

SEMIGROUP RINGS AS WEAKLY KRULL DOMAINS

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ABSTRACT. Let D be an integral domain and Γ be a torsion-free commutative cancellative (additive) semigroup with identity element and quotient group G . In this paper, we show that if $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$), then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain, Γ is a weakly Krull UMT-monoid, and G is of type $(0, 0, 0, \dots)$ (resp., type $(0, 0, 0, \dots)$ except p). Moreover, we give arithmetical applications of this result.

1. INTRODUCTION

Let D be an integral domain and $X^1(D)$ be the set of height-one prime ideals of D . We say that D is a Krull domain if D satisfies the following three properties:

- (i) $D = \bigcap_{P \in X^1(D)} D_P$,
- (ii) each nonzero nonunit of D is contained in only finitely many height-one prime ideals of D , and
- (iii) D_P is a principal ideal domain (PID) for all $P \in X^1(D)$.

Krull domains include UFDs and Dedekind domains. However, many well-studied rings are close to being Krull by satisfying (i) and (ii), but property (iii) fails, e.g. non-principal orders in number fields and $\mathbb{Q}[X^2, X^3]$ for an indeterminate X over the field \mathbb{Q} of rational numbers. An integral domain satisfying (i) and (ii) is called a weakly Krull domain. Hence, Krull domains and one-dimensional noetherian domains are weakly Krull, but the backwards implications need not hold true. The notion of weakly Krull domains was first introduced and studied by Anderson, Mott and Zafrullah [3]. A weakly factorial domain (WFD) is an integral domain whose nonzero elements can be written as finite products of primary elements. Then UFDs are WFDs, and D is a WFD if and only if D is a weakly Krull domain and each t -invertible t -ideal of D is principal [4, Theorem].

Let Γ be a monoid, i.e., a commutative cancellative (additive) semigroup with identity element and $D[\Gamma]$ be the semigroup ring of Γ over D . Then Γ has a quotient group [28, Theorem 1.2], and $D[\Gamma]$ is an integral domain if and only if Γ is torsion-free [28, Theorem 8.1]. It is well known that $D[\Gamma]$ is a Krull domain (resp., UFD) if and only if D is a Krull domain (resp., UFD), Γ is a Krull monoid (resp., factorial monoid), and $\langle \Gamma \rangle$, the quotient group of Γ , satisfies the ascending chain condition on its cyclic subgroups [28, Theorem 15.6] (resp., [28, Theorem 14.16]). In [10],

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Chang characterized when $D[\Gamma]$ is a WFD under the assumption that $\langle \Gamma \rangle$ satisfies the ascending chain condition on its cyclic subgroups. Then, he asked when $D[\Gamma]$ becomes a weakly Krull domain [10, Question 14]. Furthermore, in [13], Chang and Oh completely characterized the weakly factorial property of $D[\Gamma]$. Recently, in [16], Fadinger and Windisch gave a partial answer to Chang's question using the concept of weakly Krull monoids. These are defined analogously to weakly Krull domains by properties (i) and (ii) above, which were introduced and characterized by Halter-Koch in [30]. There he proved that D is weakly Krull if and only if its multiplicative monoid $D \setminus \{0\}$ is a weakly Krull monoid.

Now, in this paper, we give a complete characterization of weakly Krull semigroup rings $D[\Gamma]$. Precisely, we show that if $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$), then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain, Γ is a weakly Krull UMT-monoid, and the quotient group of Γ is of type $(0, 0, 0, \dots)$ (resp., type $(0, 0, 0, \dots)$ except p). As a corollary, we recover Matsuda's results [37, 38] that if $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$), then $D[\Gamma]$ is a generalized Krull domain if and only if D is a generalized Krull domain, Γ is a generalized Krull monoid, and $\langle \Gamma \rangle$ is of type $(0, 0, 0, \dots)$ (resp., type $(0, 0, 0, \dots)$ except p).

Moreover, in the final section we use the main result in order to obtain arithmetical statements on weakly Krull semigroup rings. For instance, we provide a large class of weakly Krull numerical semigroup rings that have full systems of sets of lengths. Also for a certain class of affine semigroup rings we prove a result about the connection of its class group and its system of sets of lengths. Thereby, we are the first to give a fairly broad but sufficiently concrete class of non-local weakly Krull domains that are not Krull, but whose arithmetic is still accessible.

2. DEFINITIONS RELATED TO THE t -OPERATION AND MONOIDS

Let D be an integral domain with quotient field K , Γ be a torsion-free monoid with quotient group $\langle \Gamma \rangle$; so $D[\Gamma]$ is an integral domain, \bar{D} be the integral closure of D in K , and $\bar{\Gamma}$ be the integral closure (i.e., root closure) of Γ in $\langle \Gamma \rangle$. If we say that D is local, we do not impose that D is noetherian.

2.1. The t -operation on integral domains. Let $F(D)$ be the set of nonzero fractional ideals of D . For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$. It is easy to see that $I^{-1} \in F(D)$. Hence, the v - and t -operations are well-defined as follows:

- (1) $I_v = (I^{-1})^{-1}$ and
- (2) $I_t = \bigcup \{J_v \mid J \text{ is a finitely generated subideal of } I\}$.

Let $*$ = v or $*$ = t . Then, for any nonzero $a \in K$ and $I, J \in F(D)$, (i) $aI_* = (aI)_*$, (ii) $I \subseteq I_*$; $I \subseteq J$ implies $I_* \subseteq J_*$, (iii) $(I_*)_* = I_*$, and (iv) $(IJ)_* = (IJ_*)_*$. An $I \in F(D)$ is called a $*$ -ideal if $I_* = I$.

A t -ideal is a *maximal t -ideal* of D if it is maximal among proper integral t -ideals of D . It is easy to see that each maximal t -ideal is a prime ideal, each t -ideal is contained in a maximal t -ideal, a prime ideal minimal over a t -ideal is a t -ideal, each nonzero principal ideal is a v -ideal, each v -ideal is a t -ideal, $I \subseteq I_t \subseteq I_v$ for all $I \in F(D)$, and $I_t = I_v$ if I is finitely generated. Let $\text{Max}(D)$ (resp., $t\text{-Max}(D)$) be the set of maximal ideals (resp., maximal t -ideals) of D . It is easy to see that $D = \bigcap_{M \in \text{Max}(D)} D_M = \bigcap_{P \in t\text{-Max}(D)} D_P$. By $t\text{-dim}(D) = 1$, we mean that D is not

a field and each prime t -ideal of D is a maximal t -ideal, and in this case, $X^1(D) = t\text{-Max}(D)$. It is well known that if D is not a field, then D is a weakly Krull domain if and only if $t\text{-dim}(D) = 1$ and D is of finite t -character (i.e., each nonzero nonunit of D is contained in only finitely many maximal t -ideals) [3, Lemma 2.1].

An $I \in F(D)$ is said to be invertible (resp., t -invertible, v -invertible) if $II^{-1} = D$ (resp., $(II^{-1})_t = D$, $(II^{-1})_v = D$). Let $T(D)$ be the abelian group of t -invertible fractional t -ideals of D under the t -multiplication $I * J = (IJ)_t$ and $\text{Prin}(D)$ be the set of nonzero principal fractional ideals of D . Then $\text{Prin}(D)$ is a subgroup of $T(D)$, and $\text{Cl}_t(D) = T(D)/\text{Prin}(D)$ denotes the factor group of $T(D)$ modulo $\text{Prin}(D)$. We denote the group of all v -invertible fractional v -ideals by $F_v(D)^\times$ and the monoid of all v -invertible integral v -ideals by $\mathcal{I}_v^*(D)$ where multiplication is defined via $I \cdot_v J = (I \cdot J)_v$. Then $F_v(D)^\times$ is the quotient group of $\mathcal{I}_v^*(D)$. By $\mathcal{C}_v(D)$ we denote the quotient of $F_v(D)^\times$ modulo $\text{Prin}(D)$ and call it the (v -)class group of D . If we denote the set of all non-zero (integral) principal ideals of D by $\mathcal{H}(D)$, then the embedding $\mathcal{H}(D) \rightarrow \mathcal{I}_v^*(D)$ is a cofinal divisor homomorphism and $\mathcal{I}_v^*(D)/\mathcal{H}(D) = \mathcal{C}_v(D)$ (for more in this direction see [24, Chapter 2.10]). It is easy to see that a t -invertible t -ideal is a v -invertible v -ideal. Hence, $\text{Prin}(D) \subseteq T(D) \subseteq F_v(D)^\times$, and thus $\text{Cl}_t(D)$ is a subgroup of $\mathcal{C}_v(D)$, and equality holds if D is a Mori domain (e.g., Krull domain). An integral domain is a Mori domain if it satisfies the ascending chain condition on its integral v -ideals.

