), \boldsymbol{\psi}_i^h]_{\Omega_2} - a_2[\mathbf{u}_0, \boldsymbol{\psi}_i^h], \qquad i = 1, \dots, J_h,\ t \in [0,T], \\ g_i^h(0) = d_i, \qquad i = 1, \dots, J_h. \end{cases}$$

We have the following results concerning the existence of and a priori estimates for a finite element solution $\boldsymbol{\xi}_h$ of (3.21)–(3.22). The proof is the same as that in [10] for the Galerkin approximations and thus is omitted.

THEOREM 3.1. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, *and* $\mathbf{u}_1$ *satisfy* (2.6). *Then, there exists a unique function* $\boldsymbol{\xi}_h \in C^1([0,T];\boldsymbol{\Psi}^h)$ *which satisfies* (3.21)–(3.22) *and the estimate*

$$\begin{aligned}&\|\boldsymbol{\xi}_h(t)\|^2_{0,\Omega} + \|\boldsymbol{\xi}_h\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \left\|\int_0^t \boldsymbol{\xi}_h(s)\,ds\right\|^2_{\mathbf{H}^1(\Omega_2)}\\ &\le Ce^{CT}(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{v}_{0,h}\|^2_{0,\Omega_1} + \|\mathbf{u}_{1,h}\|^2_{0,\Omega_2})\,\forall t\in[0,T].\end{aligned} \tag{3.24}$$

Setting $\mathbf{v}_h = \boldsymbol{\xi}_h|_{\Omega_1}$, $\mathbf{u}_h = \mathbf{u}_{0,h} + \int_0^t \boldsymbol{\xi}_h(s)|_{\Omega_2}\,ds$ and using (3.19), we immediately obtain the existence of a $(\mathbf{v}_h, \mathbf{u}_h)$ satisfying the discretely divergence-free version of (3.14)–(3.19), as follows.

THEOREM 3.2. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, *and* $\mathbf{u}_1$ *satisfy* (2.6). *Then, there exists a unique* $(\mathbf{v}_h, \mathbf{u}_h) \in C^1([0,T];\boldsymbol{\Psi}^h|_{\Omega_1}) \times C^2([0,T];X_2)$ *satisfying*

$$\begin{aligned}&\rho_1[\partial_t \mathbf{v}_h, \boldsymbol{\eta}_h]_{\Omega_1} + a_1[\mathbf{v}, \boldsymbol{\eta}_h] + \rho_2[\partial_{tt}\mathbf{u}_h, \boldsymbol{\eta}_h]_{\Omega_2} + a_2[\mathbf{u}_h, \boldsymbol{\eta}_h]\\ &\qquad = \rho_1[\mathbf{f}_1, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}_h]_{\Omega_2} \qquad \forall\, \boldsymbol{\eta}_h \in \boldsymbol{\Psi}^h,\ t\in[0,T]\end{aligned} \tag{3.25}$$

*and* (3.15)–(3.19). *Moreover, the following estimate holds:*

$$\begin{aligned}&\|\mathbf{v}_h(t)\|^2_{0,\Omega_1} + \|\partial_t \mathbf{u}_h(t)\|^2_{0,\Omega_2} + \|\mathbf{v}_h\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|\mathbf{u}_h\|^2_{\mathbf{H}^1(\Omega_2)}\\ &\le Ce^{CT}(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{v}_{0,h}\|^2_{0,\Omega_1} + \|\mathbf{u}_{1,h}\|^2_{0,\Omega_2})\ \forall t\in[0,T].\end{aligned} \tag{3.26}$$

**3.3. The discrete inf-sup conditions and discrete pressure fields.** We have proved the existence of a finite element solution in the discretely divergence-free formulation consisting of (3.25) and (3.15)–(3.19). We will show the existence of a discrete pressure $p_h$ such that (3.14) holds. A crucial step towards this goal is the verification of discrete inf-sup conditions. The discrete inf-sup conditions will also play a role in deriving strong energy estimates in a subsequent section.

We rewrite (3.14) as

$$\begin{aligned}b[\boldsymbol{\eta}_h, p_h] = &-\rho_1[\partial_t \mathbf{v}_h, \boldsymbol{\eta}_h]_{\Omega_1} - a_1[\mathbf{v}, \boldsymbol{\eta}_h] - \rho_2[\partial_{tt}\mathbf{u}_h, \boldsymbol{\eta}_h]_{\Omega_2} - a_2[\mathbf{u}_h, \boldsymbol{\eta}_h]\\ &+ \rho_1[\mathbf{f}_1, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\mathbf{f}_2, \boldsymbol{\eta}_h]_{\Omega_2} \qquad \forall\, \boldsymbol{\eta}_h \in X^h,\ t\in[0,T].\end{aligned} \tag{3.27}$$

In terms of the auxiliary variable $\boldsymbol{\xi}_h$, (3.27) is equivalent to

$$\begin{aligned}b[\boldsymbol{\eta}_h, p_h] = &-[[\partial_t \boldsymbol{\xi}_h, \boldsymbol{\eta}_h]] - a_1[\mathbf{v}_h, \boldsymbol{\eta}_h]\\ &- a_2[\mathbf{u}_h, \boldsymbol{\eta}_h] + [[\mathbf{f}, \boldsymbol{\eta}_h]] \qquad \forall\, \boldsymbol{\eta}_h \in X^h,\ \forall\, t\in[0,T].\end{aligned} \tag{3.28}$$

To show the existence of a $p_h \in C([0,T];Q_1^h)$ satisfying (3.27) or (3.28), we need to verify a discrete inf-sup condition for $b[\cdot,\cdot]$, which will be presented below; this will be the task of this subsection. To derive an estimate for $p_h$, we need an estimate for $\|\partial_t \boldsymbol{\xi}_h\|_{0,\Omega}$, or $\|\partial_t \mathbf{v}_h\|_{0,\Omega_1}$ and $\|\partial_{tt}\mathbf{u}_h\|_{0,\Omega_2}$; these will be derived in section 3.4.

The inf-sup condition we will verify is

$$\inf_{q_h\in Q_1^h} \sup_{\boldsymbol{\eta}_h\in X^h} \frac{b[\boldsymbol{\eta}_h, q_h]}{\|\boldsymbol{\eta}_h\|_{1,\Omega}\|q_h\|_{0,\Omega_1}} \ge C. \tag{3.29}$$

This inf-sup condition was proved in [2] for a special choice of $X_h$ and $Q_1^h$. We will establish (3.29) for the general case under assumption (3.12). To this end, we will first need the following lemma, and we will need to prove the inf-sup condition

$$\inf_{q_h\in Q_1^h}\sup_{\mathbf{v}_h\in X_1^h}\frac{b[\mathbf{v}_h,q_h]}{\|\mathbf{v}_h\|_{1,\Omega_1}\|q_h\|_{0,\Omega_1}}\geq C. \tag{3.30}$$

LEMMA 3.3. *For each constant $d$, there exists a piecewise linear function $\mathbf{v}\in X_1^{h_0}$ such that*

$$\int_{\Gamma_0}\mathbf{v}\cdot\mathbf{n}\,d\Gamma=-d,\qquad b[\mathbf{v},d]=|d|^2,\qquad \text{and}\qquad \|\mathbf{v}\|_{1,\Omega_1}\leq C|d|,$$

*where $\mathbf{n}$ denotes the unit outward-pointing normal along $\partial\Omega_1$, and the constant $C$ depends only on the coarse triangulation $\mathcal{T}^{h_0}(\Omega_1)$.*

*Proof.* We give the complete proof for the two-dimensional case and discuss the ideas for the three-dimensional case in an ensuing remark.

We choose from $\mathcal{T}^{h_0}(\Omega)$ a layer of triangles $K\equiv\cup_{j=1}^{J_0}K_j\subset\overline{\Omega}_1$ adjacent to $\Gamma_0$, i.e., each $K_j$ has either a side or a vertex on $\Gamma_0$. We denote the vertices on $\Gamma_0\cap\partial K$ by $A_j, j=0,1,\dots,J_0$. We define the $C^0$, piecewise linear vector function $\mathbf{v}=(v_1,v_2)$ on $K$ as follows:

$$\begin{cases}\mathbf{v}=\mathbf{0}\text{ at points }A_0\text{ and }A_{J_0},\\ \mathbf{v}=\mathbf{0}\text{ at all vertices of }K\text{ belonging to the interior of }\Omega_1,\\ \mathbf{v}\cdot\mathbf{n}_{j-1}=-\overline{d}\text{ and }\mathbf{v}\cdot\mathbf{n}_j=-\overline{d}\text{ at }A_j,\quad j=1,\dots,J_0-1,\ \mathbf{n}_{j-1}\neq\mathbf{n}_j,\\ \mathbf{v}\cdot\mathbf{n}_{j-1}=-\overline{d}\text{ and }\mathbf{v}\cdot\boldsymbol{\tau}_j=0\text{ at }A_j,\quad j=1,\dots,J_0-1,\ \mathbf{n}_{j-1}=\mathbf{n}_j,\end{cases}$$

where

$$\overline{d}=d\Bigg/\left(|\overline{A_0A_1}|/2+\sum_{j=2}^{J_0-1}|\overline{A_{j-1}A_j}|+|\overline{A_{J_0-1}A_{J_0}}|/2\right)$$

and $\mathbf{n}_j$ and $\boldsymbol{\tau}_j$ denote the unit, outward-pointing normal and unit tangent vectors, respectively, on $\partial\Omega_1\cap\overline{A_{j-1}A_j}$. Note that $\mathbf{n}_j$ and $\boldsymbol{\tau}_j$ are defined with respect to the *segment* $\overline{A_{j-1}A_j}$ so that they are well defined. Clearly, the values of $v_1(A_j)$ and $v_2(A_j)$ are proportional to $\overline{d}$. We can write

$$v_i(\mathbf{x})=\sum_{j=1}^{J_0-1}v_i(A_j)L_j^{h_0}(\mathbf{x}),\qquad i=1,2,$$

where for each $j$, $L_j^{h_0}(\mathbf{x})$ is the continuous piecewise linear basis function (the shape function) associated with the vertex $A_j$. Then,

$$\|v_i\|_{1,K}^2\leq C\sum_{j=1}^{J_0-1}|v_i(A_j)|^2\|L_j^{h_0}\|_{1,K}^2\leq C|\overline{d}|^2\sum_{j=1}^{J_0-1}\|L_j^{h_0}\|_{1,K}^2$$

so that

$$\|\mathbf{v}\|_{1,K}\leq C|\overline{d}|.$$

We extend $\mathbf{v}$ to $\Omega_1$ by zero outside $K$ and denote the extended function still by $\mathbf{v}$. Then we readily have $\mathbf{v} \in X_1^{h_0}$,

$$\|\mathbf{v}\|_{1,\Omega_1} = \|\mathbf{v}\|_{1,K} \le C|\overline{d}| \le C|d|,$$

and

$$\int_{\Gamma_0} \mathbf{v}\cdot\mathbf{n}\,d\Gamma = \sum_{j=1}^{J_0} \int_{\overline{A_{j-1}A_j}} \mathbf{v}\cdot\mathbf{n}\,d\Gamma$$
$$= -\overline{d}\left( |\overline{A_0A_1}|/2 + \sum_{j=2}^{J_0-1} |\overline{A_{j-1}A_j}| + |\overline{A_{J_0-1}A_{J_0}}|/2 \right) = -d.$$

Using Green's theorem and the last equality, we have

$$b[\mathbf{v}, d] = -d\int_{\Omega_1} \nabla\cdot\mathbf{v}\,d\Omega = -d\int_{\Gamma_0} \mathbf{v}\cdot\mathbf{n}\,d\Gamma = d^2. \qquad \square$$

*Remark* 1. In the three-dimensional case we merely need assume that $[\mathcal{T}^{h_0}(\Omega)]|_{\Gamma_0}$ contains a vertex $P_0$ shared by exactly three triangles. Indeed, in forming the coarse triangulation $\mathcal{T}^{h_0}(\Omega)$, we may simply choose a partition on a flat piece of $\Gamma_0$ to meet this requirement. Then, we define a $\mathbf{v}$ to satisfy $\mathbf{v}\cdot\mathbf{n} = \overline{d}$ and $\mathbf{v}\times\mathbf{n} = \mathbf{0}$ at $P_0$, and $\mathbf{v} = \mathbf{0}$ at all other vertices, where $\overline{d}$ is a suitable scaling of $d$.

Next we prove inf-sup condition (3.30) based on the inf-sup assumption (3.12) for the pair $\{\widetilde{X}_1^h, M_1^h\} \equiv \{X_1^h \cap \mathbf{H}_0^1(\Omega_1), Q_1^h \cap L_0^2(\Omega_1)\}$.

THEOREM 3.4. *The pair* $\{X_1^h, Q_1^h\}$ *satisfies inf-sup condition* (3.30).

