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SEMILINEAR ELLIPTIC EQUATIONS WITH SUBLINEAR INDEFINITE NONLINEARITIES

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Abstract. We prove existence, multiplicity and bifurcation results for a family of semilinear Neumann problems with nonlinear terms that are indefinite in sign and exhibit sublinear growth near zero. The solutions are non-negative, but the combined effect of indefiniteness and the non-Lipschitz character of the nonlinear term yields solutions which may vanish on large sets. Combining variational methods with bifurcation analysis and the sub- and super-solution technique, we produce solutions in a special class: positive on the set where the nonlinear term is positive. The family of problems we consider includes an equation used in population ecology.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-smooth $\partial\Omega$, and consider the Neumann boundary value problems,

$$\begin{cases} -\Delta u - \lambda u = a(x)u^q + \gamma u^p, \\ u \geq 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \qquad (1.1)_{\lambda,\gamma}$$

with $0 < q < 1 < p$ and parameters $\lambda \in \mathbb{R}$, $\gamma \geq 0$. We also assume $a(x) \in C^\beta(\Omega)$ for some $\beta \in (0,1]$.

In case $\lambda = \gamma = 0$, solutions u of $(1.1)_{\lambda=0,\gamma=0}$ correspond to stationary solutions of the parabolic equation

$$\frac{\partial \rho}{\partial t} = \Delta(\rho^m) + a(x)\rho, \quad m > 1, \qquad (1.2)$$

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under the transformation $\rho = u^{1/m}$. The evolution equation (1.2) has been proposed as a model for the population density $\rho(x,t)$ of a single mobile species in a heterogeneous environment. (See Namba [17] or Gurtin & MacCamy [15].) In (1.2) the coefficient $a(x)$ represents the local growth rate, and in [17], $a(x)$ is assumed to change sign and to take only negative values outside of a fixed region. Indeed, Bandle, Pozio, & Tesei [9] observed that stationary solutions to (1.2) can exist only if a changes sign, and that the condition

$$\int_\Omega a(x)\,dx < 0 \tag{1.3}$$

is necessary and sufficient for the existence of solutions which are strictly positive in the set where $a(x) > 0$.

When $\gamma > 0$, the nonlinearity in $(1.1)_{\lambda,\gamma}$ exhibits sublinear growth near $u = 0$ and superlinear growth for large u. Boundary-value problems incorporating this structure have been recently studied by many authors, beginning with the work of Ambrosetti, Brezis, & Cerami [5] on the Dirichlet problem for the constant-coefficient equation,

$$-\Delta u = \nu u^q + u^p, \tag{1.4}$$

with $0 < q < 1 < p$ and $\nu \geq 0$. In [5] it is observed that the combined effect of sublinear and superlinear terms yields existence and multiplicity results which are surprisingly different from the case when only one of the two terms is present. (See also Ambrosetti, Garcia Azorero, & Peral [6], Bartsch & Willem [10], Huang [16], and Tehrani [21] for related results.)

The novelty in problem $(1.1)_{\lambda,\gamma}$ is that the indefiniteness of $a(x)$ and the non-Lipschitz character of u^q $(0 < q < 1)$ near $u = 0$ can combine to allow a much richer solution set. From the technical point of view this is because the strong maximum principle cannot be applied in the set where $a(x) < 0$, giving rise to solutions which may vanish identically on large regions within Ω. Such solutions are expected (and even desired) in the application to population ecology. For example, Namba [17] presents an example in $\Omega = \mathbb{R}^1$ of a stationary solution with compact support in an interval containing the "viable habitat", the set where $a(x) > 0$. In this paper we will primarily consider this class of solutions (those which are positive on the viable habitat,) although the existence of solutions which vanish identically in some connected component of the set $\Omega^+ = \{x : \ a(x) >$

$0\}$ is an extremely interesting question. Some examples of solutions with "dead cores" (those which do vanish in some components of Ω^+) and an analysis of the structure of the solution sets appear in [8],[9]. However, a more complete study of existence, nonexistence, and multiplicity of solutions with dead cores is complicated by the fact that these solutions are not local minima of the usual variational functional (see Remark 4.2), and do not seem to be created by bifurcation from trivial solutions (see Theorem 3.1.)

The approach of this paper is in the spirit of the work of Alama & Tarantello [1], [4] on semilinear elliptic equations with superlinear indefinite nonlinearities. By introducing the parameters λ and γ the problem $(1.1)_{0,0}$ is embedded in a family of problems, each with similar variational structure and related by monotonicity with respect to the two parameters. The approach combines bifurcation theory, variational methods, and sub- and supersolutions. In contrast to the treatment in [5], the case $\gamma > 0$ will be treated as a perturbation of the $\gamma = 0$ case.

In order to present a precise statement concerning existence and multiplicity of solutions to $(1.1)_{\lambda,\gamma}$ we must introduce our hypotheses concerning the function $a(x)$. Define the (open) sets, $\Omega^+ = \{x \in \Omega : a(x) > 0\}$, $\Omega^- = \{x \in \Omega : \ a(x) < 0\}$, $\Omega^{0+} = \operatorname{int}(\{x \in \Omega : a(x) \geq 0\})$. We assume that a changes sign in Ω:

$$\Omega^+ \neq \emptyset, \qquad \Omega^- \neq \emptyset. \tag{1.5}$$

We must make more stringent hypotheses on the shape of the set Ω^{0+}:

$$\begin{cases} \Omega^{0+} \text{ has } m < \infty \text{ connected components, } \Omega^{0+} = \cup_{k=1}^m \Omega_k^{0+} \\ \qquad \text{and for each } k = 1, \ldots, m,\ \partial\Omega_k^{0+} \text{ is } C^2\text{-smooth.} \end{cases} \tag{1.6}$$

Finally, the zero set of $a(x)$ should always be connected to Ω^+:

$$\Omega_k^{0+} \cap \Omega^+ \neq \emptyset \ \text{ for every } k = 1, \ldots, m. \tag{1.7}$$

Assumption (1.6), which also appears in Bandle, Pozio, & Tesei [9] for the case $\lambda = 0 = \gamma$, allows us to apply the Hopf Lemma in the components Ω^{0+}. Hypothesis (1.7) will be essential in proving that solutions obtained by the sub- and supersolution technique correspond to local minima of an appropriate functional. In the case of superlinear problems (treated in [4] for example) or when $a(x) > 0$ (as in [5], [21]) this is done via the observation of Brezis & Nirenberg [11] (see also [3]) that for well-behaved functionals, local minimizers with respect to the C^1-topology are also minimizers with respect to the H^1-topology. In our situation the solutions we obtain may coincide with the subsolution or the supersolution at some points inside Ω, and hence it is not obviously true that the constrained minimizers (lying between a subsolution and a supersolution) are C^1-minimizers. A more careful direct analysis in an H^1-neighborhood is required. This task is carried out in Lemma 5.2, which is the most important step in solving the problem.

For the sublinear problem ($\gamma = 0$) we prove:

Theorem 1.1. *Assume* $0 < q < 1$, $\gamma = 0$, *and* $a \in C^\beta(\overline{\Omega})$ *satisfies* (1.5), (1.6), *and* (1.7).

(i) *There exists a number* $-\infty < \Lambda_0 < +\infty$ *such that for all* $\lambda \in (-\infty, \Lambda_0)$, $(1.1)_{\lambda,0}$ *admits a solution* $u_{\lambda,0} \geq 0$ *with the additional property*

$$u(x) > 0 \text{ for all } x \in \overline{\Omega^{0+}} \,. \tag{$*$}$$

If $\lambda \leq 0$, *this is the unique solution which is positive in* $\overline{\Omega^{0+}}$.

(ii) *If* $\int_\Omega a(x)\,dx \geq 0$, $\Lambda_0 = 0$, *and the problem* $(1.1)_{\lambda,0}$ *admits no solutions satisfying* $(*)$ *when* $\lambda \geq \Lambda_0 = 0$.

(iii) *If* $\int_\Omega a(x)\,dx < 0$, *then* $\Lambda_0 > 0$, *and*

(a) *the problem* $(1.1)_{\lambda,0}$ *admits a solution* $u_{\Lambda_0,0}$ *at* $\lambda = \Lambda_0$ *satisfying* $(*)$, *and there is no such solution for* $\lambda > \Lambda_0$.

(b) *for all* $\lambda \in (0, \Lambda_0)$ *the problem* $(1.1)_{\lambda,0}$ *admits a second solution* $w_{\lambda,0}$ *with* $w_{\lambda,0} \geq u_{\lambda,0}$ *in* Ω.

It is typical that a sign condition for some integral of $a(x)$ enter into the solvability of indefinite problems at (or near) the first eigenvalue of the linear part. Kazden and Warner arrived at (1.3) as a necessary condition for existence of solutions to the equation of prescribed scalar curvature on a compact manifold, and the same condition was found to be necessary and sufficient for existence of solutions to $(1.1)_{0,0}$ by Bandle, Pozio, and Tesei. For the Dirichlet problem with superlinear nonlinearity, an analogous condition was derived by Alama and Tarantello [1]. Here, as in the superlinear problem considered in [1], the role played by the sign of $\int_\Omega a(x)\,dx$ is made clearer by a bifurcation analysis at $\lambda = 0$, the first eigenvalue of the Neumann Laplacian in Ω. In Theorem 3.1 we adapt an argument of Ambrosetti and Hess [7] to show that solutions bifurcate from infinity at $\lambda = 0$. Just as in [1], the sign of $\int_\Omega a(x)\,dx$ determines the direction of bifurcation.

We now turn to the more general case.

Theorem 1.2. *Assume* $0 < q < 1 < p$, $p < \frac{N+2}{N-2}$ *if* $N \geq 3$, *and* $a \in C^\alpha(\overline{\Omega})$ *satisfies* (1.5), (1.6), *and* (1.7).

(i) *For every* $\gamma > 0$ *there exists a number* $-\infty < \Lambda_\gamma \leq \Lambda_0$ *such that for every* $\lambda \in (-\infty, \Lambda_\gamma]$, $(1.1)_{\lambda,\gamma}$ *admits a solution* $u_{\lambda,\gamma}$ *satisfying* $(*)$. *If* $\lambda > \Lambda_\gamma$, *no such solution exists.*

(ii) *For every* $\lambda < \Lambda_0$, *there exists a number* $0 < \Gamma_\lambda < +\infty$ *such that* $(1.1)_{\lambda,\gamma}$ *admits a solution* $\tilde{u}_{\lambda,\gamma}$ *satisfying* $(*)$. *If* $\gamma > \Gamma_\lambda$, *no such solution exists.*

(iii) *If* $\lambda > \Lambda_0$, *then there does not exist any solution to* $(1.1)_{\lambda,\gamma}$ *which is positive on* Ω^{0+}, *for any* $\gamma \geq 0$.