2.2. Monoids. Let H be a monoid with quotient group $\langle H \rangle$. As in the case of integral domains, we can define the v -operation, the t -operation, $t\text{-Max}(H)$, t -invertibility, the class groups, and the Mori monoid. The reader is referred to [31] for more on the v - and t -operation on monoids. A monoid H is a Krull monoid if and only if H is a completely integrally closed Mori monoid, if and only if each ideal of H is t -invertible [31, Theorem 22.8].

Let G be a torsion-free abelian group. We say that G is of type $(0, 0, 0, \dots)$ if G satisfies the ascending chain condition on its cyclic subgroups (equivalently, for each nonzero element $g \in G$, there exists a largest positive integer n_g such that $n_g x = g$ is solvable in G) [28, Theorem 14.10]. For a prime number p , G is said to be of type $(0, 0, 0, \dots)$ except p if G satisfies the following two conditions; for each nonzero element $g \in G$, (i) an infinite number of prime numbers do not divide g and (ii) for each prime number $q \neq p$, q^n does not divide g for some positive integer n . Clearly, a torsion-free abelian group of type $(0, 0, 0, \dots)$ is of type $(0, 0, 0, \dots)$ except p for all prime numbers p , but not vice versa. For example, let $G = \bigcup_{n=1}^{\infty} (1/p^n)\mathbb{Z}$ for a prime number p , then G is of type $(0, 0, 0, \dots)$ except p but not of type $(0, 0, 0, \dots)$. The notion of type $(0, 0, 0, \dots)$ except p was introduced by Matsuda [37, 38] in order to study when $D[G]$ is a generalized Krull domain for an integral domain D with $\text{char}(D) = p$.

Remark 1. (1) Let G be a nonzero torsion-free abelian group and $p > 0$ be a prime number. In [13], Chang and Oh say that G is of type $(0, 0, 0, \dots)$ except p if G satisfies the following two conditions for each nonzero element $g \in G$;

- (i') the number of prime numbers dividing g is finite and
- (ii) for each prime number $q \neq p$, q^n does not divide g for some integer $n \geq 1$.

Then, in [13, Theorem 4.2], they prove that if D is an integral domain with $\text{char}(D) = p > 0$, then $D[G]$ is of finite t -character if and only if D is of finite t -character and G is of type $(0, 0, 0, \dots)$ except p . But, in order to prove [13, Theorem 4.2], they actually used Matsuda's original definition of type $(0, 0, 0, \dots)$ except p . Thus, there is no problem when we cite the results of [13].

(2) Let S be an infinite set of prime numbers such that there are also infinitely many prime numbers in $\mathbb{Z} \setminus S$ and $p \notin S$. Let m be a positive integer and

$$G = \left\{ \frac{a}{p_1^{e_1} \cdots p_k^{e_k} p^n} \mid a \in \mathbb{Z}, p_i \in S, 0 \leq e_i \leq m \text{ for } i = 1, \dots, k, \text{ and } n \geq 1 \right\}.$$

Then G is a torsion-free abelian group under the usual addition. Moreover, G is of type $(0, 0, 0, \dots)$ except p , but G does not satisfy (i') (for example, $1 \in G$ and each $q \in S$ divides 1 because $1 = q \cdot \frac{1}{q}$ and $\frac{1}{q} \in G$). Thus, (i') together with (ii) is stronger than (i) and (ii).

2.3. Semigroup rings. Let Γ be a torsion-free monoid. It is well known that Γ admits a total order $<$ compatible with its monoid operation [28, Corollary 3.4]. Hence each $f \in D[\Gamma]$ is uniquely expressible in the form $f = a_1 X^{\alpha_1} + a_2 X^{\alpha_2} + \cdots + a_k X^{\alpha_k}$, where $a_i \in D$ and $\alpha_j \in \Gamma$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. For an ideal I (resp., J) of D (resp., Γ), let $I[J] = \{a_1 X^{\alpha_1} + a_2 X^{\alpha_2} + \cdots + a_k X^{\alpha_k} \in D[\Gamma] \mid a_i \in I \text{ and } \alpha_j \in J\}$. Then $I[J]$ is an ideal of $D[\Gamma]$ [15, Lemma 2.3], and $I[J]$ is a prime ideal if and only if either I is a prime ideal of D and $J = \Gamma$ or $I = D$ and J is a prime ideal of Γ (cf. [28, Corollary 8.2] and the proof of [16, Lemma 3.1]).

2.4. UMT-domains and UMT-monoids. Let X be an indeterminate over D and $D[X]$ be the polynomial ring over D . A nonzero prime ideal Q of $D[X]$ is called an upper to zero in $D[X]$ if $Q \cap D = (0)$. So Q is an upper to zero in $D[X]$ if and only if $Q = fK[X] \cap D[X]$ for some irreducible polynomial $f \in K[X]$. Following [32], we say that D is a UMT-domain if each upper to zero in $D[X]$ is a maximal t -ideal of $D[X]$. It is known that D is a UMT-domain if and only if \bar{D}_P is a Prüfer domain for all $P \in t\text{-Max}(D)$ [19, Theorem 1.5].

In [12, Theorem 17], it was shown that $D[\Gamma]$ is a UMT-domain if and only if D is a UMT-domain and $\bar{\Gamma}_S$ is a valuation monoid for all maximal t -ideals S of Γ . Hence, the following is a natural generalization of UMT-domains to monoids.

Definition 2. Let Γ be a torsion-free monoid with quotient group G and $\bar{\Gamma}$ be the integral closure (i.e., root closure) of Γ in G . We say that Γ is a *UMT-monoid* if $\bar{\Gamma}_S$ is a valuation monoid for all maximal t -ideals S of Γ .

A Prüfer v -multiplication domain (PvMD) is an integral domain whose nonzero finitely generated ideals are t -invertible. Then D is a PvMD if and only if D_P is a valuation domain for all maximal t -ideals P of D [29, Theorem 5], if and only if D is an integrally closed UMT-domain [32, Proposition 3.2]. Now, a monoid Γ is called a Prüfer v -multiplication monoid (PvMM) if every finitely generated ideal of Γ is t -invertible. It is known that Γ is a PvMM if and only if Γ_S is a valuation monoid for all $S \in t\text{-Max}(\Gamma)$ [31, Theorem 17.2], hence we have

Proposition 3. *Let Γ be an integrally closed torsion-free monoid. Then Γ is a UMT-monoid if and only if Γ is a PvMM.*

By Proposition 3, UMT-monoids include valuation monoids, PvMMs, Krull monoids, and monoids of t -dimension one whose integral closure is a PvMM.

3. WEAKLY KRULL DOMAINS; PRELIMINARY RESULTS

Weakly Krull domains are of finite t -character. The class of integral domains of finite t -character includes Krull domains and Noetherian domains. In this section, we recall a couple of results on weakly Krull domains some of which are already known. The following proposition seems to be of this kind, but we could not find a proper reference, so we provide a full proof.