*Proof.* Owing to [19, Remark 1.4, p. 118], the inf-sup condition (3.30) is equivalent to

$$\text{(3.31)} \qquad \begin{aligned} &\forall\, q_h \in Q_1^h, \quad \text{there exists} \quad \mathbf{v}_h \in X_1^h \quad \text{such that} \\ &b[\mathbf{v}_h, q_h] \ge C\|q_h\|_{0,\Omega_1}^2 \quad \text{and} \quad \|\mathbf{v}_h\|_{1,\Omega_1} \le C\|q_h\|_{0,\Omega_1}. \end{aligned}$$

Let $q_h \in Q_1^h$ be given. Set

$$\overline{q}_h = \frac{1}{|\Omega_1|}\int_{\Omega_1} q_h\,d\Omega \qquad \text{and} \qquad \widetilde{q}_h = q_h - \overline{q}_h.$$

Then $q_h = \widetilde{q}_h + \overline{q}_h$ in $\Omega_1$ and $\|q_h\|_{0,\Omega_1}^2 = \|\widetilde{q}_h\|_{0,\Omega_1}^2 + \|\overline{q}_h\|_{0,\Omega_1}^2$. Obviously, $\widetilde{q}_h \in M_1^h \equiv Q_1^h \cap L_0^2(\Omega_1)$ so that, by inf-sup condition (3.12) for the pair $\{\widetilde{X}_1^h, M_1^h\}$, we may choose a $\widetilde{\mathbf{v}}_h \in \widetilde{X}_1^h$ such that

$$b[\widetilde{\mathbf{v}}_h, \widetilde{q}_h] = \|\widetilde{q}_h\|_{0,\Omega_1}^2 \qquad \text{and} \qquad \|\widetilde{\mathbf{v}}_h\|_{1,\Omega_1} \le C\|\widetilde{q}_h\|_{0,\Omega_1}.$$

By Lemma 3.3 with $d = \|\overline{q}_h\|_{0,\Omega_1}$, we may choose a $\overline{\mathbf{v}}_h \in X_1^h$ such that

$$b[\overline{\mathbf{v}}_h, \overline{q}_h] = \|\overline{q}_h\|_{0,\Omega_1}^2 \qquad \text{and} \qquad \|\overline{\mathbf{v}}_h\|_{1,\Omega_1} \le C\|\overline{q}_h\|_{0,\Omega_1}.$$

(We recall that we assumed that $\mathcal{T}^h(\Omega_1)$ is a refinement of a coarse triangulation $\mathcal{T}^{h_0}(\Omega_1)$ so that a piecewise linear function on $\mathcal{T}^{h_0}(\Omega_1)$ belongs to $X_1^h$.) Setting

$\mathbf{v}_h = \widetilde{\mathbf{v}}_h + \alpha\overline{\mathbf{v}}_h$ for some $\alpha > 0$ (to be determined), we have

$$\begin{aligned} b[\mathbf{v}_h, q_h] &= b[\widetilde{\mathbf{v}}_h, \widetilde{q}_h] + b[\widetilde{\mathbf{v}}_h, \overline{q}_h] + \alpha b[\overline{\mathbf{v}}_h, \widetilde{q}_h] + \alpha b[\overline{\mathbf{v}}_h, \overline{q}_h] \\ &\geq \|\widetilde{q}_h\|^2_{0,\Omega_1} + 0 - C\alpha\|\widetilde{q}_h\|_{0,\Omega_1}\|\overline{\mathbf{v}}_h\|_{1,\Omega_1} + \alpha\|\overline{q}_h\|^2_{0,\Omega_1} \\ &\geq \|\widetilde{q}_h\|^2_{0,\Omega_1} - C\alpha\|\widetilde{q}_h\|_{0,\Omega_1}\|\overline{q}_h\|_{0,\Omega_1} + \alpha\|\overline{q}_h\|^2_{0,\Omega_1} \\ &\geq \|\widetilde{q}_h\|^2_{0,\Omega_1} - [C\alpha\|\widetilde{q}_h\|^2_{0,\Omega_1} + \frac{\alpha}{2}\|\overline{q}_h\|^2_{0,\Omega_1}] + \alpha\|\overline{q}_h\|^2_{0,\Omega_1} \\ &= (1 - C\alpha)\|\widetilde{q}_h\|^2_{0,\Omega_1} + \frac{\alpha}{2}\|\overline{q}_h\|^2_{0,\Omega_1} \end{aligned}$$

so that by choosing a sufficiently small $\alpha > 0$ we obtain

$$b[\mathbf{v}_h, q_h] \geq \min\{1 - C\alpha, \alpha/2\} \left(\|\widetilde{q}_h\|^2_{0,\Omega_1} + \frac{1}{2}\|\overline{q}_h\|^2_{0,\Omega_1}\right) \geq C\|q_h\|^2_{0,\Omega_1}.$$

Also,

$$\|\mathbf{v}_h\|_{1,\Omega_1} \leq \|\widetilde{\mathbf{v}}_h\|_{1,\Omega_1} + \|\overline{\mathbf{v}}_h\|_{1,\Omega_1} \leq C\|\widetilde{q}_h\|_{0,\Omega_1} + C\|\overline{q}_h\|_{0,\Omega_1} \leq C\|q_h\|_{0,\Omega_1}.$$

Hence, we have proved (3.31) which is equivalent to (3.30). ◻

We now prove inf-sup condition (3.29) for $\{X^h, Q_1^h\}$.

THEOREM 3.5. *$\{X^h, Q_1^h\}$ satisfies the inf-sup condition* (3.29).

*Proof.* Let the discrete extension operator $E^h : X_1^h \to X^h$ be defined as follows: for any $\mathbf{v}_h \in X_1^h$, $(E^h\mathbf{v}_h)|_{\overline{\Omega}_1} = \mathbf{v}_h$ and $(E^h\mathbf{v}_h)|_{\Omega_2} \in X_2^h$ is the solution of

$$[\nabla(E^h\mathbf{v}_h), \nabla\mathbf{z}_h]_{\Omega_2} = 0 \quad \forall\, \mathbf{z}_h \in X_2^h \cap \mathbf{H}_0^1(\Omega_2), \quad (E^h\mathbf{v}_h)|_{\Gamma_2} = \mathbf{0}, \quad (E^h\mathbf{v}_h)|_{\Gamma_0} = \mathbf{v}_h|_{\Gamma_0}.$$

It is well known (see, e.g., [23] and [1]) that $\|E^h\mathbf{v}_h\|_{1,\Omega_2} \leq C\|\mathbf{v}_h\|_{1/2,\Gamma_0}$ so that

$$\begin{aligned} \|E^h\mathbf{v}_h\|_{1,\Omega} &\leq C\|(E^h\mathbf{v}_h)|_{\Omega_1}\|_{1,\Omega_1} + \|(E^h\mathbf{v}_h)|_{\Omega_2}\|_{1,\Omega_2} \\ &\leq C(\|\mathbf{v}_h\|_{1,\Omega_1} + \|\mathbf{v}_h\|_{1/2,\Gamma_0}) \leq C\|\mathbf{v}_h\|_{1,\Omega_1} \qquad \forall\, \mathbf{v}_h \in X_1^h. \end{aligned}$$

Then, for every $q_h \in Q_1^h$ we have

$$\begin{aligned} \sup_{\boldsymbol{\eta}_h \in X^h} \frac{b[\boldsymbol{\eta}_h, q_h]}{\|q_h\|_{0,\Omega_1}\,\|\boldsymbol{\eta}_h\|_{1,\Omega}} &\geq \sup_{\mathbf{v}_h \in X_1^h} \frac{b[E^h\mathbf{v}_h, q_h]}{\|q_h\|_{0,\Omega_1}\,\|E^h\mathbf{v}_h\|_{1,\Omega}} \\ &\geq C \sup_{\mathbf{v}_h \in X_1^h} \frac{b[E^h\mathbf{v}_h, q_h]}{\|q_h\|_{0,\Omega_1}\,\|\mathbf{v}_h\|_{1,\Omega_1}} = C \sup_{\mathbf{v}_h \in X_1^h} \frac{b[\mathbf{v}_h, q_h]}{\|q_h\|_{0,\Omega_1}\,\|\mathbf{v}_h\|_{1,\Omega_1}} \geq C, \end{aligned}$$

where the last step is valid because of (3.30). ◻

As a direct consequence of [19, Lemma 4.1, p. 58], Theorem 3.8, and the inf-sup condition (3.29), we obtain the following theorem concerning the existence of a discrete pressure. Note that an estimate for $p_h$ will be established in section 3.4 only after we have derived strong energy estimates, particularly the estimate for $\|\partial_t\boldsymbol{\xi}_h\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$.

THEOREM 3.6. *Assume that $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, and $\mathbf{u}_1$ satisfy* (2.6)*, and let $\boldsymbol{\xi}_h \in C^1([0,T];\boldsymbol{\Psi}^h)$ be the solution of* (3.21)–(3.22)*. Let $(\mathbf{v}_h, \mathbf{u}_h) \in C^1([0,T]; X^h|_{\Omega_1}) \times C^1([0,T]; X_2^h)$ be the solution of* (3.25) *and* (3.15)–(3.19)*. Then there exists a unique $p_h \in C([0,T]; Q_1^h)$ satisfying* (3.28) *and* (3.15).

*Proof.* The existence and uniqueness of a $p_h \in C([0,T]; Q_1^h)$ satisfying (3.28) follow directly from [19, Lemma 4.1, p. 58], Theorem 3.8, and the inf-sup condition (3.29). Since (3.28) is equivalent to (3.27), we also conclude that $p_h$ satisfies (3.27) and is the unique such solution. ◻

**3.4. Strong a priori energy estimates for the finite element solutions.** In the finite element system (3.25) and (3.15)–(3.19) the discrete initial conditions are arbitrary approximations of the corresponding continuous initial data. We now make a particular choice of discrete initial data that will allow us to derive an estimate for $\|\partial_t \boldsymbol{\xi}_h\|_{0,\Omega}$ under additional assumptions on the data. Such an estimate can then be used to derive an estimate for $\|p_h\|_{L^2(0,T;L^2(\Omega_1))}$. (The existence of a discrete pressure $p_h$ satisfying (3.14) was shown in section 3.3.) The estimates on $p_h$ and $\partial_t \boldsymbol{\xi}_h$ will be needed in order to prove the convergence of finite element solutions, since finite element formulations involve the term $b[\boldsymbol{\eta}_h, p_h]$, which, in general, does not vanish for $\boldsymbol{\eta}_h \in X^h$.

We first study the approximation of the initial condition. We choose $(\mathbf{v}_{0,h}, \mathbf{u}_{1,h}) \in \boldsymbol{\Psi}^h$ and $p_{0,h} \in Q_1^h$ to be the solution of

$$
\begin{aligned}
&a_1[\mathbf{v}_{0,h}, \boldsymbol{\eta}_h] + [\mathbf{u}_{1,h}, \boldsymbol{\eta}_h]_{\Omega_2} + b[\boldsymbol{\eta}_h, p_{0,h}] \\
&\qquad = a_1[\mathbf{v}_0, \boldsymbol{\eta}_h] + [\mathbf{u}_1, \boldsymbol{\eta}_h]_{\Omega_2} + b[\boldsymbol{\eta}_h, p_0] \qquad \forall\, \boldsymbol{\eta}_h \in X^h,
\end{aligned} \tag{3.32}
$$

$$
b[\mathbf{v}_{0,h}, q_h] = 0 \quad \forall\, q_h \in Q_1^h \qquad \text{and} \qquad \mathbf{v}_{0,h}|_{\Gamma_0} = \mathbf{u}_{1,h}|_{\Gamma_0}, \tag{3.33}
$$

where $p_0$ is the initial pressure field associated with the initial velocity field $\mathbf{v}_0$.

