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SEMIGROUPS WITH MIDUNITS

BY

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ABSTRACT. A semigroup has a midunit u if $aub = ab$ for all a and b. It is the purpose of this paper to explore semigroups with midunits, and to find an analog to the group of units. In addition, two constructions will be given which produce some semigroups with midunits, and an abstract characterization will be made of certain of these semigroups.

Semigroups with a two-sided unit have often been considered in the theory of algebraic and topological semigroups. However, there are many semigroups that do not possess a two-sided unit, but do in fact contain midunits, for example, rectangular groups and $\mathfrak{M}$-inversive semigroups (see Yamada [11]).

After discussing the basic properties of algebraic semigroups with midunits, we give generalizations of the Bruck-Reilly construction and of the notion of rectangular groups, both of which produce new examples of semigroups with midunits. As a consequence of these constructions, we find that every (regular) semigroup can be embedded in a simple (regular) semigroup with midunits (which are neither left nor right units).

Finally, a characterization is given of all regular semigroups whose idempotents form a rectangular semigroup over a band with unit, that is, $E_S \cong B \times E$, where B is a rectangular band and E is a band with unit. The result is an extension of one by Lallement and Petrich [5] and by Warne [10], and essentially states that such a semigroup can be lifted from its idempotents and possibly a more accessible semigroup.

1. Properties of semigroups with midunits. In general, the notation of [2] will be followed. For a semigroup S, S^1 is the semigroup S with a unit attached; E_S is the set of idempotents of S with $\leq$ defined as the usual partial order on E_S, i.e., $e \leq f$ if $ef = fe = e$. If S is regular and E_S is a subsemigroup of S then S is called orthodox; if $S = E_S$ then S is a band. For an element x in S, x' is an inverse of x if $xx'x = x$ and $x'xx' = x'$.

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Let S be a semigroup. An element u is a midunit if $aub = ab$ for all a, b in S; u is a left (right) unit if $ua = a$ $(au = a)$ for all a in S; u is a unit if it is both a left and right unit of S. A midunit is said to be *proper* if it is neither a left nor right unit. Although one-sided units must be idempotent, in general, midunits need not be.

Lemma 1.1. *If a and b are midunits of a semigroup S then ab is an idempotent midunit of S.*

Lemma 1.2. *Let S be a regular semigroup with midunits. Then*

(i) *the midunits of S form a band, and the midunits are maximal idempotents of S;*

(ii) *if e is a midunit, then f is a midunit if and only if f is an inverse of e;*

(iii) *if x has an inverse x' with xx' $(x'x)$ a midunit, then xy (yx) is a midunit for any inverse y of x.*

Proof. (i) If a is a midunit of S, then a has an inverse a' and $a^2 = a(aa'a) = aa'a = a$, so by Lemma 1.1, the midunits form a band. If u is a midunit and $u \leq g$, then $u = ug = gu$, so $u = gug = g^2 = g$.

(ii) Let e be a midunit. If f is a midunit also, then clearly $efe = e$ and $fef = f$. On the other hand, if f is an inverse of e, then $afb = aefeb = aeb = ab$, for all a, b in S.

(iii) This follows from (ii) and the fact that if x' and y are inverses of x, then xx' $(x'x)$ is an inverse of xy (yx).

It is not difficult to see that if a semigroup has a left (right) unit, then every midunit is also a left (right) unit, and thus a semigroup with a two-sided unit has exactly one midunit. There is a relatively large class of regular semigroups for which midunits coincide with units. For x in S, let I_x denote the set of inverses of x.

Lemma 1.3. *Let S be a regular semigroup with midunit u. For e in E_S, $I_e = I_{ue} = I_{eu}$.*

Proof. Let e be an idempotent and let x be an inverse of e, i.e., $xex = x$ and $exe = e$. Then $x(ue)x = xex = x$, and $(ue)x(ue) = uexe = ue$. Thus, x is in I_{ue}. Conversely, let y be an inverse of ue. Then $yey = yuey = y$, and $eye = e(ueyue) = eue = e$. Hence $I_e = I_{ue}$.

