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SEMIGROUPS THAT ARE THE UNION OF A GROUP ON E^3 AND A PLANE(1)

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Abstract. In *Semigroups on a half-space*, Trans. Amer. Math. Soc. **147** (1970), 1–53, Horne considers semigroups that are the union of a group G and a plane L such that $G \cup L$ is a three-dimensional half-space and G is the interior. After proving a great many things about half-space semigroups, Horne introduces the notion of a radical and determines all possible multiplications in L for a half-space semigroup with empty radical. (It turns out that S has empty radical if and only if each G-orbit in L contains an idempotent.) An example is provided for each configuration in L. However, no attempt was made to show that the list of examples actually exhausted the possibilities for a half-space semigroup without radical. Another way of putting this problem is to determine when two different semigroups can have the same maximal group. In this paper we generalize Horne's results, for a semigroup without zero, by showing that if S is any locally compact semigroup in which L is the boundary of G, then S is a half-space. Moreover, we are able to answer completely, for semigroups without radical and without a zero, the question posed above. It turns out that, with one addition (which we provide), Horne's list of half-space semigroups without radical and without zero is complete.

Introduction. A semigroup on a half-space is a topological semigroup S whose underlying space is homeomorphic to the set $\{(x_1, x_2, x_3) \in E^3 : x_3 \geqq 0\}$ and which has a maximal group G corresponding to the set $\{(x_1, x_2, x_3) \in E^3 : x_3 > 0\}$. All possible multiplications in L, the boundary of G, and the corresponding maximal groups are determined in [8] for semigroups without radical, and examples are given for each allowable pair G, L. We will not define the radical of a semigroup here, but it turns out that a semigroup on a half-space has empty radical if and only if each G-orbit in L contains an idempotent. (See Theorems 6.7 and 6.8 of [8].) The question of when G, L, separately, determine $G \cup L$ was not pursued in [8], but it was conjectured there that the collection of examples given actually exhausted the possibilities for a semigroup of a half-space without radical. In this paper we answer this question for semigroups without radical and without zero, and we show that S need only be locally compact in order to be a half-space. More precisely: We assume that S is a locally compact semigroup without a zero which is the union of a Lie group G on E^3 and its boundary L which is assumed to be

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homeomorphic to E^2. (We mean by the boundary of a set A, the set $A^- \backslash A$. u is a zero for S if $uS = Su = u$.) We show that S must be a half-space. Moreover, with one addition (see Theorem 10), the examples given by Horne constitute all of the semigroups without radical and without a zero on a half-space.

There are basically three difficulties to be faced if S is no longer assumed to be a half-space topologically. We will briefly look at each of them. Let e be an idempotent in the boundary of G, and let H be the left isotropy subgroup of e. One must either prove again or manage to do without:

(i) e is in the closure of H.

(ii) Ge is simply connected.

(iii) H^- is a semigroup on a half-plane.

Horne gives a proof for (i) that is valid if Ge is locally euclidean and S is a manifold with boundary. Theorem 1 asserts that (i) is true without the latter assumption. This theorem is invoked chiefly to get us into the situation of either Theorem 3 or Lemma 11. The homeomorphisms constructed in these theorems are the principal motifs underlying the arguments that piece together G and its boundary. Moreover, the question of when G, L, separately, determine $G \cup L$ is decided by exploiting these maps.

Problem (ii) above is circumvented by considering idempotents whose orbits are closed in L. (See Preliminaries, P4. In this paper, Pn, where n is an integer, refers to a similarly-numbered paragraph in the Preliminaries.) Thus $Ge = L$ if dim $Ge = 2$, and if $Ge = 1$, it is easy to show that Ge cannot be a simple closed curve (Lemma 7). The results concerning actions of a group in the plane contained in [4] and [7] are fundamental here as they are in [8]. Finally, P7 and Lemma 11 suffice for (iii). Many of the arguments used in [8], where S is assumed to be a half-space carry over here. This is particularly true when determining the maximal groups that are possible for a given situation in L. In almost every case we include such arguments here for the sake of continuity. The Preliminaries represent our attempt to render this paper fairly self-contained and, in particular, to make the body of the paper independent of [8]. However, to make this paper *completely* independent of Horne's work would involve too much duplication and is not a desirable object anyway.

P4 states that there is an idempotent e in L such that Ge and eG are each closed subsets of L. The plan of this paper is a case-by-case analysis: dim $Ge = i$ and dim $eG = j$, where $0 \leqq i, j \leqq 2$, except for dim $Ge = 0 =$ dim eG. If e is a zero for S, then a different approach is needed. In a forthcoming paper we will examine this case in detail.

1. **Preliminaries.** All topological spaces here are assumed to be Hausdorff. Unless specified otherwise, "group" means "topological group," "semigroup" means "topological semigroup" and, in the statements of theorems, "isomorphism" denotes a one-to-one multiplicative function that is a homeomorphism onto. A double arrow $\twoheadrightarrow$ always denotes an onto function. If G is a group acting

on a space X, and $x \in X$, then $G_l(x)=\{g \in G \mid gx=x\}$ is the left isotropy subgroup of x with respect to G. $G_l(x)$ is a closed subgroup of G. The left G-orbit through x is the set $\{gx \mid g \in G\}$ which we denote by Gx. Similar remarks apply to $G_r(x)$ and xG. We sometimes say "right isotropy group of x," etc., when the group is understood.

Suppose now that G is an open, dense, connected subgroup of a semigroup, and that L is the boundary of G. Let $x \in L$. Proofs of the following two results can be found in [8, p. 4]: (i) If Gx is open in L, then $xG \subset Gx$, and $G_r(x)$ is normal. (There is a dual theorem to this.) (ii) If $x^2 \in Gx \cap xG$, then $Gx \cap xG$ is, algebraically, a group. Also, it is easily verified that if Gx is closed in L, then Gx is an ideal in the closure of G. These results, and certain easy consequences of them will be used often and without comment in what follows. (For instance, if e is an idempotent and Ge is open in L, then eG is a group, algebraically.)

If G is a Lie group, then we denote the Lie algebra of G by $L[G]$. If H is a subgroup of G, then the connected component of the identity in H is a closed normal subgroup of H and is denoted by H_0. If H is a closed subgroup of G, and V is a subspace of G such that $V \cap H=\{1\}$, and the multiplication map of G restricted to $H \times V$ is a homeomorphism onto G; then we write "$G=HV$." Clearly, if $G=HV$, then, after supplying the obvious definition, we know that $G=WH$, for some $W \subset G$.

For easy reference we will refer to certain facts by numbers. It should be understood that when a reference is made to [8] in this paper, the proof that appears in [8] may have to be rewritten somewhat to yield the result as stated here. Also, some of the results in these Preliminaries require Theorem 1.

P1. Let G be a Lie group acting on a locally compact space X, and let $x \in X$. If there is a neighborhood V of the identity of G such that, for $v \in V$, $vx=x$ implies $v=1$, then there is a compact set C in X, containing x, such that the group action $a\colon V \times C \to X$ is a homeomorphism onto a neighborhood of x in X [2, p. 314]. We will refer sometimes to C and sometimes to $a\colon V \times C \to X$ as a local cross-section to the local orbits of G at x.

P2. Let G be a simply connected Lie group, and let H be a closed subgroup of G. Then H/H_0 is isomorphic to $\pi_1(G/H)$ [16, p. 617].

P3. S is a semigroup on a half-plane if (i) S is a semigroup whose underlying space is homeomorphic to the set $\{(x, y) \mid x \geqq 0$, where x, y are real numbers$\}$, and (ii) the subset of S corresponding to $\{(x, y) \mid x>0\}$ is a group. Let $S=H \cup L$, where H is the group in (ii), and L is the boundary of H. There are only two groups on the plane, the abelian vector group and the nonabelian affine group, Af(1). The affine group can be represented by real matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$ where $x>0$. Thus there are but two possibilities for H, and the corresponding possibilities for multiplication in L are as follows:

(A) H is abelian:

1. L is a group.

2. L has a zero 0 dividing L into two components A, B such that $AB=BA=0$, and for all x in A, $Hx=xH=A$, and for all x in B, $Hx=xH=B$; and one of the

following is true: 2(i) $A^2=B^2=0$. 2(ii) A and B are groups. 2(iii) $A^2=0$, and B is a group.