LEMMA 3.7. *Assume that $\mathbf{v}_0 \in X_1$, $p_0 \in L^2(\Omega_1)$, $\mathbf{u}_1 \in X_2$, and $\mathbf{v}_0|_{\Gamma_0} = \mathbf{u}_1|_{\Gamma_0}$. Then there exists a unique triplet $(\mathbf{v}_{0,h}, p_{0,h}, \mathbf{u}_{1,h}) \in X_1^h \times Q_1^h \times X_2^h$ which satisfies (3.32)–(3.33) and*

$$
\begin{aligned}
&\|\mathbf{v}_{0,h} - \mathbf{v}_0\|_{1,\Omega_1} + \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,\Omega_2} + \|p_{0,h} - p_0\|_{0,\Omega_1} \\
&\qquad \le C(\|\boldsymbol{\eta}_h - \mathbf{v}_0\|_{1,\Omega_1} + \|\boldsymbol{\eta}_h - \mathbf{u}_1\|_{0,\Omega_2} + \|q_h - p_0\|_{0,\Omega_1}) \qquad \forall\, (\boldsymbol{\eta}_h, q_h) \in X^h \times Q_1^h.
\end{aligned} \tag{3.34}
$$

*If, in addition, $\mathbf{v}_0 \in \mathbf{H}^{r+1}(\Omega_1)$, $p_0 \in H^r(\Omega_1)$, and $\mathbf{u}_1 \in \mathbf{H}^{r+1}(\Omega_2)$ for some $r \in [0,k]$ ($k$ being the integer appearing in the approximation properties), then*

$$
\begin{aligned}
&\|\mathbf{v}_{0,h} - \mathbf{v}_0\|_{1,\Omega_1} + \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,\Omega_2} + \|p_{0,h} - p_0\|_{0,\Omega_1} \\
&\qquad \le C h^r (\|\mathbf{v}_0\|_{r+1,\Omega_1} + \|\mathbf{u}_1\|_{r+1,\Omega_2} + \|p_0\|_{r,\Omega_1}).
\end{aligned} \tag{3.35}
$$

*Proof.* We set $\widetilde{X} = \{\boldsymbol{\eta} \in \mathbf{L}^2(\Omega) : \boldsymbol{\eta}|_{\Omega_1} \in X_1, \operatorname{div} \boldsymbol{\eta}|_{\Omega_1} = 0\}$ and equip $\widetilde{X}$ with the inner product

$$
[\boldsymbol{\xi}, \boldsymbol{\eta}]_{\widetilde{X}} = a_1[\boldsymbol{\xi}, \boldsymbol{\eta}] + [\boldsymbol{\xi}, \boldsymbol{\eta}]_{\Omega_2} \qquad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \widetilde{X}.
$$

It is easy to check that $\widetilde{X}$ is a Hilbert space. The continuous inf-sup condition (2.4) implies

$$
\begin{aligned}
\inf_{q \in L^2(\Omega_1)} \sup_{\boldsymbol{\eta} \in \widetilde{X}} \frac{b[\boldsymbol{\eta}, q]}{\|\boldsymbol{\eta}\|_{\widetilde{X}} \|q\|_{0,\Omega_1}} &\ge \inf_{q \in L^2(\Omega_1)} \sup_{\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)} \frac{b[\boldsymbol{\eta}, q]}{\|\boldsymbol{\eta}\|_{\widetilde{X}} \|q\|_{0,\Omega_1}} \\
&\ge \inf_{q \in L^2(\Omega_1)} \sup_{\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)} \frac{b[\boldsymbol{\eta}, q]}{\|\boldsymbol{\eta}\|_{1,\Omega} \|q\|_{0,\Omega_1}} \ge C.
\end{aligned}
$$

Thus, by [19, Theorem 1.1, p. 114], there exists a unique $(\widetilde{\boldsymbol{\xi}}_0, \widetilde{p}_0) \in \widetilde{X} \times L^2(\Omega_1)$ satisfying

$$
[\widetilde{\boldsymbol{\xi}}_0, \boldsymbol{\eta}]_{\widetilde{X}} + b[\boldsymbol{\eta}, \widetilde{p}_0] = a_1[\mathbf{v}_0, \boldsymbol{\eta}] + [\mathbf{u}_1, \boldsymbol{\eta}]_{\Omega_2} + b[\boldsymbol{\eta}, p_0] \qquad \forall\, \boldsymbol{\eta} \in \widetilde{X}, \tag{3.36}
$$

$$b[\widetilde{\boldsymbol{\xi}}_0, q] = 0 \qquad \forall\, q \in L^2(\Omega_1). \tag{3.37}$$

As $\boldsymbol{\xi}_0$ defined by (2.11) and $p_0$ constitute an obvious solution to (3.36)–(3.37), we have

$$\widetilde{\boldsymbol{\xi}}_0 = \boldsymbol{\xi}_0 = \begin{cases} \boldsymbol{\xi}|_{\Omega_1} = \mathbf{v}_0, \\ \boldsymbol{\xi}|_{\Omega_2} = \mathbf{u}_1, \end{cases} \qquad \text{and} \qquad \widetilde{p}_0 = p_0. \tag{3.38}$$

Similarly, the discrete inf-sup condition (3.29) implies

$$\inf_{q_h \in Q_1^h} \sup_{\boldsymbol{\eta}_h \in X^h} \frac{b[\boldsymbol{\eta}_h, q_h]}{\|\boldsymbol{\eta}_h\|_{\widetilde{X}} \|q_h\|_{0,\Omega_1}} \geq \inf_{q_h \in Q_1^h} \sup_{\boldsymbol{\eta}_h \in X^h} \frac{b[\boldsymbol{\eta}_h, q_h]}{\|\boldsymbol{\eta}_h\|_{1,\Omega} \|q_h\|_{0,\Omega_1}} \geq C,$$

so that by [19, Theorem 1.1, p. 114] there exists a unique $(\boldsymbol{\xi}_{0,h}, p_{0,h}) \in X^h \times Q_1^h$ satisfying

$$[\boldsymbol{\xi}_{0,h}, \boldsymbol{\eta}_h]_{\widetilde{X}} + b[\boldsymbol{\eta}_h, p_{0,h}] = a_1[\mathbf{v}_0, \boldsymbol{\eta}_h] + [\mathbf{u}_1, \boldsymbol{\eta}_h]_{\Omega_2} + b[\boldsymbol{\eta}_h, p_0] \qquad \forall\, \boldsymbol{\eta}_h \in X^h, \tag{3.39}$$

$$b[\boldsymbol{\xi}_{0,h}, q_h] = 0 \qquad \forall\, q \in Q_1^h; \tag{3.40}$$

moreover, the following error estimate holds:

$$\begin{aligned} &\|\boldsymbol{\xi}_{0,h} - \boldsymbol{\xi}_0\|_{\widetilde{X}} + \|p_{0,h} - p_0\|_{0,\Omega_1} \\ &\quad \leq C(\|\boldsymbol{\eta}_h - \boldsymbol{\xi}_0\|_{\widetilde{X}} + \|q_h - p_0\|_{0,\Omega_1}) \qquad \forall\, (\boldsymbol{\eta}_h, q_h) \in X^h \times Q_1^h. \end{aligned} \tag{3.41}$$

By setting

$$\mathbf{v}_{0,h} = \boldsymbol{\xi}_{0,h}|_{\Omega_1} \quad \text{and} \quad \mathbf{u}_{1,h} = \boldsymbol{\xi}_{0,h}|_{\Omega_2}, \tag{3.42}$$

we see that (3.41) is equivalent to (3.34) and that (3.32)–(3.33) are satisfied. The uniqueness of the solution $(\mathbf{v}_{0,h}, p_{0,h}, \mathbf{u}_{1,h})$ for (3.32)–(3.33) follows from the uniqueness of the solution $(\boldsymbol{\xi}_{0,h}, p_{0,h})$ for (3.39)–(3.40).

Next, assuming that $\mathbf{v}_0 \in \mathbf{H}^{r+1}(\Omega_1)$, $p_0 \in H^r(\Omega_1)$, and $\mathbf{u}_1 \in \mathbf{H}^{r+1}(\Omega_2)$ for some $r \in [1, k]$, we proceed to prove (3.35) by making a particular choice of $\boldsymbol{\eta}_h$ in (3.34). Let $(\overline{\mathbf{v}}_{0,h}, \overline{p}_{0,h}) \in X_1^h \times Q_1^h$ be the unique finite element solution of the following Stokes system on $\Omega_1$:

$$a_1[\overline{\mathbf{v}}_{0,h}, \mathbf{z}_h] + b[\mathbf{z}_h, \overline{p}_{0,h}] = a_1[\mathbf{v}_0, \mathbf{z}_h] + b[\mathbf{z}_h, \overline{p_0}] \qquad \forall\, \mathbf{z}_h \in X_1^h \cap \mathbf{H}_0^1(\Omega_1),$$

$$b[\overline{\mathbf{v}}_{0,h}, q_h] = 0 \qquad \forall\, q_h \in Q_1^h \cap L_0^2(\Omega_1),$$

$$\overline{\mathbf{v}}_{0,h}|_{\Gamma_1} = \mathbf{0} \qquad \text{and} \qquad [\overline{\mathbf{v}}_{0,h}, \mathbf{s}_h]_{0,\Gamma_0} = [\mathbf{v}_0, \mathbf{s}_h]_{0,\Gamma_0} \quad \forall\, \mathbf{s}_h \in X_1^h|_{\Gamma_0},$$

where $\overline{p_0} = p_0 - (1/|\Omega_1|) \int_{\Omega_1} p_0 \, d\mathbf{x}$. Using the results of [23] concerning error estimates for the finite element approximations of the Stokes equations with inhomogeneous boundary conditions, we obtain

$$\begin{aligned} &\|\overline{\mathbf{v}}_{0,h} - \mathbf{v}_0\|_{1,\Omega_1} + \|\overline{p}_{0,h} - \overline{p_0}\|_{0,\Omega_1} \\ &\quad \leq Ch^r(\|\mathbf{v}_0\|_{r+1,\Omega_1} + \|\overline{p_0}\|_{r,\Omega_1}) \leq Ch^r(\|\mathbf{v}_0\|_{r+1,\Omega_1} + \|p_0\|_{r,\Omega_1}). \end{aligned} \tag{3.43}$$

Analogously, let $\overline{\mathbf{u}}_{1,h} \in X_2^h$ be the unique finite element solution of the following elliptic system on $\Omega_2$ with an inhomogeneous boundary condition:

$$
\begin{aligned}
&[\nabla \overline{\mathbf{u}}_{1,h}, \nabla \mathbf{w}_h]_{\Omega_2} = [\nabla \mathbf{u}_1, \nabla \mathbf{w}_h]_{\Omega_2} \qquad \forall\, \mathbf{w}_h \in X_2^h \cap \mathbf{H}_0^1(\Omega_2), \\
&\overline{\mathbf{u}}_{1,h}|_{\Gamma_2} = \mathbf{0} \qquad \text{and} \qquad [\overline{\mathbf{u}}_{1,h}, \mathbf{s}_h]_{0,\Gamma_0} = [\mathbf{u}_1, \mathbf{s}_h]_{0,\Gamma_0} \quad \forall\, \mathbf{s}_h \in X_2^h|_{\Gamma_0}.
\end{aligned} \tag{3.44}
$$

Then we have

$$
\|\overline{\mathbf{u}}_{1,h} - \mathbf{u}_1\|_{0,\Omega_2} \le \|\overline{\mathbf{u}}_{1,h} - \mathbf{u}_1\|_{1,\Omega_2} \le Ch^r \|\mathbf{u}_1\|_{r+1,\Omega_2}. \tag{3.45}
$$

The assumption $\mathbf{v}_0|_{\Gamma_0} = \mathbf{u}_1|_{\Gamma_0}$ implies $\overline{\mathbf{v}}_{0,h}|_{\Gamma_0} = \overline{\mathbf{u}}_{1,h}|_{\Gamma_0}$, so that the element $\overline{\boldsymbol{\eta}}_h$ defined by

$$
\overline{\boldsymbol{\eta}}_h|_{\Omega_1} = \begin{cases} \overline{\mathbf{v}}_{0,h} & \text{in } \Omega_1, \\ \overline{\mathbf{u}}_{1,h} & \text{in } \Omega_2 \end{cases}
$$

satisfies $\overline{\boldsymbol{\eta}}_h \in X^h$. By choosing $\boldsymbol{\eta}_h = \overline{\boldsymbol{\eta}}_h$ and $q_h = \overline{p}_{0,h} + (1/|\Omega_1|)\int_{\Omega_1} p_0 \, d\mathbf{x}$ in (3.34) and using (3.43)–(3.45), we arrive at (3.35). □

We now derive a strong a priori energy estimate for the auxiliary finite element solution $\boldsymbol{\xi}_h$.

