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Semifree circle actions, Bott towers and quasitoric manifolds

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Abstract. A *Bott tower* is the total space of a tower of fibre bundles with base $\mathbb{C}P^1$ and fibres $\mathbb{C}P^1$. Every Bott tower of height n is a smooth projective toric variety whose moment polytope is combinatorially equivalent to an n-cube. A circle action is *semifree* if it is free on the complement to the fixed points. We show that a quasitoric manifold over a combinatorial n-cube admitting a semifree action of a 1-dimensional subtorus with isolated fixed points is a Bott tower. Then we show that every Bott tower obtained in this way is topologically trivial, that is, homeomorphic to a product of 2-spheres. This extends a recent result of Il'inskiĭ, who showed that a smooth compact toric variety admitting a semifree action of a 1-dimensional subtorus with isolated fixed points is homeomorphic to a product of 2-spheres, and makes a further step towards our understanding of Hattori's problem of semifree circle actions. Finally, we show that if the cohomology ring of a quasitoric manifold is isomorphic to that of a product of 2-spheres, then the manifold is homeomorphic to this product. In the case of Bott towers the homeomorphism is actually a diffeomorphism.

Bibliography: 18 titles.

§ 1. Introduction

In their study of symmetric spaces Bott and Samelson [1] introduced a family of complex manifolds obtained as the total spaces of iterated bundles over $\mathbb{C}P^1$ with fibre $\mathbb{C}P^1$. Grossberg and Karshon [2] showed that these manifolds carry an algebraic torus action, therefore constituting an important family of smooth projective toric varieties, and called them *Bott towers*. Civan and Ray [3] developed significantly the study of Bott towers by enumerating the invariant stably complex structures and calculating their complex and real K-theory rings, and cobordisms.

An action of a group is called *semifree* if it is free in the complement to the fixed points. A particularly interesting class of *Hamiltonian* semifree circle actions was studied by Hattori, who proved in [4] that a compact symplectic manifold M carrying a semifree Hamiltonian $\mathbb{S}^1$-action with non-empty isolated fixed point set has the same cohomology ring and the same Chern classes as $S^2 \times \cdots \times S^2$, thus imposing a severe restriction on the topological structure of the manifold. Hattori's

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results were further extended by Tolman and Weitsman, who showed in [5] that a semifree symplectic $\mathbb{S}^1$-action with non-empty isolated fixed point set is automatically Hamiltonian, and that the *equivariant* cohomology ring and Chern classes of M also agree with those of $S^2 \times \cdots \times S^2$. In dimensions up to 6 it is known that a symplectic manifold with an $\mathbb{S}^1$-action having the above properties is homeomorphic to a product of 2-spheres, but in higher dimensions this remains open.

Il'inskiĭ considered in [6] an algebraic version of the above question on semifree symplectic $\mathbb{S}^1$-actions. Namely, he conjectured that a smooth compact complex algebraic variety X carrying a semifree action of the algebraic 1-torus $\mathbb{C}^*$ with positive number of isolated fixed points is homeomorphic to $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$. The algebraic and symplectic versions of the conjecture are related via the common subclass of projective varieties; a smooth projective variety is a symplectic manifold. Il'inskiĭ proved the *toric* version of his algebraic conjecture, namely, when X is a *toric manifold* (a non-singular compact *toric variety*) and the semifree 1-torus is a subgroup of the acting torus (of dimension $\dim_{\mathbb{C}} X$). The first step of Il'inskiĭ's argument was to show that if X admits a semifree action of a subcircle with isolated fixed points, then the corresponding fan is combinatorially equivalent to the fan over the faces of a cross-polytope. A result of Dobrinskaya [7] implies that such X is a Bott tower, and this was the starting point in our study of semifree circle actions on Bott towers and related classes of manifolds with torus action, such as *quasitoric manifolds*. This class of manifolds was introduced by Davis and Januszkiewicz in [8]. A quasitoric manifold is a compact manifold M of dimension $2n$ with a locally standard action of an n-dimensional torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ whose orbit space is a simple polytope P. Recently, quasitoric manifolds have attracted substantial interest in the emerging field of 'toric topology'; we review their construction in § 3; for a more detailed account see [9], Ch. 6.

A projective toric manifold with moment polytope P is a quasitoric manifold over P, and a Bott tower is a toric manifold with moment polytope combinatorially equivalent to a cube (or plainly it is a *toric manifold over a cube*). We have therefore the following hierarchy of classes of manifolds M with action of $\mathbb{T}^n$:

$$\begin{aligned}\text{Bott towers} &\subset \text{toric manifolds over cubes}\\ &\subset \text{quasitoric manifolds over cubes.}\end{aligned} \tag{1.1}$$

By the above-mentioned result of Dobrinskaya [7] the first inclusion in (1.1) is in fact an identity (we explain this in Corollary 3.5 below). We proceed in §§ 4 and 5 by obtaining two results relating semifree circle actions on Bott towers, their topological structure, and cohomology rings. In Theorem 4.4 we show that if a Bott tower admits a semifree $\mathbb{S}^1$-action with isolated fixed points, then it is $\mathbb{S}^1$-equivariantly homeomorphic to a product of 2-spheres. We also show in Theorem 5.1 that a Bott tower with cohomology ring isomorphic to that of a product of spheres is homeomorphic to this product. Both results are then extended to a much more general class of quasitoric manifolds over cubes (Theorems 4.8 and 5.7 respectively), which also allows us to deduce Il'inskiĭ's result on semifree actions on toric varieties (Corollary 4.9).

Since a cohomology isomorphism implies a homeomorphism in the case of a product of spheres, we may ask whether the cohomology ring detects the homeomorphism type of a Bott tower or a quasitoric manifold in general. Some progress in this direction has been achieved in [10] in the case of quasitoric manifolds over a product of simplices, which can be regarded as an intermediate stage between quasitorics over a cube and the full generality.

It would be interesting to obtain smooth analogues of our classification results. In the case of Bott towers our homeomorphisms are actually diffeomorphisms (see Theorems 4.4 and 5.1), but some of our key arguments for quasitoric manifolds do not work in the smooth category. Although quasitoric manifolds are necessarily smooth ([8], p. 421), the main problem here is that the original Davis-Januszkiewicz classification result ([8], Proposition 1.8) establishes only an equivariant *homeo*morphism between a quasitoric manifold and the canonical model determined by the polytope and the characteristic function. As a consequence, we do not know if there are exotic equivariant smooth structures, even on 4-dimensional quasitoric manifolds. (A canonical equivariant smooth structure coinciding with the standard one in the toric case is described in [11], § 4.)

§ 2. Bott towers

We briefly review the definitions here, following [2] and [3], where the reader may find a much more detailed account of the history and applications of Bott towers.

Definition. A *Bott tower* of height n is a sequence of manifolds $(B^{2k}\colon k \leqslant n)$ such that $B^2 = \mathbb{C}P^1$ and $B^{2k} = P(\underline{\mathbb{C}} \oplus \xi_{k-1})$ for $1 < k \leqslant n$, where $P(\,\cdot\,)$ denotes the complex projectivization of a vector bundle, ξ_{k-1} is a complex line bundle over $B^{2(k-1)}$ and $\underline{\mathbb{C}}$ is a trivial line bundle. In particular, we have a fibre bundle $B^{2k} \to B^{2(k-1)}$ with fibre $\mathbb{C}P^1$.

We shall also use the same name of 'Bott tower' for the last stage B^{2n} in the sequence; it follows from the definition that B^{2n} is a complex manifold obtained as the total space of an iterated bundle with fibre $\mathbb{C}P^1$. Bott towers of height 2 are known as *Hirzebruch surfaces*.

The standard results on the cohomology of projectivized bundles lead to the following description of the cohomology ring of a Bott tower (all the cohomology groups are taken with coefficients from $\mathbb{Z}$ unless otherwise specified).

Lemma 2.1. *$H^*(B^{2k})$ is a free module over $H^*(B^{2(k-1)})$ on the generators* 1 *and u_k, which have dimensions* 0 *and* 2 *respectively. The ring structure is determined by the single relation*

$$u_k^2 = c_1(\xi_{k-1})u_k,$$

and the restriction of u_k to the fibre $\mathbb{C}P^1 \subset B^{2k}$ is the first Chern class of the canonical line bundle over $\mathbb{C}P^1$.

Let u_1 be the canonical generator of $H^2(\mathbb{C}P^1)$ (the first Chern class of the canonical line bundle). Then the Bott tower is determined by the list of integers a_{ij} $(1 \leqslant i < j \leqslant n)$, where

$$u_k^2 = \sum_{i=1}^{k-1} a_{ik} u_i u_k, \qquad 1 \leqslant k \leqslant n. \tag{2.1}$$

The cohomology ring of B^{2n} is the quotient of the $\mathbb{Z}[u_1, \dots, u_n]$ by relations (2.1).