(B) H is nonabelian:

1. L is a group.

2(i). L has a zero, and $A^2=B^2=0$ (see above).

3. L is a left (right) zero semigroup, and every left (right) orbit of L is all of L [14], [15].

P4. Let G be a Lie group on E^3 embedded in a locally compact semigroup, and let L, the boundary of G, be homeomorphic to E^2. If H is a closed connected subgroup of G and $H \cap L \neq \varnothing$, then there is an idempotent e in $H^- \cap L$ such that He and eH are closed subsets of L [8, p. 4].

P5. Suppose a connected Lie group acts on a space X and suppose that dim $Gx = 1$ for some $x \in X$. Then there is a one-parameter subgroup P of G such that $Gx=Px$. Furthermore, a necessary and sufficient condition that a particular one-parameter subgroup P of G have the property $Gx=Px$ is that no conjugate of P is contained in the left isotropy subgroup of x. Moreover, Gx is homeomorphic to either a line or a circle if X is a plane [8, p. 4].

P6. Let P, the positive reals under multiplication, act on M, a locally compact semigroup, and let e be an idempotent such that $Pe \neq \{e\}$. If Pe is locally compact, then there is a local cross-section to the local orbits of P at e (see P1) such that right multiplication by e is one-to-one on the fibres Vc, where V is a neighborhood of the identity in P and c is any element of the cross-sectioning subset C in M [8, pp. 8–9].

P7. Let G be a Lie group on E^3 embedded in a locally compact semigroup. Let L, the boundary of G, be homeomorphic to the plane, E^2. Let e be an idempotent in L such that Ge is a line, and let $H=G_l(e)$. Then (i) if $eH \neq e$, then H^- is a semigroup on a half-plane (see P3) such that $H^- \cap L=eH$ is a closed line in L that crosses Ge at e and eH is a right zero semigroup; (ii) if $eH=e$, then $H^- \cap L$ is a half-ray with e as endpoint or a line, and H^- is topologically a plane or a half-plane, respectively [8, pp. 8–10].

P8. Let G, L be as in P7 and let e be an idempotent in L such that Ge and eG are lines in L, and $Ge \neq eG$. Then $G_l(e) \neq G_r(e)$ [8, pp. 11–12].

P9. If H is a planar group embedded in a locally compact semigroup, and if the boundary of H is a line, then H^- is a half-plane. If the boundary of H is a half-line, then H^- is a plane. In the latter case, if L is the boundary of H, then the endpoint of L is a zero 0 for H and $L^2=0$ [8], [14].

P10. Let G be a semidirect product V_2R of the two-dimensional vector group and the additive reals (V_2 is normal here). If G is isomorphic to the group of real matrices of the form

$$\begin{bmatrix} a^r & 0 & x \\ 0 & b^r & y \\ 0 & 0 & 1 \end{bmatrix}$$

where $r \in R$, $(x, y) \in V_2$, and a and b are fixed positive real numbers such that $0 < b < 1 < a$; then we say that G is a hyperbolic semidirect product. (Compare this definition with the one given in [8, p. 13]. Also, on the same page is a list of the possible semidirect products on E^3 which is somewhat finer than the list that we will give below.) The paragraph beginning at the bottom of [8, p. 15] should be amended with the result that the maximal group G of Theorem 2.5, p. 14, must be hyperbolic instead of nonhyperbolic as stated. The examples following the proof of Theorem 2.5 are hyperbolic instead of nonhyperbolic as stated. The change required in the proof is a trivial one: The boundary of H^- is a right zero semigroup, and the boundary of J^- is a left zero semigroup. Consequently, using Theorem 2 of [5], one sees that the action of p is expanding on Q_1 and contracting on Q_2, not contracting on each as stated.

We conclude these preliminaries with a description of those Lie groups whose underlying space is euclidean three-space, E^3. It can be shown, using the list of three-dimensional Lie algebras given in [11], that the following list of possibilities is complete:

(1) The three-dimensional vector group, V_3.

(2) The nonabelian nilpotent group N of 3×3 real matrices

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

(3) The direct product of the affine group Af (1) with the additive reals, Af (1) $\times R$.

(4) The semidirect products V_2R of 3×3 real matrices

$$\begin{bmatrix} p_1(t) & p_2(t) & x \\ p_3(t) & p_4(t) & y \\ 0 & 0 & 1 \end{bmatrix}$$

where the map

$$t \to P(t) = \begin{bmatrix} p_1(t) & p_2(t) \\ p_3(t) & p_4(t) \end{bmatrix}$$

is a continuous homomorphism of R into the group of nonsingular 2×2 real matrices. It should be noted at once that we reserve the term "semidirect product" for those semidirect products that do not yield groups isomorphic to either V_3 or Af (1) $\times R$.

(5) The simple group Sl (2) which is the universal covering group of the group of real 2×2 matrices of determinant 1.

It is necessary that we examine the semidirect products more closely. V_2 is normal in V_2R and, since we do not include the direct product, all normal one-parameter subgroups of V_2R are in V_2. The set of linear transformations in the plane $\{P(t) \mid t \in R\}$ is either reducible or not. If it is, then there is a one-parameter subgroup of V_2 that is invariant under the action of each $P(t)$ and this is equivalent to saying that this one-parameter subgroup of V_2 is normal in V_2R. If the set $\{P(t) \mid t \in R\}$ is irreducible, then V_2R has no normal one-parameter subgroup. Suppose that V_2R has at least one normal one-parameter subgroup. Then we may assume that $p_3(t)=0$ for all $t \in R$. It follows that, for all $t \in R$, $p_1(t)=a^t$ and $p_2(t)=b^t$, for some fixed real numbers $a, b>0$. Using this fact, one can show that if $a=b$, then $p_2(t)=Ata^t$ and, if $a \neq b$, then $p_2(t)=A(b^t-a^t)$, where A is a fixed real number. One can show, by switching bases in V_2, that the group obtained in the latter case is isomorphic to that obtained if $A=0$. Thus we may assume that $P(t)$ is given by one of the following forms, where $a, b>0$:

(i) $$\begin{bmatrix} a^t & 0 \\ 0 & b^t \end{bmatrix}, \quad a \neq b \text{ and } a, b \neq 1,$$

(ii) $$\begin{bmatrix} a^t & Ata^t \\ 0 & a^t \end{bmatrix}, \quad A \neq 0,$$

(iii) $$\begin{bmatrix} a^t & 0 \\ 0 & a^t \end{bmatrix}, \quad a \neq 1.$$

Notice that if a or b is 1 in (i) then the corresponding semidirect product is isomorphic to $\mathrm{Af}(1) \times R$. Also, the direct product V_3 is obtained by letting $a=1$ in (iii).

Consider the following special case of (i):

$$\begin{bmatrix} a^t & 0 \\ 0 & (1/a)^t \end{bmatrix}.$$

Because of Theorem 5 it is convenient to adopt the following convention. If G is of type (i) and is not the special case just above, then we say "$G=G_2$." If G is a semidirect product having at least one normal one-parameter subgroup and $G \neq G_2$, then we say "$G=G_1$." Thus, if G is a semidirect product then precisely one of the following is true: $G=G_1$, $G=G_2$, or G has no normal one-parameter subgroup.

2. THEOREM 1. *Let G be a connected, locally compact group of finite dimension embedded in a semigroup, and let e be an idempotent in the closure of G such that Ge is locally compact. Then* $e \in G_l(e)^-$.