THEOREM 3.8. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, *and* $\mathbf{u}_1$ *satisfy* (2.6) *and*

$$
\partial_t \mathbf{f}_i \in L^2(0,T;\mathbf{L}^2(\Omega_i)),\ i=1,2, \quad \mathbf{v}_0 \in \mathbf{H}^2(\Omega_1),\ \mathbf{u}_1 \in \mathbf{H}^2(\Omega_1),\ \mathbf{u}_0 \in \mathbf{H}^2(\Omega_2). \tag{3.46}
$$

*Assume further that there exists a* $p_0 \in H^1(\Omega_1)$ *such that*

$$
(p_0 \mathbf{n}_1 - \mu_1 \nabla \mathbf{v}_0 \cdot \mathbf{n}_1)|_{\Gamma_0} = (\mu_2 \nabla \mathbf{u}_0 \cdot \mathbf{n}_2 + (\lambda_2 + \mu_2)(\operatorname{div} \mathbf{u}_0)\mathbf{n}_2)|_{\Gamma_0}, \tag{3.47}
$$

*where* $\mathbf{n}_i$ *denotes the outward-pointing normal along* $\partial\Omega_i$, $i = 1, 2$. *Then there exists a unique solution* $\boldsymbol{\xi}_h \in C^1([0,T];\boldsymbol{\Psi}^h)$ *for* (3.21)–(3.22) *with the initial condition* $\boldsymbol{\xi}_{0,h}$ *defined by*

$$
\boldsymbol{\xi}_{0,h}|_{\Omega_1} = \mathbf{v}_{0,h} \quad \text{and} \quad \boldsymbol{\xi}_{0,h}|_{\Omega_2} = \mathbf{u}_{1,h}, \tag{3.48}
$$

*where* $\mathbf{v}_{0,h}$ *and* $\mathbf{u}_{1,h}$ *are determined by* (3.32)–(3.33). *Moreover,* $\boldsymbol{\xi}_h$ *satisfies the estimates*

$$
\begin{aligned}
&\|\boldsymbol{\xi}_h\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\xi}_h\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \left\| \int_0^t \boldsymbol{\xi}_h(s)\, ds \right\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_2))} \\
&\quad \le Ce^{CT} (\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{v}_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{0,\Omega_2} + \|p_0\|^2_{0,\Omega_1})
\end{aligned} \tag{3.49}
$$

*and*

$$
\begin{aligned}
&\|\partial_t\boldsymbol{\xi}_h\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\partial_t\boldsymbol{\xi}_h\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \left\| \int_0^t \partial_t\boldsymbol{\xi}_h(s)\, ds \right\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_2))} \\
&\quad \le Ce^{CT} (\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{2,\Omega_2}).
\end{aligned} \tag{3.50}
$$

*Proof.* By Theorem 3.1, there exists a unique solution $\boldsymbol{\xi}_h \in C^1([0,T];\boldsymbol{\Psi}^h)$ for (3.21)–(3.22) and (3.24). We note that, by virtue of Lemma 3.7, the initial condition $\boldsymbol{\xi}_{0,h} \in \boldsymbol{\Psi}^h$ satisfies the estimate

$$
\|\boldsymbol{\xi}_{0,h}\|_{1,\Omega_1} + \|\boldsymbol{\xi}_{0,h}\|_{0,\Omega_2} \le C(\|\mathbf{v}_0\|_{1,\Omega_1} + \|p_0\|_{0,\Omega_1} + \|\mathbf{u}_1\|_{0,\Omega_2}).
$$

Thus (3.49) follows from the last estimate and (3.24).

Defining $\boldsymbol{\zeta}_h = \partial_t \boldsymbol{\xi}_h$ and differentiating (3.21), we obtain that for each $t \in [0,T]$

$$\begin{aligned}\rho_1[\partial_t \boldsymbol{\zeta}_h, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\partial_t \boldsymbol{\zeta}_h, \boldsymbol{\eta}_h]_{\Omega_2} + a_1[\boldsymbol{\zeta}_h, \boldsymbol{\eta}_h] + a_2\left[\int_0^t \boldsymbol{\zeta}_h(s)\,ds, \boldsymbol{\eta}_h\right] \\ = \rho_1[\partial_t \mathbf{f}_1, \boldsymbol{\eta}_h]_{\Omega_1} + \rho_2[\partial_t \mathbf{f}_2, \boldsymbol{\eta}_h]_{\Omega_2} - a_2[\boldsymbol{\xi}_h(0), \boldsymbol{\eta}_h] \qquad \forall\, \boldsymbol{\eta}_h \in \boldsymbol{\Psi}^h.\end{aligned} \tag{3.51}$$

Setting $\boldsymbol{\eta}_h = \boldsymbol{\zeta}_h(t)$ in (3.51) and integrating in $t$, we obtain

$$\begin{aligned}&[[\boldsymbol{\zeta}_h(t), \boldsymbol{\zeta}_h(t)]] + \int_0^t a_1[\boldsymbol{\zeta}_h(s), \boldsymbol{\zeta}_h(s)]\,ds + a_2\left[\int_0^t \boldsymbol{\zeta}_h(s)\,ds, \int_0^t \boldsymbol{\zeta}_h(s)\,ds\right] \\ &\quad \le C(\|\boldsymbol{\zeta}_h(0)\|_{0,\Omega}^2 + \|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}) + a_2\left[\mathbf{u}_0, \int_0^t \boldsymbol{\zeta}_h(s)\,ds\right] + \int_0^t \|\boldsymbol{\zeta}_h(s)\|_{0,\Omega}^2\,ds \\ &\quad \le C(\|\boldsymbol{\zeta}_h(0)\|_{0,\Omega}^2 + \|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2}) \\ &\qquad + \frac{1}{2}a_2\left[\int_0^t \boldsymbol{\zeta}_h(s)\,ds, \textstyle\int_0^t \boldsymbol{\zeta}_h(s)\,ds\right] + \int_0^t \|\boldsymbol{\zeta}_h(s)\|^2_{0,\Omega}\,ds,\end{aligned}$$

so that

$$\begin{aligned}&\|\boldsymbol{\zeta}_h(t)\|^2_{0,\Omega} + \int_0^t a_1[\boldsymbol{\zeta}_h(t), \boldsymbol{\zeta}_h(t)]\,dt + a_2\left[\int_0^t \boldsymbol{\zeta}_h(s)\,ds, \int_0^t \boldsymbol{\zeta}_h(s)\,ds\right] \\ &\quad \le C(\|\boldsymbol{\zeta}_h(0)\|^2_{0,\Omega} + \|\partial_t \mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\xi}_h(0)\|^2_{1,\Omega_2}) + \int_0^t \|\boldsymbol{\zeta}_h(s)\|^2_{0,\Omega}\,ds.\end{aligned} \tag{3.52}$$

Dropping the second and third terms on the left-hand side of (3.52) and then applying the following version of Gronwall's inequality [12, p. 625],

$$\text{if } \; r(t) \le C_1 + C_2 \int_0^t r(s)\,ds, \;\text{ then }\; r(t) \le C_1(1 + C_2 t)e^{C_2 t}, \tag{3.53}$$

we deduce

$$\|\boldsymbol{\zeta}_h(t)\|^2_{0,\Omega} \le Ce^{CT}(\|\boldsymbol{\zeta}_h(0)\|^2_{0,\Omega} + \|\boldsymbol{\xi}_h(0)\|^2_{1,\Omega_2} + \|\partial_t \mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}).$$

The last estimate and (3.52) yield

$$\begin{aligned}&\|\boldsymbol{\zeta}_h(t)\|^2_{0,\Omega} + \int_0^t a_1[\boldsymbol{\zeta}_h(t), \boldsymbol{\zeta}_h(t)]\,dt + a_2\left[\int_0^t \boldsymbol{\zeta}_h(s)\,ds, \int_0^t \boldsymbol{\zeta}_h(s)\,ds\right] \\ &\quad \le Ce^{CT}(\|\boldsymbol{\zeta}_h(0)\|^2_{0,\Omega} + \|\partial_t \mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\xi}_h(0)\|^2_{1,\Omega_2}).\end{aligned} \tag{3.54}$$

The term $\|\boldsymbol{\xi}_h(0)\|^2_{1,\Omega_2}$ on the right-hand side of (3.54) can be estimated with the help of inverse inequalities (3.13), (3.45), and (3.35) with $r = 1$:

$$\begin{aligned}\|\boldsymbol{\xi}_h(0)\|_{1,\Omega_2} &\le \|\boldsymbol{\xi}_{0,h} - \overline{\mathbf{u}}_{1,h}\|_{1,\Omega_2} + \|\overline{\mathbf{u}}_{1,h} - \mathbf{u}_1\|_{1,\Omega_2} + \|\mathbf{u}_1\|_{1,\Omega_2} \\ &\le \frac{C}{h}\|\boldsymbol{\xi}_{0,h} - \overline{\mathbf{u}}_{1,h}\|_{0,\Omega_2} + Ch\|\mathbf{u}_1\|_{2,\Omega_2} + \|\mathbf{u}_1\|_{1,\Omega_2} \\ &\le \frac{C}{h}\|\boldsymbol{\xi}_{0,h} - \mathbf{u}_1\|_{0,\Omega_2} + \frac{C}{h}\|\mathbf{u}_1 - \overline{\mathbf{u}}_{1,h}\|_{0,\Omega_2} + Ch\|\mathbf{u}_1\|_{2,\Omega_2} + \|\mathbf{u}_1\|_{1,\Omega_2} \\ &\le C(\|\mathbf{v}_0\|_{2,\Omega_1} + \|\mathbf{u}_1\|_{2,\Omega_2} + \|p_0\|_{0,\Omega_1}),\end{aligned} \tag{3.55}$$

where $\overline{\mathbf{u}}_{1,h}$ is defined by (3.44). The term $\|\boldsymbol{\zeta}_h(0)\|^2_{0,\Omega}$ can be estimated as follows. Evaluating (3.21) at $t=0$, then setting $\boldsymbol{\eta}_h = \partial_t \boldsymbol{\xi}_h(0)$ and using (3.52), we have

$$
\begin{aligned}
&[[\partial_t \boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]] = [[\mathbf{f}(0), \partial_t \boldsymbol{\xi}_h(0)]] - a_2[\mathbf{u}_0, \partial_t \boldsymbol{\xi}_h(0)] - a_1[\boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)] \\
&= [[\mathbf{f}(0), \partial_t \boldsymbol{\xi}_h(0)]] - a_2[\mathbf{u}_0, \partial_t \boldsymbol{\xi}_h(0)] - b[\partial_t \boldsymbol{\xi}_h(0), p_0] - a_1[\mathbf{v}_0, \partial_t \boldsymbol{\xi}_h(0)] \\
&\quad - [\mathbf{u}_1, \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} + [\boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} \\
&= [[\mathbf{f}(0), \partial_t \boldsymbol{\xi}_h(0)]] + [\Delta \mathbf{u}_0 + \nabla(\operatorname{div} \mathbf{u}_0), \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} + [\Delta \mathbf{v}_0 - \nabla p_0, \partial_t \boldsymbol{\xi}_h(0)] \\
&\quad + \int_{\Gamma_0} (-\mu_2 \nabla \mathbf{u}_0 \cdot \mathbf{n}_2 - (\lambda_2 + \mu_2)(\operatorname{div} \mathbf{u}_0)\mathbf{n}_2 + p_0 \mathbf{n}_1 - \nabla \mathbf{v}_0 \cdot \mathbf{n}_1) \cdot \partial_t \boldsymbol{\xi}_h(0)\, d\Gamma \\
&\quad - [\mathbf{u}_1, \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} + [\boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2}.
\end{aligned}
$$

Applying assumption (3.47) and initial condition (3.32)–(3.33) to the last relation, we are led to

$$
\begin{aligned}
[[\partial_t \boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]] &= [[\mathbf{f}(0), \partial_t \boldsymbol{\xi}_h(0)]] + [\Delta \mathbf{u}_0 + \nabla(\operatorname{div} \mathbf{u}_0), \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} \\
&\qquad + [\Delta \mathbf{v}_0 - \nabla p_0, \partial_t \boldsymbol{\xi}_h(0)] - [\mathbf{u}_1, \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} + [\boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]_{\Omega_2} \\
&\le C(\|\mathbf{f}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_2 + \|\mathbf{u}_1\|^2_{2,\Omega_2} + \|p_0\|^2_{1,\Omega_1}) \\
&\qquad + C\|\boldsymbol{\xi}_{0,h}\|^2_{0,\Omega} + \frac{1}{2}[[\partial_t \boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]],
\end{aligned}
$$

so that, using (3.55), the last relation simplifies to

$$
\|\partial_t \boldsymbol{\xi}_h(0)\|^2_{0,\Omega} \le C(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{2,\Omega_2}).
$$

Combining (3.54), (3.55), and the last relation, we obtain (3.50). ∎

*Remark* 2. The particular choice of the initial condition (3.32)–(3.33) played a key role in the estimation of $[[\partial_t \boldsymbol{\xi}_h(0), \partial_t \boldsymbol{\xi}_h(0)]]$.

Using relation (3.20) in reverse, i.e., setting $\mathbf{u}_h = \mathbf{u}_{0,h} + \int_0^t \boldsymbol{\xi}_h(s)|_{\Omega_2}\, ds$ and $\mathbf{v}_h = \boldsymbol{\xi}_h|_{\Omega_1}$, we arrive at the following theorem.