It is convenient to organize the integers a_{ij} into an $n\times n$ integer upper triangular matrix:

$$A=\begin{pmatrix} -1 & a_{12} & \cdots & a_{1n}\\ 0 & -1 & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1 \end{pmatrix}. \tag{2.2}$$

Example 2.2. When $n=2$, the Bott tower B^4 is determined by a single line bundle ξ_1 over $\mathbb{C}P^1$. We have $\xi_1=\gamma^m$ for some $m\in\mathbb{Z}$ where γ is the canonical line bundle over $B^2=\mathbb{C}P^1$, so the cohomology ring is determined by the relations $u_1^2=0$ and $u_2^2=mu_1u_2$. It is well known that $P(\underline{\mathbb{C}}\oplus\gamma^m)\cong P(\underline{\mathbb{C}}\oplus\gamma^{m'})$ if and only if $m\equiv m'$ (mod 2), where $\cong$ means 'diffeomorphic'. The proof goes as follows. We note that $P(E)\cong P(E\otimes\eta)$ for any complex line bundle η. Let $m\equiv m'$ (mod 2). Then $m'-m=2\ell$ for some $\ell\in\mathbb{Z}$ and we have diffeomorphisms

$$P(\underline{\mathbb{C}}\oplus\gamma^m)\cong P((\underline{\mathbb{C}}\oplus\gamma^m)\otimes\gamma^\ell)=P(\gamma^\ell\oplus\gamma^{m+\ell}).$$

Here both $\gamma^\ell\oplus\gamma^{m+\ell}$ and $\underline{\mathbb{C}}\oplus\gamma^{m'}$ are over $\mathbb{C}P^1$ and have equal first Chern class, so they are isomorphic. Hence the last space above is $P(\underline{\mathbb{C}}\oplus\gamma^{m'})$. On the other hand, it is not difficult to see that if $H^*(P(\underline{\mathbb{C}}\oplus\gamma^m))\cong H^*(P(\underline{\mathbb{C}}\oplus\gamma^{m'}))$ as rings, then $m\equiv m'$ (mod 2).

This example shows that the cohomology ring determines the topological type of a Bott tower B^{2n} for $n=2$. A case-to-case analysis based on a classification result in [7], § 3 shows that this is also the case for $n=3$. So we may ask the following question.

Question 2.3. Are Bott towers B_1^{2n} and B_2^{2n} homeomorphic if their cohomology rings are isomorphic:

$$H^*(B_1^{2n})\cong H^*(B_2^{2n})\,?$$

We investigate this question further, in § 5, where a partial answer is given.

§ 3. Quasitoric manifolds

Davis and Januszkiewicz introduced in [8] a class of $2n$-dimensional manifolds M with action of an n-dimensional torus T. They required the action to be *locally standard* (locally isomorphic to the standard T-representation in $\mathbb{C}^n$) and the quotient space M/T to be a *simple* n-dimensional polytope P, so that there is a projection $\pi\colon M\to P$ whose fibres are orbits of the action. Davis and Januszkiewicz call their manifolds *toric*; more recently, the term *quasitoric* has been adopted to avoid confusion with non-singular compact toric varieties of algebraic geometry. We follow this convention below and refer to such M as *quasitoric manifolds* reserving the term 'toric manifolds' for algebraic varieties. We note that a non-singular projective toric manifold is a quasitoric manifold; for more discussion of the relationship between the two classes see [9], Ch. 6.

Every quasitoric manifold can be given a smooth structure in which the T-action is smooth; see a remark in [8], p. 421 and a more detailed exposition in [11], § 3.

Let m denote the number of *facets* (codimension-one faces) of P; we order the facets so that the first n of them meet at a vertex. We denote the facets by F_i for $1 \leqslant i \leqslant m$ and denote by $\mathscr{F}$ the set of all facets. The preimage $\pi^{-1}(F_i)$ is a connected codimension-two submanifold of M, fixed pointwise by a circle subgroup of T. We denote it by M_i and refer to it as the *characteristic submanifold* corresponding to F_i, for $1 \leqslant i \leqslant m$. An *omniorientation* of M [12] consists of a choice of orientation for M and for each characteristic submanifold.

Let N be the integer lattice of one-parameter circle subgroups in T, so that $N \cong \mathbb{Z}^n$. Given a characteristic submanifold M_j we denote by λ_j a primitive vector in N that spans the circle subgroup $T_{M_j} \subset T$ fixing M_j. The vector λ_j is determined up to a sign. The correspondence $\lambda\colon F_j \mapsto \lambda_j$ was called in [8] the *characteristic function* of the quasitoric manifold M.

The omniorientation allows us to interpret the characteristic function as a linear map $\lambda\colon \mathbb{Z}^{\mathscr{F}} \to N$. To do this we must canonically specify one of the two directions for each vector λ_j. First, we note that an action of the parametrized circle subgroup $T_{M_j} \subset T$ determines an orientation of the normal bundle ν_j of the embedding $M_j \subset M$. The omniorientation of M also provides an orientation for ν_j by means of the following decomposition of the tangent bundle:

$$\tau(M)\big|_{M_j} = \tau(M_j) \oplus \nu_j.$$

Now we choose the primitive vectors λ_j, $1 \leqslant j \leqslant m$, so that the two orientations of ν_j coincide.

In general, there is no canonical choice of omniorientation for M. However, if M admits a *T-equivariant almost complex structure*, then a choice of such a structure provides a canonical way of orienting M and normal bundles ν_j for $1 \leqslant j \leqslant m$, thereby specifying an omniorientation *associated* with the equivariant almost complex structure. In what follows we shall always choose the associated omniorientation if M is equivariantly almost complex, in particular, if M is a (non-singular compact) toric manifold; otherwise we shall fix arbitrarily an omniorientation.

By definition, the characteristic function satisfies the *non-singularity condition*: $\lambda_{j_1}, \dots, \lambda_{j_n}$ is a basis of the lattice N whenever the intersection $F_{j_1} \cap \dots \cap F_{j_n}$ is non-empty. So we may use the vectors $\lambda_1, \dots, \lambda_n$ to identify N with $\mathbb{Z}^n$ and can represent the map λ by an integral $n \times m$ matrix of the form

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix}. \tag{3.1}$$

It is often convenient to partition Λ as $(E \mid \Lambda_\star)$, where E is an identity matrix and $\Lambda_\star$ has size $n \times (m-n)$. For any vertex $F_{j_1} \cap \dots \cap F_{j_n}$ the corresponding columns $\lambda_{j_1}, \dots, \lambda_{j_n}$ form a basis of $\mathbb{Z}^n$ and the corresponding determinant is ± 1. We refer to (3.1) as the *refined form* of the characteristic matrix Λ, and call $\Lambda_\star$ its *reduced submatrix*.

Having chosen a basis for N we may identify our torus T with the standard product of unit circles in $\mathbb{C}^n$:

$$\mathbb{T}^n = \{(e^{2\pi i\varphi_1}, \dots, e^{2\pi i\varphi_n}) \in \mathbb{C}^n\}, \tag{3.2}$$

where $(\varphi_1, \ldots, \varphi_n)$ ranges over $\mathbb{R}^n$. We shall also denote a generic point of $\mathbb{T}^n$ by $(t_1, \ldots, t_n)$. The circle subgroup fixing M_j can now be written as follows:

$$T_{M_j} = \bigl\{(t^{\lambda_{1j}}, \ldots, t^{\lambda_{nj}}) = (e^{2\pi i\lambda_{1j}\varphi}, \ldots, e^{2\pi i\lambda_{nj}\varphi}) \in \mathbb{T}^n\bigr\}, \\ 1 \leqslant j \leqslant m, \quad \varphi \in \mathbb{R}, \quad t = e^{2\pi i\varphi}. \tag{3.3}$$

Remark. Not every (non-singular compact) toric manifold X is a quasitoric manifold, as the quotient X/T may fail to be a convex polytope (although it is a polytope when X is projective). Nevertheless, X has characteristic submanifolds X_j (T-invariant divisors), and there is a canonical omniorientation induced from the complex structures on X and X_j. Therefore, the characteristic matrix Λ is defined for every (non-singular compact) toric manifold X. The vectors λ_j are the primitive vectors along the edges of the fan corresponding to X.

Let v_j be the class in $H^2(M)$ dual to the fundamental class of M_j, $1 \leqslant j \leqslant m$. According to [8], Theorem 4.14, the ring $H^*(M)$ is generated by $v_1, \ldots, v_m$ modulo two sets of relations. The first set is formed by the monomial relations which arise from the *Stanley-Reisner ideal* of P; the second set consists of the linear relations determined by the characteristic matrix:

$$v_i = -\lambda_{i,n+1}v_{n+1} - \cdots - \lambda_{i,m}v_m, \qquad 1 \leqslant i \leqslant n. \tag{3.4}$$

It follows that $v_{n+1}, \ldots, v_m$ suffice to generate $H^*(M)$ multiplicatively.

Two quasitoric manifolds M_1 and M_2 are said to be *weakly T-equivariantly homeomorphic* (or simply *weakly T-homeomorphic*) if there are an automorphism $\theta\colon T \to T$ and a homeomorphism $f\colon M_1 \to M_2$ such that

$$f(t \cdot x) = \theta(t) \cdot f(x)$$

for every $t \in T$ and $x \in M_1$. If θ is the identity automorphism, then M_1 and M_2 are said to be *T-homeomorphic*. Following Davis and Januszkiewicz, we say that two quasitoric manifolds M_1 and M_2 over the same P are *equivalent* if there is a weak T-homeomorphism $f\colon M_1 \to M_2$ covering the identity on P. By [8], Proposition 1.8, a quasitoric manifold M over P is determined up to an equivalence by its characteristic function λ. This follows from the 'basic construction' providing a canonical quasitoric manifold $M(P, \lambda)$, which depends only on P and λ, together with a weak T-homeomorphism $M(P, \lambda) \to M$ covering the identity on P.