A regular semigroup S is a weakly inverse semigroup, if for x, y in S, $I_x = I_y$ implies that $x = y$. These semigroups have been discussed in [1], and of course include the class of inverse semigroups.

Proposition 1.4. *For the class of weakly inverse semigroups, every midunit is a two-sided unit.*

Proof. This follows directly from Lemma 1.3.

The following classes of bands and semigroups play an important part in the structure of semigroups with midunits. For semigroups S and T, $S \times T$ is the usual direct product of S and T.

Proposition 1.5. *Let B be a semigroup. The following are equivalent:*

(i) $xyx = x$ *for all* x, y *in* B.

(ii) *For all* x, y *in* B, $xy = yx$ *implies* $x = y$.

(iii) $B \cong L \times R$, *where* L *is a left zero semigroup and* R *is a right zero semigroup.*

A semigroup satisfying any of the properties in Proposition 1.5 is called a rectangular band. Such bands are discussed in Chapter II of [7]. On any set L (R), multiplication can be defined by $xy = x$ $(xy = y)$, making L (R) a left (right) zero semigroup. Consequently, for any cardinal number α, rectangular bands with α elements exist.

Proposition 1.6. *Let S be a semigroup. The following are equivalent:*

(i) S *is a regular semigroup with* E_S *a rectangular band.*

(ii) S *is an orthodox completely simple semigroup.*

(iii) $S \cong L \times G \times R$, *where* L *is a left zero semigroup,* R *a right zero semigroup, and* G *a group.*

(iv) S *is isomorphic to the direct product of a rectangular band and a group.*

From property (iv), such a semigroup satisfying Proposition 1.6 is called a rectangular group. For the proof of Proposition 1.6, see Chapter IV of [7].

Associated with every semigroup having at least one midunit is a special type of ideal extension. For a complete discussion, see Chapter III of [7] or §3 of [8].

Let S be a semigroup, and for every x in S associate a set Z_x with the properties: The Z_x are pairwise disjoint and $Z_x \cap S = \{x\}$. The set $V = \bigcup_{x \in S} Z_x$ with multiplication $*$ defined by $x * y = ab$ if $x \in Z_a$, $y \in Z_b$, is called an *inflation* of S.

The following lemma is a paraphrase of Proposition III 4.9 of [7].

Lemma 1.7. *A semigroup V is an inflation of a semigroup S if and only if $V = S \cup Q$ where Q is a set disjoint from S and θ: $Q \longrightarrow S$ is a mapping, with multiplication $*$ defined by*

$$a * b = \begin{cases} a(b\theta) & \text{if } a \in S,\ b \in Q, \\ (a\theta)b & \text{if } a \in Q,\ b \in S, \\ (a\theta)(b\theta) & \text{if } a,\ b \in Q, \\ ab & \text{if } a,\ b \in S. \end{cases}$$

Theorem 1.8. *Let S be a semigroup with at least one midunit. Let*

$$M_S = \{x \in S \mid x \text{ is a midunit or } x \text{ has inverse } x' \text{ with } xx',\ x'x \text{ midunits}\}.$$

Then M_S is a subsemigroup of S which is an inflation of a rectangular group.

Proof. Let R and Z be defined by

$$R = \{x \in S \mid x \text{ has an inverse } x' \text{ with } xx',\ x'x \text{ midunits}\},$$

and

$$Z = \{x \in S \mid x \text{ is a midunit, } x^2 \neq x\}.$$

Then clearly, $M_S = R \cup Z$ and $R \cap Z = \emptyset$.

Now R is a subsemigroup of S, since if x, y are in R there exist x', y' so that xx', $x'x$, yy', $y'y$ are all midunits. From this it is easy to show that $y'x'$ is an inverse of xy and further, $(xy)(y'x') = xx'$, and $(y'x')(xy) = y'y$. That is, xy is in R.