THEOREM 3.9. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$, *and* $\mathbf{u}_1$ *satisfy* (2.6) *and* (3.46). *Assume further that there exists a* $p_0 \in H^1(\Omega_1)$ *such that* (3.47) *holds. Then there exists a unique solution* $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in C^1([0,T]; X_1^h) \times C([0,T]; Q_1^h) \times C^1([0,T]; X_2^h)$ *for* (3.14)–(3.19) *with the initial conditions* $(\mathbf{v}_{0,h}, \mathbf{u}_{1,h})$ *defined by* (3.32)–(3.33). *Moreover,* $(\mathbf{v}_h, p_h, \mathbf{u}_h)$ *satisfies the estimates*

$$
\begin{aligned}
&\|\mathbf{v}_h\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_1))} + \|\partial_t \mathbf{u}_h\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_2))} \\
&\qquad + \|\mathbf{v}_h\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|\mathbf{u}_h\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_2))} \\
&\le Ce^{CT}(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{v}_0\|^2_{1,\Omega_1} + \|p_0\|^2_{0,\Omega_1} + \|\mathbf{u}_1\|^2_{0,\Omega_2})
\end{aligned}
\tag{3.56}
$$

*and*

$$
\begin{aligned}
&\|\partial_t \mathbf{v}_h\|^2_{L^\infty(0,T;L^2(\Omega_1))} + \|\partial_{tt} \mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega_2))} \\
&\qquad + \|\partial_t \mathbf{v}_h\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|\partial_t \mathbf{u}_h\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_2))} \\
&\le Ce^{CT}(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{2,\Omega_2}).
\end{aligned}
\tag{3.57}
$$

Recall that Theorem 3.6 stated only the existence and uniqueness of a discrete pressure $p_h$ satisfying (3.28), (3.27) and (3.14). By virtue of the strong energy estimates (3.57) and the discrete inf-sup conditions, we now can establish an estimate for $p_h$.

THEOREM 3.10. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$*, and* $\mathbf{u}_1$ *satisfy* (2.6) *and* (3.46)*. Assume further that there exists a* $p_0 \in H^1(\Omega_1)$ *such that* (3.47) *holds. Let* $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in C^1([0,T]; X_1^h) \times C([0,T]; Q_1^h) \times C^1([0,T]; X_2^h)$ *be the solution for* (3.14)–(3.19) *with the initial conditions* $(\mathbf{v}_{0,h}, \mathbf{u}_{1,h})$ *defined by* (3.32)–(3.33)*. Then* $p_h$ *satisfies the estimate*

$$\begin{aligned} &\|p_h\|^2_{L^2(0,T;L^2(\Omega_1))} \\ &\qquad \le Ce^{CT}\Big(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{2,\Omega_2}\Big). \end{aligned} \tag{3.58}$$

*Proof.* We observe that from (3.28) we have

$$\begin{aligned} \|p_h\|_{L^2(0,T;L^2(\Omega_1))} \le C\Big(&\|\partial_t \boldsymbol{\xi}_h\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \\ &+ \|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\xi}_h\|_{L^2(0,T;X_1)} + \|\textstyle\int_0^t \boldsymbol{\xi}_h(s)\,ds\|_{L^2(0,T;X_1)}\Big). \end{aligned}$$

Thus, (3.58) follows from the last relation and energy estimate (3.50) for $\boldsymbol{\xi}_h$. ∎

*Remark* 3. Note that Theorems 3.8, 3.9, and 3.10 require the specification of an initial pressure $p_0$ and the initial interface stress condition (3.47). From a physical point of view, these requirements are entirely reasonable.

**4. The convergence of finite element solutions and error estimates.** Having proved the existence of finite element solutions $(\mathbf{v}_h, p_h, \mathbf{u}_h)$ for problem (3.14)–(3.19) and (3.32)–(3.33), we now prove the convergence of the finite element solutions and derive error estimates.

**4.1. The convergence of finite element solutions.** We first consider the convergence of the finite element approximations.

THEOREM 4.1. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0$*, and* $\mathbf{u}_1$ *satisfy* (2.6) *and* (3.46) *and that there exists a* $p_0 \in H^1(\Omega_1)$ *such that* (3.47) *holds. Let* $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in C^1([0,T]; X_1^h) \times C([0,T]; Q_1^h) \times C^1([0,T]; X_2^h)$ *be the unique solution of* (3.14)–(3.19) *with the initial conditions* $(\mathbf{v}_{0,h}, \mathbf{u}_{1,h})$ *defined by* (3.32)–(3.33)*. Assume further that the finite element meshes are nested, i.e., that the triangulation* $\mathcal{T}^{h_2}(\Omega)$ *is a refinement of the triangulation* $\mathcal{T}^{h_1}(\Omega)$ *whenever* $h_2 < h_1$*. Then, there exists a unique* $(\mathbf{v}, p, \mathbf{u})$ *such that*

$$\begin{cases} \mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\Omega_1)) \cap L^2(0,T;X_1), \\ \partial_t \mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\Omega_1)) \cap L^2(0,T;X_1), \quad p \in L^2(0,T;L^2(\Omega_1)), \\ \mathbf{u} \in L^\infty(0,T;X_2), \quad \partial_t \mathbf{u} \in L^\infty(0,T;X_2), \quad \partial_{tt}\mathbf{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega_2)), \end{cases} \tag{4.1}$$

$$\mathbf{v}_h \rightharpoonup \mathbf{v} \quad \text{in } L^2(0,T;X_1), \qquad \mathbf{v}_h \overset{*}{\rightharpoonup} \mathbf{v} \quad \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega_1)), \tag{4.2}$$

$$\partial_t \mathbf{v}_h \overset{*}{\rightharpoonup} \partial_t \mathbf{v} \quad \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega_1)), \qquad \partial_t \mathbf{v}_h \rightharpoonup \partial_t \mathbf{v} \quad \text{in } L^2(0,T;X_1), \tag{4.3}$$

$$\mathbf{u}_h \overset{*}{\rightharpoonup} \mathbf{u} \quad \textit{in } L^\infty(0,T;X_2), \tag{4.4}$$

$$\partial_t\mathbf{u}_h \overset{*}{\rightharpoonup} \partial_t\mathbf{u} \quad \textit{in } L^\infty(0,T;\mathbf{L}^2(\Omega_1)), \qquad \partial_t\mathbf{u}_h \overset{*}{\rightharpoonup} \partial_t\mathbf{u} \quad \textit{in } L^\infty(0,T;X_2), \tag{4.5}$$

$$\partial_{tt}\mathbf{u}_h \overset{*}{\rightharpoonup} \partial_{tt}\mathbf{u} \quad \textit{in } L^\infty(0,T;\mathbf{L}^2(\Omega_2)), \tag{4.6}$$

*and*

$$p^h \rightharpoonup p \quad \textit{weakly in } L^2(0,T;L^2(\Omega_1)). \tag{4.7}$$

*Furthermore,* $(\mathbf{v},p,\mathbf{u})$ *satisfies* (3.3)–(3.6) *and the estimates*

$$\begin{aligned}\|\mathbf{v}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_1))} &+ \|\partial_t\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_2))} \\ &+ \|\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_2))} \\ \le Ce^{CT}(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} &+ \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{2,\Omega_2})\end{aligned} \tag{4.8}$$

*and*

$$\begin{aligned}\|\partial_t\mathbf{v}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_1))} &+ \|\partial_t\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_1))} + \|p\|^2_{L^2(0,T;L^2(\Omega_1))} \\ &+ \|\partial_{tt}\mathbf{u}(t)\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_2))} + \|\partial_t\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_2))} \\ \le Ce^{CT}(\|\mathbf{f}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} &+ \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|^2_{1,\Omega_1} + \|\mathbf{u}_1\|^2_{2,\Omega_2}).\end{aligned} \tag{4.9}$$

*Proof.* We have that $\{(\mathbf{v}_h,p_h,\mathbf{u}_h)\}$ satisfies the estimates (3.56)–(3.57) and (3.58). Using these estimates, we may extract a subsequence $\{(\mathbf{v}_{h_n},p_{h_n},\mathbf{u}_{h_n})\}$ of $\{(\mathbf{v}_h,p_h,\mathbf{u}_h)\}$, with $\{h_n\}$ decreasing to 0 as $n\to\infty$, such that (4.2)–(4.7) hold for the subsequence $\{(\mathbf{v}_{h_n},p_{h_n},\mathbf{u}_{h_n})\}$ for a $(\mathbf{v},p,\mathbf{u})$ satisfying (4.1).

Equation (3.17) holds for $h=h_n$, and thus, by passing to the limit as $n\to\infty$ in that equation, we obtain (3.6). Also, $\mathbf{u}(0)=\mathbf{u}_0$ trivially holds.

To prove that $(\mathbf{v},p,\mathbf{u})$ satisfies (3.3) we begin from (3.14) with $h=h_n$. We arbitrarily fix an integer $N$ and a function $\boldsymbol{\eta}\in C^1([0,T];X^{h_N})$. For each $n>N$ we obtain from (3.14) and the nesting assumption on the triangulation family $\mathcal{T}^h(\Omega)$ that

$$\begin{aligned}\int_0^T \Big(\rho_1[\partial_t\mathbf{v}_{h_n},\boldsymbol{\eta}]_{\Omega_1} + a_1[\mathbf{v}_{h_n},\boldsymbol{\eta}] + b[\boldsymbol{\eta},p_{h_n}] + \rho_2[\partial_{tt}\mathbf{u}_{h_n},\boldsymbol{\eta}]_{\Omega_2} + a_2[\mathbf{u}_{h_n},\boldsymbol{\eta}]\Big)\,dt \\ = \int_0^T \Big(\rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2}\Big)\,dt.\end{aligned} \tag{4.10}$$

Passing to the limit as $n\to\infty$, we find

$$\begin{aligned}\int_0^T \Big(\rho_1[\partial_t\mathbf{v},\boldsymbol{\eta}]_{\Omega_1} + a_1[\mathbf{v},\boldsymbol{\eta}] + b[\boldsymbol{\eta},p] + \rho_2[\partial_{tt}\mathbf{u},\boldsymbol{\eta}]_{\Omega_2} + a_2[\mathbf{u},\boldsymbol{\eta}]\Big)\,dt \\ = \int_0^T \Big(\rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2}\Big)\,dt.\end{aligned} \tag{4.11}$$

Equality (4.11) then holds for all $\boldsymbol{\eta} \in L^2(0,T;\mathbf{H}_0^1(\Omega))$, as $\bigcup_{n=N}^{\infty} C([0,T];X^{h_n})$ is dense in $L^2(0,T;\mathbf{H}_0^1(\Omega))$ for the $L^2(0,T;\mathbf{H}_0^1(\Omega))$ norm. Hence,

$$\rho_1[\partial_t \mathbf{v},\boldsymbol{\eta}]_{\Omega_1} + a_1[\mathbf{v},\boldsymbol{\eta}] + b[\boldsymbol{\eta},p] + \rho_2[\partial_{tt}\mathbf{u},\boldsymbol{\eta}]_{\Omega_2} + a_2[\mathbf{u},\boldsymbol{\eta}] \\ = \rho_1[\mathbf{f}_1,\boldsymbol{\eta}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\boldsymbol{\eta}]_{\Omega_2} \qquad \forall\,\boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega),\ \text{a.e. } t,$$

which is precisely (3.3).

From (3.15) we obtain

$$\int_0^T b[\mathbf{v}_{h_n},q]\,ds = 0$$

for all $q \in L^2(0,T;Q_1^{h_N})$ and all $n \geq N$. Passing to the limit as $n \to \infty$ leads us to

$$\int_0^T b[\mathbf{v},q]\,ds = 0 \tag{4.12}$$

for all $q \in L^2(0,T;Q_1^{h_N})$. Using the denseness (with respect to the $L^2(0,T;L^2(\Omega_1))$ norm) of $\bigcup_{n=N}^{\infty} L^2(0,T;Q_1^{h_n})$ in $L^2(0,T;L^2(\Omega_1))$, we see that (4.12) holds for all $q \in L^2(0,T;L^2(\Omega_1))$. In particular, this implies (3.4).