Let $\operatorname{ch} f(P)$ denote the set of characteristic functions on the facets of P, that is, the set of maps $\lambda\colon \mathscr{F} \to N$ satisfying the non-singularity condition. The group $\mathrm{GL}(N) \cong \mathrm{GL}(n, Z)$ of automorphisms of the lattice N acts from the left on the set $\operatorname{ch} f(P)$ (an automorphism $g\colon N \to N$ acts by composition $\lambda \mapsto g \cdot \lambda$). Since automorphisms of the lattice N correspond to automorphisms of the torus T, there is a one-to-one correspondence

$$\mathrm{GL}(N) \setminus \operatorname{ch} f(P) \longleftrightarrow \{\text{equivalence classes of } M \text{ over } P\}. \tag{3.5}$$

One may assign an $(n \times m)$-matrix Λ to each element $\lambda \in \operatorname{ch} f(P)$ by ordering the facets and choosing a basis for N, as we did above. A choice of the matrix Λ in the refined form (3.1) can now be regarded as a choice of a specific representative of the left coset class in $\mathrm{GL}(N) \setminus \mathrm{ch}\, f(P)$. If a characteristic matrix is given in an unrefined form $\Lambda = (A \mid B)$, where A is $n \times n$ and B is $n \times (m-n)$, then the refined representative in its coset class is given by $(E \mid A^{-1}B)$.

The Davis-Januszkiewicz canonical model $M(P, \lambda)$ can be obtained as the quotient of the *moment-angle manifold* $\mathscr{Z}_P$ by a freely acting $(m-n)$-dimensional torus subgroup of $\mathbb{T}^m$ determined by the kernel of the characteristic map $\lambda\colon \mathbb{Z}^m \to N$; see [8], [9], Ch. 7. The moment-angle manifold $\mathscr{Z}_P$ can be embedded in $\mathbb{C}^m$ as a complete intersection of $m-n$ real quadratic hypersurfaces ([11], § 3). It follows that both $\mathscr{Z}_P$ and $M(P, \lambda)$ are necessarily smooth. Hence every quasitoric manifold M over P with characteristic function λ acquires a *canonical equivariant smooth structure*. We do not know, however, if this equivariant smooth structure is unique (see the discussion in § 1).

We are particularly interested in the case when the quotient polytope $P = M/\mathbb{T}^n$ is the n-cube $\mathbb{I}^n$. Then $m = 2n$ and we shall additionally assume that the facets F_j and F_{n+j} are opposite (that is, disjoint) for $1 \leqslant j \leqslant n$. In the case of $P = \mathbb{I}^n$ the moment-angle manifold is the product of n three-dimensional spheres, embedded in $\mathbb{C}^{2n}$ as

$$\bigl\{(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} \colon |z_j|^2 + |z_{n+j}|^2 = 1 \text{ for } 1 \leqslant j \leqslant n\bigr\}.$$

The quotient $(S^3)^n/\mathbb{T}^{2n}$ by the coordinatewise action is a cube $\mathbb{I}^n$. The n-dimensional subtorus $T(\Lambda) \subset \mathbb{T}^{2n}$ determined by the kernel of characteristic map (3.1) is given by

$$(t_1, \dots, t_n) \mapsto (t_1^{-\lambda_{1,n+1}} t_2^{-\lambda_{1,n+2}} \cdots t_n^{-\lambda_{1,2n}}, \\ \dots, t_1^{-\lambda_{n,n+1}} t_2^{-\lambda_{n,n+2}} \cdots t_n^{-\lambda_{n,2n}}, t_1, t_2, \dots, t_n). \tag{3.6}$$

It acts freely on $(S^3)^n$, and the quotient $(S^3)^n/T(\Lambda)$ is the quasitoric manifold M determined by Λ. The torus $\mathbb{T}^{2n}/T(\Lambda) \cong \mathbb{T}^n$ acts on $(S^3)^n/T(\Lambda) \cong M$ with quotient $\mathbb{I}^n$. In coordinates, $(t_1, \dots, t_n) \in \mathbb{T}^n$ acts on an equivalence class $[z_1, \dots, z_{2n}] \in (S^3)^n/T(\Lambda)$ as multiplication by $(t_1, \dots, t_n, 1, \dots, 1)$.

Proposition 3.1. *A Bott tower carries a natural torus action turning it into a quasitoric manifold over a cube with reduced characteristic submatrix $\Lambda_\star = A^t$, see* (2.2) *and* (3.1).

Proof. As shown in [3], Proposition 3.1, the Bott tower corresponding to (2.2) can be obtained as the quotient of $(S^3)^n$ by the n-dimensional subtorus of $\mathbb{T}^{2n}$ defined by the inclusion

$$(t_1, \dots, t_n) \mapsto (t_1, t_1^{-a_{12}} t_2, \dots, t_1^{-a_{1n}} t_2^{-a_{2n}} \cdots t_{n-1}^{-a_{n-1,n}} t_n, t_1, t_2, \dots, t_n).$$

It remains to observe that this coincides with $T(\Lambda)$ from (3.6) for $\Lambda_\star = A^t$.

Remark. The Stanley-Reisner relations for the n-cube are $v_i v_{i+n} = 0$, $1 \leqslant i \leqslant n$. These relations in combination with (3.4) give us (2.1) for $\Lambda_\star = A^t$ and $u_i = v_{i+n}$.

By definition, a Bott tower is a complex manifold. An algebraic version of the above construction (formula (3.6) also describes embeddings of algebraic, that is, non-compact tori) was used in [2] to describe Bott towers as non-singular projective toric varieties. The two approaches yield the same result as shown in [3], § 2.

Given a permutation σ of n elements, we denote by $P(\sigma)$ the corresponding $n\times n$ *permutation matrix*, which contains ones at the positions $(i,\sigma(i))$ for $1\leqslant i\leqslant n$, and zeros elsewhere. There is an action of the symmetric group S_n on $n\times n$ matrices by conjugations $A\mapsto P(\sigma)^{-1}AP(\sigma)$ or, equivalently, by permutations of the rows and columns of A.

Proposition 3.2. *A quasitoric manifold M over a cube with reduced submatrix $\Lambda_\star$ is equivalent to a Bott tower if and only if $\Lambda_\star$ is conjugate by means of a permutation matrix to an upper triangular matrix.*

Proof. Assume that $\Lambda\star$ is conjugate by means of a permutation matrix to an upper triangular matrix. It is clear that this condition is equivalent to the conjugacy of $\Lambda_\star$ to a lower triangular matrix. Consider the action of S_n on the set of facets of $\mathbb{I}^n$ by permuting pairs of opposite facets. A rearrangement of facets corresponds to a rearrangement of columns in the characteristic $(n\times 2n)$-matrix Λ, so an element $\sigma\in S_n$ acts as follows:

$$\Lambda\mapsto\Lambda\cdot\begin{pmatrix}P(\sigma)&0\\0&P(\sigma)\end{pmatrix}.$$

This action does not preserve the refined form of Λ, as $(E\,|\,\Lambda_\star)$ becomes $(P(\sigma)\,|\,\Lambda_\star P(\sigma))$. The refined representative of the left coset class (3.5) of Λ is $(E\,|\,P(\sigma)^{-1}\Lambda_\star P(\sigma))$. (In other words, we must compensate for the permutation of pairs of facets by an automorphism of $\mathbb{T}^n$ permuting the coordinate subcircles to keep the characteristic matrix in the refined form.) This implies that a permutation action on pairs of opposite facets induces an action by conjugations on reduced submatrices. Hence we may assume, up to an equivalence, that M has a lower triangular reduced submatrix $\Lambda_\star$. The non-singularity condition guarantees that the diagonal entries of $\Lambda_\star$ are equal to ±1, and we can set all of them equal to -1 by changing the omniorientation of M if necessary. Now, M and the Bott tower corresponding to the matrix $A=\Lambda_\star^t$ have the same characteristic matrices Λ by Proposition 3.1, therefore they are equivalent by [8], Proposition 1.8.

The converse result follows from Proposition 3.1.

It is now clear that not all quasitoric manifolds over a cube are Bott towers. For example, a 4-dimensional quasitoric manifold over a square with reduced characteristic submatrix

$$\Lambda_\star=\begin{pmatrix}-1&-2\\-1&-1\end{pmatrix}$$

is not a Bott tower since $\Lambda_\star$ is not conjugate to an upper triangular matrix. (The corresponding manifold is homeomorphic to $\mathbb{C}P^2\,\#\,\mathbb{C}P^2$, and therefore does not even admit a complex structure [8].)

Given a k-element subset $\{i_1,\dots,i_k\}$ of an n-element set, the corresponding *principal minor* of a square n-matrix A is the determinant of the submatrix formed by the entries from the columns and rows with indices $i_1,\dots,i_k$. For Bott towers, Proposition 3.1 ensures that all the principal minors of the matrix $-\Lambda_\star$ are equal to 1, while for arbitrary quasitoric manifolds the non-singularity condition only guarantees that every principal minor of $\Lambda_\star$ is equal to ± 1.

Recall that an upper triangular matrix is *unipotent* if all its diagonal entries are ones. The following key technical lemma can be retrieved from the proof of Dobrinskaya's much more general result of [7], Theorem 6. We give a slightly more expanded proof here for completeness.

Lemma 3.3. *Let R be a commutative integral domain with identity element* 1, *and let A be an $n\times n$ matrix with entries in R. Suppose that every proper principal minor of A is equal to* 1. *If* $\det A = 1$, *then A is conjugate by means of a permutation matrix to a unipotent upper triangular matrix*, *otherwise it is conjugate to a matrix of the following form*:

$$\begin{pmatrix} 1 & b_1 & 0 & \dots & 0 \\ 0 & 1 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1} \\ b_n & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{3.7}$$

where $b_i \neq 0$ for all i [7].

Proof. By assumption the diagonal entries of A must be ones. We say that the ith row is *elementary* if its ith entry is 1 and the other entries are 0. Assuming by induction that the theorem holds for matrices of size $(n-1)$ we deduce that A is itself conjugate to a unipotent upper triangular matrix if and only if it contains an elementary row. We denote by A_i the square matrix of size $(n-1)$ obtained by removing from A the ith column and the ith row.