Proof. Let $H=G_l(e)$. Since H has a local cross-section in G [17], there is a compact neighborhood of 1 in H, V, and a compact subset K of G such that $K \cap V=\{1\}$, and the multiplication map of G restricted to $V \times K$ is a homeomorphism onto a neighborhood of 1 in G. Let d be the projection of G onto the coset space $\{gH \mid g \in G\}$. Then $d|K\colon K \twoheadrightarrow d(K)$ is a homeomorphism onto a neighborhood of H in G/H. Since Ge is locally compact, the map $gH \to ge$ is a homeomorphism of G/H onto Ge [1]. Thus the map $k \to ke$ is a homeomorphism of K onto a neighborhood of e in Ge. There is a net $\{g_i\}$ in G that converges to e. So $g_ie \to e$

and, eventually, for each i, there is a $k_i \in K$ such that $g_i e = k_i e$. Necessarily, $k_i \to 1$, and for each g_i there is an $h_i \in H$ such that $g_i = k_i h_i$. Thus we have $k_i h_i \to e$ and $h_i = k_i^{-1} k_i h_i \to 1 \cdot e = e$.

REMARKS. The conclusion of the theorem implies that $G_l(e)$ cannot be compact if $e \in G^- \backslash G$. Hence, under these hypotheses, (G, G^-) cannot be a Cartan space in the sense of Palais [*On the existence of slices for actions of noncompact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323]. The assumption "Ge is locally compact" cannot be entirely eliminated as the following example shows.

EXAMPLE. Let P^- be the nonnegative reals, let $P = P^- \backslash \{0\}$, and let T be any torus group of finite dimension greater than one. Let $f: P \to T$ be a one-to-one continuous but not open homomorphism of P into T such that $f(P)$ is a dense proper subgroup of T. It is well known that such a homomorphism exists [1]. $P^- \times T$ is a locally compact semigroup that contains the group $G = \{(p, f(p)) \mid p \in P\}$ which is isomorphic to P. Let e denote the point $(0, 1)$ in $P^- \times T$. e is the identity of the group $\{0\} \times T$, and Ge is a dense subgroup of $\{0\} \times T$ isomorphic to $f(P)$. By the lemma below, we may assume that there is a sequence $\{p_i\}$ in P such that $p_i \to 0$ in P^- and $f(p_i) \to 1$ in T. Thus $e \in G^-$. Consequently, we have a group G isomorphic to the positive reals that is dense in a locally compact semigroup G^- such that the boundary of G is a torus group isomorphic to T. Moreover, Ge is not locally compact, where e is an idempotent in the boundary of G. Clearly, $e \notin G_l(e)^-$, since $G_l(e)$ consists only of the identity of G.

LEMMA (FOR THE EXAMPLE). *Let P^- be the nonnegative reals and let $P = P^- \backslash \{0\}$. Let T be a locally compact group with a countable base for the neighborhood system of the identity* 1. *Let $f: P \to T$ be a one-to-one continuous but not open homomorphism such that $f(P)$ is a proper dense subgroup of T. Then there is a sequence $\{p_i\}$ in P such that $f(p_i) \to 1$ in T and either $p_i \to 0$ in P^- or $p_i^{-1} \to 0$ in P^-.*

Proof. Let $\{N_i\}$ be a countable base for the neighborhood system of 1 in T such that each N_i is open and $N_{i+1}^- \subset N_i$. For each i, $N_i \cap f(P)$ is not compact. Thus, for each i, there is a sequence $\{f(p_{ji})\}_j$ contained in $N_i \cap f(P)$ such that $\{p_{ji}\}_j$ has no convergent subsequence in P. Thus we may pick p_i, equal to p_{ji} for some j, such that $|p_i| > i$. The sequence $\{p_i\}$ thus obtained, or some subsequence of it, must satisfy the conclusion of the lemma.

THEOREM 2. *Let G be a connected, locally compact group embedded in a locally compact semigroup in such a way that the boundary of G is a single left G-orbit, Ge, where e is an idempotent. If there is a subspace V of G such that $V \cap G_l(e) = \{1\}$ and the multiplication map of G restricted to $V \times G_l(e)$ is a homeomorphism onto G; then* (i) *$G_l(e)^- = G_l(e) \cup \{e\}$, and* (ii) *the multiplication map $m: V \times G_l(e)^- \twoheadrightarrow G^-$ is a homeomorphism.*

Proof. The hypotheses imply that the map $v \to ve$ is a homeomorphism of V onto Ge. Let $H = G_l(e)$. Then $e \in H^-$, by Theorem 1. Since e is a right zero for H

and a right identity for Ge, it is clear that $H^- \cap Ge=\{e\}$. Thus we have (i). To prove (ii), we have only to show that m is an open map. Suppose $g_i \to e$ where $\{g_i\} \subset G$. Let $g_i = v_i h_i$ with $v_i \in V$ and $h_i \in H$. Then $v_i h_i e \to e$, $v_i e \to e$; thus $v_i \to 1$. Consequently, $h_i = v_i^{-1} v_i h_i \to e$. So we have shown that $v_i h_i \to e$ implies $(v_i, h_i) \to (1, e)$ in $V \times H^-$. A similar argument will work for any net in G^-. Thus m is open.

3. **Notation.** Throughout the remainder of this paper, the following notation will be adhered to. S is a locally compact semigroup consisting of a dense subgroup G and the boundary of G, L. G is a Lie group on E^3, and L is homeomorphic to E^2. Since G is locally compact, G is open in S. P, R, T, and Q will denote one-parameter subgroups of G isomorphic to the additive reals. H and J will denote planar subgroups of G; that is, subgroups of G which are isomorphic to V_2 or Af (1). It should be kept in mind that since G is a group on a euclidean space, any closed connected subgroup of G is also a group on a euclidean space [10]. e is an idempotent in L such that Ge and eG are closed subsets of L. (The existence of such an idempotent in S is asserted in P4.) We also remind the reader of the notation introduced in the Preliminaries for the various Lie groups on E^3.

4. THEOREM 3. *Let u be an idempotent in L such that Gu is open in L and is simply connected. Then* (i) *$G_l(u)^- = G_l(u) \cup \{u\}$ is isomorphic to the multiplicative nonnegative reals.* (ii) *There is a subspace V of G such that the multiplication map $m\colon V \times G_l(u)^- \twoheadrightarrow G \cup Gu$ is a homeomorphism.* (iii) *$G \cup Gu$ is a half-space semigroup.*

Proof. Since Gu is locally compact, $G/G_l(u)$ is homeomorphic to Gu. This implies, by P2, that $G_l(u)$ is connected, and since dim $Gu=2$ it follows that $G_l(u)$ is isomorphic to the positive reals. Theorem 1 implies that u is in the closure of $G_l(u)$. By a theorem of [6], (i) is established. Since $G_l(u)$ is a solid topological space, there is a cross-section V to the right orbits of $G_l(u)$ in G [18, p. 55]. That is, $G = VG_l(u)$. $G \cup Gu$ is an open subset of S, hence $G \cup Gu$ is a locally compact subsemigroup of S. Now, Theorem 2 establishes (ii). (iii) follows since V is homeomorphic to Gu which must be a planar subset of L [M. H. A. Newman, *Elements of the topology of plane sets of points*, 2nd ed., University Press, Cambridge, 1961, p. 149].

COROLLARY 4. *Dimension $Ge=2$, or dimension $eG=2$ implies that S is a half-space semigroup.*

THEOREM 5. *Let G be a semidirect product on euclidean three-space, V_2R.* (1) *If G has precisely two normal one-parameter subgroups, then* (1a) *these subgroups lie in V_2, and any automorphism of V_2 that leaves each of them invariant can be extended to an automorphism of G. Also,* (1b) *there is an automorphism of G that carries one* *of these subgroups onto the other if and only if G is isomorphic to the matrix group*

$$\begin{bmatrix} a^r & 0 & x \\ 0 & (1/a)^r & y \\ 0 & 0 & 1 \end{bmatrix}$$

where $r \in R$, $(x, y) \in V_2$, and a is a positive number not equal to 1.

