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SEMIGROUP OF OPERATORS ON DUAL BANACH SPACES

ANTHONY TO-MING LAU[1]

ABSTRACT. In this paper, we give a short and simple proof to a more general version of a recent result of Yeadon for semigroups of weak*-continuous operators on a dual Banach space. Our result has application to amenable groups and property P of a von Neumann algebra.

1. **Introduction.** Let E be a dual Banach space with a (fixed) predual E_*, i.e. E_* is a Banach space such that $(E_*)^* = E$. Let $\mathcal{S}$ be a semigroup of linear operators from E into E satisfying:

(1) $\|sx\| = \|x\|$ for all $s \in \mathcal{S}$ and $x \in E$.

(2) Each $s \in \mathcal{S}$ is a continuous linear map from (E, weak*) into (E, weak*).

Let $E^{\mathcal{S}} = \{x \in E;\ s(x) = x \text{ for all } s \in \mathcal{S}\}$ and K_x = weak*-closure of $\mathrm{co}\{s(x);\ s \in \mathcal{S}\}$ (here co X will denote the convex hull of a subset X of a linear space) for each $x \in E$. Then $E^{\mathcal{S}}$ is a weak*-closed linear subspace of E. Recently, Yeadon [12], ultilizing a method in [9, Lemma 5], proved that if $K_x \cap E^{\mathcal{S}}$ is nonempty for each $x \in E$, then there exists a bounded linear operator P from E onto $E^{\mathcal{S}}$ such that $P(x) \in K_x \cap E^{\mathcal{S}}$ for each $x \in E$. Furthermore, P commutes with any weak*-continuous linear operator Q commuting with each s in $\mathcal{S}$.

In this paper we prove a more general version of Yeadon's result. Our proof is simple and different from that of Yeadon. In §3, we concentrate on various applications of our theorems to obtain some of the results on invariant mean in [2], [3], [5], [6], [7], [10] and [11], and on property P of a von Neumann algebra in [9].

2. **The main theorem.** Let $\mathcal{B}(E)$ be the space of bounded linear operators from E into E. By the weak*-operator topology on $\mathcal{B}(E)$, denoted by W^*OT, we shall mean the locally convex topology determined by the family of seminorms $\{p_{x,\phi};\ x \in E \text{ and } \phi \in E_*\}$ where $p_{x,\phi}(T) = |p(Tx)|$. As known [4, p. 973], the unit ball of $\mathcal{B}(E)$ is compact in the W^*OT. Let $\overline{\mathrm{co}}\ \mathcal{S}$ denote the closure of co $\mathcal{S}$ in the W^*OT. Then ($\overline{\mathrm{co}}\ \mathcal{S}$, W^*OT) is a semigroup and a compact Hausdorff space such that for each $k \in \overline{\mathrm{co}}\ \mathcal{S}$, and each $s \in S$, the following mappings from ($\overline{\mathrm{co}}\ \mathcal{S}$, W^*OT) into ($\overline{\mathrm{co}}\ \mathcal{S}$, W^*OT) are continuous:

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(1) $h \to hk$,
(2) $h \to sh$.
Let $\mathcal{G} \subseteq \overline{\text{co}}\ \mathcal{S}$ be a closed subsemigroup of $\overline{\text{co}}\ \mathcal{S}$ and let

$$\mathcal{G}(x) = \{ g(x);\ g \in \mathcal{G} \}.$$

A subset $X \subseteq E$ is $\mathcal{G}$*-invariant* if $\mathcal{G}(x) \subseteq X$ for each $x \in X$.

If $F \subseteq E^{\mathcal{S}}$, an operator $P \in \mathfrak{B}(E)$ is *F-stationary* on X if for each $x \in X$, $P(x) \in F$.

THEOREM 2.1. *Let X be a $\mathcal{G}$-invariant subset of E and let F be a weak*-closed subset of $E^{\mathcal{S}}$. If $\mathcal{G}(x) \cap F$ is nonempty for each $x \in X$, then $\mathcal{G}$ contains an F-stationary operator on X. In this case, $\mathcal{G}(x) \cap F = \{P(x);$ $P \in \mathcal{G}$ and P is F-stationary on $X\}$ for each $x \in X$.*

PROOF. For each $x \in X$, let $\mathcal{K}(x) = \{ g \in \mathcal{G};\ g(x) \in F\}$. Then $\mathcal{K}(x)$ is a nonempty closed subset of $\mathcal{G}$. Furthermore, if $\sigma = \{x_1, \ldots, x_n\}$ is any finite subset of X and $g \in \bigcap \{\mathcal{K}(x_i);\ i = 1, \ldots, n-1\}$, pick $g' \in \mathcal{G}$ such that $g'(g(x_n)) \in F$. It follows that $(g' \cdot g)(x_i) = g(x_i)$ for all $i = 1, \ldots, n-1$ and $g' \cdot g \in \bigcap \{\mathcal{K}(x_i);\ i = 1, \ldots, n\}$. Consequently, if $x_0 \in X$ and $z \in \mathcal{G}(x_0) \cap F$, the sets $\mathcal{K}'(x) = \{ g \in \mathcal{G};\ g(x) \in F$ and $g(x_0) = z\}$ are closed in $\mathcal{G}$ and have finite intersection property. By compactness, $\bigcap \{\mathcal{K}'(x);\ x \in X\}$ is nonempty, and any P in this intersection is F-stationary and $P(x_0) = z$.