(iv) *For every* $\gamma \geq 0$ *and every* $\lambda < \Lambda_\gamma$ *there exists a second solution* $w_{\lambda,\gamma} \geq u_{\lambda,\gamma}$ *for the problem* $(1.1)_{\lambda,\gamma}$.

Note that the sign condition $\int_\Omega a(x)\,dx < 0$ enters into Theorem 1.2 as a necessary condition for solutions with $\lambda > 0$ via the relation $\Lambda_\gamma \leq \Lambda_0$.

We obtain solutions for $(1.1)_{\lambda,\gamma}$ with $\gamma > 0$ by treating the equation as a perturbation of the $\gamma = 0$ problem, then using the monotonicity of the variational functional with respect to the parameter γ to obtain global information about the solution sets. We remark that Theorem 1.2 is consistent with the results of [5] for equation (1.4) with Dirichlet conditions imposed on $\partial\Omega$. Note that in (1.4) the small parameter ν appears as the coefficient of u^q, but the change of dependent variable $u \to \tilde{u} = \gamma^{-(\frac{1}{p-1})}u$ converts a solution u of $(1.1)_{0,\gamma}$ into a solution $\tilde{u}$ of (1.4) with $\nu = \gamma^{\frac{1-q}{p-1}}$. Since $0 < q < 1 < p$, small γ in $(1.1)_{0,\gamma}$ corresponds to small ν in (1.4).

Although we do not have uniqueness for solutions of $(1.1)_{\lambda,\gamma}$ satisfying $(*)$ when either $\gamma > 0$ or $\lambda > 0$, there is a *minimal* solution for each such λ, γ. Since the result is tangential to the above theorems we have relegated it to an Appendix. (See Proposition 4.9.)

The paper is organized as follows: in section 2 we discuss nonexistence results and the consequences of $(*)$. Section 3 presents a bifurcation result for $(1.1)_{\lambda,\gamma}$ with $\gamma = 0$, including the observation that $(*)$ forces solutions to bifurcate with $\lambda > 0$. The values Λ_γ and Γ_λ are introduced in section 4, and the interval of existence is established via the variational formulation of Perron's method. In section 5 we prove that solutions defined by the variational Perron's method are in fact local minima of the associated functional, and in Section 6 the second solution is obtained as a mountain-pass from these local minima.

2. Preliminaries. Many of the statements of Theorems 1.1 and 1.2 hold for nonlinearities which are not powers, but which behave like powers near zero or infinity. In order to present some slightly more general results we introduce the following family of Neumann problems:

$$\begin{cases} -\Delta u - \lambda u = a(x) f(u) + \gamma g(u), \\ u \geq 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \qquad (N)_{\lambda,\gamma}$$

Throughout the entire paper we will assume (without further mention) the following basic hypotheses:

$$\begin{cases} a(x) \in C^{\beta}(\overline{\Omega}),\ 0 < \beta \leq 1; \\ \Omega \subset \mathbb{R}^N \text{ is a bounded domain with } C^2\text{-smooth } \partial\Omega, \\ \qquad \text{and (1.5), (1.6), (1.7) are satisfied;} \\ f : \mathbb{R} \to \mathbb{R} \text{ is differentiable on } \mathbb{R} \setminus \{0\} \\ \qquad \text{and (locally) Hölder continuous on } \mathbb{R}; \\ g : \mathbb{R} \to \mathbb{R} \text{ is differentiable on } \mathbb{R} \text{ and } g(0) = 0 = g'(0); \\ f(u) > 0, \quad g(u) > 0 \quad \text{for all } u > 0. \\ f(u) = 0 = g(u) \quad \text{for all } u \leq 0. \end{cases} \qquad (2.1)$$

Note that under the above hypotheses any (weak) solution $u \in H^1(\Omega) \cap L^\infty(\Omega)$ of $(N)_{\lambda,\gamma}$ is a classical solution: $u \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

The following lemma is a slightly stronger version of Lemma 2.2 of [9] (or Lemma 2.1 of [8].)

Lemma 2.1. *Suppose u is a classical solution of $(N)_{\lambda,\gamma}$. If Ω_k^{0+} is any connected component of Ω^{0+}, then either $u \equiv 0$ in $\overline{\Omega_k^{0+}}$, or $u(x) > 0$ for all $x \in \overline{\Omega_k^{0+}}$.*

Proof. Suppose $u \not\equiv 0$ in $\overline{\Omega_k^{0+}}$. Since $u \geq 0$ satisfies $-\Delta u - \lambda u = a^+(x) f(u) + \gamma g(u) \geq 0$ in Ω_k^{0+}, the strong Maximum Principle applies and hence $u > 0$ in Ω_k^{0+}. If u attains its minimum value of zero at $x \in \partial\Omega_k^{0+}$ we apply the Hopf Lemma, to obtain $\frac{\partial u}{\partial \nu} > 0$, with ν the exterior unit normal to $\partial\Omega_k^{0+}$ at x. Clearly $x \notin \partial\Omega_k^{0+} \cap \Omega$, since $\nabla u(x) = 0$ at an interior minimum. But, if $x \in \partial\Omega_k^{0+} \cap \partial\Omega$ the exterior normal to $\partial\Omega_k^{0+}$ coincides with the exterior normal to $\partial\Omega$, and we violate the Neumann boundary condition. Hence $u > 0$ in $\overline{\Omega_k^{0+}}$. □

Following [8], [9], the previous lemma motivates us to separate the solutions of $(N)_{\lambda,\gamma}$ into classes determined by their vanishing sets. If J is a subset of the finite set $\mathcal{M} = \{1, \ldots, m\}$,

$$\mathcal{N}_J = \left\{ \begin{array}{c} u(x) \in H^1(\Omega) \cap C^0(\overline{\Omega}) \text{ such that} \\ u(x) > 0 \text{ for all } x \in \overline{\Omega_k^{0+}},\ k \in J, \text{ and} \\ u(x) \equiv 0 \text{ for all } x \in \overline{\Omega_k^{0+}},\ k \notin J \end{array} \right\}.$$

We will mainly be concerned with solutions in $\mathcal{N}_{\mathcal{M}}$, in other words solutions which are strictly positive on all of Ω^{0+}. As we will see in Section 4, these are the solutions which arise naturally as local minima of a variational functional. However, we may prove the following property about the solutions of $(N)_{\lambda,\gamma}$ lying in $\mathcal{N}_J$, $J \neq \emptyset$:

Proposition 2.2. *Suppose*

$$\lim_{u \to 0+} \frac{f(u)}{u} = +\infty, \qquad \lim_{u \to +\infty} \frac{g(u)}{u} = +\infty. \tag{2.2}$$

Let $J \subset \mathcal{M}$, $J \neq \emptyset$.
(a) *There exists $\lambda^* \in \mathbb{R}$ such that when $\lambda > \lambda^*$, $(N)_{\lambda,\gamma}$ admits no solution in $\mathcal{N}_J$ for any $\gamma \geq 0$.*
(b) *For each $\lambda < \lambda^*$ there exists $\gamma^* = \gamma^*(\lambda) \geq 0$ such that $(N)_{\lambda,\gamma}$ admits no solution in $\mathcal{N}_J$ for any $\gamma \geq \gamma^*$.*

Proof. By Lemma 2.1, if $k \in J$, then $u > 0$ in $\overline{\Omega_k^{0+}}$. Choose a disk D with $\overline{D} \subset \Omega_k^{0+}$, with the additional property that $a(x) \geq a_0$ in D, for $a_0 > 0$ a constant. Let ψ be the principal eigenfunction of the following eigenvalue problem:

$$\begin{cases} \Delta\psi = \lambda^*\psi, & \text{in } D; \\ \psi = 0, & \text{on } \partial D; \\ \psi > 0, & \text{in } D. \end{cases}$$

A simple computation shows that

$$(\lambda^* - \lambda) \int_D \psi u \, dx > \int_D (a_0 f(u) + \gamma g(u)) \, \psi \, dx. \tag{2.3}$$

Since the right side is positive, we must have $\lambda \leq \lambda^*$.

Now for fixed λ, by our hypotheses on f, g we may choose γ^* so that

$$(\lambda - \lambda^*) < a_0 \frac{f(u)}{u} + \gamma^* \frac{g(u)}{u}$$

holds for all $u > 0$. So if $\gamma \geq \gamma^*$ the inequality (2.3) can never hold. □

From the previous lemma we see that solutions to $(N)_{\lambda,\gamma}$ cannot exist for λ, γ too large, which will motivate our definition (in Section 4) of Λ_γ and Γ_λ as the least upper bound of solvability of $(N)_{\lambda,\gamma}$. We finish this section with the proof that (1.3) is necessary for the existence of solutions to $(N)_{\lambda,\gamma}$ for any $\lambda \geq 0$. This fact was already proven for $\lambda = 0 = \gamma$ in [9], and the proof presented here is a very simple extension:

Proposition 2.3. *Assume*

$$f'(u) > 0 \text{ for all } u > 0; \tag{2.4}$$

$$\frac{u}{f(u)} \text{ is bounded near } u = 0. \tag{2.5}$$

If $u \not\equiv 0$, $u \geq 0$ *is a solution to* $(P)_{\lambda,\gamma}$ *with* $\lambda \geq 0$, *then*

$$\int_{\operatorname{supp} u} a(x)\, dx < 0.$$

Proof. Divide the equation by $f(u + \varepsilon)$, $\varepsilon > 0$, and integrate by parts:

$$\begin{aligned} - \int_{\operatorname{supp} u} a(x) \frac{f(u)}{f(u+\varepsilon)}\, dx &= \lambda \int_\Omega \frac{u}{f(u+\varepsilon)}\, dx + \gamma \int_\Omega \frac{g(u)}{f(u+\varepsilon)}\, dx \\ + \int_\Omega \frac{|\nabla u|^2 f'(u+\varepsilon)}{[f(u+\varepsilon)]^2}\, dx &\geq \lambda \int_\Omega \frac{u}{f(u+\varepsilon)}\, dx + \int_{\{u \geq \delta\}} \frac{|\nabla u|^2 f'(u+\varepsilon)}{[f(u+\varepsilon)]^2}\, dx, \end{aligned}$$

for any $\delta > 0$. By Dominated Convergence we may pass to the limit to obtain

$$\int_{\operatorname{supp} u} a(x)\, dx \leq -\lambda \int_\Omega \frac{u}{f(u)}\, dx - \int_{\{u \geq \delta\}} \frac{|\nabla u|^2 f'(u)}{[f(u)]^2}\, dx \leq 0.$$

If equality holds in the above, we must have that $\nabla u \equiv 0$ in the set $\{x : u(x) \geq \delta\}$. Since $\delta > 0$ is arbitrary and u is smooth, it then follows that $u \equiv c$, a constant, in Ω. But no nonzero constant solves the equation, and hence we obtain strict inequality, as desired. □

3. Bifurcation from infinity. In this section we consider only the sublinear problem $(\mathrm{N})_{\lambda,0}$, so $\gamma = 0$. We will show that there exist large solutions near $\lambda = 0$ by means of a bifurcation argument used by Ambrosetti and Hess [7]. Note that one may prove existence of solutions with $\lambda < 0$ by minimization of an appropriate functional on $H^1(\Omega)$ (see (4.1)), and the existence of a solution satisfying $(*)$ at $\lambda = 0$ was derived in [9] by means of a sub- and supersolution construction. However, the bifurcation analysis reveals the connection between the sign of $\int_\Omega a(x)\,dx$ and the existence of solutions with $\lambda \geq 0$, and justifies the inclusion of the linear term λu in our equations.