(2) *If G has more than two normal one-parameter subgroups, then any automorphism of V_2 can be extended to an automorphism of G.*

(3) *If P is a one-parameter subgroup of G that is not contained in V_2, then there is an automorphism of G that carries P to R and leaves each one-parameter subgroup of V_2 invariant.*

(4) *$G = G_2$ (see Preliminaries) if and only if there are precisely two normal one-parameter subgroups of G, and there is no automorphism of G that carries one of these onto the other.*

Proof. We may assume that G is the group of 3×3 matrices

$$\begin{bmatrix} a^r & 0 & x \\ 0 & b^r & y \\ 0 & 0 & 1 \end{bmatrix}$$

where $r \in R$, $(x, y) \in V_2$, and a, b are fixed positive numbers not equal to 1. Let R, T, and Q denote the subgroups consisting, respectively, of matrices of the form

$$\begin{bmatrix} a^r & 0 & 0 \\ 0 & b^r & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

T and Q are normal in G and $V_2 = TQ$. If $a \neq b$, then T and Q are the only normal one-parameter subgroups of G. Let $a \neq b$, and let the matrix $\left[\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right]$ represent a nonsingular linear transformation F of V_2 onto V_2 that leaves T and Q invariant. The map

$$\begin{bmatrix} a^r & 0 & x \\ 0 & b^r & y \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} a^r & 0 & cx \\ 0 & b^r & dy \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism of G that is also an extension of F. This establishes (1a). Suppose now that $a = b$ and that A is the matrix of an automorphism F of V_2. Then the following map is an automorphism of G that is also an extension of F:

$$\begin{bmatrix} a^r & 0 & x \\ 0 & a^r & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} \multicolumn{2}{c}{A} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a^r & 0 & x \\ 0 & a^r & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \multicolumn{2}{c}{A^{-1}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This gives us (1b).

We now show (3). Let $L[G]$, the Lie algebra of G have the basis (e, f, g) where $L[V_2]=\langle e, f\rangle$, the vector subspace generated by (e, f), and $L[R]=\langle g\rangle$. Then $[L[G], L[G]]=\langle e, f\rangle$, and ad g, restricted to $\langle e, f\rangle$, is a nonsingular linear transformation (compare [11, p. 12]). Let $cg+x \in L[G]$ such that $\exp(\langle cg+x\rangle)=P$, where $c\neq 0$ and $x\in\langle e, f\rangle$. Consider the linear map b defined by $b(e)=ce$, $b(f)=cf$, and $b(g)=(1/c)g'$, where $g'=cg+x$. b carries $\langle g\rangle$ onto $\langle g'\rangle$ and is a Lie algebra automorphism of $L[G]$. Since G is simply connected there is a Lie group automorphism a of G such that $a^0=b$. Consequently, $a(R)=P$ and, since all subspaces of $L[V_2]$ are invariant under a^0, it follows that all one-parameter subgroups of V_2 are invariant under a. This gives us (3).

We are ready now to prove (1b). G is the matrix group in the proof of (1a) and $a\neq b$. By (3) we may assume that d is an automorphism of G that switches the two normal one-parameter subgroups and leaves R invariant. Clearly d has the form

$$\begin{bmatrix} a^r & 0 & x \\ 0 & b^r & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} a^{r'} & 0 & By \\ 0 & b^{r'} & Ax \\ 0 & 0 & 1 \end{bmatrix}$$

where A and B are fixed nonzero real numbers. It is easily verified that $(r_1+r_2)' = r_1'+r_2'$, $A(a^{r_1}x_2+x_1)=A(b^{r_1'}x_2+x_1)$, and $B(b^{r_1}y_2+y_1)=B(a^{r_1'}y_2+y_1)$. Thus $a^{r_1}= b^{r_1'}$, and $b^{r_1}=a^{r_1'}$. This implies that for all $r\in R\backslash\{0\}$, $(\log a)/(\log b)=r'/r$ and $(\log b)/(\log a)=r'/r$. It follows then that $r'=-r$ and $b=1/a$. On the other hand, if $b=1/a$, then the map above is an automorphism of G that maps one normal one-parameter subgroup of G onto the other one. Thus (1b) is proven. (4) follows from the definition of "$G=G_2$," as given in the Preliminaries, and (1b).

We will frequently encounter the following situation: $a\colon S \twoheadrightarrow S'$ is a homeomorphism, where S, S' are semigroups. G is a dense subset of S, and $a(g_1g_2) =a(g_1)a(g_2)$, for all g_1, g_2 in G. It is straightforward to show that this implies that $a(s_1s_2)=a(s_1)a(s_2)$, for all s_1, s_2 in S. Thus we have the following result:

SUBLEMMA. *Let $a\colon S \twoheadrightarrow S'$ be a homeomorphism, where S, S' are semigroups, and G is a dense subset of S. If $a(g_1g_2)=a(g_1)a(g_2)$, for all g_1, g_2 in G, then $a(s_1s_2) =a(s_1)a(s_2)$, for all s_1, s_2 in S. Thus an isomorphism between the maximal groups of two half-space semigroups extends to an isomorphism of the semigroups if it extends to a homeomorphism.*

THEOREM 6. *Dimension Ge = dimension $eG=2$ implies that L is a group and S is isomorphic to a semigroup $G\cup G/Q$ as constructed in* [8], *where $Q=G_l(e)$ is a normal one-parameter subgroup of G. Moreover,* (i) *if G is nilpotent, then G determines S.* (ii) *If $G=\mathrm{Af}(1)\times R$, then there are precisely two semigroups $G\cup G/Q$.* (iii) $G=G_i$ *(see Preliminaries) implies that there exist i nonisomorphic semigroups $G\cup G/Q$.*

Proof. Both Ge and eG are open and closed in L, so each must be L. Consequently, L is a group and $G_l(e)$ is a normal one-parameter subgroup of G isomorphic to the additive reals. Let $Q=G_i(e)$ and let $m\colon V\times Q^- \twoheadrightarrow S$ be as in Theorem 3. Let $G\cup G/Q$ denote the disjoint union of G and the coset space $\{gQ \mid g\in G\}$. We define a multiplication by retaining the multiplication in G and letting $x(yQ)=(xy)Q$, $(xQ)y=(xy)Q$, and $(xQ)(yQ)=(xy)Q$, for all $x, y\in G$. We define the topology on $G\cup G/Q$ as that topology generated by the topology of G together with all sets of the form $W_x(e, q)\cup h(W_x)$, where W_x is an open neighborhood of x in G; (e, q) is the open interval of Q^- with endpoints e and q; and h is the projection of G onto G/Q. We mean by $W_x(e, q)$, the set $\{wp \mid w\in W_x, p\in(e, q)\}$. It is immediate that G and G/Q, as subspaces of $G\cup G/Q$, have their original topologies and that G is an open dense subset. Since Q is normal and $G=VQ$, it is not difficult to see that $G\cup G/Q$ is a locally compact, Hausdorff topological semigroup. This construction is due to Horne (see [8, pp. 46–47]).

We now show that S is isomorphic to $G\cup G/Q$. Consider the map $k\colon VQ^-\to G\cup G/Q$ defined by $k(vq)=vq$ and $k(ve)=vQ$, for all $v\in V$, $q\in Q$. Clearly k is a topological group isomorphism when restricted to G. Suppose $v_iq_i\to ve$ in VQ^-. Then $v_i\to v, q_i\to e$, so $v_i\to v$ and $q_i\to Q$ in $G\cup G/Q$. Thus $v_iq_i\to vQ$ in $G\cup G/Q$, and k is continuous. It is not difficult to see that a basic neighborhood of ve in VQ^- maps onto a basic neighborhood of vQ in $G\cup G/Q$. (Let N_1, N_2, N_3, N_4 be, respectively, an open neighborhood of v in V, of 1 in Q, of 1 in V, of e in Q^-. Then $N_1N_2N_3=W_v$, and $k(N_1N_2N_3N_4)=W_v(e, q)\cup h(W_v)$, where $N_4=[e, q)$.) We have shown that k is a homeomorphism. Thus S is isomorphic to $G\cup G/Q$.