REMARK. If $\mathcal{G} = \overline{\text{co}}\ \mathcal{S}$ and $E^{\mathcal{S}} = F$, then $\mathcal{G}(x) =$ weak*-closure of co$\{sx;$ $s \in \mathcal{S}\}$. Hence, the first part of Theorem 2.1 yields the proposition in [12].

3. Applications.

A. *Invariant means on discrete semigroups*. Let S be a semigroup and $l_\infty(S)$ be the Banach space of bounded real-valued (or complex-valued) functions on S with supremum norm. For each $a \in S$, define the left and right translation operators on $l_\infty(S)$ by $(l_a f)(t) = f(as)$ and $(r_a f)(s) = f(sa)$ for all $s \in S$, $f \in l_\infty(S)$. Let X be a closed translation invariant linear subspace of $l_\infty(S)$ containing constant one function 1. A linear functional ϕ on X is called a *mean* if $\|\phi\| = 1$ and $\phi \geqslant 0$. A mean on X is called a *left invariant mean* if $\phi(l_a f) = \phi(f)$ for all $a \in S$ and all $f \in X$ (see Day [1]).

The subspace X is called *left introverted* [respectively, *left m-introverted*] if for each mean [respectively, multiplicative mean] ϕ on $l_\infty(S)$, the function $s \to \phi(l_s f)$ is in X. (See [1, p. 540] and [8, p. 121].)

Let $\mathcal{S} = \{r_a;\ a \in S\}$. Since the weak*-topology and the topology of pointwise convergence agree on bounded subsets of $l_\infty(S)$, it follows from [3, Lemma 2 and the proposition following] that X is left introverted [respectively, left m-introverted] if and only if X is $\overline{\text{co}}\ \mathcal{S}$- [respectively, $\overline{\mathcal{S}}$-] invariant.

THEOREM 3.1 (Granirer and Lau [3]). *If X is left introverted and for each $f \in X$, $\overline{\text{co}}\ \mathcal{S}(f)$ contains a constant function, then X has a left invariant mean. In this case for each $f \in X$, $\alpha \cdot 1 \in \overline{\text{co}}\ \mathcal{S}(f)$ if and only if $m(f) = \alpha$ for some left invariant mean m on X.*

PROOF. Let F be the one-dimensional subspace of $l_\infty\ (S)$ consisting of constant functions. If $P \in \overline{\text{co}}\ \mathcal{S}$ is F-stationary on X, define $m(f) = (Pf)(a)$

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for some fixed $a \in S$. Since left and right translations commute, m is a left invariant mean on X. Now use Theorem 2.1.

Similarly we have

THEOREM 3.2 (Granirer and Lau [3]). *If X is a left m-introverted subalgebra of $l_\infty(S)$ and for each $f \in X$, $\bar{\mathcal{S}}(f)$ contains a constant function, then X has a multiplicative left invariant mean. In this case for each $f \in X$, $\lambda \cdot 1 \in \bar{\mathcal{S}}(f)$ if and only if $m(f) = \lambda$ for some multiplicative left invariant mean m on X.*

REMARK. 1. Theorem 3.1 is proved by Mitchell [5, Theorem 3] for the case $X = l_\infty(S)$ and Theorem 3.2 is proved by Granirer [2, Theorem 1] for the case $X = l_\infty(S)$. Their proofs are completely different.

2 (Granirer and Lau [3]). If X is introverted and for each $f \in X$ and each $a \in S$, $\overline{\text{co}}\ \mathcal{S}(f - l_a f)$ contains the zero function, then X has a left invariant mean. To see this, let $F = \{0\}$. Apply Theorem 2.1; we obtain $T \in \overline{\text{co}}\ \mathcal{S}$ such that $T(f - l_a f) = 0$ for all $f \in X$ and $a \in S$.

3. If for each $A \subseteq S$, $\bar{\mathcal{S}}(1_A)$ contains either the zero function or the one function, then $l_\infty(S)$ has a multiplicative left invariant mean. Indeed, let $F = \{0,1\}$. By Theorem 2.1, there exists $T \in \bar{\mathcal{S}}$ such that $T(1_A) \in F$ for all $A \subseteq S$. Then T defines a multiplicative left invariant mean on $l_\infty(S)$.