Proposition 4. *Let D be an integral domain with quotient field K , and assume that $D \neq K$.*

- (1) *Let D be a weakly Krull domain and S be a multiplicative subset of D . Then D_S is also a weakly Krull domain.*
- (2) *Let $\{S_\lambda\}$ be a set of multiplicative subsets of D such that $D = \bigcap_\lambda D_{S_\lambda}$ and D_{S_λ} is a weakly Krull domain for all λ . If $D = \bigcap_\lambda D_{S_\lambda}$ has finite character, then D is a weakly Krull domain.*

Proof. (1) [2, Proposition 4.7].

(2) Let $X^1(D_{S_\lambda})$ be the set of height-one prime ideals of D_{S_λ} . Then

$$D = \bigcap_\lambda \left(\bigcap_{P \in X^1(D_{S_\lambda})} (D_{S_\lambda})_P \right)$$

and this intersection has finite character. Now, for $P \in X^1(D_{S_\lambda})$, let $Q = P \cap D$. Then $D_Q = (D_{S_\lambda})_P$, and hence $D = \bigcap_{Q \in T} D_Q$ for some $T \subseteq X^1(D)$ and this intersection has finite character. Next, let Q' be a height-one prime ideal of D . Then $D_{Q'} = \bigcap_{Q \in T} (D_Q)_{D \setminus Q'}$ because the intersection has finite character. Since D_Q is a one-dimensional local domain, $(D_Q)_{D \setminus Q'} = K$ or $(D_Q)_{D \setminus Q'} = D_Q$. Also, since $D_{Q'}$ is a one-dimensional local domain, $D_{Q'} = D_Q$ for some $Q \in T$. Thus, $T = X^1(D)$, so D is a weakly Krull domain. \square

Let D be an integral domain with quotient field K and Γ be a torsion-free monoid with with quotient group G . For any $f = a_1X^{\alpha_1} + a_2X^{\alpha_2} + \dots + a_kX^{\alpha_k} \in D[\Gamma]$ with $\alpha_1 < \dots < \alpha_k$, let $C(f)$ be the ideal of $D[\Gamma]$ generated by $a_1X^{\alpha_1}, a_2X^{\alpha_2}, \dots, a_kX^{\alpha_k}$ and $c(f)$ be the ideal of D generated by a_1, \dots, a_k , so $C(f) \subseteq c(f)D[\Gamma]$. For convenience, we always assume that $f \neq 0$ when we study the v -closure $C(f)_v$ of $C(f)$. Let $N(H) = \{f \in D[\Gamma] \mid C(f)_v = D[\Gamma]\}$ and $H = \{aX^\alpha \mid 0 \neq a \in D \text{ and } \alpha \in \Gamma\}$. It is easy to see that H and $N(H)$ are saturated multiplicative subsets of $D[\Gamma]$, $D[\Gamma]_H = K[G]$, and $D[\Gamma] = D[\Gamma]_{N(H)} \cap D[\Gamma]_H$.

Lemma 5. *Let D be an integral domain with quotient field K , Γ be a torsion-free monoid with quotient group G , and $N(H) = \{f \in D[\Gamma] \mid C(f)_v = D[\Gamma]\}$.*

(1)

$$\begin{aligned} t\text{-Max}(D[\Gamma]) &= \{P[\Gamma] \mid P \in t\text{-Max}(D)\} \\ &\cup \{D[S] \mid S \in t\text{-Max}(\Gamma)\} \\ &\cup \{Q \in t\text{-Max}(D[\Gamma]) \mid QK[G] \subsetneq K[G]\}. \end{aligned}$$

(2) $\text{Max}(D[\Gamma]_{N(H)}) = \{P[\Gamma]_{N(H)} \mid P \in t\text{-Max}(D)\} \cup \{D[S]_{N(H)} \mid S \in t\text{-Max}(\Gamma)\}$.

$$(3) \ t\text{-Max}(D[\Gamma]_{N(H)}) = \text{Max}(D[\Gamma]_{N(H)}).$$

Proof. (1) [6, Corollary 1.3] and [15, Corollary 2.4]. (2) [7, Proposition 1.4 and Example 1.6]. (3) [7, Example 1.6 and Proposition 1.8]. \square

Proposition 6. *Let D be an integral domain with quotient field K , Γ be a torsion-free monoid with quotient group G , and $N(H) = \{f \in D[\Gamma] \mid C(f)_v = D[\Gamma]\}$. Then $D[\Gamma]$ is a weakly Krull domain if and only if $D[\Gamma]_{N(H)}$ is a one-dimensional weakly Krull domain and $K[G]$ is a weakly Krull domain.*

Proof. Let $H = \{aX^\alpha \mid 0 \neq a \in D \text{ and } \alpha \in \Gamma\}$ and $N = N(H)$. Then $K[G] = D[\Gamma]_H$ and $D[\Gamma] = D[\Gamma]_N \cap D[\Gamma]_H$, and hence $D[\Gamma]$ is a weakly Krull domain if and only if both $D[\Gamma]_N$ and $K[G]$ are weakly Krull domains by Proposition 4.

Now, by Lemma 5(3), $t\text{-Max}(D[\Gamma]_{N(H)}) = \text{Max}(D[\Gamma]_{N(H)})$. Note that if $D[\Gamma]$ is a weakly Krull domain, then $t\text{-dim}(D[\Gamma]) = 1$, thus $D[\Gamma]_N$ is a one-dimensional weakly Krull domain. \square

Let K be a field and G be an additive torsion-free abelian group. Then $K[G]$ is a Krull domain if and only if G satisfies the ascending chain condition on its cyclic subgroups [14, Theorem 1]. Hence, the following result recovers the result by Fadinger and Windisch [16, Theorem 3.7].

Corollary 7. *Let D be an integral domain with quotient field K , Γ be a torsion-free monoid with quotient group G , and assume that $K[G]$ is a weakly Krull domain. Then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull domain with $\text{ht}(P[\Gamma]) = 1$ for all $P \in t\text{-Max}(D)$ and Γ is a weakly Krull monoid with $\text{ht}(D[S]) = 1$ for all $S \in t\text{-Max}(\Gamma)$.*

Proof. By Lemma 5(2), $D[\Gamma]_{N(H)}$ is one-dimensional if and only if $\text{ht}(P[\Gamma]) = 1$ for all $P \in t\text{-Max}(D)$ and $\text{ht}(D[S]) = 1$ for all $S \in t\text{-Max}(\Gamma)$. Thus, the result follows directly from Lemma 5(2), Proposition 6 and the fact that $D[\Gamma] = D[\Gamma]_{N(H)} \cap K[G]$. \square

Corollary 8. [2, Proposition 4.11] *Let $D[X]$ be the polynomial ring over an integral domain D . Then $D[X]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain.*

Proof. Let K be the quotient field of D . We may assume that D is not a field, since if $D = K$ the statement is trivial. Let $\Gamma = \{0, 1, 2, \dots\}$. Then Γ is a torsion-free monoid under addition, $D[X] = D[\Gamma]$, and $S := \Gamma \setminus \{0\}$ is the unique nonempty prime ideal of Γ . Furthermore, Γ is a weakly Krull monoid, $\text{ht}(D[S]) = 1$, and $K[\langle \Gamma \rangle]$ is a UFD. Hence, by Corollary 7, $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull domain with $\text{ht}(P[\Gamma]) = 1$ for all $P \in t\text{-Max}(D)$, if and only if D is a weakly Krull UMT-domain (because $t\text{-dim}(D) = 1$). \square

4. WEAKLY KRULL SEMIGROUP RINGS

In this section, we completely characterize when $D[\Gamma]$ is a weakly Krull domain. Let H be a monoid and S be the set of non-invertible elements. As in [31, Theorem 15.4], we say that H is primary if S is the only non-empty prime ideal of H .