If a and b are both in Z then by Lemma 1.1, ab is an idempotent midunit in R. If x is in R and a is in Z, then xx' is a midunit, and hence $a(xx')$ is an idempotent midunit of R. Consequently, $ax = (axx')x$ is in R. Similarly, it can be shown that xa is in R. Therefore, M_S is a subsemigroup of S which is an ideal extension of R.

That R is a rectangular group can be shown by using a proof similar to that of Yamada in [11], or by using Lemma 1.2 and Proposition 1.6 that R is indeed a completely simple semigroup whose idempotents form a band.

Define θ: $Z \longrightarrow R$ by $a\theta = a^2$. Then for a, b in Z, $(a\theta)(b\theta) = a^2b^2 = ab$, since ab is a midunit. For x in R, $ax = aax = (a\theta)x$ and $xa = x(a\theta)$. Therefore, M_S is an inflation of the rectangular group R.

The subsemigroup M_S will henceforth be called the *semigroup of midunits* of S due to its obvious analogy to the group of units of a semigroup with a two-sided unit.

Corollary 1.9. *For S a regular semigroup,*

$$M_S = \{x \in S \mid \textit{for an inverse } x' \textit{ of } x,\ xx',\ x'x \textit{ are midunits}\}$$

is a subsemigroup of S which is a rectangular group.

The next characterization was given by Yamada in [11].

Corollary 1.10. *In a semigroup S every element is a midunit if and only if S is an inflation of a rectangular band.*

Proof. The direct part follows from Theorem 1.8 and Lemma 1.2(i); the converse is not difficult, using the fact that an inflation of a rectangular band B is completely determined by a mapping of the set Q into B.

Corollary 1.11. *Every idempotent of a regular semigroup S is a midunit if and only if S is a rectangular group.*

2. Construction of some semigroups with midunits. In this section we will give two constructions which yield semigroups with midunits. In the first case, a generalization of the Bruck-Reilly construction will be given to obtain a class of simple semigroups with midunits; in the second, we will define a generalization of the notion of rectangular group. Using these two types of semigroups, it can be shown that every (regular) semigroup can be embedded in a simple (regular) semigroup with proper midunits. It will also be proved that if the two constructions are applied consecutively, the operations are in a sense commutative.

Let A be a semigroup with midunits, M_A its semigroup of midunits, and θ: $A \longrightarrow M_A$ a homomorphism ($\theta^0 = i_A$). Let S be the set $I \times A \times I$, where I is the set of nonnegative integers, and define multiplication on S by

$$(i, a, j)(m, b, n) = \begin{cases} (i, a(b\theta^{j-m}), j+n-m) & \text{if } m \leq j, \\ (i+m-j, (a\theta^{m-j})b, n) & \text{if } m > j. \end{cases}$$

Then $S = \mathfrak{B}(A, \theta)$ is called a *Bruck-Reilly extension on* A.

The proof given by Reilly [9] that S is a semigroup in case A is a group, carries over to this more general situation. Clearly, for any semigroup S with at least one midunit u, such a construction always exists, since θ: $A \longrightarrow \{u^2\}$ is a satisfactory homomorphism. In addition, ψ: $A \longrightarrow \mathfrak{B}(A, \theta)$ defined by ψ: $a \longrightarrow (0, a, 0)$ is an embedding map. The following properties hold (see [6]).

Proposition 2.1. *Let* $S = \mathfrak{B}(A, \theta)$. *Then*:

(i) $E_S = \{(m, e, m) | m \in I, e \in E_A\}$.

(ii) $\{(0, u, 0) | u$ *is a midunit of* $A\}$ *is the set of midunits of* S.

(iii) S *is simple if* $A^2 = A$.

(iv) S *is bisimple if and only if* A *is bisimple.*

(v) S *is regular* (*inverse*) *if and only if* A *is regular* (*inverse*).