Note that $\bar{\mathcal{S}}(1_A)$ contains the one function [respectively, the zero function], if and only if A [respectively, $S - A$] is left thick (see [7, p. 256] for definition). Hence, this yields [6, Theorem 1(f) $\to$ (a)] and [5, Theorem 3].

B. *Invariant means on locally compact groups*. Let G be a locally compact group with a fixed left Haar measure λ. Let $P(G) = \{\phi \in L_1(G);\ \|\phi\| = 1$ and $\phi \geqslant 0\}$. For each $\phi \in P(G)$, define the translation operators on $L_\infty(G)$ by $l_\phi(f) = \phi * f$ and $r_\phi(f) = f * \tilde{\phi}$ for all $f \in L_\infty(G)$. Let X be a closed linear subspace of $L_\infty(G)$ which is invariant under r_ϕ and l_ϕ for each $\phi \in P(G)$ and containing constants. An element $m \in X^*$ is called a *mean* if $\|m\| = 1$ and $m \geqslant 0$. A mean m on X is called a *topological left invariant mean* if $m(l_\phi f) = m(f)$ for each $\phi \in P(G)$ and each $f \in X$.

Let $\mathcal{S} = \{r_\phi;\ \phi \in P(G)\}$. Then $\mathcal{S}$ is a semigroup of weak*-continuous contractions on $L_\infty(G)$. Furthermore, X is $\bar{\mathcal{S}}$-invariant (note that $\mathcal{S} = \text{co}\ \mathcal{S}$) if and only if X is topologically left introverted, i.e. for each mean m on $L_\infty(G)$ and each $f \in X$, the functional $\phi \to m(\Delta^{-1}\tilde{\phi} * f)$ defined on $L_1(G)$ is in X (see Wong [10, p. 356]). Indeed, if X is $\bar{\mathcal{S}}$-invariant, and m is a mean on $L_\infty(G)$, let $\{\phi_\alpha\}$ be a net in $P(G)$ such that ϕ_α converges to m in the weak*-topology. By compactness of $\bar{\mathcal{S}}$, we may assume that the net $\{r_{\phi_\alpha}\}$ converges to some $T \in \bar{\mathcal{S}}$ in the W^*OT. Hence, for each $\phi \in L_1(G)$,

$$\begin{aligned} m(\Delta^{-1}\tilde{\phi} * f) &= \lim_\alpha \phi_\alpha(\Delta^{-1}\tilde{\phi} * f) \\ &= \lim_\alpha \phi(f * \tilde{\phi}_\alpha) \qquad (\text{see } [10, \text{Lemma } 3.1(\text{c})]) \\ &= \lim_\alpha \phi(r_{\phi_\alpha} f) = \phi(Tf). \end{aligned}$$

Hence, X is topologically left introverted. The other direction can be proved similarly.

THEOREM (Wong [10, Theorem 5.4]). *Assume that X is topologically left*

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introverted. If for each $f \in X$, $\bar{\mathfrak{S}}(f)$ contains a constant function, then X has a topological left invariant mean. In this case for each $f \in X$, $\beta \cdot 1 \in \bar{\mathfrak{S}}(f)$ if and only if there is a topological left invariant mean m on X such that $m(f) = \beta$.

PROOF. Apply Theorem 2.1 with F being the one-dimensional subspace of F consisting of the constant functions.

REMARK. Recently, Wong [11, Theorem 3.1(1) $\Rightarrow$ (4)] proved a similar theorem for locally compact topological semigroups. It is easy to see that this result is also a consequence of Theorem 2.1. We omit the details.

C. *von Neumann algebras with property* P. Let M be a von Neumann algebra acting on a Hilbert space H and let M' be the commutant of M. Assume that M contains the identity operator on H. Let M^u be the group of unitary elements in M. For each $u \in M^u$, define $T_u(x) = u^* x u$ for each $x \in \mathfrak{B}(H)$. Let $\mathfrak{S} = \{T_u;\ u \in M^u\}$. Then M is said to have property P if for each $x \in \mathfrak{B}(H)$ $\overline{\text{co}}\ \mathfrak{S}(x) \cap M'$ is nonempty. It is easy to see that the following slight improvement of Schwartz's result [9, Lemma 5] is also a consequence of Theorem 2.1.

THEOREM. *Let $\mathfrak{G}$ be a closed subsemigroup of $\overline{\text{co}}\ \mathfrak{S}$ containing $\mathfrak{S}$. If $\mathfrak{G}(x) \cap M' \neq \emptyset$ for each $x \in \mathfrak{B}(H)$, there exist $P \in \mathfrak{G}$ such that*

$$(*) \qquad P(x) \in M' \quad \text{for each } x \in \mathfrak{B}(H).$$

In this case, for each $x \in \mathfrak{B}(H)$, $z \in \mathfrak{G}(x) \cap M'$ if and only if there exist $P \in \mathfrak{G}$ satisfying $(*)$ and $P(x) = z$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA

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