Lemma 9. *Let Γ be a primary monoid with quotient group G , S be the maximal ideal of Γ , and $\bar{\Gamma}$ be the integral closure of Γ in G . Let K be a field, and assume that $K[G]$ is a weakly Krull domain. Then $\text{ht}(K[S]) = 1$ if and only if $\bar{\Gamma}$ is a valuation monoid.*

Proof. Note that $\bar{\Gamma} = \{\alpha \in G \mid n\alpha \in \Gamma \text{ for some integer } n \geq 1\}$, so $\bar{\Gamma}$ is a primary monoid. Hence, if we let \bar{S} be the set of nonunits of $\bar{\Gamma}$, then \bar{S} is the unique non-empty prime ideal of $\bar{\Gamma}$ and $\bar{S} \cap \Gamma = S$.

Claim. $\text{ht}(K[S]) = \text{ht}(K[\bar{S}])$. (Proof. Let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n = K[\bar{S}]$ be a chain of prime ideals of $K[\bar{\Gamma}]$. Then, since $K[\bar{\Gamma}]$ is integral over $K[\Gamma]$,

$$Q_0 \cap K[\Gamma] \subsetneq Q_1 \cap K[\Gamma] \subsetneq \cdots \subsetneq Q_n \cap K[\Gamma] = K[S]$$

is a chain of prime ideals of $K[\Gamma]$ [35, Theorem 44]. Hence $\text{ht}(K[S]) \geq \text{ht}(K[\bar{S}])$. Next, let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m = K[S]$ be a chain of prime ideals of $K[\Gamma]$. Then there is a chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m$ of prime ideals of $K[\bar{\Gamma}]$ such that $M_i \cap K[\Gamma] = P_i$ for $i = 0, 1, \dots, m$ [35, Theorem 44]. Note that $M_m \cap K[\Gamma] = K[S]$, so $K[\bar{S}] \subseteq M_m$. Note also that $K[\bar{S}] \cap K[\Gamma] = K[S]$. Thus, $M_m = K[\bar{S}]$ [35, Theorem 44], so $\text{ht}(K[\bar{S}]) \geq \text{ht}(K[S])$. Hence, $\text{ht}(K[\bar{S}]) = \text{ht}(K[S])$.)

Now, by Claim, we may assume that Γ is integrally closed.

(\Rightarrow) Assume to the contrary that Γ is not a valuation monoid. Then there are $a, b \in \Gamma$ such that neither a divides b nor b divides a in Γ . Now, let $f = X^a + X^b \in K[\Gamma]$. Since $K[G]$ is a weakly Krull domain, $fK[G] = Q_1 \cap \cdots \cap Q_k$ for some primary ideals Q_i of $K[G]$ with $\text{ht}(\sqrt{Q_i}) = 1$ [3, Theorem 3.1]. Hence,

$$fK[G] \cap K[\Gamma] = \bigcap_{i=1}^k (Q_i \cap K[\Gamma])$$

and each $Q_i \cap K[\Gamma]$ is a primary ideal.

Now, assume $fK[G] \cap K[\Gamma] \not\subseteq K[S]$. Then there is an element $g \in K[G]$ such that $fg \in K[\Gamma] \setminus K[S]$. Hence, $C(fg) \not\subseteq K[S]$, and since $(C(f)C(g))_v = C(fg)_v$ [1, Corollary 3.9], we have $C(f)C(g) \not\subseteq K[S]$. Note that K is a field, so $C(f) = (X^a, X^b)K[\Gamma]$ and $C(g) = (X^{\alpha_1}, \dots, X^{\alpha_l})K[\Gamma]$ for some $\alpha_i \in G$, whence either $X^{a+\alpha_i} = X^a X^{\alpha_i} \notin K[S]$ for some i or $X^{b+\alpha_j} = X^b X^{\alpha_j} \notin K[S]$ for some j . We may assume that $X^{a+\alpha_i} \notin K[S]$. Then $a + \alpha_i \in \Gamma \setminus S$, so $a + \alpha_i$ is a unit of Γ , whence $a + \alpha_i + \beta = 0$ for some $\beta \in \Gamma$. Thus, $b = a + (b + \alpha_i) + \beta$ and $(b + \alpha_i) + \beta \in \Gamma$, which means that a divides b in Γ , a contradiction.

Hence, $fK[G] \cap K[\Gamma] \subseteq K[S]$, and thus $Q_i \cap K[\Gamma] \subseteq K[S]$ for some i with $1 \leq i \leq k$. Note that $\text{ht}(K[S]) = 1$ by assumption, so $K[S] = \sqrt{Q_i} \cap K[\Gamma]$, which implies $(K[S])K[G] \subsetneq K[G]$, a contradiction. Thus, Γ is a valuation monoid.

(\Leftarrow) Assume that $(0) \neq Q \subsetneq K[S]$ is a chain of prime ideals of $K[\Gamma]$. Then, for $0 \neq f \in Q$, there is an $\alpha \in \Gamma$ such that $f = X^\alpha g$ for some $g \in K[\Gamma]$ with $C(g) = K[\Gamma]$. Hence, $g \notin K[S]$, so $g \notin Q$, and thus $X^\alpha \in Q$. But, in this case, if $S_1 = \{\alpha \in \Gamma \mid X^\alpha \in Q\}$, then S_1 is a non-empty prime ideal of Γ and $K[S_1] \subseteq Q \subsetneq K[S]$. Thus, $S_1 \subsetneq S$, a contradiction. Therefore, $\text{ht}(K[S]) = 1$. \square

Lemma 10. *Let G be a torsion-free abelian group, D be a one-dimensional local domain with maximal ideal P , K be the quotient field of D , and \bar{D} be the integral*

closure of D in K . Assume that $K[G]$ is a weakly Krull domain. Then $\text{ht}(P[G]) = 1$ if and only if \bar{D} is a Prüfer domain.

Proof. The proof of this lemma is similar to that of Lemma 9, but we give the proof for the convenience of the reader.

Claim. $\text{ht}(P[G]) = \max\{\text{ht}(Q[G]) \mid Q \text{ is a maximal ideal of } \bar{D}\}$. (Proof. Let Q be a maximal ideal of \bar{D} , and let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n = Q[G]$ be a chain of prime ideals of $\bar{D}[G]$. Then, since $\bar{D}[G]$ is integral over $D[G]$,

$$Q_0 \cap D[G] \subsetneq Q_1 \cap D[G] \subsetneq \cdots \subsetneq Q_n \cap D[G] = P[G]$$

is a chain of prime ideals of $D[\Gamma]$. Hence $\text{ht}(P[G]) \geq \text{ht}(Q[G])$. Next, let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_m = P[G]$ be a chain of prime ideals of $D[G]$. Then there is a chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m$ of prime ideals of $\bar{D}[G]$ such that $M_i \cap D[G] = P_i$ for $i = 0, 1, \dots, m$. Note that $M_m \cap D[G] = P[G]$, so if we let $Q' = M_m \cap \bar{D}$, then Q' is a maximal ideal of \bar{D} and $Q'[G] \subseteq M_m$. Note also that $Q'[G] \cap D[G] = P[G]$, thus $Q'[G] = M_m$ [35, Theorem 44]. Hence, $\text{ht}(P[G]) \leq \text{ht}(Q'[G])$.)

Now, by Claim, we may assume that D is an integrally closed one-dimensional domain (which need not be local). Then it suffices to show that $\text{ht}(P[G]) = 1$ for all maximal ideals P of D if and only if D is a Prüfer domain.

