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SEMILINEAR PARABOLIC PROBLEMS DEFINE SEMIFLOWS ON C^k SPACES

BY

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ABSTRACT. Linear parabolic problems of a general class are proved to determine analytic semigroups on certain closed subspaces of $C^k(\overline{\Omega})$ (k integer); $C^k(\overline{\Omega})$ denotes the space of functions whose derivatives or order $\leqslant k$ are bounded and uniformly continuous, with the usual supremum norm; the closed subspaces where the semigroups are obtained, denoted by $\hat{C}^k(\overline{\Omega})$, are determined by the boundary conditions and a possible condition at infinity. One also obtains certain embedding relations concerning the fractional power spaces associated to these semigroups. Usually, results of this type are based upon the theory of solution of elliptic problems, while this work uses the corresponding theory for parabolic problems. The preceding results are applied to show that certain semilinear parabolic problems define semiflows on spaces of the type $\hat{C}^k(\overline{\Omega})$.

In this paper we show that certain general semilinear parabolic systems of partial differential equations define semiflows on spaces of k times continuously differentiable functions. For the resulting semiflows we also prove the validity of the "principle of linearized stability", i.e. that the stability of a stationary state u_0 is determined by the spectrum of the linearization about u_0 (more precisely, by the location of this spectrum with respect to the imaginary axis).

Let us indicate a typical example where our results will be applicable. The equations will describe the temporal evolution of a system distributed over a certain spatial region $\Omega \subset \mathbf{R}^n$, possibly unbounded. The state of the system is specified by a collection of $N + N'$ variables, functions of position $x \in \Omega$ and time $t \in \mathbf{R}$. We shall assemble these variables into two vectors $u = (u^1,\ldots,u^N)$ and $v = (v^1,\ldots,v^{N'})$, although in general we allow that $N' = 0$, in which case the vector v is empty. The temporal evolution of the system is considered to obey a system of differential equations of the following form:

$$\text{(Xa)} \qquad \begin{aligned} D_t u &= \nabla \cdot (K(x)\nabla u) + f(x; u, \nabla u, v), \\ D_t v &= g(x; u, \nabla u, v), \end{aligned}$$

where D_t represents the time derivative, and ∇ and $\nabla\cdot$ are the gradient and divergence operators in $\mathbf{R}^n$. Here, K is a positive definite $N \times N$ matrix, and f, g are nonlinear functions of u, ∇u, v which, in general, can also depend on $x \in \Omega$. The

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temporal evolution is also restricted by certain boundary conditions on the variables u^i; we shall assume that they are of the following form:

$$\text{(Xb)}\qquad \begin{aligned} u^i\big|_{\partial\Omega} &= b^i(x) \qquad (i = 1,\ldots,S),\\ D_\nu u^i\big|_{\partial\Omega} &= p^i(x)\big(b^i(x) - u^i\big|_{\partial\Omega}\big) \qquad (i = S+1,\ldots,N), \end{aligned}$$

where $\cdot\big|_{\partial\Omega}$ denotes the restriction to $\partial\Omega$, the boundary of Ω, and D_ν denotes the spatial derivative in the direction of the outward normal to $\partial\Omega$. Here, the b^i can be considered as forming a vector of N components, which in general might depend on $x \in \partial\Omega$, and the coefficients p^i, which also might depend on $x \in \partial\Omega$, will be assumed to be positive: $p^i(x) \geqslant 0$ ($\forall x \in \partial\Omega$, $i = S+1,\ldots,N$). Lastly, in the case of unbounded domains, the problem can optionally include a condition at infinity of the form

$$\text{(Xc)}\qquad u\big|_\infty = b_\infty,$$

where $\cdot\big|_\infty$ denotes the limit as $|x| \to \infty$, and b_∞ is a constant N-component vector. In general, when $\partial\Omega$ has several components, then the boundary conditions (Xb) could be different on each one, and when the domain Ω has several "exits to infinity" (for example a tube), then the condition at infinity (Xc) could be different for each one of them. In what follows, "problem (X)" means either (Xa), (Xb), (Xc), or (Xa), (Xb).

In particular, the problem just described includes the case of the so-called reaction-diffusion equations, which model very interesting phenomena of morphogenesis and propagation of waves in chemistry and several fields of biology (see for instance Nicolis and Prigogine **[16]** and Murray **[15]**).

To provide a framework for the mathematical study of these phenomena, it is interesting to show that the corresponding equations determine semiflows on certain function spaces, and to verify fundamental properties like the "principle of linearized stability".

Questions of this type have been studied by a number of authors in the general context of abstract evolution equations of semilinear type; i.e. equations of the form

$$\text{(II)}\qquad \dot{u} = Au + F(u),$$

where u is now a variable taking values in a Banach space E, $\dot{u}$ denotes the time derivative of u in the sense $\dot{u}(t) \equiv E\text{-}\lim_{h\to 0} h^{-1}(u(t+h) - u(t))$, A is a linear operator on E, with domain D, and F is a nonlinear operator $X \to E$, where X is a Banach space between D and E. The linear operator A will be assumed to be the generator of a semigroup e^{At} on the space E. A basic reference on the study of abstract evolution equations like (II) from the qualitative-dynamical point of view is Henry **[5]**. That author shows that, under certain conditions relative to A and F, the solutions of (II) define semiflows on some of the fractional power spaces D^α ($0 \leqslant \alpha < 1$), where D^α is defined as the domain of the fractional power B^α ($B \equiv \omega I - A$, ω sufficiently large) endowed with the graph norm. In §1.5 we collect the main results in a slightly generalized form, where the semiflow is obtained on spaces X that are not necessarily of the form D^α. The proofs for this general case are easy generalizations of the ones in the case $X = D^\alpha$; they are included in Mora **[14]**

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and the proof of Theorem 1.1 can also be found in Weissler [**30**]. We remark here that in the general case one needs to assume that the semigroup e^{At} is analytic and that it restricts itself to a semigroup on the space X; for the analysis of the stability of stationary states it is also desirable that the semigroup restricted to X be analytic too, or at least differentiable.