Since we are interested in nonnegative solutions, we consider the modified equation,

$$\begin{cases} -\Delta u + u = (1+\lambda)u^+ + a(x)f(u), \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1$_\lambda$}$$

As usual, $u^+(x) = \max\{u(x), 0\}$. A simple and familiar calculation shows that weak $H^1(\Omega)$ solutions to $(3.1)_\lambda$ are nonnegative and hence also solve $(\mathrm{N})_{\lambda,0}$. (Recall that $f(u) = 0$ when $u \leq 0$, by hypothesis (2.1).)

We choose as our space $X = C^0(\overline{\Omega})$ with the supremum norm, $\|\cdot\|_\infty$, and prove the following result:

Theorem 3.1. *Suppose*

$$\frac{f(u)}{u} \to 0 \quad \text{as} \quad u \to +\infty. \tag{3.2}$$

Then $\lambda = 0$ is a bifurcation point from infinity for $(3.1)_\lambda$. More precisely, there exists a continuum Σ_∞ of <u>positive</u> *solutions of $(3.1)_\lambda$ which meets $(0, \infty) \in \mathbb{R} \times X$. In addition, if $\lambda \neq 0$, then λ is* not *a point of bifurcation from infinity.*

Proof. First, denote by $K : X \to X$ the solution operator for the Neumann operator $Lu = -\Delta u + u$. In other words, $u = Kh$ is the unique (weak) solution to

$$\int_\Omega (\nabla u \cdot \nabla \varphi + u\varphi)\,dx = \int_\Omega h\varphi\,dx \quad \text{for all } \varphi \in H^1(\Omega).$$

Since $h \in X$ is bounded, by elliptic regularity theory we obtain $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1]$, and $\|u\|_{C^{1,\alpha}} \leq C\|h\|_X$ for a constant $C > 0$. In particular, K is a compact operator on X. If in addition $h \in C^\beta(\overline{\Omega})$, then u will be a classical solution to $Lu = -\Delta u + u = h$ satisfying Neumann boundary conditions, and $u \in C^{2,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$

Define the family of maps $\Phi_\lambda : \ X \times [0,1] \to X$,

$$\Phi_\lambda(u,t) = u - K\left[(\lambda+1)u^+ + ta(x)f(u)\right].$$

If $\Phi_\lambda(u,t) = 0$ for some $u \in E$, then by the above observations $u \in C^{2,\alpha}(\overline{\Omega})$ solves $(3.1)_\lambda$.

Claim 1: If J is a compact interval and $0 \notin J$, then there exists $R > 0$ (independent of $t \in [0,1]$) so that $\Phi_\lambda(u,t) \neq 0$ for all $\lambda \in I$, $t \in [0,1]$, and all $u \in X$ with $\|u\|_\infty > R$.

Note: As a corollary of Claim 1, $\lambda = 0$ is the only point of bifurcation from infinity for $(3.1)_\lambda$.

To prove Claim 1 we suppose the contrary: there exist sequences $\lambda_n \to \lambda' \neq 0$, $t_n \in [0,1]$, and $u_n \in X$ with $\|u_n\|_\infty \to \infty$ and $\Phi_{\lambda_n,t_n}(u_n) = 0$. Set $v_n = u_n/\|u_n\|_\infty$, so v_n is bounded in X and

$$v_n = K\Big[(1+\lambda_n)v_n + t_n a(x)\frac{f(u_n)}{\|u_n\|_\infty}\Big]$$

By hypothesis (3.2), $f(u_n)/\|u_n\|_\infty \to 0$ in X, and the compactness of the operator K implies that (for some subsequence) $v_n \to v_0$ in X with $\|v_0\|_\infty = 1$ and v_0 is a weak solution of $Lv_0 = (1+\lambda')v_0^+$. Using v_0^- as a test function in this equation, we obtain $\|v_0^-\|_H = 0$, so $v_0 \geq 0$ and solves $Lv_0 = (1+\lambda')v_0$, $v_0 \not\equiv 0$. This is impossible, since $\lambda' \neq 0 = \lambda_1$. This completes the proof of Claim 1.

To study the behavior of solutions near infinity, we perform an inversion: for $u \neq 0$ let $z = u/\|u\|_\infty^2$, and for $z \neq 0$,

$$\Psi_\lambda(z,t) = z - K\Big[(1+\lambda)z^+ + ta(x)\|z\|_\infty^2 f\big(\frac{z}{\|z\|_\infty^2}\big)\Big].$$

When $z = 0$ we set $\Psi_\lambda(0,t) = 0$. Clearly $\Psi_\lambda(z,t) = 0$ has small solutions exactly when $\Phi_\lambda(u,t) = 0$ has solutions near infinity. We denote the degree of $\Psi_\lambda(\cdot,t)$ in the ball B_R with respect to zero by $\deg(\Psi_\lambda(\cdot,t), B_R, 0)$, and $i(\Psi_\lambda, z_0, 0)$ the Leray–Schauder index of the solution z_0 to $\Psi_\lambda(z_0,1) = 0$.

Claim 2: If $\lambda < 0$, then $i(\Psi_\lambda, 0, 0) = 1$.

When $\lambda < 0$, $\Psi_\lambda(z,\lambda,0) = 0$ if and only if $Lz = (1+\lambda)z^+$, which (as in the proof of Claim 1) implies $z \geq 0$ is an eigenfunction of L with eigenvalue

$\lambda < 0$. Hence the only solution is $z \equiv 0$, and for any ball B we have $\deg(\Psi_\lambda(\cdot, 0), B, 0) = 1$.

By Claim 1, there exists $R > 0$ independent of t so that $\Psi_\lambda(z,t) \neq 0$ for all z with $0 < \|z\|_\infty < 1/R$ and all $t \in [0,1]$. Consequently, for any $\varepsilon \in (0, 1/R)$, $\deg(\Psi_\lambda(\cdot,1), B_\varepsilon, 0) = \deg(\Psi_\lambda(\cdot, 0), B_\varepsilon, 0) = 1$, which implies Claim 2.

Denote by λ_2 the first non-zero eigenvalue of the Laplacian in Ω with Neumann boundary condition.

Claim 3: For every $\lambda \in (0, \lambda_2)$ there exists $R > 0$ (possibly depending on λ,) so that for any $\tau \geq 0$, the equation $\Phi_\lambda(u, 1) = \tau$ admits no solutions $u \in X$ with $\|u\|_\infty > R$.

Suppose the contrary: there exist sequences u_n with $\|u_n\|_\infty \to \infty$, $\tau_n > 0$ so that $\Phi_\lambda(u_n, 1) = \tau_n$. First, note that $u_n \geq 0$ in Ω. Indeed, if $\Phi_\lambda(u_n, 1) = \tau_n$, then $\tau_n = Lu_n - (1+\lambda)u_n^+ - a(x)f(u_n)$, and using $-u_n^-$ as a test function in the equation,

$$0 \geq -\tau_n \int_\Omega u_n^- = \|u_n\|_H^2.$$

We now decompose X into two components, along the eigenfunction $\varphi_1 = 1$ and along its complement. For $v \in X$, define projections

$$Pv = \frac{1}{\text{meas}\,(\Omega)} \int_\Omega v\,dx, \quad P^\perp v = v - Pv.$$

Note that both projections commute with L and K. Write $u_n = s_n + w_n$, with $s_n = Pu_n \in \mathbb{R}$, $w_n = P^\perp u_n$ with average value zero. Note that $u_n \geq 0$ implies that $s_n = Pu_n > 0$. Applying either P or $P^\perp$ to the equation $\Phi_\lambda(u_n, 1) = \tau_n$, we obtain the following two equations:

$$\tau_n = s_n - K\left[(1+\lambda)s_n - P[a(x)f(u_n)]\right] = -\lambda s_n + K\left[P[a(x)f(u_n)]\right] \quad (3.3)$$

$$0 = w_n - k\big[(1+\lambda)w_n - P^\perp[a(x)f(u_n)]\big]. \quad (3.4)$$

We now distinguish two cases: either
(i) $\lim_{n\to\infty} s_n/\|w_n\|_\infty = 0$, or
(ii) $\limsup_{n\to\infty} \|w_n\|_\infty/s_n \leq C$.

In case (i) we have $\|u_n\|_\infty \leq s_n + \|w_n\|_\infty = \|w_n\|_\infty(1+o(1))$. In particular, $\|w_n\|_\infty \to \infty$, so (3.4) implies

$$0 = \frac{w_n}{\|w_n\|_\infty} - K\Big[(1+\lambda)\frac{w_n}{\|w_n\|_\infty} + P^\perp\Big(a(x)\frac{f(u_n)}{\|u_n\|_\infty}\frac{\|u_n\|_\infty}{\|w_n\|_\infty}\Big)\Big].$$

Since K is compact we may pass to the limit: $\frac{w_n}{\|w_n\|_\infty} \to \tilde{w}$ in X with $\tilde{w} \in E^\perp$, $\|\tilde{w}\|_\infty = 1$ and $\tilde{w} = K[(1+\lambda)\tilde{w}]$. Since by hypothesis $(1+\lambda)$ is not an eigenvalue of L, this is impossible, and case (i) cannot occur.

In case (ii) we have $\|u_n\|_\infty \le s_n(1+O(1))$, so $s_n \to \infty$. Then from (3.3) we obtain

$$0 \le \frac{\tau_n}{s_n} = -\lambda + K\Big[P\Big(a(x)\frac{f(u_n)}{\|u_n\|_\infty}\frac{\|u_n\|_\infty}{s_n}\Big)\Big] = -\lambda + o(1).$$

Since by hypothesis $\lambda > 0$ this is also a contradiction, and hence the statement of the claim must hold.

Note that Claim 3 directly implies the following statement:

Claim 3′. For every $\lambda \in (0, \lambda_2)$ there exists $R > 0$ so that for any $\sigma \in [0,1]$, the equation $\Psi_\lambda(z,1) = \sigma$ admits no solutions $z \in X$ with $0 < \|z\|_\infty < 1/R$.