(\Rightarrow) Assume to the contrary that D is not a Prüfer domain. Then there are $a, b \in D \setminus \{0\}$ such that (a, b) is not an invertible ideal of D . Hence $(a, b)(a, b)^{-1} \subseteq Q$ for some maximal ideal Q of D . For $0 \neq \alpha \in G$, let $f = a + bX^\alpha$. Then, by [1, Corollary 3.9],

$$fK[G] \cap D[G] = fc(f)^{-1}[G],$$

hence $fK[G] \cap D[G] \subsetneq Q[G]$. Since $K[G]$ is a weakly Krull domain, $fK[G]$ has a primary decomposition whose associated prime ideals have height-one [3, Theorem 3.1], say, $fK[G] = Q_1 \cap \cdots \cap Q_k$. Then

$$fK[G] \cap D[G] = \bigcap_{i=1}^k (Q_i \cap D[G])$$

and each $Q_i \cap D[G]$ is a primary ideal. Moreover, at least one of the $Q_i \cap D[G]$ is contained in $Q[G]$, so $\text{ht}(Q[G]) \geq 2$, a contradiction. Thus, D is a Prüfer domain.

(\Leftarrow) Let Q be a maximal ideal of D . Then $\text{ht}(Q[G]) = \text{ht}(QD_Q[G])$ and D_Q is a valuation domain. Hence, we may assume that D is a valuation domain. Then it is easy to see that $\text{ht}(Q[G]) = 1$ as in the proof (\Leftarrow) of Lemma 9. \square

Theorem 11. *Let D be an integral domain with quotient field K and Γ be a torsion-free monoid with quotient group G . Then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain, Γ is a weakly Krull UMT-monoid, and $K[G]$ is a weakly Krull domain.*

Proof. (\Rightarrow) By Proposition 6, $D[\Gamma]_{N(H)}$ is a one-dimensional weakly Krull domain and $K[G]$ is a weakly Krull domain. Also, by Corollary 7, D and Γ are weakly Krull. Now, if $P \in t\text{-Max}(D)$, then $1 = \text{ht}(P) = \text{ht}(P[\Gamma]) = \text{ht}(P[G]) = \text{ht}(PD_P[G])$ (see Corollary 7 for the second equality), thus \bar{D}_P is a Prüfer domain by Lemma 10. Thus, D is a UMT-domain. Next, if $S \in t\text{-Max}(\Gamma)$, then $\text{ht}(K[S + \Gamma_S]) =$

$\text{ht}(K[S]) = \text{ht}(S) = 1$ by Corollary 7, whence $\bar{\Gamma}_S$ is a valuation monoid by Lemma 9. Thus, Γ is a UMT-monoid.

(\Leftarrow) Note that $\text{ht}(P[\Gamma]) = \text{ht}(P[G]) = \text{ht}(PD_P[G]) = 1$ for all $P \in t\text{-Max}(D)$ by Lemma 10 and $\text{ht}(D[S]) = \text{ht}(K[S]) = \text{ht}(K[S + \Gamma_S]) = 1$ for all $S \in t\text{-Max}(\Gamma)$ by Lemma 9. Recall that

$$\text{Max}(D[\Gamma]_{N(H)}) = \{P[\Gamma]_{N(H)} \mid P \in t\text{-Max}(D)\} \cup \{D[S]_{N(H)} \mid S \in t\text{-Max}(\Gamma)\}.$$

Thus, $D[\Gamma]_{N(H)}$ is a one-dimensional weakly Krull domain. Therefore, $D[\Gamma]$ is a weakly Krull domain by Proposition 6. \square

The following lemma is from [13, Corollaries 3.2 and 4.3].

Lemma 12. *Let D be an integral domain with quotient field K and $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$). Then $K[G]$ is a weakly Krull domain if and only if G is of type $(0, 0, 0, \dots)$ (resp., of type $(0, 0, 0, \dots)$ except p).*

By Theorem 11 and Lemma 12, we have the following two corollaries which are complete characterizations of semigroup rings $D[\Gamma]$ that are weakly Krull domains.

Corollary 13. *Let D be an integral domain, Γ be a torsion-free monoid with quotient group G , and assume $\text{char}(D) = 0$. Then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain, Γ is a weakly Krull UMT-monoid, and G is of type $(0, 0, 0, \dots)$.*

Corollary 14. *Let D be an integral domain, Γ be a torsion-free monoid with quotient group G , and assume $\text{char}(D) = p > 0$. Then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain, Γ is a weakly Krull UMT-monoid, and G is of type $(0, 0, 0, \dots)$ except p .*

Let \mathbb{N}_0 be the additive monoid of nonnegative integers under the usual addition. Then \mathbb{N}_0 is a torsion-free monoid with quotient group \mathbb{Z} . A numerical monoid Γ is a submonoid of \mathbb{N}_0 such that $0 \in \Gamma$ and $\mathbb{N} \setminus \Gamma$ is finite. Hence, Γ is a torsion-free monoid with quotient group \mathbb{Z} .

Corollary 15. [36, Theorem 1.3] *Let D be an integral domain and Γ be a numerical monoid with $\Gamma \subseteq \mathbb{N}_0$. Then $D[\Gamma]$ is a weakly Krull domain if and only if D is a weakly Krull UMT-domain.*

Proof. Clearly, \mathbb{N}_0 is the integral closure of Γ in \mathbb{Z} and \mathbb{N}_0 is a valuation monoid. Moreover, $\Gamma \setminus \{0\}$ is the unique nonempty prime ideal of Γ , so Γ is a weakly Krull UMT-monoid. Note that if K is the quotient field of D , then $K[\mathbb{Z}]$ is a UFD, and hence a weakly Krull domain. Thus, the proof is completed by Theorem 11. \square

A generalized Krull domain D is a weakly Krull domain such that D_P is a valuation domain for all $P \in t\text{-Max}(D)$. The next result was proved by Matsuda ([37, Proposition 10.7] for the case of $\text{char}(D) = 0$ and [38, Theorems 1.5 and 4.3] for the case of $\text{char}(D) = p > 0$).

Corollary 16. *Let D be an integral domain, Γ be a torsion-free monoid with quotient group G , and assume $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$). Then $D[\Gamma]$ is a generalized Krull domain if and only if D is a generalized Krull domain, Γ is*

a generalized Krull monoid, and G is of type $(0, 0, 0, \dots)$ (resp., type $(0, 0, 0, \dots)$ except p).

Proof. Let P be a prime ideal of D . Then D_P is a valuation domain if and only if $D[\Gamma]_{P[\Gamma]}$ is a valuation domain. Also, if S is a prime ideal of Γ , then Γ_S is a valuation monoid if and only if $D[\Gamma]_{D[S]}$ is a valuation domain. Thus, the result follows directly from Corollaries 13 and 14. \square

It is well known that a domain D is a WFD if and only if it is weakly Krull with $Cl_t(D) = \{0\}$ [4, Theorem]. We next use the main result of this section to recover Chang and Oh's result [13] which completely characterizes when $D[\Gamma]$ is a WFD.

Corollary 17. [13, Theorems 3.4 and 4.5] *Let D be an integral domain, Γ be a torsion-free monoid with quotient group G , and assume $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$). Then $D[\Gamma]$ is a WFD if and only if D is a weakly factorial GCD-domain, Γ is a weakly factorial GCD-monoid, and G is of type $(0, 0, 0, \dots)$ (resp., $(0, 0, 0, \dots)$ except p).*

Proof. This follows directly from Proposition 3, Corollaries 13, 14, and the following observations: (i) If D and Γ are integrally closed, then $Cl_t(D[\Gamma]) = Cl_t(D) \times Cl_t(\Gamma)$ [15, Lemma 2.1 and Corollary 2.11], (ii) if $Cl_t(D[\Gamma]) = \{0\}$, then D and Γ are integrally closed and $Cl_t(D) = Cl_t(\Gamma) = \{0\}$ by [15, Theorems 2.6 and 2.7], (iii) $D[\Gamma]$ is a PvMD if and only if D is a PvMD and Γ is a PvMM [1, Proposition 6.5], (iv) D is a PvMD if and only if D is an integrally closed UMT-domain [32, Proposition 3.2], (v) D is a GCD-domain if and only if D is a PvMD with $Cl_t(D) = \{0\}$ [8, Proposition 2], (vi) Γ is a GCD-monoid if and only if Γ is a PvMM with $Cl_t(\Gamma) = \{0\}$ [31, Theorem 11.5], and (vii) Γ is a weakly factorial monoid if and only if Γ is a weakly Krull monoid with $Cl_t(\Gamma) = \{0\}$ [31, p. 258]. \square

5. ARITHMETICAL APPLICATIONS OF THE MAIN RESULT

Building on the algebraic results of the previous section and on a recent work on the distribution of prime divisors in the class groups of affine semigroup rings [17], we will study the factorization theory of weakly Krull semigroup rings. What is known up to now concerning factorizations in weakly Krull domains are on the one hand a few very general results lacking examples and on the other hand very concrete examples lacking generality. This is mainly due to the fact that the class group and the distribution of the prime divisors play a key role in the investigation of the factorization behaviour of a weakly Krull domain. But determining class groups and prime divisors in the classes is in general very hard. Nevertheless, the structure of sets of lengths of one-dimensional local Mori domains (equivalently local weakly Krull Mori domains) is given in [23].