(vi) S *is orthodox if and only if* A *is orthodox.*

Note that under the embedding of A into $\mathfrak{B}(A, \theta)$, the midunits of $\mathfrak{B}(A, \theta)$ are contained in that of A. The construction yields the next result.

Theorem 2.2. *Every* (*regular*) *semigroup with proper midunits can be embedded in a simple* (*regular*) *semigroup with proper midunits.*

Recalling that a rectangular band is closely connected with the possession of midunits, we next construct semigroups with midunits by attaching a rectangular band to any semigroup with midunit.

Let S be a semigroup, L a left zero semigroup, R a right zero semigroup. The direct product $L \times S \times R$ is called a *rectangular semigroup over* S.

There is an obvious embedding of S into $L \times S \times R$. For the remainder of this paper, L will always denote a left zero semigroup, and R a right zero semigroup. If S is a group, $L \times S \times R$ is a rectangular group.

Proposition 2.3. *Let* $T = L \times S \times R$. *Then*:

(i) T *is regular if and only if* S *is regular.*

(ii) T *is (bi)simple if and only if* S *is (bi)simple.*

(iii) $E_T = \{(a, e, \alpha) | a \in L,\ e \in E_S,\ \alpha \in R\}$.

(iv) *If* S *has midunits, then* $\{(a, u, \alpha) | a \in L,\ \alpha \in R,\ u$ *midunit of* $S\}$ *is the set of all midunits of* T, *and* $M_T = L \times M_S \times R$. *Furthermore, the midunits are proper if* $|L| > 1$, $|R| > 1$.

Proof. The proofs are direct, using the fact that L and R are bisimple bands.

We may now state the analog to the theorem that every semigroup can be embedded in a simple semigroup with unit (see Corollary 3.2 of [6]).

Theorem 2.4. *Every (regular) semigroup can be embedded in a simple (regular) semigroup with proper midunits.*

Proof. Let S be a semigroup, and let θ be the mapping of S^1 into $\{1\}$. Then $\mathfrak{B}(S^1, \theta)$ is a simple semigroup with unit. Let L and R be two sets, each with more than one element, with left zero and right zero multiplication, respectively. The semigroup $L \times \mathfrak{B}(S^1, \theta) \times R$ is a simple semigroup with proper midunits, and

$$S \longrightarrow S^1 \longrightarrow \mathfrak{B}(S^1, \theta) \longrightarrow L \times \mathfrak{B}(S^1, \theta) \times R$$

is an embedding map, where each individual map is the canonical one. By Proposition 2.1(v) and Proposition 2.3(i), S is regular if and only if $L \times \mathfrak{B}(S^1, \theta) \times R$ is regular.

In the statement of the above theorem, midunit may be replaced by left or right unit, with only the minor alteration in the proof that L or R be trivial.

Theorem 2.5. *Let* S *be a semigroup with midunits,* θ *a homomorphism of* S *into* M_S. *The semigroups* $U = L \times \mathfrak{B}(S, \theta) \times R$ *and* $V = \mathfrak{B}(L \times S \times R, \theta_0)$, *where* $(a, g, \alpha)\theta_0 = (a, g\theta, \alpha)$, *are isomorphic to one another.*

Proof. First the semigroups U and V can be defined since S has at least one midunit. Now define ψ: $U \longrightarrow V$ by $(a, (i, s, j), \alpha)\psi = (i(a, s, \alpha), j)$. Obviously, ψ is a one-to-one mapping of U onto V. Let $(a, (i, s, j), \alpha)$, $(b, (m, t, n), \beta)$ be in U. Then, if $m \leq j$,

$$[(a, (i, s, j), \alpha)(b, (m, t, n), \beta)]\psi = (a, (i, s, j)(m, t, n), \beta)\psi$$

$$= (a, (i, s(t\theta^{j-m}), j + n - m), \beta)\psi = (i, (a, s(t\theta^{j-m}), \beta), j + n - m)$$

$$= (i, (a, s, \alpha)(b, t\theta^{j-m}, \beta), j + n - m)$$

$$= (i, (a, s, \alpha)[(b, t, \beta)\theta_0^{j-m}], j + n - m) = (i, (a, s, \alpha), j)(m, (b, t, \beta), n)$$

$$= (a, (i, s, j), \alpha)\psi(b, (m, t, n), \beta)\psi.$$

If $m > j$, the proof is analogous. Therefore, ψ is a homomorphism.