Suppose now that S_1 and S_2 are two semigroups that satisfy the hypotheses of this theorem concerning S. If $a\colon G_1\to G_2$ is an isomorphism of the maximal group of S_1 onto the maximal group of S_2, and $a(Q_1)=Q_2$, then we may assume that a maps Q_1 onto Q_2 in such a way that if we define $a(e_1)$ to be e_2, then a maps Q_1^- homeomorphically onto $a(Q_1^-)$. Also, $G_2=a(V_1)a(Q_1)$. Thus the map that completes the following diagram is an extension of a and a homeomorphism of S_1 onto S_2:

$$\begin{array}{ccc} V_1\times Q_1^- & \overset{m}{\twoheadrightarrow} & S_1 \\ {\scriptstyle a|V_1\times a|Q_1^-}\downarrow & & \vdots\downarrow \\ a(V_1)\times a(Q)_1^- & \overset{m_2}{\twoheadrightarrow} & S_2 \end{array}$$

We now apply this observation to each of the possibilities for G. Clearly, (i) is true if $G=V_3$. If one carries out the following multiplication then it will be clear that there is only one normal one-parameter subgroup of N, and, in fact, it is the center:

$$\begin{bmatrix}1 & a & 0\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & x(t) & y(t)\\ 0 & 1 & z(t)\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & -a & 0\\ 0 & 1 & -b\\ 0 & 0 & 1\end{bmatrix}.$$

(ii) follows since $\mathrm{Af}(1) \times R$ has precisely two normal one-parameter subgroups, one of which is the center. (iii) follows from Theorem 5.

LEMMA 7. (i) *Suppose P is a subgroup of G isomorphic to the additive reals such that $Px \neq \{x\}$, for all x in L. Suppose that C is a line in L such that k is one-to-one on C and $k(C) = L/P$, where k is the projection of L onto L/P. Then the multiplication map $m: P \times C \twoheadrightarrow L$ is a homeomorphism.*

(ii) *If $x \in L$, then Gx cannot be a simple closed curve. Consequently, if* $\dim Gx = 1$, *then $G_l(x)$ is a planar subgroup of G.*

Proof. (i) will be a consequence of [13] if we show that L/P is Hausdorff. Let Px, Py be distinct points of L/P. We may assume $x, y \in C$. Let A_1, A_2 be disjoint subarcs of C which are neighborhoods in C of x and y, respectively. Let V be a compact neighborhood of the identity in P. The multiplication map $m: V \times A_1 \twoheadrightarrow VA_1$ is a homeomorphism since it is one-to-one. This implies, by a dimension argument, that VA_1 is a neighborhood of x in L. Clearly then, $k(PA_1)$ and $k(PA_2)$ are disjoint neighborhoods of Px and Py, respectively, in L/P.

To show (ii), suppose that Gx is a simple closed curve. Then Gx is a compact semigroup (since it is an ideal), so there is an idempotent f such that $Gf = Gx$ [2, p. 15]. Now, the map $y \to yf$ is a retraction of L onto Gf which is impossible. The last statement of the lemma is clear.

THEOREM 8. *If dimension $Ge = 2$ and dimension $eG = 1$, then S is isomorphic to $H^- \times R$, where R is the additive reals and H^- is the half-plane semigroup whose boundary is a left-zero semigroup. Of course,* $G = \mathrm{Af}(1) \times R$.

Proof. Let $H = G_r(e)$. H is normal in G since $eG \subset Ge$. Now, $e \in H^-$, and $He \neq eH$. Thus H is not abelian. Since $\mathrm{Af}(1)$ is a complete group (trivial center and all automorphisms are inner), it follows that $G = \mathrm{Af}(1) \times R$. By P7, H^- is a half-plane semigroup and $H^- \cap L = He$ is a left-zero semigroup. Let $Q = H_l(e)$. If Q were normal in H then $hQ^-h^{-1} = hQh^{-1} \cup he$, for all $h \in H$. This contradicts the fact that $Q^- = Q \cup \{e\}$ [9]. Thus we see that $He = Pe$, where P is the normal one-parameter subgroup of H, and $Ge = RPe = (Pe)R$ since R is the center of G. It follows that the line Pe satisfies the hypotheses of Lemma 7(i), so $m: Pe \times R \twoheadrightarrow L$ is a homeomorphism. Since R commutes with all elements of S, this map is also an algebraic isomorphism. To show that the multiplication map $m: H^- \times R \twoheadrightarrow S$ is a topological semigroup isomorphism, we have only to show now that it is an open map. Let $h_i r_i \to zr$, where $h_i \in H$, $r_i, r \in R$ and $z \in H^- \cap L$. Then, multiplying on the left by e, we get $er_i \to er$ and thus $r_i \to r$. Consequently $h_i \to z$, and we are through.

THEOREM 9. *If dimension $Ge = 2$ and dimension $eG = 0$, then G is a semidirect product and S is a semigroup on a half-space. Moreover, if S, S' are two such semigroups with the same maximal group, then S is isomorphic to S'.*

Proof. $Ge=L$ and $eG=e$. Let $P=G_l(e)$. P is isomorphic to the additive reals, and the center C of G must be a closed subgroup of P. Theorem 3 implies that $P^-=P \cup \{e\}$. Consequently, if C were dense in P, then e would be in the center of S, a contradiction. Thus $C=\{1\}$. This rules out all of the possibilities for G except the semidirect products. Since, for any $g \in G$, $gP^-g^{-1}=gPg^{-1} \cup \{ge\}$, it is clear that P cannot be a subgroup of V_2. Hence, by Theorem 5, $G=V_2P$. Now, by Theorem 3, we know that $m: V_2 \times P^- \twoheadrightarrow S$ is a homeomorphism. We consider now the question of when two such semigroups S, S' are isomorphic assuming that G is isomorphic to G'. Again, Theorem 5 implies that we may assume that $a: G \twoheadrightarrow G'$ is an isomorphism such that $a(P)=P'$. Clearly we could follow a by an automorphism of G', if necessary, to insure that the half-line P^- is carried homeomorphically onto $(P')^-$ where we define $a(e)$ to be e'. Also, since $G'=a(V_2)a(P)$, we know that $m': V' \times (P')^- \twoheadrightarrow S'$ is a homeomorphism, where $V'=a(V_2)$. We define $a(ve)=a(v)e'$, for all $v \in V_2$. Clearly, this extension of a is the homeomorphism that completes the following diagram:

$$\begin{array}{ccc} V_2 \times P^- & \xrightarrow{\ m\ } & S \\ {\scriptstyle a|V_2 \times a|P^-}\downarrow & & \downarrow{\scriptstyle a} \\ V' \times (P')^- & \xrightarrow{\ m'\ } & S' \end{array}$$

Thus a is a homeomorphism of S onto S'. By the sublemma, a is an isomorphism.

THEOREM 10. *If dimension eG = dimension $Ge=1$ and $eG \neq Ge$, then S is a semigroup on a half-space and G is a hyperbolic semidirect product. In fact, $G=G_2$ (see Preliminaries for an explanation of this notation and for the definition of a hyperbolic semidirect product), and there are two nonisomorphic semigroups S with the same maximal group G.*

Proof. By P8, $G_l(e) \neq G_r(e)$. Let $H=G_l(e)$ and $J=G_r(e)$. Since $e \in H^- \cap J^-$, H and J must be noncommutative planar subgroups of G. By P7, $H^- \cap L=eH$ is a right-zero semigroup, and $J^- \cap L=Je$ is a left-zero semigroup. Since eH is a closed line in eG, $eH=eG$. Similarly, $Je=Ge$. Consequently, $Ge \cap eG=\{e\}$. Clearly no inner automorphism of G may carry H onto J since this would extend to an automorphism of S that would map H^- isomorphically onto J^-, an impossibility. Let P, Q be the normal one-parameter subgroups of J, H, respectively. Then $Ge=Pe$ and $eG=eQ$ (P5). Let $q \in Q$. Then $q^{-1}J^-q=q^{-1}Jq \cup \{eq\}$, and $eq \notin J^-$ if $q \neq 1$. Thus J cannot be normal in G. Similarly, H is also nonnormal. Consequently, G is a group on E^3 with at least two conjugacy classes of nonnormal, noncommutative planar subgroups. G cannot be nilpotent since H and J are not nilpotent groups. Each copy of Af(1) in Af(1)$\times R$ is normal. All noncommutative planar subgroups of Sl(2) are conjugate to each other [8, p. 28]. Thus G must be a semidirect product. Moreover, the argument of Theorem 2.5 of [8], as amended by

P10, shows that G must be hyperbolic. Thus we have $G=(PQ)R$, where $PQ=V_2$, $R=H\cap J$, and P, Q are normal one-parameter subgroups of G, and are normal in J, H respectively.