We are the first who give a fairly broad but sufficiently concrete class of non-local weakly Krull domains that are not Krull domains, namely certain affine semigroup rings, where we can understand the arithmetic. For example, up to now the knowledge of domains having full system of sets of lengths was restricted to a class of certain Krull domains (see [33]) and a class of integer-valued polynomial rings (see [21, 22]). Our results show that there is also a class of weakly Krull domains, which are not Krull but have full system of sets of lengths.

We recall some concepts from factorization theory (for details see [24, Chapter 1]). Let H be a monoid and let $\mathcal{A}(H)$ be its set of atoms. We denote by $H_{red} = H/H^\times$ the associated reduced monoid. Consider the free abelian monoid $Z(H) = \mathcal{F}(\mathcal{A}(H_{red}))$ with the epimorphism $\pi : Z(H) \rightarrow H_{red}$ via $\pi(uH^\times) = uH^\times$ for all $u \in \mathcal{A}(H)$. For $a \in H$,

- $Z(a) = \pi^{-1}(\{aH^\times\})$ is the *set of factorizations* of a , and
- $L(a) = \{|z| : z \in Z(a)\}$ is the *set of lengths* of a , where $|z| = m$ if $z = u_1 \cdots u_m$ for $u_i \in \mathcal{A}(H_{red})$.

Then H is said to be *atomic* if $L(a)$ is non-empty for all $a \in H$ and is said to be a *BF-monoid* if H is atomic and $L(a)$ is finite for all $a \in H$. Note that if R is a Mori domain, then the multiplicative monoid $R \setminus \{0\}$ is always a BF-monoid. We consider the system $\mathcal{L}(H) = \{L(a) : a \in H\}$ of all sets of lengths of H . For convenience, we denote $\mathcal{L}(R \setminus \{0\})$ by $\mathcal{L}(R)$ for an integral domain R . A system of sets of lengths of a BF-monoid is called *full* if it equals $\{\{0\}, \{1\}\} \cup \{L \subseteq \mathbb{N}_{\geq 2} \mid L \text{ finite non-empty}\}$. The *set of distances* of H is $\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L)$, where $\Delta(L) = \{d \in \mathbb{N} : \text{there is } l \in L \text{ such that } L \cap \{l, l+1, \dots, l+d\} = \{l, l+d\}\}$. For $k \in \mathbb{N}$, we define $\mathcal{U}_k(H) = \bigcup_{k \in L \in \mathcal{L}(H)} L$.

Let R be a weakly Krull Mori domain with non-zero conductor $\mathfrak{f}_R = (R : \widehat{R}) \neq (0)$. Then $H = R \setminus \{0\}$ is a weakly Krull Mori monoid with non-empty conductor $\mathfrak{f}_H = (H : \widehat{H}) \neq \emptyset$. It follows by [24, Proposition 2.4.5.1] that the canonical map $H \rightarrow \mathcal{I}_v^*(H) \cong \prod_{\mathfrak{p} \in X^1(H)} (H_{\mathfrak{p}})_{red} \cong \mathcal{F}(P) \times D_1 \times \dots \times D_n$ is a divisor homomorphism, where $P = \{\mathfrak{p} \in X^1(H) \mid \mathfrak{p} \not\supseteq \mathfrak{f}_H\}$, $\mathcal{F}(P)$ is the free abelian monoid with basis P , and $D_i = (H_{\mathfrak{q}_i})_{red}$ for $i \in \{1, \dots, n\}$ and $X^1(H) \setminus P = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$. The existence of the first isomorphism is proven in [26, Proposition 5.3.4] and the second isomorphism as well as the finiteness of $X^1(H) \setminus P$ follows from [24, Theorem 2.6.5]. Let $G = \mathcal{C}_v(H) \cong \mathcal{C}_v(R)$ be the divisor-class group of H and let G_0 be the set of all classes containing prime divisors, that is, prime ideals $\mathfrak{p} \in P$. Let $T = D_1 \times \dots \times D_n$ (so T is a reduced monoid, i.e. T^\times is trivial) and let $\iota : T \rightarrow G$ be the canonical map induced by the isomorphisms from above and the projection $\mathcal{I}_v^*(H) \rightarrow G$. The *T-block monoid* over G_0 defined by ι is

$$B = \mathcal{B}(G_0, T, \iota) = \{(g_1 \cdots g_k, t) \in \mathcal{F}(G_0) \times T \mid g_1 + \dots + g_k + \iota(t) = 0\}.$$

Then the monoid $\mathcal{B}(G_0) = \{g_1 \cdots g_k \in \mathcal{F}(G_0) \mid g_1 + \dots + g_k = 0\}$ is a divisor-closed submonoid of B . By [26, Lemma 4.3], there exists a transfer homomorphism $\beta : H \rightarrow B$. Thus, $\mathcal{L}(R) = \mathcal{L}(H) = \mathcal{L}(B)$ by [24, Proposition 3.2.3.5]. Moreover, it follows that $\mathcal{L}(\mathcal{B}(G_0)) \subseteq \mathcal{L}(R)$ by the previous equality and $\mathcal{B}(G_0) \subseteq B$ being divisor-closed. It is easy to see that $\mathcal{B}(G_1) \subseteq \mathcal{B}(G_0)$ is a divisor closed submonoid for every subset $G_1 \subseteq G_0$, whence $\mathcal{L}(\mathcal{B}(G_1)) \subseteq \mathcal{L}(\mathcal{B}(G_0))$.

The notion of Hilbertian fields is a classical one whose origin lies in Galois theory and is to be found in [20]. For our purpose, we need a generalization of it.

Definition 18. *A field K is called pseudo-Hilbertian if, for all $n \in \mathbb{N}_0$ and for all $a_0, \dots, a_n \in K$ with $a_0 \neq 0$, there exists an irreducible polynomial in $K[X]$ whose coefficient at the monomial X^i equals a_i for all $i \in \{0, \dots, n\}$.*

Note that every Hilbertian field is an infinite pseudo-Hilbertian field. In particular, algebraic function fields over an arbitrary field and algebraic number fields are

pseudo-Hilbertian. See [20] for more on Hilbertian fields. Moreover, finite fields are pseudo-Hilbertian [39].

For the following theorem, note that if D is noetherian and Γ is a numerical monoid, then $D[\Gamma]$ is noetherian and hence Mori.