Claim 4. If $\lambda \in (0, \lambda_2)$ is fixed, then $i(\Psi_\lambda(\cdot,1), 0, 0) = 0$.

Consider the family of maps $H(z,\sigma) = \Psi_\lambda(z,1) - \sigma$, $z \in X$, $\sigma \in [0,1]$. Then for every $t \in (0,1]$, the equation $H(z,\sigma) = 0$ has no solutions in the ball $B_{1/R}$, and no solution in $B_{1/R} \setminus \{0\}$ when $\sigma \in [0,1]$. Hence, for any ball B_ε with radius $\varepsilon < 1/R$ we have $\deg(\Psi_\lambda(\cdot,1), B_\varepsilon, 0) = \deg(H(\cdot,\sigma), B_\varepsilon, 0) = 0$.

Claim 5. $\lambda = 0$ is a bifurcation point for $\Psi_\lambda(z,1)$.

This claim follows directly from Lemma 1.2 and Theorem 1.3 of Rabinowitz [18]. First, note that by Claim 1, $\lambda = 0$ is the only possible bifurcation point (from 0) for Ψ_λ, and hence we may already discard possibility (ii) of Lemma 1.2 and Theorem 1.3 of [18]. Then the arguments of [18] may be repeated exactly, with function Ψ_λ rather than $\Phi(\lambda)$, and with $\mu = 0$, up to line (1.11). The contradiction is then obtained by comparing (1.11) with Claim 2 and Claim 4 above.

Finally, we show that the solutions obtained are actually *positive* in Ω. More precisely, we prove that there exists $R > 0$ such that any solution u to $(3.1)_\lambda$ with $\lambda \in [-1,1]$ and $\|u\|_\infty > R$ must satisfy $\inf_\Omega u(x) > 0$.

We suppose the contrary: that there exists a sequence of solutions u_n to $(3.1)_{\lambda_n}$ with $\lambda_n \in [-1,1]$ and $\|u_n\|_\infty \to \infty$, such that $\inf_\Omega u(x) = 0$. From Claim 1, we must have $\lambda_n \to 0$. Decompose $u_n = s_n + w_n$, with $s_n = Pu_n > 0$, $w_n = P^\perp u_n$. Note that it suffices to prove that:

Claim 6. $\lim_{n\to\infty} \|w_n\|_\infty / s_n = 0$.

Suppose the contrary. Then $s_n \le C\|w_n\|_\infty$, so $\|u_n\|_\infty \le C\|w_n\|_\infty$. Applying (3.4), we have

$$\frac{w_n}{\|w_n\|_\infty} = K\Big[(1+\lambda_n)\frac{w_n}{\|w_n\|_\infty}\Big] + o(1),$$

and (as before), $w_n/\|w_n\|_\infty \to \tilde{w}$, with $\|\tilde{w}\|_\infty = 1$, $\tilde{w} \in P^\perp$, and $\tilde{w} = K[\tilde{w}]$. But this last equation is equivalent to $-\Delta\tilde{w} = 0$ with Neumann boundary condition in Ω, so $\tilde{w}$ is constant. This contradicts the fact that $\tilde{w} \in P^\perp$, and so $\|w_n\|/s_n \to 0$, as claimed.

Now we write $u_n = s_n\big(1 + \frac{w_n}{s_n}\big)$. Since $w_n/s_n \to 0$ uniformly in Ω we see that in fact $\inf_\Omega u_n \to \infty$ as $n \to \infty$, which contradicts our choice of the sequence u_n. This completes the proof of Theorem 3.1. □

Now that we know that there do exist large positive solutions to $(3.1)_\lambda$ for λ near 0, we must determine from which side of $\lambda = 0$ do these solutions connect to infinity. In particular, we would like to know under which conditions on f is the condition $\int_\Omega a(x)\,dx < 0$ a necessary and sufficient condition for solutions to the right of $\lambda = 0$. We prove the following:

Theorem 3.2. *Suppose*

$$\textit{There exists } q \in (0,1) \textit{ such that } \lim_{u\to\infty}\frac{f(u)}{u^q} = 1. \tag{3.5}$$

(a) *If $\int_\Omega a(x)\,dx > 0$, then Σ_∞ bifurcates to the left of $\lambda = 0$. In other words, there exists $R > 0$ such that if (u_λ, λ) solve $(3.1)_\lambda$ with $\lambda \in [-1,1]$ and $\|u\|_\infty > R$, then $\lambda < 0$.*
(b) *If $\int_\Omega a(x)\,dx < 0$, then Σ_∞ bifurcates to the right of $\lambda = 0$. In other words, there exists $R > 0$ such that if (u_λ, λ) solve $(3.1)_\lambda$ with $\lambda \in [-1,1]$ and $\|u\|_\infty > R$, then $\lambda > 0$.*

Remark 3.3. Comparing the above result with the nonexistence result of Proposition 2.3, we note that in Theorem 3.2 we require $f \sim u^q$ near infinity, but in Proposition 2.3 the same condition is imposed near $u = 0$.

Proof. We treat case (b); the proof for case (a) is identical. As in Claim 6, we suppose that there is no such R, and derive a contradiction. In such a case there would then exist a sequence (u_n, λ_n) with u_n solutions to $(3.1)_{\lambda_n}$ with $\lambda_n \to 0$ and $\|u_n\|_\infty \to \infty$, for which $\lambda_n \le 0$. Decompose u_n as in

Claim 6 above, $u_n = s_n + w_n$, $s_n = Pu_n > 0$, $w_n = P^{\perp}u_n$. As we just showed, $s_n/\|w_n\|_\infty \to \infty$. Integrating the equation over Ω we obtain:

$$-\frac{\lambda_n}{s_n^q} = \int_\Omega a(x)\frac{f(u_n)}{\|u_n\|_\infty^q}\frac{\|u_n\|_\infty^q}{s_n^q}\,dx.$$

Since $\frac{u_n}{s_n} = 1 + \frac{w_n}{s_n} \to 1$ uniformly, and $\frac{f(u_n)}{\|u_n\|_\infty^q}$ is bounded and also tends to 1, the right-hand side converges to $\int_\Omega a(x)\,dx$ by Dominated Convergence. Hence the quantity $\frac{\lambda_n}{s_n}$ must have (for all large n) the opposite sign of $\int_\Omega a(x)\,dx$. This contradicts our choice of u_n. □

4. The interval of existence. We now introduce variational methods in order to obtain some global information about the solution families of $(N)_{\lambda,\gamma}$. For $v \in H^1(\Omega) \cap L^\infty(\Omega)$ we define the functional

$$I_{\lambda,\gamma}(v) = \int_\Omega \left[\tfrac{1}{2}(|\nabla v|^2 + v^2) - \frac{(1+\lambda)}{2}(v^+)^2 - a(x)F(v) - \gamma G(v)\right]dx, \quad (4.1)$$

where $F(v) = \int_0^v f(s)\,ds$ and $G(v) = \int_0^v g(x)\,ds$. (Recall that we assume that $f(v), g(v) = 0$ for $v \le 0$.)

Our main tool in this section is the variational formulation of Perron's method, as presented by Struwe [20] (Theorem I.2.4). We will use it in this form:

Lemma 4.1. *Assume f and g satisfy* (2.2). *Let $J \subset \mathcal{M}$. Suppose $\overline{u} \in \mathcal{N}_J$ is a solution to $(N)_{\overline{\lambda},\overline{\gamma}}$ for some $\overline{\lambda} \in \mathbb{R}$ and $\overline{\gamma} \ge 0$. Then, $(N)_{\lambda,\gamma}$ admits a solution $u \in \mathcal{N}_J$ for every $\lambda \le \overline{\lambda}$ and $0 \le \gamma \le \overline{\gamma}$. Moreover, $u(x) \le \overline{u}(x)$ in Ω and $I_{\lambda,\gamma}(u) < 0$.*

Proof. When $\lambda \le \overline{\lambda}$ and $0 \le \gamma \le \overline{\gamma}$, $\overline{u}$ is a supersolution for the problem $(N)_{\lambda,\gamma}$. We consider the following minimization problem in a convex constraint set, $\inf_{v\in M} I_{\lambda,\gamma}(v)$, $M = \{v \in H^1(\Omega) :\ 0 \le v(x) \le \overline{u}(x) \text{ a.e.}\}$. By trivial modifications to Theorem I.2.4 of [20], the infimum is achieved at some $u \in M$ and $(\varphi, I'_{\lambda,\gamma}(u)) = 0$ for all $\varphi \in H^1(\Omega)$. By routine regularity arguments u is a solution to $(N)_{\lambda,\gamma}$. Since $u \in M$, it vanishes on the components Ω_k^{0+}, $k \notin J$. It remains to show that it does not vanish on Ω_k^{0+}, $k \in J$. (At the same time this will also show that $u \not\equiv 0$, the trivial solution.)

Suppose that for some $k \in J$, $u \not> 0$ in Ω_k^{0+}. By Lemma 2.1 we have $u \equiv 0$ on $\overline{\Omega_k^{0+}}$. Choose a ball $D \subset \Omega^+ \cap \Omega_k^{0+}$, and $\psi(x)$ with $0 \le \psi \in C_0^\infty(D)$. Then, for $t > 0$ small, $u + t\psi \in M$ and $I_{\lambda,\gamma}(u + t\psi) = I_{\lambda,\gamma}(u) + I_{\lambda,\gamma}(t\psi) < I_{\lambda,\gamma}(u)$ since $I_{\lambda,\gamma}(t\psi) < 0$ for all t sufficiently small. This contradicts the choice of u as an infimum of $I_{\lambda,\gamma}$, so we must have $u \in \mathcal{N}_J$. By the same argument, we see that $I_{\lambda,\gamma}(u) < I_{\lambda,\gamma}(0) = 0$. □

Remark 4.2. Given the variational formulation of the problem as an infimum, it is natural to ask whether the solutions obtained by Lemma 4.1 are local minima of $I_{\lambda,\gamma}$ in any sense. Note that this cannot be the case when $J \neq \mathcal{M}$! Indeed, following the argument of the last part of the proof, we can decrease the value of $I_{\lambda,\gamma}$ near such solutions by arbitrary local perturbations in each Ω_k^{0+}, $k \notin J$, so these solutions are very far from being minimizers in $H^1(\Omega)$. Note that even when $0 < u < \overline{u}$ in all of Ω^{0+} we cannot rule out vanishing in Ω^-, so the solution u may not be an interior point of M, even in the stronger C^1-topology. Nevertheless, in the next section we will show that (under appropriate hypotheses on f, g) the solutions in $\mathcal{N}_{\mathcal{M}}$ will be local H^1-minimizers for $I_{\lambda,\gamma}$.