To verify the initial condition (3.5) we first note that the regularity results (4.1) imply that $\mathbf{v} \in C([0,T];\mathbf{L}^2(\Omega_1)) \cap C([0,T;X_1)$, $\mathbf{u} \in C([0,T];\mathbf{L}^2(\Omega_2)) \cap C([0,T;X_2)$, and $\partial_t\mathbf{u} \in C([0,T];\mathbf{L}^2(\Omega_2))$. For each $\boldsymbol{\eta} \in C^1([0,T];\mathbf{H}_0^1(\Omega))$ with $\boldsymbol{\eta}(T) = \mathbf{0}$ we obtain, from (4.11), by integration by parts that

$$\int_0^T \Big( -\rho_1[\mathbf{v},\partial_t\boldsymbol{\eta}]_{\Omega_1} - \rho_2[\partial_t\mathbf{u},\partial_t\boldsymbol{\eta}]_{\Omega_2} + a_1[\mathbf{v},\boldsymbol{\eta}] + b[\boldsymbol{\eta},\widehat{p}] + a_2[\mathbf{u},\boldsymbol{\eta}]\Big)\,dt \\ = \int_0^T [[\mathbf{f},\boldsymbol{\eta}]]\,dt + \rho_1[\mathbf{v}(0),\boldsymbol{\eta}(0)]_{\Omega_1} + \rho_2[\partial_t\mathbf{u}(0),\boldsymbol{\eta}(0)]_{\Omega_2}. \tag{4.13}$$

On the other hand, from (4.10), we deduce that for all $\boldsymbol{\eta} \in C^1([0,T];X^{h_N})$ and all $n > N$,

$$\int_0^T \Big( -\rho_1[\mathbf{v}_{h_n},\partial_t\boldsymbol{\eta}]_{\Omega_1} - \rho_2[\partial_t\mathbf{u}_{h_n},\partial_t\boldsymbol{\eta}]_{\Omega_2} \\ + a_1[\mathbf{v}_{h_n},\boldsymbol{\eta}] + b[\boldsymbol{\eta},p_{h_n}] + a_2[\mathbf{u}_{h_n},\boldsymbol{\eta}]\Big)\,dt \\ = \int_0^T [[\mathbf{f},\boldsymbol{\eta}]]\,dt + \rho_1[\mathbf{v}_{h_n}(0),\boldsymbol{\eta}(0)]_{\Omega_1} + \rho_2[\partial_t\mathbf{u}_{h_n}(0),\boldsymbol{\eta}(0)]_{\Omega_2}. \tag{4.14}$$

Holding $N$ fixed and passing to the limit as $n \to \infty$ in (4.14) and utilizing (3.35), we arrive at

$$\int_0^T \Big( -\rho_1[\mathbf{v},\partial_t\boldsymbol{\eta}]_{\Omega_1} - \rho_2[\partial_t\mathbf{u},\partial_t\boldsymbol{\eta}]_{\Omega_2} + a_1[\mathbf{v},\boldsymbol{\eta}] + b[\boldsymbol{\eta},\widehat{p}] + a_2[\mathbf{u},\boldsymbol{\eta}]\Big)\,dt \\ = \int_0^T [[\mathbf{f},\boldsymbol{\eta}]]\,dt + \rho_1[\mathbf{v}_0,\boldsymbol{\eta}(0)]_{\Omega_1} + \rho_2[\mathbf{u}_1,\boldsymbol{\eta}(0)]_{\Omega_2} \tag{4.15}$$

for all $\boldsymbol{\eta} \in C^1([0,T];X^{h_N})$. Comparing (4.13) and (4.15), we obtain

$$\rho_1[\mathbf{v}(0) - \mathbf{v}_0,\boldsymbol{\eta}(0)]_{\Omega_1} + \rho_2[\partial_t\mathbf{u}(0) - \mathbf{u}_1,\boldsymbol{\eta}(0)]_{\Omega_2} = 0 \tag{4.16}$$

for all $\boldsymbol{\eta}(0) \in X^{h_N}$. Since $\bigcup_{n=N}^{\infty} X^{h_n}$ is dense in $\mathbf{L}^2(\Omega)$ for the $\mathbf{L}^2(\Omega)$ norm, we derive

$$\mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \mathbf{L}^2(\Omega_1) \qquad \text{and} \qquad \partial_t \mathbf{u}(0) = \mathbf{u}_1 \quad \text{in } \mathbf{L}^2(\Omega_2).$$

To check $\mathbf{u}(0) = \mathbf{u}_0$ we first note that with regularity (4.1) we are justified to write

$$\mathbf{u} = \mathbf{u}(0) + \int_0^t \partial_t \mathbf{u}(s)\, ds. \tag{4.17}$$

From the compact embedding $H^1(0,T;B) \hookrightarrow\hookrightarrow L^2(0,T;B)$ for any Banach space $B$ and the weak convergence (4.2)–(4.5) we deduce that for a further subsequence $h_{n_j}$ we have

$$\partial_t \mathbf{u}_{h_{n_j}} \to \partial_t \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega_2)) \qquad \text{and} \qquad \mathbf{u}_{h_{n_j}} \to \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega_2)),$$

so that, passing to the limit in the relation

$$\mathbf{u}_{h_n} = \mathbf{u}_{0,h_n} + \int_0^t \partial_t \mathbf{u}_{h_n}(s)\, ds$$

and noting that $\|\mathbf{u}_{0,h} - \mathbf{u}_0\|_{0,\Omega_2} \to 0$ as $h \to 0$, we obtain

$$\mathbf{u} = \mathbf{u}_0 + \int_0^t \partial_t \mathbf{u}(s)\, ds. \tag{4.18}$$

A comparison of (4.17) and (4.18) yields $\mathbf{u}(0) = \mathbf{u}_0$.

Hence we have verified that $(\mathbf{v}, p, \mathbf{u})$ satisfies (3.1)–(3.6). Of course, $(\mathbf{v}, \mathbf{u})$ is also a solution for (2.7)–(2.10), so that, by Theorem 2.2, $(\mathbf{v}, \mathbf{u})$ is the unique solution of (2.7)–(2.10) and estimate (4.8) holds. Then, by Theorem 2.3, we obtain the uniqueness of $p$. Estimate (4.9) follows from (3.57) and (3.58).

Finally, it follows from the uniqueness of the limit $(\mathbf{v}, p, \mathbf{u})$ that the entire family of finite element solutions $(\mathbf{v}_h, p_h, \mathbf{u}_h)$ satisfies (4.2)–(4.7) as $h \to 0$. ∎

We also have the following strong convergence, the proof of which is contained in that of Theorem 4.1.

Corollary 4.2. *Assume that all hypotheses of Theorem* 4.1 *hold. Then*

$$\mathbf{v}_h \to \mathbf{v} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega_1)), \qquad \mathbf{u}_h \to \mathbf{u} \quad \text{in } L^2(0,T;X_2)$$

*and*

$$\partial_t \mathbf{u}_h \to \partial_t \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega_2)).$$

**4.2. Error estimates for finite element approximations.** We will estimate the error between the continuous solution defined by (3.3)–(3.6) and the finite element solution defined by (3.14)–(3.19) and (3.32)–(3.33). To this end we introduce the weighted $\mathbf{L}^2(\Omega)$ projection operator onto the discretely divergence-free space $\boldsymbol{\Psi}^h$. ($\boldsymbol{\Psi}^h$ is discretely divergence-free in $\Omega_1$.)

The projection operator $\mathcal{P}^h : \mathbf{L}^2(\Omega) \to \boldsymbol{\Psi}^h$ with respect to the weighted $\mathbf{L}^2(\Omega)$ inner product is defined as follows: for every $\boldsymbol{\eta} \in \mathbf{L}^2(\Omega)$, $\mathcal{P}^h\boldsymbol{\eta} \in \boldsymbol{\Psi}^h$ is the solution of

$$[[\mathcal{P}^h\boldsymbol{\eta}, \mathbf{z}^h]] = [[\boldsymbol{\eta}, \mathbf{z}^h]] \qquad \forall \mathbf{z}^h \in \boldsymbol{\Psi}^h. \tag{4.19}$$

Note that the definition of $\mathbf{\Psi}^h$ implies

$$b[\mathcal{P}^h \boldsymbol{\eta}, q^h] = 0 \qquad \forall\, q^h \in Q_1^h. \tag{4.20}$$

We assume that the domains $\Omega_1$ and $\Omega_2$ satisfy the following regularity assumptions.

*Hypothesis* (H1). The problem

$$\begin{cases} (\bar{\mathbf{v}}, \bar{p}) \in \mathbf{H}_0^1(\Omega_1) \times L_0^2(\Omega_1), \\ [\nabla \bar{\mathbf{v}}, \nabla \mathbf{z}]_{\Omega_1} + b[\mathbf{z}, \bar{p}] = [\bar{\mathbf{f}}_1, \mathbf{z}]_{\Omega_1} \qquad \forall\, \mathbf{z} \in \mathbf{H}_0^1(\Omega_1), \\ b[\mathbf{v}, q] = 0 \qquad \forall\, q \in L^2(\Omega_1) \end{cases} \tag{4.21}$$

is $\mathbf{H}^{2-\epsilon_1}$ regular for an $\epsilon_1 \in (0,1)$; i.e., for every $\bar{\mathbf{f}}_1 \in \mathbf{L}^2(\Omega_1)$, the solution $(\bar{\mathbf{v}}, \bar{p})$ to problem (4.21) belongs to $\mathbf{H}^{2-\epsilon_1}(\Omega_1) \times H^{1-\epsilon_1}(\Omega_1)$, $-\bar{p}\mathbf{n}_1 + (\nabla \bar{\mathbf{v}} + \nabla \bar{\mathbf{v}}^T)\mathbf{n}_1 \in \mathbf{H}^{1/2-\epsilon_1}(\Gamma_0)$, and

$$\|\bar{\mathbf{v}}\|_{\mathbf{H}^{2-\epsilon_1}(\Omega_1)} + \|\bar{p}\|_{H^{1-\epsilon_1}(\Omega_1)} \| - \bar{p}\mathbf{n}_1 + (\nabla \bar{\mathbf{v}} + \nabla \bar{\mathbf{v}}^T)\mathbf{n}_1\|_{1/2-\epsilon_1,\Gamma_0} \le C \|\bar{\mathbf{f}}_1\|_{0,\Omega_1}.$$

*Hypothesis* (H2). The problem

$$\begin{cases} \bar{\mathbf{u}} \in \mathbf{H}_0^1(\Omega_1), \\ [\nabla \bar{\mathbf{u}}, \nabla \mathbf{w}]_{\Omega_1} = [\bar{\mathbf{f}}_2, \mathbf{w}]_{\Omega_1} \qquad \forall\, \mathbf{w} \in \mathbf{H}_0^1(\Omega_2) \end{cases} \tag{4.22}$$

is $\mathbf{H}^{2-\epsilon_2}$ regular for an $\epsilon_2 \in (0,1)$; i.e., for every $\bar{\mathbf{f}}_2 \in \mathbf{L}^2(\Omega_2)$, the solution $\bar{\mathbf{u}}$ to problem (4.22) belongs to $\mathbf{H}^{2-\epsilon_2}(\Omega_2)$, $\nabla \bar{\mathbf{u}} \cdot \mathbf{n}_2 \in \mathbf{H}^{1/2-\epsilon_2}(\Gamma_0)$, and

$$\|\bar{\mathbf{u}}\|_{\mathbf{H}^{2-\epsilon_2}(\Omega_2)} \|\nabla \bar{\mathbf{u}} \cdot \mathbf{n}_2\|_{1/2-\epsilon_2,\Gamma_0} \le C \|\bar{\mathbf{f}}_2\|_{0,\Omega_2}.$$

*Remark* 4. Hypotheses (H1)–(H2) are simply equivalent to angle conditions on $\Omega_1$ and $\Omega_2$ owing to the well-known regularity results on polygonal domains for boundary value problems (4.21) and (4.22); see [24] and [19]. In particular, if both $\Omega_1$ and $\Omega_2$ are convex (in which case $\Gamma_0$ is necessarily a straight line), then $\epsilon_1$ and $\epsilon_2$ can be chosen arbitrarily small.

Under Hypotheses (H1)–(H2), we may prove the following error estimates for the projection operator $\mathcal{P}^h$:

$$\begin{aligned} &\|\boldsymbol{\zeta} - \mathcal{P}^h \boldsymbol{\zeta}\|_{1,\Omega} \le C h^{r-\epsilon} (\|\boldsymbol{\zeta}\|_{r+1,\Omega_1} + \|\boldsymbol{\zeta}\|_{r+1,\Omega_2}) \\ &\qquad \forall\, \boldsymbol{\zeta} \in \mathbf{\Psi} \quad \text{with } \boldsymbol{\zeta}|_{\Omega_i} \in \mathbf{H}^{r+1}(\Omega_i), \quad i = 1,2, r \in [0,k], \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} &\|\boldsymbol{\zeta} - \mathcal{P}^h \boldsymbol{\zeta}\|_{0,\Omega} \le C h^{r+1-\epsilon} (\|\boldsymbol{\zeta}\|_{r+1,\Omega_1} + \|\boldsymbol{\zeta}\|_{r+1,\Omega_2}) \\ &\qquad \forall\, \boldsymbol{\zeta} \in \mathbf{\Psi} \quad \text{with } \boldsymbol{\zeta}|_{\Omega_i} \in \mathbf{H}^{r+1}(\Omega_i), \quad i = 1,2, r \in [0,k]. \end{aligned} \tag{4.24}$$

The proof of (4.23)–(4.24) will be given in the appendix, Theorem A.3.

Now we prove the following error estimates for the semidiscrete finite element approximations of the fluid-solid interaction problem.