Theorem 19. *Let D be a weakly Krull UMT-domain with non-zero conductor $\mathfrak{f}_D = (D : \widehat{D}) \neq (0)$ and infinite pseudo-Hilbertian quotient field K . Let $\Gamma \neq \mathbb{N}_0$ be a numerical monoid and suppose that $D[\Gamma]$ is a Mori domain. Then $\mathcal{L}(D[\Gamma])$ is full.*

Proof. By Corollary 15 and assumption, $D[\Gamma]$ is a weakly Krull Mori domain. Then

$$\mathfrak{f}_{D[\Gamma]} = (D[\Gamma] : \widehat{D[\Gamma]}) \neq (0)$$

by [17, Lemma 3.1], whence we are in the situation that we explained at the beginning of this section. The class group of $D[\Gamma]$ is of the form $\mathcal{C}_v(D[\Gamma]) \cong \mathcal{C}_v(D[X]) \oplus \text{Pic}(K[\Gamma])$ [5, Theorem 5]. Since K is infinite and pseudo-Hilbertian, $K[\Gamma]$ has infinitely many prime divisors in every class by [17, Theorem 1] and $\text{Pic}(K[\Gamma])$ is infinite by [17, Propositions 3.4 & 3.7]. So if $G_0 \subseteq \mathcal{C}_v(D[\Gamma])$ denotes the set of classes containing prime divisors, then G_0 contains the infinite abelian group $\text{Pic}(K[\Gamma])$. Hence $\mathcal{B}(\text{Pic}(K[\Gamma])) \subseteq \mathcal{B}(G_0) \subseteq \mathcal{B}(G_0, T, \iota)$. Therefore $\mathcal{L}(\mathcal{B}(\text{Pic}(K[\Gamma]))) \subseteq \mathcal{L}(D[\Gamma])$ and the statement follows by Kainrath's Theorem [33]. \square

To give two important special cases of Theorem 19, we apply it to orders in algebraic number fields and to polynomial rings.

Corollary 20. *Let $\Gamma \neq \mathbb{N}_0$ be a numerical monoid.*

- (1) *If D is an order in an algebraic number field, then $\mathcal{L}(D[\Gamma])$ is full.*
- (2) *If D is a noetherian weakly Krull UMT-domain with non-zero conductor, then $\mathcal{L}(D[X][\Gamma])$ is full.*

Proof. (1) If D is an order in an algebraic number field, then D is a noetherian weakly Krull UMT-domain (the integral closures of the localizations at maximal t -ideals are one-dimensional Krull by Mori-Nagata Theorem, whence Prüfer) with non-zero conductor and infinite pseudo-Hilbertian quotient field. Now the assertion follows from Theorem 19.

(2) If D is a noetherian weakly Krull UMT-domain with non-zero conductor, then $D[X]$ is noetherian and a UMT-domain [19, Theorem 2.4]. It follows from Corollary 15 that $D[X]$ is weakly Krull and from [17, Lemma 3.1] that the conductor $\mathfrak{f}_{D[X]}$ is non-zero. Since the quotient field $K(X)$ of $D[X]$ is infinite and pseudo-Hilbertian, we can apply Theorem 19. \square

Recall that a monoid H with quotient group $\langle H \rangle$ is said to be *seminormal* if for all $x \in \langle H \rangle$ we have that $x^2, x^3 \in H$ implies $x \in H$. For the next theorem, note that seminormal affine monoids are characterized in terms of their geometry, e.g. see [9, Proposition 2.42]. Also, for seminormal weakly Krull affine monoids, either statement (1) or statements (2) and (3) of the next theorem are true always. The theorem is particularly interesting because even in the case of orders \mathcal{O} in algebraic number fields, $\min(\Delta(\mathcal{O})) > 1$ can occur.

Theorem 21. *Let K be a field, Γ be a weakly Krull affine monoid that is not a numerical monoid, and assume that the root closure $\bar{\Gamma}$ is factorial. Then either $K[\Gamma]$ is half-factorial or $\min(\Delta(K[\Gamma])) = 1$. Moreover, the following hold true.*

- (1) *If $\mathcal{C}_v(K[\Gamma])$ is infinite, then $\mathcal{L}(K[\Gamma])$ is full.*
- (2) *If $\mathcal{C}_v(K[\Gamma])$ is finite, then $K[\Gamma]$ satisfies the Structure Theorem for Sets of Lengths (see [24, Definition 4.7.1]).*
- (3) *If $\mathcal{C}_v(K[\Gamma])$ is finite and Γ is seminormal, then both $\Delta(K[\Gamma])$ and $\mathcal{U}_k(K[\Gamma])$ are finite intervals for all $k \geq 2$.*

Proof. Note that every weakly Krull affine monoid is a UMT-monoid, because its localizations at maximal t -ideals are finitely generated primary monoids, whence their integral closures are primary Krull monoids, that is, discrete rank one valuation monoids. Thus, $K[\Gamma]$ is a weakly Krull domain by Corollary 15. Moreover, $K[\Gamma]$ is noetherian, $K[\bar{\Gamma}] = K[\widehat{\Gamma}] = \widehat{K[\Gamma]}$ and there is a one-to-one correspondence of height-one prime ideals $X^1(K[\bar{\Gamma}]) \rightarrow X^1(K[\Gamma])$ given by $P \mapsto P \cap K[\Gamma]$ using a combination of Lemma 5 and [40, Proposition 2.7]. Since every finitely generated monoid always has a non-empty conductor [24, Theorem 2.7.13], it follows from [17, Lemma 3.1] that $K[\Gamma]$ has a non-zero conductor. Thus, we are in the situation that we explained at the beginning of this section. Moreover, by [17, Theorem 2], there are infinitely many prime divisors in all classes of $\mathcal{C}_v(K[\Gamma])$.

The statement on the half-factoriality and the minimum of $\Delta(K[\Gamma])$ follows from [27, Theorem 1.1].

(1) Let $\mathcal{C}_v(K[\Gamma])$ be infinite. Then $\mathcal{B}(\mathcal{C}_v(K[\Gamma])) \subseteq \mathcal{B}(\mathcal{C}_v(K[\Gamma]), T, \iota)$. Therefore $\mathcal{L}(\mathcal{B}(\mathcal{C}_v(K[\Gamma]))) \subseteq \mathcal{L}(K[\Gamma])$ and the statement follows by Kainrath's Theorem [33].

(2) This is immediate by [24, Chapter 4.7].

(3) If Γ is seminormal, then $K[\Gamma]$ is seminormal by [9, Theorem 4.75]. Thus $\mathcal{U}_k(K[\Gamma])$ (resp., $\Delta(K[\Gamma])$) is a finite interval for all $k \geq 2$ by [26, Theorem 5.8.2 (a)] (resp., [27, Theorem 1.1]). \square

Remark 22. Let R be a weakly Krull Mori domain. Then the monoid $H = \mathcal{I}^*(R)$ is a weakly Krull Mori monoid. If R has a nonzero conductor, then H has a nonzero conductor; if R is seminormal, then H is seminormal; if the v -class group $\mathcal{C}_v(R)$ of R has (infinitely) many prime divisors in the classes, then the same is true for H (see [25, Theorem 4.4 & Corollary 4.7]). Thus, all the mentioned arithmetical properties for R hold true for H too.

We close this section with an application of Theorem 21(1). We first need the following lemma.