For the remainder of the paper we restrict our attention to solutions in the class $\mathcal{N}_{\mathcal{M}}$. For convenience, we redefine our problem:

Definition 4.3. We say u solves $(\mathrm{P})_{\lambda,\gamma}$ if u solves $(N)_{\lambda,\gamma}$ and $u(x) > 0$ for all $x \in \overline{\Omega^{0+}}$.

Given the monotonicity in the parameters λ and γ implied by Lemma 4.1 it is natural to define:

$$\Lambda_\gamma = \sup\{\lambda : \ (\mathrm{P})_{\lambda,\gamma} \text{ admits a solution}\}, \text{ and}$$
$$\Gamma_\lambda = \sup\{\gamma \geq 0 : \ (\mathrm{P})_{\lambda,\gamma} \text{ admits a solution}\}.$$

It is a direct corollary of Lemma 4.1 that Λ_γ is nonincreasing as a function of γ, and that Γ_λ is also nonincreasing as a function of λ.

Lemma 4.4. *Assume* (2.2) *and*

$$\frac{g(u)}{u} \to 0 \quad \text{as } u \to 0^+. \tag{4.2}$$

Then
(i) *for every* $\gamma \geq 0$, $-\infty < \Lambda_\gamma < \infty$;
(ii) *for each* $\lambda < \Lambda_0$, $0 < \Gamma_\lambda < \infty$;
(iii) *If* $\lambda > \Lambda_0$, *then* $\Gamma_\lambda = 0$.

Proof. From Theorem 3.1 we know that $\Lambda_0 > -\infty$, and from Proposition 2.2 $\Lambda_0 < \lambda^* < \infty$. Moreover, Lemma 4.1 implies that $(P)_{\lambda,0}$ admits a solution for all $\lambda < \Lambda_0$. For any $\lambda < \Lambda_0$, take $\lambda' \in (\lambda, \Lambda_0)$ and the corresponding solution u' of $(P)_{\lambda',0}$. Then, for

$$\gamma' = \frac{(\lambda' - \lambda)}{\sup_{x\in\Omega}[g(u'(x))/u'(x)]} \tag{4.3}$$

and $\gamma \leq \gamma'$, we have

$$\int_\Omega [\nabla u' \cdot \nabla\varphi - (\lambda u' - a(x)f(u') - \gamma g(u'))\varphi] = \int_\Omega [(\lambda' - \lambda)u' - \gamma g(u')]\varphi \geq 0 \tag{4.4}$$

for all $0 \leq \varphi \in H^1(\Omega)$. Therefore, u' is a supersolution for the problem $(P)_{\lambda,\gamma}$ with $0 < \gamma \leq \gamma'$. Repeating the proof of Lemma 4.1, we obtain solutions to $(P)_{\lambda,\gamma}$ with $0 < \gamma < \gamma'$, and hence $\Gamma_\lambda \geq \gamma' > 0$ for all $\lambda < \Lambda_0$. This proves (ii).

To prove (i), fix a $\lambda' < \Lambda_0$ with solution u' to $(P)_{\lambda',0}$. Then for any $\gamma > 0$, choose $\lambda < \lambda'$ sufficiently negative such that $\gamma < \gamma'$ with γ' defined as in (4.3). Again, u' is a supersolution for the problem $(P)_{\lambda,\gamma}$, and the same argument shows that $\Lambda_\gamma > -\infty$. The fact that $\Lambda_\gamma < \infty$ has already been proven in Proposition 2.2.

Finally, if $(P)_{\lambda,\gamma}$ admits a solution with $\gamma > 0$ and $\lambda > \Lambda_0$, by Lemma 4.1 the problem $(P)_{\lambda,0}$ would admit a solution, contradicting the definition of Λ_0. This completes the proof of the lemma. □

Remark 4.5. The question of whether the problem $(P)_{\Lambda_0,\gamma}$ admits solutions for some $\gamma > 0$ is more delicate, and remains an open question.

As a result of the previous lemmas, we have existence of solutions of $(P)_{\lambda,\gamma}$ on intervals in λ and γ. Our next task is to study the structure of the solution sets. The following comparison lemma is based on a uniqueness proof by Bandle, Pozio, and Tesei [9] for the sublinear problem at $\lambda = 0$.

Lemma 4.6. *Assume that*

$$\begin{cases} f(u) \text{ is nondecreasing,} & 1/f(u) \text{ is integrable near zero,} \\ f'(u) \text{ is strictly decreasing,} & f(u)/u \text{ is nonincreasing.} \end{cases} \tag{4.5}$$

If $\lambda \le 0$ and $v, w \in C^{2,\alpha}(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v \le \lambda v + a(x)f(v), & \text{in } \Omega, \\ -\Delta w \ge \lambda w + a(x)f(w), & \text{in } \Omega, \end{cases} \tag{4.6}$$

then $v \le w$ in Ω.

The proof is a simple extension of Theorem 3.1 of [9], and is left to the reader.

We now present some applications of Lemma 4.6.

Corollary 4.7. *Assume* (4.5). *Then for all $\lambda \le 0$, problem $(P)_{\lambda,0}$ admits a unique solution $v_{\lambda,0}$. Moreover, if $\lambda < \lambda' \le 0$, then $v_{\lambda,0} \le v_{\lambda',0}$.*

Lemma 2.1 and Corollary 4.7 can both fail to hold if Ω^{0+} has (countably) infinitely many connected components. See [8] for an example of how this may occur.

Next we consider the limiting values, $\lambda = \Lambda_\gamma$ and $\gamma = \Gamma_\lambda$ and show that $(\mathrm{P})_{\lambda,\gamma}$ does admit solutions there.

Lemma 4.8. *Assume f, g satisfy* (2.2), (3.5), (4.5), *and*

$$g(u) \le C(1 + u^p), \quad \text{for all } u > 0; \tag{4.7}$$

$$0 \le \theta G(u) \le g(u)u \quad \text{for all } u \ge R, \tag{4.8}$$

for constants $C > 0$, $p \in (1, \frac{N+2}{N-2}]$, $R > 0$, and $\theta > 2$. Then
(a) *For all $\gamma > 0$, $(P)_{\lambda,\gamma}$ admits a solution at $\lambda = \Lambda_\gamma$;*
(b) *There exists a solution to $(P)_{\Lambda_0,0}$ if and only if $\int_\Omega a(x)\,dx < 0$.*
(c) *For all $\lambda < \Lambda_0$, $(P)_{\lambda,\gamma}$ admits a solution at $\gamma = \Gamma_\lambda$.*

Proof. Fix $\gamma \ge 0$ and suppose $\lambda_n \to \Lambda_\gamma$, $\lambda_n < \Lambda_\gamma$. The solutions u_n to $(\mathrm{P})_{\lambda_n,\gamma}$ obtained by Lemma 4.1 satisfy $I_{\lambda_n,\gamma}(u_n) < 0$, and by Lemma 4.6 each is bounded below by a fixed solution $v_{\lambda'}$ of $(\mathrm{P})_{\lambda',0}$ with $\lambda' \le \min\{0, \lambda\} \le 0$.

Using the equation and the negativity of $I_{\lambda_n,\gamma}(u_n)$ we have two relations,

$$\int_\Omega \Big\{\frac{1}{2}|\nabla u_n|^2 - \frac{\lambda}{2}u_n^2 - a(x)F(u_n) - \gamma G(u_n)\Big\}\,dx < 0, \tag{4.9}$$

$$\int_\Omega \big\{|\nabla u_n|^2 - \lambda u_n^2 - a(x)f(u_n)u_n - \gamma g(u_n)u_n\big\}\,dx = 0. \tag{4.10}$$

Using the hypotheses (3.5), (4.7), and (4.8), we have

$$\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|_H^2 < \Big(\frac{1}{2}-\frac{1}{\theta}\Big)(1+\lambda)\|u_n\|_2^2 + \int_\Omega a(x)\big[F(u_n)-\frac{1}{\theta}f(u_n)u_n\big] + \gamma\int_\Omega\big[G(u_n)-\frac{1}{\theta}g(u_n)u_n\big] \le C_1\|u_n\|_2^2 + C_2. \tag{4.11}$$

We claim that $\|u_n\|_2$ is uniformly bounded. Suppose the contrary, $\|u_n\|_2 \to \infty$. Then, by (4.11) $v_n = u_n/\|u_n\|_2$ is a bounded sequence in $H^1(\Omega)$, with a weakly convergent subsequence (which we continue to denote v_n), $v_n \rightharpoonup v_0$, $\|v_0\|_2 = 1$, with strong convergence in $L^p(\Omega)$ for $1 \le p < \frac{2N}{N-2}$. Applying a nonnegative test function $\varphi \in C^1(\Omega)$ to the equation and dividing by $\|u_n\|_2$ we obtain:

$$\begin{aligned}\gamma\|u_n\|_2^{\theta-2}\int_\Omega v_n^{\theta-1}\varphi\,dx &\le \gamma\|u_n\|_2^{\theta-2}\Big(\int_\Omega \frac{g(u_n)\varphi}{\|u_n\|_2^{\theta-1}}\,dx\Big)+o(1) \qquad (4.12)\\ &= \int_\Omega\big[\nabla v_n\cdot\varphi-\lambda_n v_n\varphi-a(x)\frac{f(u_n)\varphi}{\|u_n\|_2}\big]\,dx+o(1)\\ &= \int_\Omega\left[\nabla v_0\cdot\varphi-\Lambda_\gamma v_0\varphi\right]\,dx+o(1).\end{aligned}$$

If $\gamma > 0$, take $\varphi = v_0$ so that

$$\int_\Omega v_0^\theta\,dx \le \liminf_{n\to\infty}\int_\Omega v_n^\theta\,dx = 0,$$

which contradicts $\|v_0\|_2 = 1$. If $\gamma = 0$ and $\int_\Omega a(x)\,dx < 0$, then by Theorem 3.2 we know $\Lambda_0 > 0$. In that case, (4.12) simplifies to

$$\int_\Omega\left(\nabla v_0\cdot\nabla\varphi-\Lambda_0 v_0\varphi\right)\,dx = 0$$

for each $\varphi \in C^1(\Omega)$. Since $0 \le v_0 \not\equiv 0$ in Ω, this can not occur since $\Lambda_0 > 0 = \lambda_1$. In conclusion, we must have $\|u_n\|_H \le C_1\|u_n\|_2 + C_2 \le C$.

Extracting a subsequence (still denoted by u_n), we have $u_n \rightharpoonup u^*$ in $H^1(\Omega)$ and pointwise almost everywhere. In particular, since each $u_n \ge v_{\lambda'}$ we have $u^* \ge u_0$ in Ω. By weak convergence we may pass to the limit in the equation to obtain the desired solution.