We may assume by induction that A_n is a unipotent upper triangular matrix. Next we apply to A_1 the induction assumption. The permutation of rows and columns transforming A_1 into a unipotent upper triangular matrix turns A into an 'almost' unipotent upper triangular matrix; the latter may have only one non-zero entry below the diagonal, which must be in the first column. If this non-zero entry is distinct from a_{n1}, then the nth row of A is elementary and A is conjugate to a unipotent upper triangular matrix. Otherwise we have

$$A = \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1} \\ b_n & 0 & \dots & 0 & 1 \end{pmatrix},$$

where $b_{n-1} \neq 0$ and $b_n \neq 0$ (otherwise A contains an elementary row). Now let a_{1j_1} be the last non-zero entry in the first row of A. If A does not contain an elementary row, then we may define by induction $a_{j_i j_{i+1}}$ as the last non-zero non-diagonal entry in the j_ith row of A. Clearly, we have

$$1 < j_1 < \dots < j_i < j_{i+1} < \dots < j_k = n$$

for some $k < n$. Now, if $j_i = i + 1$ for $1 \leqslant i \leqslant n-1$, then A is the matrix (3.7) with $b_i = a_{j_{i-1}j_i}$, $1 \leqslant i \leqslant n-1$. Otherwise, the submatrix

$$S = \begin{pmatrix} 1 & a_{1j_1} & 0 & \dots & 0 \\ 0 & 1 & a_{j_1j_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{j_{k-1}n} \\ b_n & 0 & \dots & 0 & 1 \end{pmatrix},$$

of A formed by the columns and rows with indices $1, j_1, \dots, j_k$ is proper and has determinant $1 \pm b_n \prod a_{j_i j_{i+1}} \neq 1$. This contradiction finishes the proof.

The following theorem is not new; the equivalence of a) and b) is a particular case of [7], Theorem 6 and the equivalence of b) and c) follows from [13] and [14], Proposition 5.53. We give here a proof because we need it in the next sections.

Theorem 3.4. *Let $M = M(I^n, \Lambda)$ be a quasitoric manifold over a cube with canonical equivariant smooth structure, and $\Lambda_\star$ the corresponding reduced submatrix. Then the following conditions are equivalent*:

a) *M is equivalent to a Bott tower*;
b) *all the principal minors of $-\Lambda_\star$ are equal to* 1;
c) *M has a $\mathbb{T}^n$-equivariant almost complex structure* (*with the associated omniorientation*).

Proof. The implication b) $\Rightarrow$ a) follows from Lemma 3.3 and Proposition 3.2. The implication b) $\Rightarrow$ c) is obvious. Let us prove c) $\Rightarrow$ b). Recall the definition of the *sign* $\sigma(p)$ of a fixed point of a $\mathbb{T}^n$-action on M from [13], § 4, [7] and [15]. Every fixed point p can be obtained as the intersection $M_{j_1} \cap \dots \cap M_{j_n}$ of n characteristic submanifolds and corresponds to the vertex of P obtained as the intersection $F_{j_1} \cap \dots \cap F_{j_n}$ of the corresponding facets. The tangent space to M at p therefore decomposes into the sum of normal subspaces to the M_{j_k} for $1 \leqslant k \leqslant n$:

$$\tau_p(M) = \nu_{j_1}|_p \oplus \dots \oplus \nu_{j_n}|_p. \tag{3.8}$$

The omniorientation of M provides two different ways of orienting the above space; we set $\sigma(p) = 1$ if these two orientations coincide and $\sigma(p) = -1$ otherwise. This sign can be calculated as follows in terms of P and the characteristic matrix Λ:

$$\sigma(p) = \operatorname{sign}\bigl(\det(\lambda_{j_1}, \dots, \lambda_{j_n}) \cdot \det(a_{j_1}, \dots, a_{j_n})\bigr) \tag{3.9}$$

(see [15], § 1), where a_i is the normal vector to the facet F_i pointing inside the polytope. If $P = \mathbb{I}^n$, then every fixed point p corresponds to a vertex defined as

$$F_{i_1} \cap \dots \cap F_{i_k} \cap F_{n+l_1} \cap \dots \cap F_{n+l_{n-k}}$$

for some $1 \leqslant i_1 < \dots < i_k \leqslant n$ and $1 \leqslant l_1 < \dots < l_{n-k} \leqslant n$, and we have $a_i = e_i$ (the ith basis vector) for $1 \leqslant i \leqslant n$ and $a_j = -e_j$ for $n+1 \leqslant j \leqslant 2n$. Thus, the expression on the right-hand side of (3.9) is equal to the principal minor of $-\Lambda_\star$ formed by the columns and rows with indices $l_1, \dots, l_{n-k}$. It remains to note that in the almost complex case the two orientations in (3.8) coincide, so the sign of every fixed point is 1.

A *cross-polytope* is a regular polytope dual to the cube (in particular, a 3-dimensional cross-polytope is an octahedron).

Corollary 3.5. *Let X be a toric manifold whose associated fan is combinatorially equivalent to the fan consisting of cones over the faces of a cross-polytope. Then X is a Bott tower.*

Proof. The reduced matrix $\Lambda_\star$ of X has size $n \times n$ and all the principal minors of $-\Lambda_\star$ are equal to 1 by the same reason as in the proof of Theorem 3.4. By Lemma 3.3, $-\Lambda_\star$ is conjugate to a unipotent upper triangular matrix, so Λ has the same form as the characteristic matrix of a Bott tower. In the toric-manifold case the columns of Λ are primitive vectors along edges of the fan, so the combinatorial type of the fan and Λ determine the fan completely. It follows that the fan of X is the same as the fan of some Bott tower, which implies that X is a Bott tower by the one-to-one correspondence between fans and toric manifolds.

A toric manifold over a cube satisfies the assumption of Corollary 3.5. Hence the class of Bott towers coincides with the class of toric manifolds over a cube, and the first inclusion in (1.1) is an identity. Similarly to Lemma 3.3, Corollary 3.5 is a particular case of a more general result of Dobrinskaya [7], Corollary 7, which gives a criterion for a quasitoric manifold over a product of simplices to be decomposable into a tower of fibre bundles.

§ 4. Semifree circle actions

An action of a group on a topological space is said to be *semifree* if it is free on the complement to the fixed points. We first show (Theorem 4.3 below) that if the torus $\mathbb{T}^n$ acting on a quasitoric manifold M over a cube has a circle subgroup acting semifreely and with isolated fixed points, then M is a Bott tower. Then we prove that all these Bott towers are $\mathbb{S}^1$-equivariantly homeomorphic to a product of 2-dimensional spheres (with the diagonal $\mathbb{S}^1$-action).

A complex n-dimensional representation of $\mathbb{S}^1$ is determined by a set of weights $k_j \in \mathbb{Z}$ for $1 \leqslant j \leqslant n$. In appropriate coordinates an element $s = e^{2\pi i\varphi} \in \mathbb{S}^1$ acts as follows:

$$s \cdot (z_1, \dots, z_n) = (e^{2\pi i k_1 \varphi} z_1, \dots, e^{2\pi i k_n \varphi} z_n). \tag{4.1}$$

The following result is straightforward.

Proposition 4.1. *A representation of $\mathbb{S}^1$ in $\mathbb{C}^n$ is semifree if and only if $k_j = \pm 1$ for $1 \leqslant j \leqslant n$.*

A closed circle subgroup S^1_ν of $\mathbb{T}^n$ is determined by a primitive integer vector $\nu = (\nu_1, \dots, \nu_n)$:

$$S^1_\nu = \{(e^{2\pi i \nu_1 \varphi}, \dots, e^{2\pi i \nu_n \varphi})\} \subseteq \mathbb{T}^n. \tag{4.2}$$

We shall consider the tangential representations of $\mathbb{T}^n$ and its circle subgroups at fixed points of M. The representation of $\mathbb{T}^n$ in the tangent space $\tau_p(M)$ at a fixed point $p = M_{j_1} \cap \dots \cap M_{j_n}$ decomposes into the sum of non-trivial real two-dimensional representations in the normal subspaces of the M_{j_k} in M. The omniorientation endows each of these normal subspaces with a complex structure, thereby identifying it with $\mathbb{C}$ and $\tau_p M$ with $\mathbb{C}^n$. In these coordinates the weights of the representation of the circle subgroup (4.2) in τ_pM can be identified with the coefficients $k_i = k_i(\nu, p)$, $1 \leqslant i \leqslant n$, of the decomposition of ν in terms of $\lambda_{j_1}, \dots, \lambda_{j_n}$:

$$\nu = k_1(\nu, p)\lambda_{j_1} + \cdots + k_n(\nu, p)\lambda_{j_n} \tag{4.3}$$

(see, for instance, [15], Lemma 2.3).

Corollary 4.2. *A subcircle $S^1_\nu \subseteq \mathbb{T}^n$ acts on a quasitoric manifold M semifreely and with isolated fixed points if and only if for every vertex $p = F_{j_1} \cap \cdots \cap F_{j_n}$ the coefficients in (4.3) satisfy $k_i(\nu, p) = \pm 1$ for $1 \leqslant i \leqslant n$.*

Theorem 4.3. *Let M be a quasitoric manifold over a cube with reduced submatrix $\Lambda_\star$. Assume that M admits a semifree circle subgroup with isolated fixed points. Then M is equivalent to a Bott tower.*

Proof. We may assume by induction that every characteristic submanifold is a Bott tower, so that every proper principal minor of $-\Lambda_\star$ is 1. Therefore, we are in the situation of Lemma 3.3 and $-\Lambda_\star$ is a matrix of one of the two types described there. The second type is ruled out because of the semifreeness assumption. Indeed, let $\Lambda = (E \mid -B)$, where B is the matrix (3.7) and assume that $S^1_\nu \subseteq \mathbb{T}^n$ acts semifreely with isolated fixed points. Applying the criterion from Corollary 4.2 to the vertex $p = F_1 \cap \cdots \cap F_n$ we obtain $\nu_i = \pm 1$ for $1 \leqslant i \leqslant n$. Now we apply the same criterion to the vertex $p' = F_{n+1} \cap \cdots \cap F_{2n}$. Since the submatrix formed by the corresponding columns of Λ is precisely $-B$, it follows that $\det B = \pm 1$. This implies that at least one of the b_i is equal to ± 1, that is, at least one of the rows of B contains just two ± 1's and zeros. Therefore, if all the coefficients $k_i(\nu, p')$ in the expression $\nu = k_1(\nu, p')\lambda_{n+1} + \cdots + k_n(\nu, p')\lambda_{2n}$ are equal to ± 1, then at least one component ν_j of ν is even: a contradiction. The proof is complete.