3. **Rectangular semigroups.** In this section a characterization will be given of all regular semigroups whose idempotents form a rectangular semigroup over a band with unit. However, several preliminary results are needed first.

Lemma 3.1. *Let S be a regular semigroup and x be in S. If e and f are idempotents with $x\mathfrak{R}e$, $x\mathfrak{L}f$, then there exists an inverse x' of x with $xx' = e$, $x'x = f$.*

Proof. Let a be an inverse of x. Then $x' = fae$ satisfies the desired conditions.

Lemma 3.2. *On a rectangular band, $\mathfrak{L}$ and $\mathfrak{R}$ are congruences with $S/\mathfrak{L}$ a right zero semigroup and $S/\mathfrak{R}$ a left zero semigroup.*

Proof. This is clear by considering the characterization of a rectangular band as the direct product $L \times R$, where $\mathfrak{R}$-classes are of the form $\{(a, \alpha) | \alpha \in R\}$, $a \in L$ and $\mathfrak{L}$-classes are the sets $\{(a, \alpha) | a \in L\}$, $\alpha \in R$.

Lemma 3.3. *Let F be a band that is the direct product of a rectangular band B and E, a band with unit u. The following properties hold:*

(i) $(\alpha, s)\mathfrak{R}(\beta, t)$ *if and only if* $\alpha \mathfrak{R} \beta$ *and* $s \mathfrak{R} t$;

(ii) $(\alpha, u)\mathfrak{R}(\beta, u)$ *if and only if* $(\alpha, s)\mathfrak{R}(\beta, s)$ *for all* s *in* E;

(iii) $(\alpha, s) \leq (\beta, t)$ *if and only if* $\alpha = \beta$ *and* $s \leq t$.

Proof. Parts (i) and (ii) are true in general for direct products, and part (iii) follows from the fact that idempotents of a rectangular band are primitive.

Theorem 3.4. *A semigroup S is isomorphic to a rectangular semigroup over an orthodox semigroup with unit if and only if S is regular with E_S isomorphic to a rectangular semigroup over a band with unit.*

Proof. If S is isomorphic to $L \times T \times R$, T an orthodox semigroup with unit, then certainly, E_S is isomorphic to $L \times E_T \times R$.

Conversely, let S be regular with E_S a rectangular semigroup over a band E with unit. That is, let $E_S = \{(\alpha, a) | \alpha \in B, a \in E\}$, where B is a rectangular band. The set of midunits of S, $P = \{(\alpha, u) | \alpha \in B\}$, where u is the unit of E, is a rectangular band, and hence on P, $\mathfrak{R}$ and $\mathfrak{L}$ are congruences. We pick a representative from each $\mathfrak{R}$ ($\mathfrak{L}$) class and call the collection L (R). By Lemma 3.2, under the induced multiplication, L is a left zero semigroup, and R is a right zero semigroup.

We fix α in B. Then let C be defined by

$$C = \{x \in S |\text{ for some inverse } x',\ xx' = (\alpha, a),\ x'x = (\alpha, b), \text{ some } a, b\}.$$

We claim that C is an orthodox regular semigroup with unit (α, u). For, if x and y are in C, then for some inverse x' of x, y' of y, xx', $x'x$, yy', $y'y$ are all in C. By [3] since S is orthodox, $y'x'$ is an inverse of xy and $y'x'xy \leq y'y$, $xyy'x' \leq xx'$. By Lemma 3.3(iii), xy is thus in C. That C is regular and in fact orthodox follows from the fact that S is orthodox and the definition of C.