Consider the multiplication map $m\colon P\times eQ\to L$. If $p_1eq_1=p_2eq_2$, then $p_2^{-1}p_1e=eq_2q_1^{-1}$ which implies that $p_2=p_1$ and $q_2=q_1$. Thus m is one-to-one. Suppose $p_ieq_i\to x\in L$. Then $ep_ieq_i\to ex=eq$, for some $q\in Q$, since eG is a right ideal. $ep_ie\in(eG)^-\cap(Ge)^-=\{e\}$. Thus $eq_i\to eq$ and $q_i\to q$. Similarly, $p_i\to p$. Thus PeQ is closed in L, and we have shown that m is open onto PeQ. PeQ is two-dimensional and homogeneous. Thus PeQ is open and closed in L; $PeQ=L$. Thus $m\colon P\times eQ\twoheadrightarrow L$ is a homeomorphism. Suppose $p_ih_i\to peq$, where $p_i,p\in P$, and $h_i\in H$, $q\in Q$. Then $p_ie\to p(eqe)=pe$, and thus $p_i\to p$. So, $h_i\to eq$. This suffices to show that the multiplication map $m\colon P\times H^-\twoheadrightarrow S$ is a homeomorphism. Consequently, S is a half-space semigroup.

Let S_1, S_2 be two semigroups satisfying the hypotheses of S. Assume that S_1 and S_2 have the same maximal group G. Then $G=P_1R_1Q_1=P_2R_2Q_2$, where $R_iQ_i=H_i=G_l(e_i)$, $R_iP_i=J_i=G_r(e_i)$, and $R_i=H_i\cap J_i$. Also, $R_i^-=R_i\cup\{e_i\}$. Suppose that a is an automorphism of G such that $a(H_1)=H_2$, and $a(J_1)=J_2$. By Theorem 5, this could happen if and only if G is hyperbolic and $G=G_1$. We may assume that a carries R_1^- homeomorphically onto R_2^-, where we define $a(e_1)$ to be e_2. We define $a(e_1q)$ to be $e_2a(q)$, where $q\in Q_1$. (Notice that a must map Q_1 to Q_2.) Clearly, a now maps $H_1^-\cap L_1$ homeomorphically onto $H_2^-\cap L_2$. We now show that $a|H_1^-$ is a homeomorphism of H_1^- onto H_2^-. Let $r_iq_i\to e_1q_1$. Then $e_1q_i\to e_1q_1$, and thus $q_i\to q_1$, and $r_i\to e_1$. Consequently, $a(q_i)\to a(q_1)$, $a(r_i)\to e_2$, and $a(r_iq_i)=a(r_i)a(q_i)\to e_2a(q_1)=a(e_1q_1)$. Thus a is continuous on H_1^-. A similar argument will show that a^{-1} is continuous on H_2^-. Thus $a|H_1^-\colon H_1^-\twoheadrightarrow H_2^-$ is a homeomorphism. Clearly, the map that completes the diagram below is an extension of a and is a homeomorphism of S_1 onto S_2.

$$\begin{array}{rccc} & P_1\times H_1^- & \xrightarrow{m_1} & S_1 \\ a|P_1\times a|H_1^- & \Big\downarrow & & \vdots\downarrow \\ & P_2\times H_2^- & \xrightarrow{m_2} & S_2 \end{array}$$

Thus S_1 is isomorphic to S_2.

If G is any hyperbolic semidirect product, then G has a representation as a matrix group of 4×4 real matrices of the form

$$\begin{bmatrix} 1 & a & & \\ 0 & t^c & \multicolumn{2}{c}{0} \\ & & t^d & b \\ \multicolumn{2}{c}{0} & 0 & 1 \end{bmatrix},$$

where t is any element of the multiplicative positive reals, $(a, b) \in V_2$, and, since G is hyperbolic, both c and d are fixed positive real numbers. The closure of G in the semigroup of 4×4 real matrices consists of all those matrices above with $t \geqq 0$. The boundary of G is a plane (here, we mean by "the boundary of G," the set $G^-\backslash G$) L, and it is easy to verify that H, the subgroup of matrices of the form

$$\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & t^c & 0 & 0 \\ 0 & 0 & t^d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

is $G_l(e)$, and that J, the subgroup of matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t^c & 0 & 0 \\ 0 & 0 & t^d & b \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

is $G_r(e)$, where e is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Ge and eG are closed lines in L intersecting in e. This construction is given in the proof of Theorem 2.5, referred to earlier. If $G=G_2$, then we may obtain a second semigroup, not isomorphic to the one above but with a maximal group isomorphic to G. We proceed as follows. The map $x \to (x^T)^{-1} = x^*$ is an automorphism of $GL(R, 4)$. The closure of G^* in the semigroup of 4×4 real matrices is a locally compact semigroup, and the boundary K of G^* is a plane. e is in K, $H^*=G_r^*(e)$, $J^*=G_l^*(e)$, and G^*e, eG^* are closed lines in K intersecting in e. Thus $G^* \cup K$ is a semigroup that satisfies the hypotheses of this theorem, and G^* is isomorphic to G. If $b\colon G^* \cup K \twoheadrightarrow G \cup L$ were an isomorphism, then $b(H^*)$ would have to be $G_r(e)$ which is J. Consequently, the automorphism of G, $x \to x^* \to b(x^*)$, would map H to J, which is impossible if $G=G_2$. This concludes the proof.

For the definition of the radical of a half-space semigroup and a discussion of it, one may consult [8, pp. 40–44]. It suffices for us to know that S has empty radical if and only if each right G-orbit and each left G-orbit in L contains an idempotent. Also, if H^- is a half-plane semigroup with a zero 0 and x is in the boundary of H, then x is a nilpotent element of H^- if $x \neq 0$ and $x^2=0$ [5].

LEMMA 11. *Let $eG=eP$, where P is a one-parameter subgroup of G, and let $eH=He$, where $H=G_r(e)$. Then the multiplication map $m\colon H^-\times P\twoheadrightarrow S$ is a homeomorphism, and H^- is a half-plane. Each component of $L\backslash eG$ is a right orbit, and S has no radical if and only if H^- has no nilpotent elements.*