Our next result shows that a Bott tower with semifree circle subgroup and isolated fixed points is topologically (or even $\mathbb{S}^1$-equivariantly) trivial, that is, homeomorphic to a product of 2-spheres. Let t (respectively, $\mathbb{C}$) be the standard (respectively, the trivial) complex one-dimensional $\mathbb{S}^1$-representation. The product bundle with fibre V and fixed base will be denoted by $\underline{V}$. We say that an action of a group G on a Bott tower B^{2n} *preserves the tower structure* if for each stage $B^{2k} = P(\underline{\mathbb{C}} \oplus \xi_{k-1})$, $k \leqslant n$, the line bundle ξ_{k-1} is G-equivariant. The intrinsic $\mathbb{T}^n$-action obviously preserves the tower structure.

Theorem 4.4. *Assume that a Bott tower B^{2n} admits a semifree $\mathbb{S}^1$-action with isolated fixed points preserving the tower structure. Then B^{2n} is $\mathbb{S}^1$-equivariantly diffeomorphic to the product $(P(\mathbb{C} \oplus t))^n$.*

Proof. We may assume by induction that the $(n-1)$th stage of the Bott tower is $(P(\mathbb{C} \oplus t))^{n-1}$ and $B^{2n} = P(\underline{\mathbb{C}} \oplus \xi)$ for some $\mathbb{S}^1$-line bundle ξ over $(P(\mathbb{C} \oplus t))^{n-1}$.

Let γ be the canonical line bundle over $P(\mathbb{C} \oplus t) \cong \mathbb{C}P^1$. It carries a unique structure of an $\mathbb{S}^1$-line bundle and we have

$$\gamma\big|_{(1:0)} = \mathbb{C}, \qquad \gamma\big|_{(0:1)} = t. \tag{4.4}$$

We denote by

$$x \in H^2(P(\mathbb{C} \oplus t))$$

the first Chern class of γ, and let

$$x_i \in H^2(P(\mathbb{C}\oplus t)^{n-1})$$

be the pullback of x by the projection π_i onto the ith factor. Then the first Chern class of ξ may be written as $\sum_{i=1}^{n-1} a_i x_i$ with $a_i \in \mathbb{Z}$. The $\mathbb{S}^1$-line bundles ξ and $\bigotimes_{i=1}^{n-1} \pi_i^*(\gamma^{a_i})$ have the same underlying bundles; so there is an integer k such that

$$\xi = t^k \bigotimes_{i=1}^{n-1} \pi_i^*(\gamma^{a_i}) \tag{4.5}$$

as $\mathbb{S}^1$-line bundles ([16], Corollary 4.2).

We encode fixed points in $P(\mathbb{C}\oplus t)^{n-1}$ by sequences $(p_1^{\varepsilon_1}, \ldots, p_{n-1}^{\varepsilon_{n-1}})$, where $\varepsilon_i = 0$ or $\varepsilon_i = 1$, and $p_i^{\varepsilon_i}$ denotes $(1:0)$ if $\varepsilon_i = 0$ or $(0:1)$ if $\varepsilon_i = 1$. Then it follows from (4.4) and (4.5) that

$$\xi\big|_{(p_1^{\varepsilon_1},\ldots,p_{n-1}^{\varepsilon_{n-1}})} = t^{k+\sum_{i=1}^{n-1}\varepsilon_i a_i}.$$

The $\mathbb{S}^1$-action on $B^{2n} = P(\underline{\mathbb{C}}\oplus\xi)$ is semifree if and only if $|k + \sum_{i=1}^{n-1}\varepsilon_i a_i| = 1$ for all possible values of ε_i. Setting $\varepsilon_i = 0$ for all i we obtain $|k| = 1$. Let $k = 1$ (the case $k = -1$ is treated similarly). Then $(a_1, \ldots, a_{n-1}) = (0, \ldots, 0)$ or $(0, \ldots, 0, -2, 0, \ldots, 0)$. In the former case, $\xi = \underline{t}$ and $B^{2n} = P(\underline{\mathbb{C}}\oplus\xi) \cong P(\mathbb{C}\oplus t)^n$. In the latter case we have $\xi = t\pi_i^*(\gamma^{-2})$ for some i, so that $B^{2n} = P(\underline{\mathbb{C}}\oplus\xi)$ (as a projectivized vector bundle) is the pullback of $P(\underline{\mathbb{C}}\oplus t\gamma^{-2})$ by means of the projection π_i. Since for any $\mathbb{S}^1$-vector bundle E and $\mathbb{S}^1$-line bundle η the projectivizations $P(E)$ and $P(E\otimes\eta)$ are $\mathbb{S}^1$-diffeomorphic, it follows that $P(\underline{\mathbb{C}}\oplus t\gamma^{-2}) \cong P(\gamma\oplus t\gamma^{-1})$. The first Chern class of $\gamma\oplus t\gamma^{-1}$ is zero, so its underlying bundle is trivial. The $\mathbb{S}^1$-representation in the fibre of $\gamma\oplus t\gamma^{-1}$ over a fixed point is $\mathbb{C}\oplus t$ by (4.4). Therefore, $\gamma\oplus t\gamma^{-1} = \underline{\mathbb{C}}\oplus\underline{t}$ as $\mathbb{S}^1$-bundles. It follows that $P(\underline{\mathbb{C}}\oplus t\gamma^{-2}) \cong P(\underline{\mathbb{C}}\oplus\underline{t})$, which finishes the proof.

Remark. The diffeomorphism of Theorem 4.4 is not a diffeomorphism of $\mathbb{T}^n$-manifolds.

Our next aim is to generalize Theorem 4.4 to quasitoric manifolds. Although its result does not hold for all quasitoric manifolds (see Example 4.5), it remains true if we additionally assume the quotient polytope to be a cube.

Example 4.5. Let M be the 4-dimensional quasitoric manifold over a $2k$-gon with characteristic matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 1 \end{pmatrix}.$$

Corollary 4.2 shows that the circle subgroup determined by the vector $\nu = (1, 1)$ acts semifreely on M, but the quotient of M is not a 2-cube if $k > 2$, so M cannot be homeomorphic to the product of spheres (it can be shown that M is the connected sum of $k-1$ copies of $S^2 \times S^2$).

Surprisingly, quasitoric manifolds over polygons provide the only essential source of counterexamples; see Theorem 4.8 below for the precise statement.

Lemma 4.6. *A simple polytope P of dimension $n \geqslant 2$ all of whose 2-faces are 4-gons is combinatorially equivalent to a cube.*

Proof. We may assume by induction that all the facets of P are cubes. We claim that P is a cube. This result can be found in [17], Exercise 0.1, but we include the proof for completeness. We shall prove the dual statement about simplicial polytopes. The simplicial polytope dual to P is a cross-polytope; we call its boundary K (which is a sphere triangulation) a *cross complex*. Recall that the *star* of a vertex v in a simplicial complex K is the subcomplex $\mathrm{st}\,v$ consisting of all simplices containing v and all the faces of these simplices. The *link* of v is the subcomplex $\mathrm{lk}\,v \subset \mathrm{st}\,v$ consisting of the simplices not containing v. The duality between P and K extends to the duality between the facets of P (which are $(n-1)$-dimensional simple polytopes) and the links of vertices of K (which are triangulations of $(n-2)$-spheres). The dual statement follows from the next lemma.

Lemma 4.7. *Let K be a connected simplicial complex of dimension $k \geqslant 2$. If the link of each vertex of K is a cross complex of dimension $k-1$, then K is a cross complex.*

Proof. Let v be a vertex of K. By assumption $\mathrm{lk}\,v$ is a cross complex of dimension $k-1$, so for every vertex $p \in \mathrm{lk}\,v$ there is a unique vertex $q \in \mathrm{lk}\,v$ not joined to p by an edge in $\mathrm{lk}\,v$. Still, p and q may be joined by an edge in K, so we consider two cases.

Case 1. Suppose that there is a pair of vertices p, q in $\mathrm{lk}\,v$ not joined by an edge in K. Let $\mathscr{R}$ be the set of other vertices of $\mathrm{lk}\,v$. The cardinality of $\mathscr{R}$ is $2(k-1)$. The link $\mathrm{lk}\,p$ is a cross complex, therefore it has $2k$ vertices and contains v and all the elements of $\mathscr{R}$. Since q is not joined to p by an edge in K, q is not in $\mathrm{lk}\,p$; so there is another vertex $p' \in \mathrm{lk}\,p$, $p' \notin v \cup \mathscr{R}$. Similarly, we have $q' \in \mathrm{lk}\,q$, $q' \notin v \cup \mathscr{R}$. Now take any $r \in \mathscr{R}$ and consider $\mathrm{lk}\,r$. Since $\mathrm{lk}\,v$ is a $(k-1)$-dimensional cross complex, r is joined to $2(k-1)$ vertices of $\mathrm{lk}\,v$ by edges in $\mathrm{lk}\,v$. We also know that r is joined to v, p' and q'. But since $\mathrm{lk}\,r$ is also a $(k-1)$-dimensional cross complex, r may be joined only to $2k$ vertices. Therefore $p' = q'$, which implies that K is a cross complex.