When $\gamma = 0$ and $\int_\Omega a(x)\,dx \le 0$, we have already shown that $(\mathrm{P})_{0,0}$ admits no solution (Proposition 2.2). We have therefore proven (a) and (b).

If we fix $\lambda < \Lambda_0$, we have (from Lemma 4.4) that $0 < \Gamma_\lambda < +\infty$. By repeating the exact same argument as above we arrive at the conclusion (c). □

When $\lambda > 0$ or $\gamma > 0$ we do not expect uniqueness. In fact we will find a second solution to problem $(\mathrm{P})_{\lambda,\gamma}$ in such a case. But we do have a minimal solution for all λ, γ:

Proposition 4.9. *Assume* (4.5) *holds, and $g(u)$ is increasing. Then for all λ, γ for which* $(\mathrm{P})_{\lambda,\gamma}$ *admits a solution, there is a minimal solution, $v_{\lambda,\gamma}$ with $v_{\lambda,\gamma}(x) \le u(x)$ for any solution of* $(\mathrm{P})_{\lambda,\gamma}$. *Moreover, $v_{\lambda,\gamma} \le v_{\lambda',\gamma'}$ whenever $\lambda \le \lambda'$ and $\gamma \le \gamma'$.*

The proof involves the usual monotone iteration, with multiple appeals to the weak maximum principle. For completeness we provide the details in the Appendix.

5. Solutions as local minima. In this section we prove the important and nontrivial fact that the solutions of $(\mathrm{P})_{\lambda,\gamma}$ obtained via Lemma 4.1 define local minima for the functional $I_{\lambda,\gamma}$ in the H^1-topology. Recall that this will *not* be the case for solutions which vanish in some component of Ω^{0+}. (See Remark 4.2.)

For simplicity we specialize to the case

$$f(u) = u^q, \quad g(u) = u^p, \quad 0 < q < 1 < p, \text{ and } p \le \tfrac{N+2}{N-2} \text{ if } N \ge 3.$$

The results in the this section are true for somewhat more general f, g, and the essential ideas are the same as in the case of powers.

First, we require the following consequence of the strong maximum principle:

Lemma 5.1. *Suppose $\lambda \le \overline{\lambda}$, $0 \le \gamma \le \overline{\gamma}$, and either $\lambda < \overline{\lambda}$ or $\gamma < \overline{\gamma}$. If u is a solution of $(P)_{\lambda,\gamma}$ and $\overline{u}$ is a solution of $(P)_{\overline{\lambda},\overline{\gamma}}$ with $0 \le u(x) \le \overline{u}(x)$ in Ω, then $u(x) < \overline{u}(x)$ for all $x \in A = \{x \in \overline{\Omega} : \overline{u} > 0\}$.*

Proof. Let $v = \overline{u} - u \ge 0$ in Ω. Then v satisfies

$$-\Delta v + a^- \Big[\frac{f(\overline{u}) - f(u)}{\overline{u} - u}\Big] v \ge 0.$$

To derive a contradiction, we suppose that $v(x_0) = 0$ for some $x_0 \in A$. First, we show that x_0 cannot be an interior point of Ω. Indeed, if $x_0 \in A \cap \Omega$ it is an interior point of A, and we may choose a ball $B = B_r(x_0)$ with $\overline{B} \subset A$. On B, $\overline{u}$ is bounded below away from zero and hence the quotient

$$0 \leq \frac{f(\overline{u}) - f(u)}{\overline{u} - u} \leq \frac{f(\overline{u})}{\overline{u}}$$

is uniformly bounded in B. We may now apply the strong Maximum Principle in B to conclude that $v \equiv 0$ in B. That is, $u(x) = \overline{u}(x)$ for all $x \in B$, which leads to a contradiction when comparing the equations satisfied by these functions.

It remains to show that x_0 cannot lie on $\partial\Omega$ either. Since A is relatively open in $\overline{\Omega}$, we can choose $r > 0$ so that $B' = B_r(x_0) \cap \overline{\Omega} \subset A$. Since $\partial\Omega$ is smooth, the exterior normal to B' at x_0 coincides with the normal to Ω at x_0. Applying the Hopf Lemma to $v \geq 0$ at the point $x_0 \in \partial B'$, we have $\frac{\partial v}{\partial \nu}(x_0) < 0$, which contradicts the Neumann boundary condition satisfied by v on all of $\partial\Omega$. □

Now we may prove the key fact which relates sub- and supersolutions to local minima:

Proposition 5.2. *Assume that* $\lambda \leq \overline{\lambda}$, $0 \leq \gamma \leq \overline{\gamma}$, *and either* $\lambda < \overline{\lambda}$ *or* $\gamma < \overline{\gamma}$. *Let* $\overline{u}$ *be a solution to* $(P)_{\overline{\lambda},\overline{\gamma}}$ *suppose that* v_0 *attains*

$$I_{\lambda,\gamma}(v_0) = \inf\{I(v) : \ v \in H^1(\Omega), \ 0 \leq v(x) \leq \overline{u} \ a.e.\}.$$

Then v_0 *is a local minimizer for* $I_{\lambda,\gamma}$ *in* $H^1(\Omega)$; *that is, there exists* $\delta > 0$ *such that*

$$I_{\lambda,\gamma}(v_0) \leq I_{\lambda,\gamma}(w) \quad \text{for all } w \in H^1(\Omega) \text{ with } \|w - v_0\|_H < \delta.$$

Proof. Define $M = \{v \in H^1(\Omega) : \ 0 \leq v(x) \leq \overline{u}(x), \text{ a.e. in } \Omega\}$. Recall that $\overline{u}$ is a supersolution for the problem $(\mathrm{P})_{\lambda,\gamma}$, and Lemma 4.1 ensures that there exists (at least one) minimizer v_0 for $I_{\lambda,\gamma}$ in the set M, which solves $(\mathrm{P})_{\lambda,\gamma}$.

Suppose that there exists a sequence $\{u_n\} \in H^1(\Omega)$ with $u_n \to v_0$ and $I_{\lambda,\gamma}(u_n) < I_{\lambda,\gamma}(v_0)$. Let

$$v_n = \max\{0, \min\{u_n, \overline{u}\}\}, \quad u_n^- = \max\{-u_n, 0\} \geq 0, \quad w_n = (u_n - \overline{u})^+ \geq 0,$$

so that $u_n = v_n - u_n^- + w_n$, $v_n \in M$, and u_n^- and w_n have disjoint supports. Define the measurable sets $R_n = \{x \in \Omega : 0 \leq u_n(x) \leq \overline{u}(x)\}$, $S_n = \operatorname{supp} w_n$,

$T_n = \operatorname{supp} u_n^-$, $A = \{x \in \Omega : \overline{u}(x) > 0\}$, $B = \Omega \setminus A$. From Lemma 5.1 we have $\overline{u}(x) > v_0(x)$ for all $x \in A$, and Lemma 2.1 implies that $B \subset \Omega^-$.

Claim 1. meas $(A \cap S_n) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. For $\delta > 0$ (to be chosen later), set

$$E_n = \{x \in A : u_n(x) > \overline{u}(x) > v_0 + \delta\},$$
$$F_n = \{x \in A : u_n(x) > \overline{u} \text{ and } \overline{u} \le v_0 + \delta\}.$$

Since

$$\begin{aligned} 0 &= \text{meas } (\{x \in A : \overline{u}(x) \le v_0(x)\}) \\ &= \text{meas } \big(\cap_{j=1}^{\infty} \{x \in A : \overline{u}(x) \le v_0(x) + 1/j\}\big) \\ &= \lim_{j\to\infty} \text{meas } (\{x \in A : \overline{u}(x) \le v_0(x) + 1/j\}) \,, \end{aligned}$$

there exists $\delta_0 > 0$ so that meas $(F_n) \le$ meas $(\{x \in A : \ \overline{u} \le v_0 + \delta_0\}) < \frac{\varepsilon}{2}$. On the other hand, since $u_n \to v_0$ there exists $n_0 > 0$ so that for all $n \ge n_0$,

$$\frac{\varepsilon\delta_0^2}{2} \ge \int_\Omega (u_n - v_0)^2 \, dx \ge \int_{E_n} \delta_0^2 \, dx = \delta_0^2 \, \text{meas} \, (E_n).$$

Hence meas $(E_n) \le \frac{\varepsilon}{2}$, and meas $(A \cap S_n) \le$ meas $(E_n)+$meas $(F_n) < \varepsilon$. This completes the proof of the claim.

For convenience, set $H(x,u) = (\lambda - 1)(u^+)^2/2 + a(x)F(u) + G(u)$. We calculate:

$$\begin{aligned} I_{\lambda,\gamma}(u_n) &= \int_{R_n} \big(\frac{1}{2}|\nabla v_n|^2 + \frac{1}{2}v_n^2 - H(x,v_n)\big)\, dx + \\ &+ \int_{S_n} \big(\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}u_n^2 - H(x,u_n)\big)\, dx + \int_{T_n} \frac{1}{2}|\nabla u_n^-|^2 + \frac{1}{2}(u_n^-)^2 \, dx \\ &= \int_{S_n} \big\{\frac{1}{2}\left(|\nabla(\overline{u} + w_n)|^2 - |\nabla \overline{u}|^2 + (\overline{u} + w_n)^2 - \overline{u}^2\right) \\ &- [H(x, \overline{u} + w_n) - H(x,\overline{u})]\big\} + I_{\lambda,\gamma}(v_n) + \|u_n^-\|_H^2 \\ &= \int_{S_n} \frac{1}{2}\left(|\nabla w_n|^2 + w_n^2\right) + (\nabla \overline{u} \cdot \nabla w_n + \overline{u} w_n) - [H(x,\overline{u} + w_n) - H(x,\overline{u})] \\ &+ I_{\lambda,\gamma}(v_n) + \|u_n^-\|_H^2 \\ &\ge \frac{1}{2}\|w_n\|_H^2 + I_{\lambda,\gamma}(v_0) + \frac{1}{2}\|u_n^-\|_H^2 \\ &- \int_{S_n} \left(H(x,\overline{u} + w_n) - H(x,\overline{u}) - H_u(x,\overline{u})w_n\right)\, dx \qquad (5.1) \end{aligned}$$

where we have used the fact that v_0 minimizes $I_{\lambda,\gamma}$ in M and that $\overline{u}$ is a supersolution.