Proof. Let C_1 and C_2 be the components of $L\backslash eG$. By Theorem 1, $e\in H^-$, where $H=G_r(e)$. Also, $G=HP$, where P is a one-parameter subgroup of G such that $eG=eP$. (That $G=HP$ is easy to show since for each $g\in G$, there is a $p\in P$ such that $eg=ep$, so $g\in Hp$, etc.) By P7, $H^-\cap L$ is either a line or a half-line. In either case, $H^-\cap eG=\{e\}$. Let us assume that $H^-\cap C_1\neq\varnothing$, and let $x\in C_1\cap H^-$, $x\neq e$. Each point of H lies on a one-parameter subgroup of H, and e is a zero for H that is not a limit point of idempotents in H^- (P3, P9). Consequently (see Lemma 2.9 of [8]), there must be a one-parameter subgroup R of H such that $R^-=R\cup\{e\}$. xR is one component of $(H^-\cap L)\backslash\{e\}$. Let $Q=H_r(x)$. Clearly, $Q\neq R$, and since $Hx=xH$, Q is a normal subgroup of H. Thus $H=QR$. $G=HP=QRP$, and $xG=xRP$. If $xr_1p_1=xr_2p_2$, where $r_1, r_2\in R$ and $p_1, p_2\in P$, then $ep_1=ep_2$, so $p_1=p_2$. Thus $r_1=r_2$ and the map $rp\to xrp$ is one-to-one. This implies that xG is two-dimensional, and being homogeneous it is open in L. We now show that $xRP=C_1$. Suppose $xr_ip_i\to y\in L$. Then $ep_i\to ey=ep$, for some $p\in P$ since eG is a right ideal. Thus $p_i\to p$ and $xr_i\to yp^{-1}$. This implies that either $yp^{-1}=e$ or $yp^{-1}=xr$ for some $r\in R$. In either case, we see that xRP is an open subset of C_1 whose boundary is contained in eG. This implies that $xRP=C_1$. Since $Q\subset G_r(x)$, it follows that $Q=G_r(x)$, and thus the map $rp\to xrp$ is a homeomorphism. At this point, one should notice that if $H^-\cap L$ is a line, then the argument above implies that $C_2=yRP$, and the map $rp\to yrp$ is a homeomorphism, where y is any point of $H^-\cap C_2$, and $r\in R$, $p\in P$.

We now show that the multiplication map $m\colon[H\cup(xR)^-]\times P\twoheadrightarrow[G\cup xG\cup eG]$ is a homeomorphism. Each of the following numbered statements has a quick proof if one "multiplies on the left by e." Together, they are sufficient to show that m is open and thus a homeomorphism: (1) $q_ir_ip_i\to xrp\Rightarrow q_ir_i\to xr$ and $p_i\to p$. (2) $q_ir_ip_i\to ep\Rightarrow q_ir_i\to e$ and $p_i\to p$. (3) $xr_ip_i\to xrp\Rightarrow xr_i\to xr$ and $p_i\to p$. (4) $xr_ip_i\to ep\Rightarrow xr_i\to e$ and $p_i\to p$. We are now ready to show that $H^-\cap L$ cannot be a half-line. Suppose that $H^-=H\cup(xR)^-$. Then H^- is homeomorphic to E^2 (P9), and thus $G\cup xR\cup eG$ must be homeomorphic to E^3. This is impossible since $G\cup xR\cup eG$ is not locally compact at any point of eG. Thus $H^-\cap L$ is a line that crosses eG at e (P7). By an obvious elaboration of an argument above, it follows that the multiplication map $m\colon H^-\times P\twoheadrightarrow S$ is a homeomorphism. The last assertion of the lemma follows easily from the following two facts: First, if C_1 contains an idempotent u, then uG is a subsemigroup of S, which implies that $x^2\neq e$. On the other hand, if $x^2\neq e$, then xR must be a group.

THEOREM 12. *If S does not have a zero, then S is a half-space semigroup.*

Proof. Recall that u is a zero for S if and only if $Su=uS=u$. The remaining possibilities to be considered for eG and Ge—leaving aside the obvious dual cases—are (i) $Ge=eG$ is a closed line; (ii) $Ge=e$ and eG is a closed line; (iii) $Ge=eG=e$. Let $H=G_r(e)$. Clearly (ii) implies that $He=eH$, and (iii) implies that e is a zero for S. Now consider (i). $e \in H^-$, and H is a planar subgroup of G. If H is commutative or $G_l(e)=G_r(e)$, then $He=eH$, and we are through. Suppose then that $J=G_r(e)$, $H \neq J$, and H, J are isomorphic to Af(1). By P7, $H^- \cap L=He$ is a left-zero semigroup and $J^- \cap L=eJ$ is a right-zero semigroup. Since He is a closed line in Ge, $He=Ge$. Similarly, $eJ=eG$. Thus $He=eJ$, an impossibility. Thus $He=eH$. Now the theorem follows from Lemma 11.

THEOREM 13. *Let dimension eG = dimension $Ge=1$, and $Ge=eG$. Let P be a one-parameter subgroup of G such that $eG=eP$, and let $H=G_l(e)$. Then H is a normal planar subgroup of G, and the multiplication map $m\colon H^- \times P \twoheadrightarrow S$ is a homeomorphism.*

Let S' be a half-space semigroup with maximal group G'. Let e' be an idempotent in the boundary of G' such that dimension $e'G'$ = dimension $G'e'=1$, and $G'e'=e'G'$. Assume that G' is isomorphic to G. Then

(i) *If G is nilpotent, or H is nonabelian, then S is isomorphic to S' if and only if H^- is isomorphic to $(H')^-$.*

(ii) *S has empty radical if and only if H^- has no nilpotent elements. If both S and S' have empty radical, then S is isomorphic to S'. Thus, if S has empty radical, G determines G^-.*

(iii) *If S has empty radical, then H is abelian and G has at least two normal one-parameter subgroups. (Thus $G \neq N$.)*

(iv) *In any case, G must have at least one normal one-parameter subgroup. (Thus $G \neq SL(2)$.)*

Proof. The dimension of eG and the fact that $eG=Ge$ together imply that eG is a group isomorphic to the additive reals and that $H=G_l(e)$ is a normal planar subgroup of G. By Lemma 11, the multiplication map $m\colon H^- \times P \twoheadrightarrow S$ is a homeomorphism and H^- is a semigroup on a half-plane. If $G=V_3$, m is an isomorphism. If H is nonabelian, then $G=\mathrm{Af}(1) \times R$, and we may take P to be the center of G. Hence, again, m is an isomorphism. If $G=N$, the nonabelian nilpotent group on E^3, or if G is a semidirect product, then H must be abelian. Clearly G is not $SL(2)$ since H is normal.

Let C_1, C_2 be the components of $L\backslash eG$, and let P be a one-parameter subgroup of G such that $Ge=Pe$. e is a zero for H^-, so there is a one-parameter subgroup T of H such that $T^-=T \cup \{e\}$. (See the argument of Lemma 11.) Let $T_i=H_l(x_i)$, where $x_i \in C_i \cap H^-$, for $i=1, 2$. T_i must be normal in H since $Hx_i=x_iH$ (P3). If x_1 (say) is an idempotent, then $T_1^-=T_1 \cup \{x_1\}$ which implies that $T_1 \neq T_2$. Thus, in this case, H must be abelian. Notice that for any $g \in G$, $geg^{-1} \in Ge$ and is an idempotent. Hence $ge=eg$, and e is in the center of S. Hence $Pe=eP$ and the dual of Lemma 11 tells us that $Gx_i = x_iG = C_i$, for $i = 1, 2$. Thus T_1 and T_2 are, in fact, normal in G. So G must have at least one normal one-parameter subgroup, and if S has no radical, G must have two. This gives us (iii) and (iv).