Case 2. Now suppose that every pair of vertices in $\mathrm{lk}\,v$ is joined by an edge in K. This will lead us to a contradiction. Each vertex u in $\mathrm{lk}\,v$ is joined to v and all the vertices in $\mathrm{lk}\,v$ apart from u itself. There are no other vertices joined to u by edges because $\mathrm{lk}\,u$ contains $2k$ vertices. This means that any pair of vertices in K is joined by an edge and K has precisely $2k+1$ vertices. The number of k-simplices meeting at each vertex is 2^k, and a k-simplex has $k+1$ vertices. Hence the total number of k-simplices in K is $2^k(2k+1)/(k+1)$.

Now we calculate the total number of k-simplices in K in a different way. Let σ be a k-simplex in K not containing v. Then σ contains a pair of vertices, say p and q, that are not joined by an edge in $\mathrm{lk}\,v$ (otherwise σ itself must be in $\mathrm{lk}\,v$ since $\mathrm{lk}\,v$ is a cross complex). Let L be the link of p in $\mathrm{lk}\,v$. Then L is a cross complex of dimension $k-2$, and it also coincides with the link of q in $\mathrm{lk}\,v$. We obtain that $\mathrm{lk}\,p$ is the join $L * \{v, q\}$ because both subcomplexes have the same vertex sets and both are cross complexes; in a similar way $\mathrm{lk}\,q = L * \{v, p\}$. Since σ contains p and q, it follows that $\sigma = \tau * p * q$ for some $(k-2)$-simplex $\tau \in L$. Therefore, σ has at least two faces of dimension $(k-1)$ in $\operatorname{lk} v$, namely, $\tau * p$ and $\tau * q$. Neither of these can be a face of another k-simplex not containing v because every $(k-1)$-simplex in K is a face of precisely two k-simplices and because $\tau * p$ is also a face of $\tau * p * v$, while $\tau * q$ is also a face of $\tau * q * v$. It follows that the number of k-simplices not containing v is at most half the number of $(k-1)$-simplices in $\operatorname{lk} v$. The number of k-simplices containing v is equal to the number of $(k-1)$-simplices in $\operatorname{lk} v$. The latter is 2^k, so the total number of k-simplices in K is at most $2^{k-1} + 2^k$, which is less than $2^k(2k+1)/(k+1)$ if $k \geqslant 2$. This contradiction finishes the proof.

Remark. Another proof of Lemma 4.6 can be given by producing a non-degenerate simplicial map from K onto a cross complex. Such a map is a topological (non-ramified) cover of a sphere by a sphere, so it must be an isomorphism for $n \geqslant 3$. This approach was used in [6].

Theorem 4.8. *Assume that a quasitoric manifold M admits a semifree action of a subcircle with isolated fixed points, and every 2-face of the quotient polytope P is a 4-gon. Then M is $\mathbb{S}^1$-equivariantly homeomorphic to a product of 2-dimensional spheres.*

Proof. By Lemma 4.6 the orbit polytope is a cube. By Theorem 4.3, M is equivalent to a Bott tower. Finally, by Theorem 4.4 it is $\mathbb{S}^1$-homeomorphic to a product of spheres.

We can also deduce the main result of Il'inskiĭ [6].

Corollary 4.9. *A (compact non-singular) toric manifold X carrying a semifree action of a circle subgroup with isolated fixed points is diffeomorphic to a product of 2-spheres.*

Proof. By Theorem 4.4 it is sufficient to show that X is a Bott tower. To this end we show that the fan corresponding to X is combinatorially equivalent to the fan over a cross-polytope and use Corollary 3.5 after that. The semifree circle subgroup acting on X also acts semifreely and with isolated fixed points on every characteristic submanifold X_j of X. Using induction on the dimension and Lemma 4.7 we reduce the statement to the 2-dimensional case, so that we merely have to show that the quotient polytope of a complex 2-dimensional toric manifold with semifree circle subgroup action and isolated fixed points is a 4-gon. (Note that a (non-singular compact) complex 2-dimensional toric manifold is always projective, so we can work with polytopes instead of fans.) The following case-by-case analysis is just a reformulation of the argument from [6], § 3.

Let Σ be the fan corresponding to our complex 2-dimensional toric manifold. One-dimensional cones of Σ correspond to facets (or edges) of the quotient polygon P^2. We must show that there are precisely 4 one-dimensional cones. The values of the characteristic function at the facets of P^2 are given by the primitive vectors generating the corresponding one-dimensional cones of Σ. Let ν be the vector generating the semifree circle subgroup. We may choose an initial vertex p of P^2 such that ν belongs to the 2-dimensional cone of Σ corresponding to p. Then we index the primitive vectors λ_i, $1 \leqslant i \leqslant m$, so that ν is in the cone generated by λ_1 and λ_2, and any two consecutive vectors span a two-dimensional cone (see Fig. 1). This provides us with a refined $2 \times m$ characteristic matrix Λ. We have

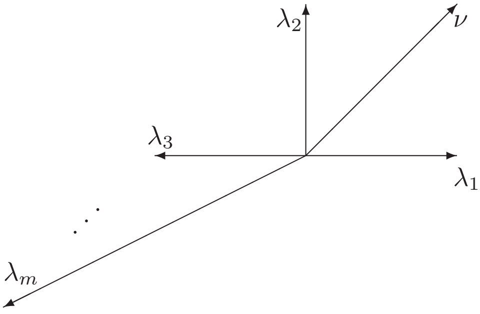

Figure 1

$\lambda_1 = (1, 0)$ and $\lambda_2 = (0, 1)$, and applying the criterion from Corollary 4.2 to the first cone $\langle\lambda_1, \lambda_2\rangle$ (that is, to the initial vertex of the polygon) we obtain $\nu = (1, 1)$.

Consider now the second cone. The non-singularity condition gives us $\det(\lambda_2, \lambda_3) = 1$, therefore $\lambda_3 = (-1, *)$. Writing $\nu = k_1\lambda_2 + k_2\lambda_3$ and applying Corollary 4.2 to the second cone $\langle\lambda_2, \lambda_3\rangle$ we obtain

$$(1, 1) = \pm(0, 1) \pm (-1, *).$$

Therefore, $\lambda_3 = (-1, 0)$ or $\lambda_3 = (-1, -2)$. Similarly, considering the last cone $\langle\lambda_m, \lambda_1\rangle$ we obtain $\lambda_m = (*, -1)$, and then, applying Corollary 4.2, we see that $\lambda_m = (0, -1)$ or $\lambda_m = (-2, -1)$. The case when $\lambda_3 = (-1, -2)$ and $\lambda_m = (-2, -1)$ is impossible since then the second and the last cones overlap.

Let $\lambda_3 = (-1, 0)$. Then applying a similar analysis to the third cone $\langle\lambda_3, \lambda_4\rangle$ shows that $\lambda_4 = (0, -1)$ or $\lambda_4 = (-2, -1)$. Therefore, $\lambda_4 = \lambda_m$ (otherwise cones overlap).

Similarly, if $\lambda_m = (0, -1)$, then we obtain $\lambda_{m-1} = (-1, 0)$ or $\lambda_{m-1} = (-1, -2)$. Therefore, $\lambda_{m-1} = \lambda_3$.

In any case, we have $m = 4$ and P^2 is a 4-gon. This completes the proof.

Note that the proof above leaves three possibilities for the vectors λ_3 and λ_4 of the corresponding 2-dimensional fan: $(-1, 0)$ and $(0, -1)$, or $(-1, 0)$ and $(-2, -1)$, or $(-1, -2)$ and $(0, -1)$, and the last two pairs are equivalent by means of an orientation-reversing automorphism of T^2. The corresponding reduced submatrices are

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

The first corresponds to $\mathbb{C}P^1 \times \mathbb{C}P^1$, and the second to a non-trivial Bott tower (Hirzebruch surface) with $a_{12} = -2$.

We finish this section by describing explicitly the class of matrices (2.2) corresponding to our specific class of Bott towers, for arbitrary dimension.

Theorem 4.10. *A Bott tower B^{2n} admits a semifree circle subgroup with isolated fixed points if and only if its matrix (2.2) satisfies the identity*

$$\frac{1}{2}(E-A)=C_1C_2\cdots C_n,$$

where C_k, for $1\leqslant k\leqslant n$, is either the identity matrix or a unipotent upper triangular matrix with only one non-zero element $c_{i_kk}=1$ positioned in the kth column above the diagonal.