Now we estimate each term in $H(x,u)$ on the set A, using the fact that this set is very small:

$$\int_{S_n \cap A} w_n^2\, dx \le (\text{meas}\,(S_n \cap A))^{2/N} \left(\int_\Omega w_n^{\frac{2N}{N-2}}\, dx \right)^{\frac{N-2}{N}} \le o(1)\, \|w_n\|_H^2, \tag{5.2}$$

where $o(1) \to 0$ as $n \to \infty$. Applying Lemma 5.1, there exists $k > 0$ so that $\overline{u}(x) \ge k$ for $x \in \Omega^{0+}$. Hence,

$$0 \le F(\overline{u}(x) + w_n(x)) - F(\overline{u}(x)) - f(\overline{u}(x))w_n(x) \le f'(k)w_n^2(x)$$

for all $x \in \Omega^{0+}$. Using the estimate (5.2) again, we have:

$$0 \le \int_{A\cap\Omega^{0+}} a(x)[F(\overline{u} + w_n) - F(\overline{u}) - f(\overline{u})w_n] \le o(1)\, \|w_n\|_H^2. \tag{5.3}$$

On $A \cap \Omega^-$, we note that

$$\int_{A\cap\Omega^-} a(x)[F(\overline{u} + w_n) - F(\overline{u}) - f(\overline{u})w_n] \le 0. \tag{5.4}$$

To estimate the other term, we note that there exists $\theta = \theta(x) \in (0,1)$ so that

$$\begin{aligned} 0 &\le G(\overline{u}(x) + w_n(x)) - G(\overline{u}(x)) - g(\overline{u}(x))w_n(x) \\ &= g'(\overline{u}(x) + \theta w_n(x))w_n(x)^2/2 \le C(1 + w_n^{p-1})w_n^2. \end{aligned} \tag{5.5}$$

As a consequence of this estimate,

$$\begin{aligned} &\int_A [G(\overline{u}(x) + w_n(x)) - G(\overline{u}(x)) - g(\overline{u}(x))w_n(x)] \le C \int_A w_n^2 + w_n^{p+1} \\ &\quad \le o(1)\|w_n\|_H^2 + C\|w_n\|_H^{p+1} \le o(1)\|w_n\|_H^2. \end{aligned}$$

To estimate the terms on the set B we must use that fact that $B \subset \Omega^-$. Since $\overline{u} = 0$ on B, we have

$$0 \ge \int_B a(x)[F(\overline{u} + w_n) - F(\overline{u}) - f(\overline{u})w_n] = -\int_B a^-(x)F(w_n). \tag{5.6}$$

Now we must try to make the right-hand side of (5.6) dominate the remaining terms in B. Since $w_n \to 0$ in $H^1(\Omega)$ and $a^-(x) > 0$ almost everywhere in Ω^-, by repeating the argument in the Claim we have

$$\operatorname{meas}(P_n) = \operatorname{meas}\left(\{x \in \Omega^- : \; w_n(x) \geq (a^-(x)/10(q+1))^{\frac{1}{1-q}}\}\right) \to 0$$

as $n \to \infty$. Therefore we may estimate:

$$\begin{aligned} \int_B w_n^2 &\leq \int_{B\setminus P_n} \frac{a^-(x)}{10(q+1)} w_n^{q+1} + \int_{P_n} w_n^2 \qquad (5.7)\\ &\leq \frac{1}{10}\int_B a^-(x)F(w_n) + (\operatorname{meas}(P_n))^{2/N} \|w_n\|_H^2 \\ &\leq \frac{1}{10}\int_B a^-(x)F(w_n) + o(1)\|w_n\|_H^2. \end{aligned}$$

We estimate the remaining term in a similar way,

$$\int_B G(w_n)\,dx \leq \frac{1}{10}\int_B a^-(x)F(w_n)\,dx + o(1)\|w_n\|_H^2. \qquad (5.8)$$

Inserting the above estimates into (5.1), we arrive at the inequality:

$$0 > I_{\lambda,\gamma}(u_n) - I_{\lambda,\gamma}(v_0) \geq \frac{1}{2}\left[(1-o(1))\|w_n\|_H^2 + \|u_n^-\|_H^2\right].$$

For n sufficiently large, the coefficient of $\|w_n\|_H^2$ is positive, and hence $w_n, u_n^- \equiv 0$ for large n. In other words, $u_n = v_n \in M$, and necessarily $I_{\lambda,\gamma}(u_n) \geq I_{\lambda,\gamma}(v_0) = \inf_M I_{\lambda,\gamma}$, which contradicts our choice of u_n. $\square$

Remark 5.3. One can also replace 0 with a non-negative subsolution $\underline{u}$ and obtain the same conclusion for the minimization problem

$$I_{\lambda,\gamma}(v_0) = \inf\{I_{\lambda,\gamma}(v) : \; \underline{u}(x) \leq v(x) \leq \overline{u}(x)\}.$$

For our purposes we will only need the result for $\underline{u} = 0$, and so we leave the (straightforward) details of the extension to the reader.

6. The second solution. When either $\lambda > 0$ or $\gamma > 0$ the functional $I_{\lambda,\gamma}$ is unbounded below. If in addition, $(\mathrm{P})_{\lambda,\gamma}$ admits a local minimizer for such λ, γ we may expect that $(\mathrm{P})_{\lambda,\gamma}$ admits a second solution via the Mountain-Pass Theorem. Indeed, let us seek such a solution in the form $u = u_{\lambda,\gamma} + v$, with $v \geq 0$ and $u_{\lambda,\gamma}$ the solution of $(\mathrm{P})_{\lambda,\gamma}$ obtained by minimization:

$$I_{\lambda,\gamma}(u_{\lambda,\gamma}) = \inf\{I_{\lambda,\gamma}(v) : \ v \in H^1(\Omega), \ 0 \leq v(x) \leq u_{\Lambda_\gamma,\gamma}\}$$

where $u_{\Lambda_\gamma,\gamma}$ is any solution of $(\mathrm{P})_{\Lambda_\gamma,\gamma}$, whose existence is guaranteed by Lemma 4.8. If u is to solve $(\mathrm{P})_{\lambda,\gamma}$, then v should solve

$$-\Delta v = \lambda v + a(x)\left(f(u_{\lambda,\gamma} + v) - f(u_{\lambda,\gamma})\right) + \gamma\left(g(u_{\lambda,\gamma} + v) - g(u_{\lambda,\gamma})\right).$$

Set

$$h(x,v) = a(x)\left(f(u_{\lambda,\gamma} + v^+) - f(u_{\lambda,\gamma})\right) + \gamma\left(g(u_{\lambda,\gamma} + v^+) - g(u_{\lambda,\gamma})\right),$$
$$H(x,v) = \int_0^v h(x,s)\,ds,$$

and define for $v \in H^1(\Omega)$ the functional

$$J_{\lambda,\gamma}(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}v^2 - \frac{1+\lambda}{2}(v^+)^2 - H(x,v)\right) dx.$$

A straightforward calculation shows that

$$J_{\lambda,\gamma}(v) = I_{\lambda,\gamma}(u_{\lambda,\gamma} + v^+) - I_{\lambda,\gamma}(u_{\lambda,\gamma}) + \frac{1}{2}\|v^-\|_H^2. \tag{6.1}$$

In this section (as in the previous one) we make the simplifying assumption that $f(u) = u^q$, $g(u) = u^p$, with $0 < q < 1 < p < \frac{N+2}{N-2}$. With this choice we apply Lemma 5.2 to the identity (6.1) to conclude that $v = 0$ is a local minimum (in $H^1(\Omega)$) for $J_{\lambda,\gamma}$:

Lemma 6.1. *There exists $r > 0$ such that $J_{\lambda,\gamma}(v) \geq 0 = J_{\lambda,\gamma}(0)$ for all $v \in H^1(\Omega)$ with $\|v\|_H < r$.*

We also have:

Lemma 6.2. *Assume either $\gamma > 0$ or $\lambda > 0$. There exists a constant $t > 0$ such that $J_{\lambda,\gamma}(t) < 0$.*

Proof. For $C_0 > 0$ and $s > 1$ there exists $t_s > 0$ so that

$$(t+u)^{s+1} - u^{s+1} \geq \frac{1}{2}t^{s+1},$$

for all $t \geq t_s$ and for all any $u \in \mathbb{R}$ with $0 \leq u \leq C_0$. Since (for λ, γ fixed) $u_{\lambda,\gamma} \geq 0$ is a fixed bounded function, we conclude that

$$G(u_{\lambda,\gamma}+t) - G(u_{\lambda,\gamma}) \geq \frac{1}{2}t^{p+1}, \qquad F(u_{\lambda,\gamma}+t) - F(u_{\lambda,\gamma}) \geq \frac{1}{2}t^{q+1}$$

for t sufficiently large.

Now we calculate

$$\begin{aligned} J_{\lambda,\gamma}(t) &= I_{\lambda,\gamma}(u_{\lambda,\gamma}+t) - I_{\lambda,\gamma}(u_{\lambda,\gamma}) \\ &= \int_\Omega \bigl(-\lambda t u - \frac{\lambda t^2}{2} - a(x)[F(u_{\lambda,\gamma}+t) - F(u_{\lambda,\gamma})] \\ &\qquad - \gamma[G(u_{\lambda,\gamma}+t) - G(u_{\lambda,\gamma})]\bigr) \\ &\leq -\lambda t \int_\Omega u\,dx - \frac{\lambda t^2}{2}\operatorname{meas}(\Omega) + Ct^{q+1} - \gamma C t^{p+1}. \end{aligned} \tag{6.2}$$

If $\lambda > 0$, we simply drop the (nonpositive) term containing γ, and obtain

$$J_{\lambda,\gamma}(t) \leq -\frac{\lambda t^2}{2}\operatorname{meas}(\Omega) + o(t^2),$$

with $o(t^2)/t^2 \to 0$ as $t \to \infty$ uniformly in γ. In this case we may choose t_λ (independently of γ) so that $J_{\lambda,\gamma}(t_\lambda) < 0$.

If $\lambda \leq 0$ but $\gamma > 0$, the highest order term in (6.2) is the term $-\gamma C t^{p+1}$, and again we may choose $t = t(\lambda, \gamma)$ large enough to make $J_{\lambda,\gamma}(t) < 0$. $\square$

Lemma 6.3. *Suppose $\gamma \geq 0$, $\lambda \in \mathbb{R}$, and λ, γ are not both zero. If $\lambda_n \to \lambda$, $\gamma_n \to \gamma$, and v_n is a sequence in $H^1(\Omega)$ such that $J_{\lambda_n,\gamma_n}(v_n) \to c$ and $J'_{\lambda_n,\gamma_n}(v_n) \to 0$, then v_n contains an $H^1(\Omega)$-convergent subsequence. Moreover, $v_n \to v_0 \geq 0$, with $u_0 = u_{\lambda,\gamma} + v_0$ a solution to (P)$_{\lambda,\gamma}$.*

Note that when $\gamma = 0$ solutions of (P)$_{\lambda,0}$ are not *a priori* bounded as $\lambda \to 0$, and so the restriction on λ, γ is to be expected.