Let x be in S. Then by Lemma 3.3 there is a unique $\beta \in L$, $\gamma \in R$ so that $x\mathfrak{R}(\beta, a)$, $x\mathfrak{L}(\gamma, b)$ for some a, b in E. Thus, x has an inverse x' so that $xx' = (\beta, a)$, $x'x = (\gamma, b)$. Moreover, $x = (\beta, u)(\alpha, a)x(\alpha, b)(\gamma, u)$ and $(\alpha, a)x(\alpha, b)$ is in C. For,

$$\begin{aligned}(\beta, u)(\alpha, a)x(\alpha, b)(\gamma, u) &= (\beta, u)(\alpha, a)(\beta, a)x(\gamma, b)(\alpha, b)(\gamma, u)\\ &= (\beta\alpha\beta, a)x(\gamma\alpha\gamma, b) = (\beta, a)x(\gamma, b) = x,\end{aligned}$$

and $y' = (\alpha, b)x'(\alpha, a)$ is an inverse of $y = (\alpha, a)x(\alpha, b)$, with yy', $y'y$ in C.

Define $\psi\colon S \longrightarrow L \times C \times R$ by $\psi\colon x \longrightarrow (\beta, (\alpha, a)x(\alpha, b), \gamma)$ where $x\mathfrak{R}(\beta, a)$, $x\mathfrak{L}(\gamma, b)$, $\beta \in L$, $\gamma \in R$. Now β and γ are uniquely defined since L and R are partitions of the $\mathfrak{R}$ and $\mathfrak{L}$ classes, respectively, of P, which carries over to S, via Lemma 3.3(ii). Moreover, if $x\mathfrak{R}(\beta, a)$, $x\mathfrak{R}(\beta, c)$, and $x\mathfrak{L}(\gamma, b)$, $x\mathfrak{L}(\gamma, d)$, then by Lemma 3.3(i), $a\mathfrak{R}c$ and $b\mathfrak{L}d$ in E. Hence $(\alpha, a)\mathfrak{R}(\alpha, c)$ and $(\alpha, b)\mathfrak{L}(\alpha, d)$. Consequently,

$$(\alpha, c)(\alpha, a) = (\alpha, a), \qquad (\alpha, b)(\alpha, d) = (\alpha, b),$$

and

$$\begin{aligned}(\alpha, a)x(\alpha, b) &= (\alpha, c)(\alpha, a)x(\alpha, b)(\alpha, d) = (\alpha, c)(\alpha, a)(\beta, a)x(\gamma, b)(\alpha, b)(\alpha, d)\\ &= (\alpha\beta, ca)x(\gamma\alpha, bd) = (\alpha, c)(\beta, a)x(\gamma, b)(\alpha, d) = (\alpha, c)x(\alpha, d).\end{aligned}$$

That is, ψ is a well-defined mapping into $L \times C \times R$.

To see that ψ is a homomorphism, let $xx' = (\beta, a)$, $x'x = (\gamma, b)$, $yy' = (\delta, c)$, $y'y = (\epsilon, d)$. Now $y'x'$ is an inverse of xy and $xyy'x' = (\beta, r)$, $y'x'xy = (\epsilon, s)$ for some $r \leq a$, $s \leq d$. Thus, $(xy)\psi = (\beta, (\alpha, r)xy(\alpha, s), \epsilon)$. On the other hand,

$$\begin{aligned}(x\psi)(y\psi) &= (\beta, (\alpha, a)x(\alpha, b), \gamma)(\delta, (\alpha, c)y(\alpha, d), \epsilon)\\ &= (\beta, (\alpha, a)x(\alpha, b)(\alpha, c)y(\alpha, d), \epsilon)\\ &= (\beta, (\alpha, a)x(\gamma, b)(\alpha, b)(\alpha, c)(\delta, c)y(\alpha, d), \epsilon)\\ &= (\beta, (\alpha, a)x(\gamma\delta, bc)y(\alpha, d), \epsilon) = (\beta, (\alpha, a)x(\gamma, b)(\delta, c)y(\alpha, d), \epsilon)\\ &= (\beta, (\alpha, a)xy(\alpha, d), \epsilon) = (\beta, (\alpha, a)(\beta, r)xy(\epsilon, s)(\alpha, d), \epsilon)\\ &= (\beta, (\alpha\beta, r)xy(\epsilon\alpha, s), \epsilon) = (\beta, (\alpha, r)(\beta, r)xy(\epsilon, s)(\alpha, s), \epsilon)\\ &= (\beta, (\alpha, r)xy(\alpha, s), \epsilon) = (xy)\psi.\end{aligned}$$