Proof. Assume first B^{2n} admits a semifree circle subgroup with isolated fixed points. We have two sets of multiplicative generators for $H^*(B^{2n})$: the set $\{u_1,\ldots,u_n\}$ from Lemma 2.1 satisfying (2.1) and the set $\{x_1,\ldots,x_n\}$ satisfying $x_i^2=0$, $1\leqslant i\leqslant n$. The reduced sets with $i\leqslant k$ can be regarded as the corresponding sets of generators for the kth stage B^{2k}. As is clear from the proof of Theorem 4.4, we have $c_1(\xi_{k-1})=-2c_{i_kk}x_{i_k}$ for some $i_k<k$, where $c_{i_kk}=1$ or $c_{i_kk}=0$. From $u_k^2+2c_{i_kk}x_{i_k}u_k=0$ we obtain $x_k=u_k+c_{i_kk}x_{i_k}$. In other words, the transition matrix C_k from the basis $x_1,\ldots,x_{k-1},u_k,\ldots,u_n$ of $H^2(B^{2n})$ to $x_1,\ldots,x_k,u_{k+1},\ldots,u_n$ may have only one non-zero entry off the diagonal, which is c_{i_kk}. The transition matrix from $u_1,\ldots,u_n$ to $x_1,\ldots,x_n$ is therefore the product $D=C_1C_2\cdots C_n$ (here C_1 is the identity matrix since $x_1=u_1$). Then $D=(d_{jk})$ is a unipotent upper triangular matrix consisting of zeros and ones, $x_k=\sum_{j=1}^n d_{jk}u_j$, and

$$0=x_k^2=\left(u_k+\sum_{j=1}^{k-1}d_{jk}u_j\right)^2=u_k^2+2\sum_{j=1}^{k-1}d_{jk}u_ju_k+\cdots,\qquad 1\leqslant k\leqslant n.$$

On the other hand, $0=u_k^2-\sum_{j=1}^{k-1}a_{jk}u_ju_k$ by (2.1). Comparing the coefficients of u_ju_k for $1\leqslant j\leqslant k-1$ in the last two equations and observing that these elements are linearly independent in $H^4(B^{2k})$ we obtain $2d_{jk}=-a_{jk}$ for $1\leqslant j<k\leqslant n$. As both D and $-A$ are unipotent upper triangular matrices, this implies $2D=E-A$.

Assume now that the matrix A satisfies

$$E-A=2C_1C_2\cdots C_n.$$

Then for the corresponding Bott tower we have $\xi_{k-1}=\pi_{i_k}^*(\gamma^{-2c_{i_kk}})$. Therefore, we may choose a circle subgroup such that ξ_{k-1} becomes $t\pi_{i_k}^*(\gamma^{-2c_{i_kk}})$ (as an $\mathbb{S}^1$-bundle) for $1<k\leqslant n$. This circle subgroup acts semifreely and with isolated fixed points as seen from the same argument as in the proof of Theorem 4.4.

Example 4.11. It follows from Theorem 4.10 that if a Bott tower admits a semifree circle subgroup with isolated fixed points, then the matrix (2.2) may have only entries equal to 0 or -2 above the diagonal. However, the hypothesis of Theorem 4.10 is stronger. For instance, if $n=3$, then the Bott tower corresponding to the matrix

$$A=\begin{pmatrix}-1&0&-2\\0&-1&-2\\0&0&-1\end{pmatrix}$$

does not belong to the class under consideration since $(E-A)/2$ cannot be factored as $C_1C_2C_3$. On the other hand, any other matrix with zeros and -2's off the diagonal will do. For example, if

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then we have

$$\frac{1}{2}(E-A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that not every Bott tower homeomorphic to a product of spheres admits a semifree subcircle action with isolated fixed points (the latter condition is stronger even for $n = 2$). We shall consider the class of Bott towers that are homeomorphic to a product of spheres in the next section.

§ 5. Topological classification and cohomology

The following statement shows that Bott towers diffeomorphic to products of spheres can be detected by their cohomology rings, thereby providing a partial answer to Question 2.3 posed in § 2.

Theorem 5.1. *A Bott tower B^{2n} is isomorphic to the product $(\mathbb{C}P^1)^n$ if and only if $H^*(B^{2n}) \cong H^*((\mathbb{C}P^1)^n)$ as graded rings.*

Proof. From Lemma 2.1 we obtain

$$H^*(B^{2n}) = H^*\bigl(B^{2n-2}\bigr)[u_n]/\bigl(u_n^2 - c_1(\xi_{n-1})u_n\bigr).$$

We may therefore write any element of $H^2(B^{2n})$ as $x + bu_n$, where $x \in H^2(B^{2n-2})$ and $b \in \mathbb{Z}$. Since

$$(x + bu_n)^2 = x^2 + 2bxu_n + b^2u_n^2 = x^2 + b(2x + bc_1(\xi_{n-1}))u_n,$$

the square of $x + bu_n$ with $b \neq 0$ is zero if and only if $x^2 = 0$ and $2x + bc_1(\xi_{n-1}) = 0$. This shows that the elements $x + bu_n$ with $b \neq 0$ whose squares are zero generate a rank-one free subgroup of $H^2(B^{2n})$.

Assume that $H^*(B^{2n}) \cong H^*((\mathbb{C}P^1)^n)$. Then there exists a basis $\{x_1, \dots, x_n\}$ in $H^2(B^{2n})$ such that $x_i^2 = 0$ for all i. By the observation above we may assume that $x_1, \dots, x_{n-1}$ are in $H^2(B^{2n-2})$. Because $\{x_1, \dots, x_n\}$ is a basis, x_n is not in $H^2(B^{2n-2})$ and we may assume that $x_n = \sum_{i=1}^{n-1} b_ix_i + u_n$ with some $b_i \in \mathbb{Z}$. A product of the form $\prod_{i\in I} x_i$, where I is a subset of $\{1, \dots, n\}$, belongs to $H^*(B^{2n-2})$ if and only if $n \notin I$. This shows that $H^*(B^{2n-2})$ is generated by $x_1, \dots, x_{n-1}$ and is isomorphic to the cohomology ring of $(\mathbb{C}P^1)^{n-1}$. Therefore, we may assume by induction that $B^{2n-2} \cong (\mathbb{C}P^1)^{n-1}$.

Writing $c_1(\xi_{n-1}) = \sum_{i=1}^{n-1} a_ix_i$ we see that

$$0 = x_n^2 = \left(u_n + \sum_{i=1}^{n-1} b_ix_i\right)^2 = \sum_{i=1}^{n-1}(a_i + 2b_i)x_iu_n + \left(\sum_{i=1}^{n-1} b_ix_i\right)^2.$$

This may hold only if at most one of the a_i is non-zero because the elements x_ix_j $(i < j)$ and x_iu_n form an additive basis of $H^4(B^{2n})$. Therefore, ξ_{n-1} is the pullback of γ^{-2b_i} over $\mathbb{C}P^1$ by the projection $B^{2n-2} = (\mathbb{C}P^1)^{n-1} \to \mathbb{C}P^1$. Since $P(\underline{\mathbb{C}} \oplus \gamma^{-2b_i})$ is a product bundle (see Example 2.2), so is $B^{2n} = P(\underline{\mathbb{C}} \oplus \xi_{n-1})$.

We can now also effectively describe the class of matrices (2.2) corresponding to Bott towers that are diffeomorphic to a product of 2-spheres.

Theorem 5.2. *A Bott tower B^{2n} is diffeomorphic to $(\mathbb{C}P^1)^n$ if and only if the corresponding matrix* (2.2) *satisfies the identity*

$$\frac{1}{2}(E - A) = C_1C_2\cdots C_n,$$

where each C_k, $1 \leqslant k \leqslant n$, is a unipotent upper triangular matrix that may have only one non-zero element above the diagonal; this element lies in the kth column.

Proof. We use the same argument as in the proof of Theorem 4.10. The only difference is that the number c_{i_kk} in the formula $c_1(\xi_{k-1}) = -2c_{i_kk}x_{i_k}$ is now an arbitrary integer.

In the rest of this section we generalize the result of Theorem 5.1 to an arbitrary quasitoric manifold, but only in the topological category (see Theorem 5.7).

We start by analysing the algebraic structure of the cohomology of quasitoric manifolds over a cube. Although it is possible to make this analysis over $\mathbb{Z}$, it is more convenient for our purposes to reduce the coefficients modulo 2. Let S be a graded algebra over $\mathbb{Z}/2$ generated by degree-one elements $x_1, \dots, x_n$. We refer to S as a *Bott quadratic algebra* (or simply a *BQ-algebra*) of rank n if it satisfies the following two properties:

(C1) $x_k^2 = \sum_{i<k} a_{ik}x_ix_k$ with $a_{ik} \in \mathbb{Z}/2$ for $1 \leqslant k \leqslant n$ (in particular, $x_1^2 = 0$);

(C2) $\prod_{i=1}^n x_i \neq 0$.

If B^{2n} is a Bott tower, then (2.1) implies that $H^*(B^{2n}; \mathbb{Z}/2)$ is a BQ-algebra with double grading, which explains our terminology. Our arguments below work for a wider class of algebras with (C1) replaced by the weaker property:

(P1′) $x_k^2 = \sum_{i<j\leqslant k} a_{ijk}x_ix_j$ with $a_{ijk} \in \mathbb{Z}/2$ for $1 \leqslant k \leqslant n$.

Because of (C1) we can express any element of S as a linear combination of square-free monomials. We denote such a monomial $x_{i_1} \dots x_{i_s}$ by x_I, where $I = \{i_1, \dots, i_s\}$.

Lemma 5.3. *The elements x_I for all the subsets $I \in \{1, \dots, n\}$ form an additive basis of S. In particular, $\dim S_q = \binom{n}{q}$ where S_q denotes the graded component of degree q.*

Proof. (C1) implies that the set $\{x_I\}$ generates S additively. We order monomials x_I using the reverse lexicographic ordering of the subsets of $\{1, \dots, n\}$. Namely, if $I = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$ and similarly $J = \{j_1, \dots, j_s\}$, then we set $x_I < x_J$ if $i_k < j_k$ and $i_q = j_q$ for $k + 1 \leqslant q \leqslant s$.

Suppose that there is a non-trivial linear relation for the x_I, and let x_J be the maximal monomial appearing in this relation. Then we may use this relation to replace the subfactor x_J in $\prod_{i=1}^n x_i$, and then use (C1) repeatedly whenever x_k^2

occurs. At the end we obtain zero, which contradicts (C2). Therefore, there exist no non-trivial linear relations among the x_I.