Proof. For simplicity, denote $J_n = J_{\lambda_n,\gamma_n}$, $I_n = I_{\lambda_n,\gamma_n}$. First,

$$J'_n(v_n)v_n^- = -\|v_n\|_H^2,$$

and hence $v_n^- \to 0$ in $H^1(\Omega)$. Therefore, if we write $u_n = u_{\lambda_n,\gamma_n} + v_n^+$ we have

$$I_n(u_n) \to I_{\lambda,\gamma}(u_{\lambda,\gamma}) + c, \qquad I_n'(u_n)\varphi = o(1)\|\varphi\|_H, \tag{6.3}$$

for any $\varphi \in H^1(\Omega)$. We claim that $\|u_n\|_2$ is uniformly bounded. Suppose the contrary, $\|u_n\|_2 \to \infty$, then set $w_n = u_n/\|u_n\|_2$.

First, consider the case $\gamma > 0$. Proceeding as in Lemma 4.8, by (6.3) we obtain

$$\|u_n\|_H^2 \le (\lambda+1)\|u_n\|_2^2 + o(1)\|u_n\|_H. \tag{6.4}$$

Then (6.4) implies that $\|w_n\|_H \le C$, and therefore a subsequence (still denoted by w_n) converges weakly, $w_n \rightharpoonup w_0$ in $H^1(\Omega)$, with $\|w_0\|_2 = 1$.

Since $I_n(u_n)$ is bounded,

$$\begin{aligned}
&\frac{\gamma}{p+1}\int_\Omega w_0^{p+1}\,dx = \lim_{n\to\infty}\int_\Omega \frac{\gamma G(u_n)}{\|u_n\|_2^{p+1}}\,dx \\
&\le \lim_{n\to\infty}\frac{1}{\|u_n\|_2^{p-1}}\Big(\frac{1}{2}\|\nabla w_n\|_2^2 - \frac{\lambda}{2}\|w_n\|_2^2 - \int_\Omega a(x)\frac{|w_n|^{q+1}}{(q+1)\|u_n\|_2^{1-q}}\,dx\Big) \le 0,
\end{aligned}$$

which is impossible since $\|w_0\|_2 = 1$. This proves that $\|u_n\|_2$ is bounded, so by (6.4) we have $\|u_n\|_H \le C$, in the case $\gamma > 0$.

If $\gamma = 0$ but $\lambda \neq 0$, we obtain (6.4) by dividing $I_n(u_n)$ by $\|u_n\|_2^2$. Again we may conclude that a subsequence of $w_n \rightharpoonup w_0$ (weakly) in $H^1(\Omega)$, with $\|w_0\|_2 = 1$. For any $\varphi \in H^1(\Omega)$, $I_n'(u_n)\varphi = o(1)$ implies that

$$\int_\Omega \big(\nabla w_0 \cdot \nabla\varphi + [w_0 - (\lambda+1)w_0^+]\varphi\big)\,dx = 0.$$

Hence $w_0 \ge 0$ and λ is an eigenvalue of the Neumann problem for $-\Delta$ in Ω, a contradiction when $\lambda \neq 0$.

In conclusion, $\|u_n\|_2$ is uniformly bounded, and by (6.4) so is $\|u_n\|_H$ uniformly bounded. Extracting a subsequence (again denoted by u_n) we may pass to a weak limit $u_n \rightharpoonup u_0$ in $H^1(\Omega)$, with strong convergence in $L^s(\Omega)$ for $s \in [1, \frac{2N}{N-2})$. Therefore, as $n, m \to \infty$,

$$o(1) = \big(I_n'(u_n) - I_{\lambda,\gamma}'(u_m)\big)\,(u_n - u_m) = \int_\Omega |\nabla(u_n - u_m)|^2 + (u_n + u_m)^2 + o(1),$$

which proves that $u_n \to u_0$ in $H^1(\Omega)$, and consequently $v_n \to v_0$ as promised.

Proposition 6.4. *Suppose $\lambda \in (0, \Lambda_0)$ and $\gamma \in [0, \Gamma_\lambda)$, or $\lambda \le 0$ and $\gamma \in (0, \Gamma_\lambda)$. Then $(P)_{\lambda,\gamma}$ admits a second solution, $w_{\lambda,\gamma} = u_{\lambda,\gamma} + v_{\lambda,\gamma}$ with $0 \not\equiv v_{\lambda,\gamma} \ge 0$.*

Proof. Let $t = t_{\lambda,\gamma}$ be the value given by Lemma 6.2. We set

$$\mathcal{S} = \mathcal{S}_{\lambda,\gamma} = \{\sigma \in C([0,1]; H^1(\Omega)) : \ \sigma(0) = 0, \ \sigma(1) = t_{\lambda,\gamma}\},$$

and $c_{\lambda,\gamma} = \inf_{\sigma\in\mathcal{S}} \max_{s\in[0,1]} J_{\lambda,\gamma}(\sigma(s))$. Lemma 6.1 implies that $c_{\lambda,\gamma} \ge 0$, and the Palais-Smale condition has been verified in Lemma 6.3. If there exists $\rho \in (0, r)$ so that $\inf\{J_{\lambda,\gamma}(v) : \ \|v\|_H = \rho\} > 0$, then $c_{\lambda,\gamma} > 0$ and existence of a second solution follows from the celebrated Mountain-Pass Theorem of Ambrosetti and Rabinowitz. If $\inf\{J_{\lambda,\gamma}(v) : \ \|v\|_H = \rho\} = 0$ for all $\rho \in (0, r)$, then for any fixed $\rho \in (0, r)$ the set $F = \partial B_\rho(0)$ satisfies the hypotheses of Theorem (1) in Ghoussoub and Preiss [13], and their Theorem (1.bis) asserts the existence of a solution for each $\rho \in (0, r)$. □

A. Appendix: Minimal solutions. Here we prove Proposition 4.9. Suppose λ, $\gamma \ge 0$ are such that $(P)_{\lambda,\gamma}$ admits at least one solution. We will construct the minimal solution via the following monotone iteration: Take as v_0 the (unique) solution of $(P)_{\tilde\lambda,0}$, where $\tilde\lambda = \min\{0, \lambda\} \le 0$. If $\lambda \ge -1$, take $\mu = 1$; otherwise choose $\mu = -\lambda$. If $v_0, \dots, v_n$ are already determined, then v_{n+1} should solve

$$\begin{cases} -\Delta v_{n+1} + \mu v_{n+1} + a^-(x) f(v_{n+1}) \\ \qquad\qquad = (\mu + \lambda) v_n + a^+(x) f(v_n) + \gamma g(v_n), \ \text{in } \Omega, \\ \frac{\partial v_{n+1}}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{A.1}$$

Claim1: For any fixed $h \in C^\alpha(\overline{\Omega})$ there exists a unique solution $w \in C^{1,\alpha}(\overline{\Omega})$ to

$$\begin{cases} -\Delta w + \mu w + a^-(x) f(w) = h(x), \ \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{A.2}$$

Indeed, the functional

$$J(w) = \int_\Omega \big(\frac{1}{2}|\nabla w|^2 + \frac{\mu}{2} w^2 + a^-(x) F(w) - h(x) w\big)\, dx$$

is bounded below, coercive, and lower semi-continuous on $H^1(\Omega)$. It therefore attains its minimum value in $H^1(\Omega)$ at a solution of (A.2). By the usual bootstrap argument, $w \in C^{1,\alpha}(\overline{\Omega})$. If there were two solutions w_1, w_2 of (A.2), then subtracting the equations and integrating against $(w_2 - w_1)^+$ gives:

$$\int_\Omega \Big(|\nabla (w_2 - w_1)^+|^2 + \mu [(w_2 - w_1)^+]^2 \\ + a^-(x)(f(w_2) - f(w_1))(w_2 - w_1)^+ \Big)\, dx = 0.$$

Since f is an increasing function, each term is nonnegative and we conclude $w_2 \le w_1$. Reversing the roles of w_2, w_1 we see that $w_2 = w_1$, and Claim 1 is verified.

Claim 2: For each n, $v_n \le v_{n+1}$.

By induction: first take $n = 0$. Subtract the equations to obtain:

$$-\Delta(v_1 - v_0) + \mu(v_1 - v_0) + a^-(x)(f(v_1) - f(v_0)) = (\lambda - \tilde{\lambda})v_0 + \gamma g(v_0) \ge 0.$$

Now if we multiply by $-(v_1 - v_0)^- \le 0$ and integrate, we obtain

$$\int_\Omega |\nabla (v_1 - v_0)^-|^2 + \mu[(v_1 - v_0)^-]^2 + a^-(x)(f(v_0) - f(v_1))(v_0 - v_1)^+ \le 0.$$

As all of the terms in the integral are nonnegative, we must have $v_0 \le v_1$. If $v_{n-1} \le v_n$, then proceeding as above, (A.1) implies:

$$\begin{aligned} &-\Delta(v_{n+1} - v_n) + \mu(v_{n+1} - v_n) + a^-(x)[f(v_{n+1}) - f(v_n)] \\ &= (\mu + \lambda)(v_n - v_{n-1}) + a^+(x)[f(v_n) - f(v_{n-1})] + \gamma[g(v_n) - g(v_{n-1})] \ge 0. \end{aligned}$$

As above, we multiply by $-(v_{n+1} - v_n)^-$ to complete the argument.

Claim 3: If u is any solution to $(\mathrm{P})_{\lambda,\gamma}$, then for all $n \ge 0$, $v_n \le u$.

The fact that $u \ge v_0$ follows from Lemma 4.6. For $n \ge 1$ we use induction as in the proof of Claim 2. We leave the details to the interest reader.

As a result of the three Claims, there exists a monotone sequence, $v_0 \le v_n \le u$ for any solution u of $(\mathrm{P})_{\lambda,\gamma}$. In particular, $v_n \to u_{\lambda,\gamma} \le u$ pointwise in Ω. Since v_n are uniformly bounded by any fixed solution u, from the equations (A.1) we see that $\|v_n\|_{C^{1,\alpha}}$ is bounded as well. In particular, we may pass to the limit in the weak form of (A.1) to obtain that $u_{\gamma,\lambda}$ solves $(\mathrm{P})_{\lambda,\gamma}$. Hence $u_{\gamma,\lambda}$ is the desired minimal solution.

To obtain monotonicity in λ, γ, note that when $\lambda \le \lambda'$ and $\gamma \le \gamma'$, the minimal solution $u_{\lambda',\gamma'}$ is a supersolution for $(\mathrm{P})_{\lambda,\gamma}$. By Lemma 4.1 there exists a solution u for $(\mathrm{P})_{\lambda,\gamma}$, with $u_{\gamma,\lambda} \le u \le u_{\lambda',\gamma'}$. $\square$

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