We assume, for the remainder of the proof, that S and S' are half-space semigroups that satisfy the hypotheses mentioned in the theorem. To simplify the notation a bit, we let G denote the maximal group of each of the semigroups S, S'. Suppose now that a is an automorphism of G such that $a(H) = H'$, where $H' = G_l(e')$. Then we may assume that $a(P) = P'$. (P, P' are assumed merely to have no conjugate in the appropriate isotropy subgroups.) If a can be extended to an isomorphism of H^- onto $(H')^-$, then the diagram below implies that S is isomorphic to S':

$$\begin{array}{ccc} H^- \times P & \xrightarrow{m} & S \\ {\scriptstyle a|H^- \times a|P}\downarrow & & \vdots \\ (H')^- \times P' & \xrightarrow{m'} & S' \end{array}$$

Let $G = V_3$, or $H = \mathrm{Af}(1)$. If $k\colon H^- \twoheadrightarrow (H')^-$ is an isomorphism, then $k|H$ can be extended to an isomorphism of G. Consequently, by the remarks above, S is isomorphic to S' if and only if H^- is isomorphic to $(H')^-$. Now let $G = N$ and let $k\colon H^- \twoheadrightarrow (H')^-$ be an isomorphism. We show now that $k|H$ can be extended to an automorphism of G. Thus, in this case also, S is isomorphic to S' if and only if H^- is isomorphic to $(H')^-$. If $x \in H^- \backslash H$ and $x \neq e$, then $H_r(x) = G_r(x)$ is a normal one-parameter subgroup of G. Thus $G_r(x)$ is the center of G and is invariant under $k|H$. It suffices now, since G is simply connected, to show the following proposition: *If* $L[H]$, $L[H']$ *are two-dimensional ideals in* $L[G]$, *and b is an isomorphism of one onto the other that leaves invariant the center of* $L[G]$, *then b extends to an automorphism of* $L[G]$. We prove this now. Let $L[G] = \langle e, f, g\rangle$, where $fg = e$, $ef = eg = 0$, and $L[H] = \langle e, g\rangle$. Let $k(e) = ce$ and $k(g) = a_1e + a_2f + a_3g$, where c, a_1, a_2, a_3 are fixed real numbers, and $c \neq 0$. It is easy to extend k to all of $L[G]$. (If $a_3 \neq 0$, let $k(f) = (b/a_3)f$; if $a_3 = 0$, let $k(f) = (-b/a_2)g$.) This gives us (i).

We now prove (ii). Let S and S' have empty radical. Then H^- and $(H')^-$ are abelian half-plane semigroups with zero and without nilpotent elements. This implies that there is an isomorphism $a\colon H^- \twoheadrightarrow (H')^-$ [Horne, *Real commutative semigroups on the plane*, Pacific J. Math. **11** (1961), pp. 981–997]. It will suffice now to show that $a|H$ can be extended to an isomorphism of G. This is clear if $G = V_3$. If G is a semidirect product, then $H = H' = V_2$. The list of semidirect products on E^3, given in the Preliminaries, and Theorem 5 shows us that a difficulty arises if $G = G_2$ and a switches the two normal one-parameter subgroups of G. However, using the technique of the paper cited just above, it is not difficult to show that we may assume that a leaves both normal one-parameter subgroups of G invariant.

(*Remark*. One can show that any automorphism of H that either switches the isotropy subgroups of the nonzero idempotents in the boundary or leaves them invariant will extend to an automorphism of H^-, provided that the map does not reverse the orientation of the isotropy subgroup.) Theorem 5 implies now that a extends to an automorphism of G. Let $G=\mathrm{Af}(1)\times R$, the only remaining possibility for G. Then $H=H'=RQ$, where R is the center of G, and Q is the normal one-parameter subgroup of Af (1). By the Remark above, we may assume that $a(R)=R$, $a(Q)=Q$, and $P=P'$. There is an extension of $a|Q$ to an automorphism of Af (1) [5]. Thus a extends to an automorphism of G.

THEOREM 14. *Let dimension $Ge=0$, dimension $eG=1$, and assume that S has empty radical. Then $G=\mathrm{Af}(1)\times R$, and there are precisely two isomorphism classes of semigroups that satisfy these hypotheses. One class consists of those semigroups in which the center of G is not closed in S, which implies that one component of $L\backslash eG$ is a group; and the other class consists of those semigroups in which neither component of $L\backslash eG$ is a group.*

Proof. $Ge=e$ and there is a one-parameter subgroup P of G such that $eG=eP$. Let $H=G_r(e)$. By Lemma 11, the multiplication map $m\colon H^-\times P\twoheadrightarrow S$ is a homeomorphism, and H^- is a half-plane semigroup. Let C_1, C_2 be the components of $L\backslash eG$ and let e_1, e_2 be idempotents in the boundary of H such that $e_i\in H_i^-\cap C_i$, $i=1,2$. Since S is a half-space semigroup, Theorem 2.12 of [8] applies. We list some facts that follow from Theorem 2.12: (1) $G=\mathrm{Af}(1)\times R$. (2) H is not normal and is abelian; hence $R\subset H$. (3) Dimension $Ge_i=2$ if and only if C_i is a group. (4) Not both C_1, C_2 are groups. Thus we may assume that C_1 is not a group. (5) Therefore, $G_l(e_1)$ is a noncommutative planar subgroup H_1 of G, and $G=H_1\times R$. (6) $H_1\cap H=T_1=G_r(e_1)$, and $T_1^-=T_1\cup\{e_1\}$. (7) We may assume that P is the noncentral normal one-parameter subgroup of G. Thus $H_1=PT_1$.

An example of each type of semigroup mentioned in the theorem is given in [8, p. 47]. We must show that any two semigroups of the same type are isomorphic. If C_2 is a group, then $G_l(e_2)$ is a normal one-parameter subgroup of G in H. So $G_l(e_2)=R$, since if P were in H, H would be the normal abelian planar subgroup of G. Thus $R^-=R\cup\{e_2\}$. On the other hand, if C_2 is not a group, then $G_l(e_2)=H_2$ is isomorphic to Af (1). In this case, the right isotropy subgroup of e_2 relative to H_2 is nonnormal and is also $G_r(e_2)$. Let $G_r(e_2)=T_2$. Similarly, $G_r(e_1)=T_1$ is nonnormal. Since $e_iR=e_iH$, for $i=1,2$, it follows that if $R^-\cap L\neq\varnothing$, then $R^-=R\cup\{e\}$. But this implies that $p^{-1}R^-p=R\cup\{ep\}$, for any $p\in P$. This is impossible [6] since e is a zero for R. Thus $R^-\cap L=\varnothing$, if C_2 is not a group. We have shown, then, that $H=T_1\times T_2=T_1\times R$, and C_2 is a group if and only if $R=G_r(e_2)=H_r(e_2)$.

Suppose now that S, S' are two semigroups satisfying the hypotheses of S in the theorem. If a is an isomorphism of G onto G' such that $a|H$ extends to an isomorphism of H^- onto $(H')^-$, then the diagram below shows that S is isomorphic to S' (we may assume $a(P)=P'$):

$$\begin{array}{ccc} & H^{-}\times P & \overset{m}{\twoheadrightarrow} & S \\ a|H^{-}\times a|P & \downarrow & & \vdots\downarrow \\ & (H')^{-}\times P' & \overset{m'}{\twoheadrightarrow} & S' \end{array}$$

We consider now the case where C_2 and C_2' are not groups. The argument for the other possibility is similar (and simpler), and we omit it. Let $G_r(e_2)=T_2$, and $G_r'(e_2')=T_2'$. Let $d\colon T_2 \twoheadrightarrow T_2'$ be an isomorphism with the property: $t_i \rightarrow e_2 \Rightarrow d(t_i) \rightarrow e_2'$, for $\{t_i\}\subset T_2$. Let $c\colon H_1 \twoheadrightarrow H_1'$ be an isomorphism, where $H_1=G_l(e_1)$, $H_1'=G_l'(e_1')$, and let c have the property: $t_i \rightarrow e_1 \Rightarrow c(t_i) \rightarrow e_1'$, where $\{t_i\}\subset T_1$. Then the map, $c|T_1\times d$ is an isomorphism of H onto H' which induces an isomorphism, $b\colon R \twoheadrightarrow R'$. Consequently, the map, $c\times b$ is an isomorphism of G onto G'. We must show that $c|T_1\times d$ extends to an isomorphism of H_1^- onto $(H_1')^-$. However, this follows from the Remark in the proof of Theorem 13 and the fact that H_1^- is isomorphic to $(H_1')^-$.

THEOREM 15. *If S has no zero, then S is a half-space semigroup. If, in addition, S has empty radical, then S is isomorphic either to one of the semigroups mentioned in Theorem* 10, *or to one of the examples constructed in the proof of Theorem* 7.1 *of* [8].

Proof. The first statement is Theorem 12. The rest of the theorem is merely the result of checking the statements of Theorems 6, 8 through 10, 13, 14; and the collection of examples referred to in [8].

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