THEOREM 4.3. *Assume that* $\mathbf{f}_1, \mathbf{v}_0, \mathbf{f}_2, \mathbf{u}_0,$ *and* $\mathbf{u}_1$ *satisfy* (2.6) *and* (3.46) *and that there exists a* $p_0 \in H^1(\Omega_1)$ *such that* (3.47) *holds. Assume also that* (H1)–(H2) *hold.*

*Let* $(\mathbf{v}, p, \mathbf{u})$ *be the solution of* (3.1)–(3.6)*, and* $(\mathbf{v}_h, p_h, \mathbf{u}_h)$ *be the solution of* (3.14)–(3.19) *and* (3.32)–(3.33)*. Assume that for some* $r \in [1, k]$, $\mathbf{v} \in L^2(0,T;\mathbf{H}^{r+1}(\Omega_1))$, $\partial_t \mathbf{v} \in L^2(0,T;\mathbf{H}^{r-1}(\Omega_1))$, $p \in L^2(0,T;H^r(\Omega_1))$, $\partial_t \mathbf{u} \in L^2(0,T;\mathbf{H}^{r+1}(\Omega_2))$, $\partial_{tt}\mathbf{u} \in L^2(0,T;\mathbf{H}^{r-1}(\Omega_2))$, $\mathbf{v}_0 \in \mathbf{H}^{r+1}(\Omega_1)$, $\mathbf{u}_1 \in \mathbf{H}^{r+1}(\Omega_2)$, $\mathbf{u}_0 \in \mathbf{H}^{r+1}(\Omega_2)$*, and* $p_0 \in H^r(\Omega_1)$*. Then,*

$$
\begin{aligned}
&\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{0,\Omega_1} + \|\mathbf{v} - \mathbf{v}_h\|^2_{L^2(0,T;X_1)} \\
&\quad + \|\partial_t \mathbf{u}(t) - \partial_t \mathbf{u}_h(t)\|^2_{0,\Omega_2} + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{1,\Omega_2} \\
&\leq Ce^{CT}h^{2r}\big(\|\mathbf{v}_0\|^2_{r+1,\Omega_1} + \|\mathbf{u}_1\|^2_{r+1,\Omega_2} + \|\mathbf{u}_0\|^2_{r+1,\Omega_2} + \|p_0\|^2_{r,\Omega_1} \\
&\quad + \|p\|^2_{L^2(0,T;H^r(\Omega_1))}\big) + Ce^{CT}h^{2(r-\epsilon)}\big(\|\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_1))} \\
&\quad + \|\mathbf{u}_t\|^2_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_2))} + \|\partial_t \mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_1))} + \|\partial_{tt}\mathbf{u}\|^2_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_2))}\big)
\end{aligned} \tag{4.25}
$$

*for all* $t \in [0, T]$*.*

*Proof.* Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_h$ be defined by (2.11) and (3.20), respectively. We set $\widetilde{\mathbf{v}}_h(t) = [\mathcal{P}^h\boldsymbol{\xi}(t)]|_{\Omega_1}$ and $\widetilde{\mathbf{w}}_h(t) = [\mathcal{P}^h\boldsymbol{\xi}(t)]|_{\Omega_2}$.

By subtracting (3.14)–(3.15) from the corresponding equations of (3.3)–(3.4), we obtain the following "orthogonality conditions":

$$
\begin{aligned}
&\rho_1[\partial_t \mathbf{v} - \partial_t \mathbf{v}_h, \boldsymbol{\eta}_h]_{\Omega_1} + b[\boldsymbol{\eta}_h, p - p_h] + a_1[\mathbf{v} - \mathbf{v}_h, \boldsymbol{\eta}_h] \\
&\qquad + \rho_2[\mathbf{u}_{tt} - \partial_{tt}\mathbf{u}_h, \boldsymbol{\eta}_h]_{\Omega_2} + a_2[\mathbf{u} - \mathbf{u}_h, \boldsymbol{\eta}_h] = 0 \qquad \forall\, \boldsymbol{\eta}_h \in X^h, \text{ a.e. } t,
\end{aligned} \tag{4.26}
$$

$$
b[\mathbf{v} - \mathbf{v}_h, q_h] = 0 \qquad \forall\, q_h \in Q_1^h, \text{ a.e. } t. \tag{4.27}
$$

By adding/subtracting terms and using (4.26)–(4.27), we deduce that

$$
\begin{aligned}
&\rho_1[\partial_t \mathbf{v}_h - \partial_t \mathbf{v}_h, \mathbf{v} - \mathbf{v}_h]_{\Omega_1} + a_1[\mathbf{v} - \mathbf{v}_h, \mathbf{v} - \mathbf{v}_h] \\
&\qquad + \rho_2[\partial_{tt}\mathbf{u} - \partial_{tt}\mathbf{u}_h, \partial_t \mathbf{u} - \partial_t \mathbf{u}_h]_{\Omega_2} + a_2[\mathbf{u} - \mathbf{u}_h, \partial_t \mathbf{u} - \mathbf{u}_h] \\
&\quad = \rho_1[\partial_t \mathbf{v} - \partial_t \mathbf{v}_h, \mathbf{v} - \widetilde{\mathbf{v}}_h]_{\Omega_1} + a_1[\mathbf{v} - \mathbf{v}_h, \mathbf{v} - \widetilde{\mathbf{v}}_h] \\
&\qquad + \rho_2[\partial_{tt}\mathbf{u} - \partial_{tt}\mathbf{u}_h, \partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h]_{\Omega_2} + a_2[\mathbf{u} - \mathbf{u}_h, \partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h] \\
&\qquad - b[\widetilde{\mathbf{v}}_h - \mathbf{v}_h, p - p_h] + \rho_1[\partial_t \mathbf{v} - \partial_t \mathbf{v}_h, \widetilde{\mathbf{v}}_h - \mathbf{v}_h]_{\Omega_1} \\
&\qquad + a_1[\mathbf{v} - \mathbf{v}_h, \widetilde{\mathbf{v}} - \mathbf{v}_h] + \rho_2[\partial_{tt}\mathbf{u} - \partial_{tt}\mathbf{u}_h, \widetilde{\mathbf{w}}_h - \partial_t \mathbf{u}_h]_{\Omega_2} \\
&\qquad + a_2[\mathbf{u} - \mathbf{u}_h, \widetilde{\mathbf{w}}_h - \partial_t \mathbf{u}_h] + b[\widetilde{\mathbf{v}}_h - \mathbf{v}_h, p - p_h] \\
&\quad = \rho_1[\partial_t \mathbf{v} - \partial_t \mathbf{v}_h, \mathbf{v} - \widetilde{\mathbf{v}}_h]_{\Omega_1} + a_1[\mathbf{v} - \mathbf{v}_h, \mathbf{v} - \widetilde{\mathbf{v}}_h] \\
&\qquad + \rho_2[\partial_{tt}\mathbf{u} - \partial_{tt}\mathbf{u}_h, \partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h]_{\Omega_2} + a_2[\mathbf{u} - \mathbf{u}_h, \partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h] \\
&\qquad + b[\mathbf{v}_h - \widetilde{\mathbf{v}}_h, p - p_h].
\end{aligned} \tag{4.28}
$$

By the definition of $\widetilde{\mathbf{v}}_h$ and (4.20), we obtain

$$
b[\widetilde{\mathbf{v}}_h(t), p_h] = b[\mathcal{P}^h\boldsymbol{\xi}(t), p_h] = 0 = b[\mathcal{P}^h\boldsymbol{\xi}(t), q_h] = b[\widetilde{\mathbf{v}}_h(t), q_h] \qquad \forall\, q_h \in Q_1^h\,. \tag{4.29}
$$

Utilizing (3.15), we have

$$b[\mathbf{v}_h(t), p_h] = 0 = b[\mathbf{v}_h(t), q_h] \qquad \forall\, q_h \in Q_1^h. \tag{4.30}$$

Additionally,

$$\begin{aligned}
&\rho_1[\partial_t \mathbf{v} - \partial_t \mathbf{v}_h, \mathbf{v} - \widetilde{\mathbf{v}}_h]_{\Omega_1} + \rho_2[\partial_{tt}\mathbf{u} - \partial_{tt}\mathbf{u}_h, \partial_t \mathbf{u} - \widetilde{\mathbf{w}}]_{\Omega_2} \\
&\quad = [[\partial_t \boldsymbol{\xi}(t) - \partial_t \boldsymbol{\xi}_h(t), \boldsymbol{\xi}(t) - \mathcal{P}^h \boldsymbol{\xi}(t)]] = [[\partial_t \boldsymbol{\xi}(t), \boldsymbol{\xi}(t) - \mathcal{P}^h \boldsymbol{\xi}(t)]] \\
&\quad = [[\partial_t \boldsymbol{\xi}(t) - \partial_t \mathcal{P}^h \boldsymbol{\xi}(t), \boldsymbol{\xi}(t) - \mathcal{P}^h \boldsymbol{\xi}(t)]] \\
&\quad = \frac{1}{2}\frac{d}{dt}[[\boldsymbol{\xi}(t) - \mathcal{P}^h \boldsymbol{\xi}(t), \boldsymbol{\xi}(t) - \mathcal{P}^h \boldsymbol{\xi}(t)]] \\
&\quad = \frac{\rho_1}{2}\frac{d}{dt}\|\mathbf{v} - \widetilde{\mathbf{v}}_h\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\frac{d}{dt}\|\partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h\|_{0,\Omega_2}^2.
\end{aligned} \tag{4.31}$$

Combining (4.28)–(4.31), we deduce that for all $q_h \in L^2(0,T;Q_1^h)$

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\|\mathbf{v} - \mathbf{v}_h\|_{0,\Omega_1}^2 + a_1[\mathbf{v} - \mathbf{v}_h, \mathbf{v} - \mathbf{v}_h] + \frac{\rho_2}{2}\frac{d}{dt}\|\partial_t \mathbf{u} - \partial_t \mathbf{u}_h\|_{0,\Omega_2}^2 \\
&\qquad + \frac{1}{2}\frac{d}{dt} a_2[\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h] \\
&= \frac{\rho_1}{2}\frac{d}{dt}\|\mathbf{v} - \widetilde{\mathbf{v}}_h\|_{0,\Omega_1}^2 + a_1[\mathbf{v} - \mathbf{v}_h, \mathbf{v} - \widetilde{\mathbf{v}}_h] + \frac{\rho_2}{2}\frac{d}{dt}\|\partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h\|_{0,\Omega_2}^2 \\
&\qquad + a_2[\mathbf{u} - \mathbf{u}_h, \partial_t \mathbf{u} - \widetilde{\mathbf{w}}^h] + b[\mathbf{v}_h - \widetilde{\mathbf{v}}_h, p - q_h] \\
&\le \frac{\rho_1}{2}\frac{d}{dt}\|\mathbf{v} - \widetilde{\mathbf{v}}_h\|_{0,\Omega_1}^2 + \frac{k_1}{4}\|\mathbf{v}(t) - \mathbf{v}_h(t)\|_{1,\Omega_1}^2 + C\|\mathbf{v}(t) - \widetilde{\mathbf{v}}_h(t)\|_{1,\Omega_1}^2 \\
&\qquad + \frac{\rho_2}{2}\frac{d}{dt}\|\partial_t \mathbf{u} - \widetilde{\mathbf{w}}_h\|_{0,\Omega_2}^2 + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{0,\Omega_2}^2 + C\|\partial_t \mathbf{u}(t) - \widetilde{\mathbf{w}}_h(t)\|_{1,\Omega_2}^2 \\
&\qquad + C\|\mathbf{v}(t) - \widetilde{\mathbf{v}}_h(t)\|_{1,\Omega_2}^2 + \frac{k_1}{4}\|\mathbf{v}(t) - \mathbf{v}_h(t)\|_{1,\Omega_2}^2 + C\|p(t) - q_h\|_{0,\Omega_1}^2.
\end{aligned}$$

Applying (2.2)–(2.3) to the last relation and integrating in $t$, we obtain

$$\begin{aligned}
&\rho_1\|\mathbf{v}(t) - \mathbf{v}_h(t)\|_{0,\Omega_1}^2 + k_1\|\mathbf{v} - \mathbf{v}_h\|_{L^2(0,T;\mathbf{H}^1(\Omega_1))}^2 \\
&\qquad + \rho_2\|\partial_t \mathbf{u}(t) - \partial_t \mathbf{u}_h(t)\|_{0,\Omega_2}^2 + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{1,\Omega_2}^2 \\
&\le C\Big(\|\mathbf{v}(0) - \mathbf{v}_{0,h}\|_{0,\Omega_1}^2 + \|\partial_t \mathbf{u}(0) - \mathbf{u}_{1,h}\|_{0,\Omega_2}^2 + \|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega_2}^2 \\
&\qquad + \|\boldsymbol{\xi}_0 - \mathcal{P}^h \boldsymbol{\xi}_0\|_{0,\Omega}^2 + \|\boldsymbol{\xi}(t_0) - \mathcal{P}^h \boldsymbol{\xi}(t_0)\|_{0,\Omega_1}^2 + \|\boldsymbol{\xi} - \mathcal{P}^h \boldsymbol{\xi}\|_{L^2(0,T;\mathbf{H}^1(\Omega))}^2 \\
&\qquad + \|p - q_h\|_{L^2(0,T;L^2(\Omega_1))}^2\Big) + \int_0^t \|\mathbf{u}(s) - \mathbf{u}_h(s)\|_{1,\Omega_2}^2 ds
\end{aligned} \tag{4.32}$$

for all $q_h \in L^2(0,T;Q_1^h)$, where $t_0 \in [0,T]$ is such that

$$\|\boldsymbol{\xi}(t_0) - \mathcal{P}^h \boldsymbol{\xi}(t_0)\|_{0,\Omega}^2 = \max_{t\in[0,T]} \|\boldsymbol{\xi}(t) - \mathcal{P}^h \boldsymbol{\xi}(t)\|_{0,\Omega}^2 .$$