Lemma 5.4. *Suppose we have a surjective graded homomorphism f from S into a graded algebra S' over $\mathbb{Z}/2$ satisfying $S'_{n-1} \neq 0$. Then the dimension of the kernel of $f\colon S_1 \to S'_1$ is at most one. Moreover, if the dimension of the kernel is precisely one, then S' is a BQ-algebra of rank $n-1$.*

Proof. We denote $f(x_i)$ by $\bar{x}_i$. Then (C1) holds for $\bar{x}_1, \ldots, \bar{x}_n$. Suppose the dimension of the kernel is at least two. Then there exist p and q, $p > q \geqslant 1$, such that

$$\bar{x}_p = \sum_{i<p} b_i \bar{x}_i, \qquad \bar{x}_q = \sum_{j<q} c_j \bar{x}_j, \tag{5.1}$$

where $b_i, c_j \in \mathbb{Z}/2$. By Lemma 5.3, S_{n-1} is generated by the elements x_I with $|I| = n-1$. We shall show that $\bar{x}_I = 0$ for any such I, which contradicts the assumption $S'_{n-1} \neq 0$.

Assume first that $q \geqslant 2$. Since $|I| = n-1$, I contains p or q. We replace $\bar{x}_p$ and $\bar{x}_q$ in $\bar{x}_I$ by (5.1) and use (C1) repeatedly whenever $\bar{x}_k^2$ occurs. Then we end up at zero.

Now if $q = 1$, that is, $\bar{x}_1 = 0$, then it is sufficient to show that $\bar{x}_I = 0$ for $I = \{2, 3, \ldots, n\}$. We replace $\bar{x}_p$ in $\bar{x}_I$ with the help of (5.1) and use (C1) repeatedly whenever $\bar{x}_k^2$ with $k \geqslant 2$ occurs. Then each term in the final expression contains $\bar{x}_1$, which is zero.

Now we prove the second statement of the lemma. By assumption the elements $\bar{x}_i$ satisfy a non-trivial linear relation. Let $\bar{x}_j$ be the maximal element occurring in this relation. We can eliminate $\bar{x}_j$ in S' using this linear relation and (C1). Then (C1) holds for the $\bar{x}_i$ with $i \neq j$. Therefore, S'_{n-1} is generated by $\prod_{i\neq j} \bar{x}_i$. This element is distinct from zero because $S'_{n-1} \neq 0$. Hence S' is a BQ-algebra of rank $n-1$.

Theorem 5.5. *Let M be a quasitoric manifold with quotient polytope P. Then $H^{2*}(M; \mathbb{Z}/2)$ is a BQ-algebra of rank n if and only if P is an n-cube.*

Proof. Assume that P is an n-cube. Since every principal minor of $\Lambda_\star$ is 1 modulo 2, it follows by the same argument as in the proof of Lemma 3.3 that $\Lambda_\star$ is conjugate to a unipotent upper triangular matrix. (A matrix (3.7) in which all the b_i, $1 \leqslant i \leqslant n$, are non-zero modulo 2 cannot occur because it has determinant zero modulo 2.) Then it follows from (3.4) that $H^{2*}(M; \mathbb{Z}/2)$ is a BQ-algebra of rank n.

Now assume that $H^{2*}(M; \mathbb{Z}/2)$ is a BQ-algebra. Let $b_r(M)$ be the rth Betti number of M and let $f_s(P)$ be the number of faces of P of codimension $s+1$. Then

$$b_2(M) = f_0(P) - n, \qquad b_4(M) = f_1(P) - (n-1) f_0(P) + \binom{n}{n-2}$$

(see [8], Theorem 3.1) and we obtain from Lemma 5.3 that

$$f_0(P) = 2n, \qquad f_1(P) = 2n(n-1). \tag{5.2}$$

For every characteristic submanifold M_i the restriction map

$$H^*(M;\mathbb{Z}/2) \to H^*(M_i;\mathbb{Z}/2)$$

is surjective ([18], Lemma 2.3). It follows from Lemmas 5.3 and 5.4 that $b_2(M_i) \geqslant b_2(M) - 1 = n - 1$. Therefore,

$$f_0(F_i) = (n-1) + b_2(M_i) \geqslant 2(n-1), \tag{5.3}$$

where F_i is the facet corresponding to M_i and

$$f_1(P) = \frac{1}{2}\sum_{i=1}^{2n} f_0(F_i) \geqslant 2n(n-1).$$

Comparing this with (5.2) we see that we have the equality in (5.3) for every i, that is, $b_2(M_i) = n-1$. This implies that the kernel of $H^2(M;\mathbb{Z}/2) \to H^2(M_i;\mathbb{Z}/2)$ is one-dimensional, so $H^{2*}(M_i;\mathbb{Z}/2)$ is a BQ-algebra of rank $n-1$ by Lemma 5.4. This enables us to use induction on n.

When $n = 2$, equations (5.2) imply that combinatorially P is a square. Suppose the theorem holds for $n-1$, where $n \geqslant 3$. Since $H^{2*}(M_i)$ is a BQ-algebra of rank $n-1$, every facet of P is an $(n-1)$-cube; in particular, every 2-face of P is a square. Then P is an n-cube by Lemma 4.6.

Lemma 5.6. *Let M be a quasitoric manifold over an n-cube. If*

$$H^*(M;\mathbb{Q}) \cong H^*((\mathbb{C}P^1)^n;\mathbb{Q})$$

is an isomorphism of graded rings, then M is equivalent to a Bott tower.

Proof. By assumption there exist elements $y_1,\dots,y_n$ in $H^2(M;\mathbb{Q})$ generating $H^*(M;\mathbb{Q})$ and satisfying $y_i^2 = 0$ for $1 \leqslant i \leqslant n$. Let $M_i \subset M$ be a characteristic submanifold; we denote the restriction of y_k to $H^2(M_i;\mathbb{Q})$ by $\bar{y}_k$. Then $\bar{y}_1,\dots,\bar{y}_n$ generate $H^*(M_i;\mathbb{Q})$ as a ring because $H^*(M;\mathbb{Q}) \to H^*(M_i;\mathbb{Q})$ is surjective. Since $b_2(M_i) = n-1$, there is a non-trivial linear relation for the elements $\bar{y}_k$. Using this linear relation we can eliminate one generator, for instance, $\bar{y}_n$, and obtain a surjective map $\mathbb{Q}[\bar{y}_1,\dots,\bar{y}_{n-1}]/(\bar{y}_1^2,\dots,\bar{y}_{n-1}^2) \to H^*(M_i;\mathbb{Q})$. Since the components of degree $2q$ in both rings have dimension $\binom{n-1}{q}$, this surjective map is an isomorphism. Therefore, $H^*(M_i;\mathbb{Q}) \cong H^*((\mathbb{C}P^1)^{n-1};\mathbb{Q})$, so we may use an induction argument and assume that every M_i is a Bott tower.

Let $\Lambda_\star$ be the reduced submatrix of M. It follows from Lemma 3.3 that $-\Lambda_\star$ is conjugate to a unipotent upper triangular matrix or to a matrix (3.7) with non-zero entries b_i, $1 \leqslant i \leqslant n$. It is sufficient to exclude the second case. Suppose that $-\Lambda_\star$ is given by (3.7). Then $\det(-\Lambda_\star) = -1$, that is,

$$\prod_{i=1}^{n} b_i = (-1)^n 2. \tag{5.4}$$

Using (3.4) we obtain

$$H^*(M) = \mathbb{Z}[x_1,\dots,x_n]/\big(x_1(x_1+b_1x_2), x_2(x_2+b_2x_3),\dots,x_n(b_nx_1+x_n)\big),$$

where we set $x_i = v_{i+n}$ for $1 \leqslant i \leqslant n$. By assumption there is a non-zero element $x \in H^2(M, \mathbb{Q}$ whose square is zero. We write $x = \sum_{i=1}^n a_i x_i$ for some $a_i \in \mathbb{Q}$; then

$$0 = \left(\sum_{i=1}^n a_i x_i\right)^2 = \sum_{i=1}^n a_i^2 x_i^2 + 2\sum_{i<j} a_i a_j x_i x_j$$
$$= -a_1^2 b_1 x_1 x_2 - a_2^2 b_2 x_2 x_3 - \dots - a_n^2 b_n x_n x_1 + 2\sum_{i<j} a_i a_j x_i x_j,$$

which implies that

$$a_1^2 b_1 = 2a_1 a_2, \quad a_2^2 b_2 = 2a_2 a_3, \quad \dots, \quad a_n^2 b_n = 2a_n a_1. \tag{5.5}$$

Suppose $a_i \neq 0$ for every i. Multiplying the above identities we obtain $\prod_{i=1}^n b_i = 2^n$, which contradicts (5.4). Therefore, $a_i = 0$ for some i, but in combination with (5.5) this implies that $a_i = 0$ for every i. This contradicts the assumption that $x = \sum_{i=1}^n a_i x_i \neq 0$. Therefore, (3.7) cannot occur as a reduced characteristic matrix and M is a Bott tower.

We are now ready to prove the following final result.

Theorem 5.7. *A quasitoric manifold M is homeomorphic to $(\mathbb{C}P^1)^n$ if and only if $H^*(M) \cong H^*((\mathbb{C}P^1)^n)$ is an isomorphism of graded rings.*

Proof. Since $H^*((\mathbb{C}P^1)^n; \mathbb{Z}/2)$ is a BQ-algebra of rank n, the quotient polytope of M is an n-cube by Theorem 5.5. Then M is a Bott tower by Lemma 5.6. Finally, M is homeomorphic to $(\mathbb{C}P^1)^n$ by Theorem 5.1.

We now put forward the following quasitoric analogue of Problem 2.3.

Question 5.8. Does an isomorphism of graded rings

$$H^*(M_1) \cong H^*(M_2)$$

imply a homeomorphism of quasitoric manifolds M_1 and M_2?

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