Lemma 23. *Let K be a field and S_1, \dots, S_n be numerical monoids with $n > 1$. Then*

$$\mathcal{C}_v(K[\bigoplus_{i=1}^n S_i]) \cong \bigoplus_{i=1}^n \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_i]).$$

Proof. Note that $\mathcal{C}_v(K[\mathbb{Z}^m])$ is trivial for all integers $m \geq 1$, so it suffices to show that for all positive integers m with $m \leq n$,

$$\mathcal{C}_v(K[\bigoplus_{i=1}^n S_i]) \cong \mathcal{C}_v(K[\mathbb{Z}^m \oplus S_{m+1} \oplus \dots \oplus S_n]) \oplus \bigoplus_{i=1}^m \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_i]).$$

We prove the isomorphism by induction on $m \leq n$. First if $m = 1$, then

$$\begin{aligned} \mathcal{C}_v(K[\bigoplus_{i=1}^n S_i]) &\cong \mathcal{C}_v(K[\bigoplus_{i=2}^n S_i][S_1]) \\ &\cong \mathcal{C}_v(K[\bigoplus_{i=2}^n S_i][\mathbb{Z}]) \oplus \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_1]) \\ &\cong \mathcal{C}_v(K[\mathbb{Z} \oplus \bigoplus_{i=2}^n S_i]) \oplus \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_1]), \end{aligned}$$

where the second isomorphism follows from [5, Theorem 5]. Now assume that $m > 1$. Then by the induction hypothesis,

$$\begin{aligned} \mathcal{C}_v(K[\bigoplus_{i=1}^n S_i]) &\cong \mathcal{C}_v(K[\mathbb{Z}^{m-1} \oplus S_m \oplus \dots \oplus S_n]) \oplus \bigoplus_{i=1}^{m-1} \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_i]) \\ &\cong \mathcal{C}_v(K[\mathbb{Z}^{m-1} \oplus \bigoplus_{i=m+1}^n S_i][S_m]) \oplus \bigoplus_{i=1}^{m-1} \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_i]) \\ &\cong \mathcal{C}_v(K[\mathbb{Z}^{m-1} \oplus \bigoplus_{i=m+1}^n S_i][\mathbb{Z}]) \oplus \bigoplus_{i=1}^m \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_i]) \\ &\cong \mathcal{C}_v(K[\mathbb{Z}^m \oplus S_{m+1} \oplus \dots \oplus S_n]) \oplus \bigoplus_{i=1}^m \mathcal{C}_v(K(X_1, \dots, X_{n-1})[S_i]). \end{aligned}$$

Thus, the isomorphism holds for all positive integers m with $m \leq n$. \square

Corollary 24. *Let K be a field, S_1, \dots, S_n be numerical monoids with $n > 1$ such that $S_i \neq \mathbb{N}_0$ for at least one of the S_i , and $\Gamma = \bigoplus_{i=1}^n S_i$. Then $\mathcal{L}(K[\Gamma])$ is full.*

Proof. Clearly, Γ is an affine monoid and by [18, Proposition 5.8] it is weakly Krull, so we only need to show that $K[\Gamma]$ has an infinite class group. Then we can apply Theorem 21(1). But this follows immediately from Lemma 23 in combination with [17, Propositions 3.4 and 3.7]. \square

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REFERENCES

- [1] D.D. Anderson and D.F. Anderson, *Divisorial ideals and invertible ideals in a graded integral domain*, J. Algebra 76 (1982), 549-569.
- [2] D.D. Anderson, E. Houston, and M. Zafrullah, *t-linked extensions, the t-class group, and Nagata's theorem*, J. Pure Appl. Algebra 86 (1993), 109-124.
- [3] D.D. Anderson, J.L. Mott, and M. Zafrullah, *Finite character representations for integral domains*, Boll. UMI. (7), 6-B (1992), 613-630.
- [4] D.D. Anderson and M. Zafrullah, *Weakly factorial domains and groups of divisibility*, Proc. Amer. Math. Soc. 109 (1990) 907-913.
- [5] D. F. Anderson and G. W. Chang, *The Class Group of $D[\Gamma]$ for D an Integral Domain and Γ a Numerical Semigroup*, Commun. Algebra, 32 (2004), 787-792.
- [6] ———, *Homogeneous splitting sets of a graded integral domain*, J. Algebra 288 (2005), 527-544.

- [7] ———, *Graded integral domains and Nagata rings*, J. Algebra 387 (2013), 169-184.
- [8] A. Bouvier, *Le groupe des classes d'un anneau intgre*, 107 eme Congres des Societes Savantes, Brest fasc. IV (1982), 85-92.
- [9] W. Bruns and J. Gubeladze, *Polytopes, rings, and K-theory*, Springer-Verlag, 2008.
- [10] G.W. Chang, *Semigroup rings as weakly factorial domains*, Comm. Algebra 37 (2009), 3278-3287.
- [11] ———, *Graded integral domains and Prüfer-like domains*, J. Korean Math. Soc., 54 (2017), 1733-1757.
- [12] G.W. Chang and P. Sahandi, *Graded integral domains whhich are UMT-domains*, Comm. Algebra 46 (2018), 2742-2752.
- [13] G.W. Chang and D.Y. Oh, *Semigroup rings as weakly factorial domains, II*, Internat. J. Algebra Comput. 29 (2019), 407-418.
- [14] L.G. Chouinard II, *Krull semigroups and divisor class groups*, Canad. J. Math. 33 (1981), 1459-1468.
- [15] S. El Baghdadi, L. Izelgue, and S. Kabbaj, *On the class group of a graded domain*, J. Pure Appl. Algebra 171 (2002) 171-184.
- [16] V. Fadinger and D. Windisch, *A characterization of weakly Krull monoid algebras*, J. Algebra 590 (2022), 277-292.
- [17] ———, *On the distribution of prime divisors in class groups of affine monoid algebras*, Comm. Algebra, to appear.
- [18] V. Fadinger and Q. Zhong, *On product-one sequences over subsets of groups*, Periodica Math. Hungarica, to appear.
- [19] M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra 25 (1998), 1017-1039.
- [20] M.D. Fried and M. Jarden, *Field Arithmetic*, 3rd edition, Springer, 1986.
- [21] S. Frisch, *A construction of integer-valued polynomials with prescribed sets of lengths of factorizations*, Monatsh. Math. **171** (2013), 341-350.
- [22] S. Frisch, S. Nakato and R. Rissner, *Sets of lengths of factorizations of integer-valued polynomials on Dedekind domains with finite residue fields*, J. Algebra 528 (2019), 231-249.
- [23] A. Geroldinger, F. Gotti and S. Tringali, *On strongly primary monoids with a focus on Puiseux monoids*, J. Algebra 567 (2021) 310-345.
- [24] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.
- [25] A. Geroldinger and M.A. Khadam, *On the arithmetic of monoids of ideals*, Arkiv for Matematik, to appear.
- [26] A. Geroldinger, F. Kainrath and A. Reinhart, *Arithmetic of seminormal weakly Krull monoids and domains*, J. Algebra **444** (2015), 201-245.
- [27] A. Geroldinger and Q. Zhong, *The set of distances in seminormal weakly Krull monoids*, J. Pure Appl. Algebra **220** (2016), 3713-3732.
- [28] R. Gilmer, *Commutative Semigroup Rings*, The Univ. of Chicago, Chicago, 1984.
- [29] M. Griffin, *Some results on v-multiplication rings*, Canad. Math. J. 19 (1967), 710-722.
- [30] F. Halter-Koch, *Divisor Theories with Primary Elements and Weakly Krull Domains*, Boll. U.M.I. (7) 9-B (1995), 417-441.
- [31] F. Halter-Koch, *Ideal Systems*, Monographs and Textbooks in Pure and Appl. Math., vol. 211, Marcel Dekker, New York, 1998.
- [32] E. Houston and M. Zafrullah, *On t-invertibility II*, Comm. Algebra **17** (1989), 1955-1969.
- [33] F. Kainrath, *Factorization in Krull monoids with infinite class group*, Colloquium Mathematicae, **80** (1999), 23-30.
- [34] B.G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra 124 (1989), 284-299.
- [35] I. Kaplansky, *Commutative Rings*, rev.ed., The Univ. of Chicago, Chicago, 1974.
- [36] J.W. Lim, *Weakly Krull domains and the composite numerical semigroup ring $D + E[\Gamma^*]$* , Pacific J. Math. 257 (2012), 227-242.
- [37] R. Matsuda, *Torsion-free abelian semigroup rings III*, Bull. Fac. Sci. Ibaraki Univ. 9 (1977) 1-49.

- [38] R. Matsuda, *Krull properties of semigroup rings*, Bull. Fac. Sci. Ibaraki Univ. 14 (1982) 1-12.
- [39] P. Pollack, *Irreducible polynomials with several prescribed coefficients*, Finite Fields Appl. **22** (2013), 70–78.
- [40] A. Reinhart, *On integral domains that are C-monoids*, Houst. J. Math. **39** (2013), 1095–1116.

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