The error estimate (3.34) yields

$$\|\mathbf{v}_0 - \mathbf{v}_{0,h}\|_{0,\Omega_1}^2 + \|\mathbf{u}_1 - \mathbf{u}_{1,h}\|_{0,\Omega_2}^2 = Ch^{2r}(\|\mathbf{v}_0\|_{r+1,\Omega_1}^2 + \|\mathbf{u}_1\|_{r+1,\Omega_2}^2 + \|p_0\|_{r,\Omega_1}^2). \tag{4.33}$$

Equation (3.19) and the approximation properties imply

$$\|\mathbf{u}_0 - \mathbf{u}_{0,h}\|_{1,\Omega_2}^2 \le Ch^{2r}\|\mathbf{u}_0\|_{r+1,\Omega_1}^2. \tag{4.34}$$

Also, by virtue of (4.24), we have

$$\begin{aligned}\|\boldsymbol{\xi}(t_0) - \mathcal{P}^h\boldsymbol{\xi}(t_0)\|_{0,\Omega}^2 &\le Ch^{2r-2\epsilon}\Big(\|\mathbf{v}(t_0)\|_{r,\Omega_1}^2 + \|\partial_t\mathbf{u}(t_0)\|_{r,\Omega_2}^2\Big)\\ &\le Ch^{2r-2\epsilon}\Big(\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_1))}^2 + \|\partial_t\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_1))}^2\\ &\qquad + \|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_2))}^2 + \|\partial_t\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_2))}^2\Big).\end{aligned} \tag{4.35}$$

Thus, utilizing (4.33)–(4.35), (4.23), and (3.9), we may simplify (4.32) to

$$\begin{aligned}&\rho_1\|\mathbf{v}(t) - \mathbf{v}_h(t)\|_{0,\Omega_1}^2 + k_1\|\mathbf{v} - \mathbf{v}_h\|_{L^2(0,T;\mathbf{H}^1(\Omega_1))}^2\\ &\qquad + \rho_2\|\partial_t\mathbf{u}(t) - \partial_t\mathbf{u}_h(t)\|_{0,\Omega_2}^2 + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{1,\Omega_2}^2\\ &\le Ch^{2r}\Big(\|\mathbf{v}_0\|_{r+1,\Omega_1}^2 + \|\mathbf{u}_1\|_{r+1,\Omega_2}^2 + \|p_0\|_{r,\Omega_1}^2 + \|\mathbf{u}_0\|_{r+1,\Omega_1}^2\\ &\qquad + \|p\|_{L^2(0,T;H^r(\Omega_1))}^2\Big) + Ch^{2r-2\epsilon}\Big(\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_1))}^2\\ &\qquad + \|\partial_t\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_1))}^2 + \|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_2))}^2\\ &\qquad + \|\partial_t\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_2))}^2\Big) + \int_0^t \|\mathbf{u}(s) - \mathbf{u}_h(s)\|_{1,\Omega_2}^2 ds.\end{aligned} \tag{4.36}$$

By dropping the first three terms on the left-hand side of (4.36) and applying the Gronwall's inequality (3.53), we obtain

$$\begin{aligned}\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{1,\Omega_2}^2 &\le Ce^{CT}h^{2r}\Big[\|\mathbf{v}_0\|_{r+1,\Omega_1}^2 + \|\mathbf{u}_1\|_{r+1,\Omega_2}^2\\ &\qquad + \|p_0\|_{r,\Omega_1}^2 + \|\mathbf{u}_0\|_{r+1,\Omega_1}^2 + \|p\|_{L^2(0,T;H^r(\Omega_1))}^2\Big]\\ &\qquad + Ce^{CT}h^{2r-2\epsilon}\Big(\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_1))}^2 + \|\partial_t\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_1))}^2\\ &\qquad + \|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega_2))}^2 + \|\partial_t\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{r-1}(\Omega_2))}^2\Big).\end{aligned} \tag{4.37}$$

Hence, (4.25) follows from (4.36)–(4.37). □

**Appendix. Error estimates for the weighted $L^2$ projection onto $\Psi^h$.** The objective of this subsection is to prove error estimates (4.23)–(4.24) for the weighted $L^2$ projection operator $\mathcal{P}^h$ defined by (4.19).

We introduce an operator $\mathcal{S}^h : \boldsymbol{\Psi} \to \boldsymbol{\Psi}^h$ as follows. For each $\boldsymbol{\zeta} \in \boldsymbol{\Psi} \subset \mathbf{H}_0^1(\Omega)$,

$$\mathcal{S}^h\boldsymbol{\zeta} = \begin{cases} \boldsymbol{\zeta}_{1,h} & \text{in } \Omega_1,\\ \boldsymbol{\zeta}_{2,h} & \text{in } \Omega_2,\end{cases} \tag{A.1}$$

where $\boldsymbol{\zeta}_{1,h} \in X_1^h$ together with some $\sigma_h \in Q_1^h$ is the finite element solution of

$$\begin{cases} a_1[\boldsymbol{\zeta}_{1,h}, \mathbf{z}_h] + b[\mathbf{z}_h, \overline{\sigma}_h] = [\boldsymbol{\zeta}, \mathbf{z}_h] \qquad \forall\, \mathbf{z}_h \in X_1^h \cap \mathbf{H}_0^1(\Omega_1), \\ b[\boldsymbol{\zeta}_{1,h}, q_h] = 0 \qquad \forall\, q_h \in Q_1^h \cap L_0^2(\Omega_1), \\ \boldsymbol{\zeta}_{1,h}|_{\Gamma_1} = \mathbf{0} \qquad \text{and} \qquad [\boldsymbol{\zeta}_{1,h}, \mathbf{s}_h]_{0,\Gamma_0} = [\boldsymbol{\zeta}, \mathbf{s}_h]_{0,\Gamma_0} \quad \forall\, \mathbf{s}_h \in X_1^h|_{\Gamma_0}, \end{cases}$$

and $\boldsymbol{\zeta}_{2,h} \in X_2^h$ is the finite element solution of

$$\begin{cases} [\nabla\boldsymbol{\zeta}_{2,h}, \nabla\mathbf{w}_h]_{\Omega_2} = [\nabla\boldsymbol{\zeta}, \nabla\mathbf{w}_h]_{\Omega_2} \qquad \forall\, \mathbf{w}_h \in X_2^h \cap \mathbf{H}_0^1(\Omega_2), \\ \boldsymbol{\zeta}_{2,h}|_{\Gamma_2} = \mathbf{0} \qquad \text{and} \qquad [\boldsymbol{\zeta}_{2,h}, \mathbf{s}_h]_{0,\Gamma_0} = [\boldsymbol{\zeta}, \mathbf{s}_h]_{0,\Gamma_0} \quad \forall\, \mathbf{s}_h \in X_2^h|_{\Gamma_0}. \end{cases}$$

Evidently, $\boldsymbol{\zeta}_{1,h}|_{\Gamma_0} = \boldsymbol{\zeta}_{2,h}|_{\Gamma_0}$, so that $\mathcal{S}^h\boldsymbol{\zeta}$ defined by (A.1) indeed satisfies $\mathcal{S}^h\boldsymbol{\zeta} \in \boldsymbol{\Psi}^h$.

Using the results of [23, 25] concerning error estimates for the finite element approximations of the Stokes equations (noting that $\operatorname{div}\boldsymbol{\zeta}|_{\Omega_1} = 0$) with inhomogeneous boundary conditions, we obtain

$$\text{(A.2)} \qquad \|\boldsymbol{\zeta}_{1,h} - \boldsymbol{\zeta}\|_{1,\Omega_1} \le Ch^r \|\boldsymbol{\zeta}\|_{r+1,\Omega_1} \quad \text{if } \boldsymbol{\zeta}|_{\Omega_1} \in \mathbf{H}^{r+1}(\Omega_1).$$

Furthermore, under assumption (H1), we may adapt straightforwardly the proof in [23] for an Aubin–Nitsche-type result to obtain

$$\text{(A.3)} \qquad \|\boldsymbol{\zeta}_{1,h} - \boldsymbol{\zeta}\|_{0,\Omega_1} \le Ch^{1-\epsilon_1} \|\boldsymbol{\zeta}_{1,h} - \boldsymbol{\zeta}\|_{1,\Omega_1}.$$

Likewise,

$$\text{(A.4)} \qquad \|\boldsymbol{\zeta}_{2,h} - \boldsymbol{\zeta}\|_{1,\Omega_2} \le Ch^r \|\boldsymbol{\zeta}\|_{r+1,\Omega_2} \quad \text{if } \boldsymbol{\zeta}|_{\Omega_2} \in \mathbf{H}^{r+1}(\Omega_2),$$

and, under assumption (H2),

$$\text{(A.5)} \qquad \|\boldsymbol{\zeta}_{2,h} - \boldsymbol{\zeta}\|_{0,\Omega_2} \le Ch^{1-\epsilon_2} \|\boldsymbol{\zeta}_{2,h} - \boldsymbol{\zeta}\|_{1,\Omega_2}.$$

To summarize, we have the following results.

PROPOSITION A.1. *If* $\boldsymbol{\zeta} \in \boldsymbol{\Psi}$ *and* $\boldsymbol{\zeta}|_{\Omega_i} \in \mathbf{H}^{r+1}(\Omega_i)$ $(i = 1, 2)$ *for some* $r \in [0, k]$, *then*

$$\text{(A.6)} \qquad \|\mathcal{S}^h\boldsymbol{\zeta} - \boldsymbol{\zeta}\|_{1,\Omega} \le Ch^r (\|\boldsymbol{\zeta}\|_{r+1,\Omega_1} + \|\boldsymbol{\zeta}\|_{r+1,\Omega_2}).$$

*If, in addition, assumptions* (H1)–(H2) *hold, then*

$$\text{(A.7)} \qquad \|\mathcal{S}^h\boldsymbol{\zeta} - \boldsymbol{\zeta}\|_{0,\Omega} \le Ch^{1-\epsilon} \|\mathcal{S}^h\boldsymbol{\zeta} - \boldsymbol{\zeta}\|_{1,\Omega},$$

*where* $\epsilon = \max\{\epsilon_1, \epsilon_2\}$

The following proposition establishes relationships between approximation properties for the operator $\mathcal{P}^h$ and those for the operator $\mathcal{S}^h$.

PROPOSITION A.2. *Assume that* (H1)–(H2) *hold. Then,*

$$\text{(A.8)} \qquad \|\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}\|_{1,\Omega} \le Ch^{-\epsilon} \|\boldsymbol{\zeta} - \mathcal{S}^h\boldsymbol{\zeta}\|_{1,\Omega} \qquad \forall\, \boldsymbol{\zeta} \in \boldsymbol{\Psi}.$$

*Proof.* Let $\boldsymbol{\zeta} \in \boldsymbol{\Psi}$ be given. The best approximation property of a projection operator implies that

$$\text{(A.9)} \qquad \|\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}\|_{0,\Omega} \le \|\mathcal{S}^h\boldsymbol{\zeta} - \boldsymbol{\zeta}\|_{0,\Omega}.$$

Using the triangle inequality, the inverse inequality (3.13), and inequality (A.9), we deduce that

$$\begin{aligned}\|\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}\|_{1,\Omega} &\leq \|\boldsymbol{\zeta} - \mathcal{S}^h\boldsymbol{\zeta}\|_{1,\Omega} + \|\mathcal{S}^h\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}\|_{1,\Omega} \\ &\leq \|\boldsymbol{\zeta} - \mathcal{S}^h\boldsymbol{\zeta}\|_{1,\Omega} + \frac{C}{h}\|\mathcal{S}^h\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}\|_{0,\Omega} \\ &\leq \|\boldsymbol{\zeta} - \mathcal{S}^h\boldsymbol{\zeta}\|_{1,\Omega} + \frac{C}{h}\|\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}\|_{0,\Omega} + \frac{C}{h}\|\mathcal{S}^h\boldsymbol{\zeta} - \boldsymbol{\zeta}\|_{0,\Omega} \\ &\leq \|\boldsymbol{\zeta} - \mathcal{S}^h\boldsymbol{\zeta}\|_{1,\Omega} + \frac{C}{h}\|\mathcal{S}^h\boldsymbol{\zeta} - \boldsymbol{\zeta}\|_{0,\Omega}.\end{aligned}$$

Thus, (A.8) follows from the last inequality and (A.7). □

Finally, as obvious consequences of (A.8) and (A.6)–(A.7), we obtain the following error estimates for $\boldsymbol{\zeta} - \mathcal{P}^h\boldsymbol{\zeta}$:

THEOREM A.3. *Assume that* (H1)–(H2) *hold. Then the operator* $\mathcal{P}^h$ *satisfies the error estimates* (4.23) *and* (4.24).

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