Now if $x\psi = y\psi$, then $(\beta, (\alpha, a)x(\alpha, b), \gamma) = (\delta, (\alpha, c)y(\alpha, d), \epsilon)$. That is, $\beta = \delta$, $\gamma = \epsilon$, and $z = (\alpha, a)x(\alpha, b) = (\alpha, c)y(\alpha, d) = w$. Since $zz' = (\alpha, a)$, $ww' = (\alpha, c)$, and $z'z = (\alpha, b)$, $w'w = (\alpha, d)$, it must be that $(\alpha, a) = (\alpha, c)$ and $(\alpha, b) = (\alpha, d)$. That is, $a = c$ and $b = d$. Hence,

$$\begin{aligned} x &= (\beta, a)(\alpha, a)(\beta, a)x(\gamma, b)(\alpha, b)(\gamma, b) = (\beta, a)(\alpha, a)x(\alpha, b)(\gamma, b) \\ &= (\beta, a)(\alpha, a)y(\alpha, b)(\gamma, b) = (\beta, a)(\alpha, a)(\beta, a)y(\gamma, b)(\alpha, b)(\gamma, b) \\ &= (\beta, a)y(\gamma, b) = y. \end{aligned}$$

Therefore, ψ is one-to-one.

For (β, t, γ) in $L \times C \times R$, t has an inverse t' in C with $tt' = (\alpha, a)$, $t't = (\alpha, b)$ for some a, b in E. Let $x = (\beta, a)t(\gamma, b)$. It is not difficult to see that $xx' = (\beta, a)$, $x'x = (\gamma, b)$, where $x' = (\gamma, b)t'(\beta, a)$, and $x\psi = (\beta, t, \gamma)$.

It is clear from the proof of Theorem 3.4 that $E_C \simeq E$, and thus,

(1) $$S \simeq L \times C \times R \text{ if and only if } E_S \simeq L \times E_C \times R,$$

where C is an orthodox regular semigroup with unit.

For the special case that S is bisimple and E is a semilattice with unit, the above theorem was proved by Lallement and Petrich in [5], and by Warne in [10] in case S is regular and E is a semilattice (not necessarily with unit).

Corollary 3.5. *A semigroup is isomorphic to a rectangular semigroup over an orthodox (bi)simple regular semigroup with unit if and only if S is a (bi)simple regular semigroup with E_S isomorphic to a rectangular semigroup over a band with unit.*

Proof. This follows directly from (1) and Proposition 2.3.

Recall that an ω-chain is a semilattice of the form $\{e_0 > e_1 > \cdots\}$.

Corollary 3.6. *The following are equivalent for a semigroup S:*

(i) *S is a simple regular semigroup with E_S isomorphic to a rectangular semigroup over an ω-chain;*

(ii) *$S \simeq L \times \mathfrak{B}(V, \theta) \times R$, where V is a finite chain of groups;*

(iii) *$S \simeq \mathfrak{B}(L \times V \times R, \theta_0)$, where V is a finite chain of groups and θ_0 as in Theorem* 2.5.

Proof. The equivalence of the first two statements follows from Corollary 3.5 and Theorem 3 of [4] (see also Theorem 3.6 of [6]). Due to Theorem 2.5, (ii) and (iii) are equivalent.

Other properties such as complete regularity also carry over from C to S in the same manner as in Corollary 3.5.

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