To apply these results to problems like (X) one is faced with the question of verifying that strongly elliptic operators with homogeneous boundary conditions are generators of (analytic) semigroups on suitable function spaces. For example, in the case of problem (X) we have to consider the matrix operator $\nabla \cdot (K \nabla)$ with the homogeneous boundary conditions corresponding to (Xb) (for the moment we shall forget the condition at infinity). When K is a diagonal matrix, the question of generating an (analytic) semigroup reduces to the scalar case ($N = 1$), where it has been proved on the spaces $L_p(\Omega)$ $(1 < p < \infty)$ (see for example Tanabe [**26**, §3.8]), and also on spaces of continuous functions vanishing at infinity (see Stewart [**24**, **25**]). Let us remark that this property is not true in the case of Hölder spaces, where von Wähl [**29**] and Kielhöfer [**8**, **9**] have shown that the corresponding semigroups are not of class C_0. Most of the present paper is spent in showing the analytic semigroup property on spaces of k times continuously differentiable functions, either vanishing at infinity or not. In general, these results will be obtained for operators A corresponding to general strongly elliptic systems, in particular including the case corresponding to problem (X) (with general nondiagonal K).

Usually, the (analytic) semigroup property is obtained by working with the resolvent operator $(\lambda I - A)^{-1}$, for which one uses the theory of solution of the corresponding elliptic problem. In this paper we shall use a different approach, namely to work directly on the evolution operator e^{At} by using the theory of solution of the corresponding parabolic problem (some precedent of this approach can be found in Sobolevskiĭ [**19**, **20**]). In particular, this method is specially suitable for C^k-spaces (with or without the condition of vanishing at infinity). In addition to the analytic semigroup property, it also provides very useful relations between the fractional power spaces D^α $(0 \leqslant \alpha \leqslant 1)$ and the spaces $C^{k+\eta}(\overline{\Omega})$ $(0 \leqslant \eta \leqslant 2m)$ ($2m$ is the order of the elliptic operator).

These results will be used to treat semilinear parabolic systems like (X) (and much more general ones). We shall show that they define semiflows on certain spaces of the type $C^k(\overline{\Omega})$, and we shall obtain a corresponding version of the principle of linearized stability. The general class of problems where our results will be applicable is described in §1.3, and the main results of the paper are Theorems 2.4, 2.5, 3.1, 4.1 and 4.2; Corollaries 4.1 and 4.2 illustrate the application of Theorems 4.1 and 4.2 to certain particular cases of problem (X).

1. Some definitions and other preliminaries.

1.1. By a semiflow we mean what is usually called a "local semiflow" or a "local semidynamical system"; i.e. we restrict our consideration to positive times, and we allow trajectories going to infinity in a finite time. This situation is really sufficient for all the essential notions and results of qualitative dynamics to be applicable (see for example Bhatia and Hájek [**3**]). Specifically, given a topological space W, by a

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semiflow on W we mean a mapping

$$\phi : \Gamma \subset (\mathbf{R}_+ \times W) \to W, \\ (t, u) \to \phi_t u,$$

with Γ of the form $\{(t, u) \mid u \in W,\ t \in [0, \omega(u))\}$ $(0 < \omega(u) \leqslant \infty)$, and satisfying the following properties $(\forall u \in W)$:

(a) Semigroup property $\phi_0 u = u$, and $\phi_{t+s} u = \phi_t \phi_s u$; this second equality is understood in the sense that, if one of the sides is defined, then the other is defined also and the equality holds.

(b) Continuity with respect to the time: the mapping $t \mapsto \phi_t u$ is continuous from $[0, \omega(u))$ to W.

(c) Continuity with respect to the initial state: the mapping $u \mapsto \omega(u)$ is lower semicontinuous from W to $\overline{\mathbf{R}}_+$ or, equivalently, for any $t > 0$, the set $\Gamma_t \equiv \{u \mid t < \omega(u)\}$ is open in W; furthermore, the mapping $u \mapsto \phi_t u$ is continuous from Γ_t to W $(\forall t > 0)$.

(d) Maximality: if $\omega(u) < \infty$, then the orbit of u, i.e. the set $\{\phi_t u \mid 0 \leqslant t < \omega(u)\}$, is not contained in any compact subset of W.

In fact one can prove that the maximality property (d) follows automatically from the other conditions (see Bhatia and Hájek [**3**, §1.10]). When $\omega(u) = \infty$ for every $u \in W$, we will say that the semiflow is global. If W is an open set in a Banach space and the mappings $u \mapsto \phi_t u$ are of class C^r, then the semiflow is said to be of class C^r.

1.2. In what follows, Ω denotes a domain (i.e. an open connected set) in the n-dimensional euclidean space $\mathbf{R}^n$ $(n \geqslant 1)$. In general, both Ω and $\partial\Omega$ may be unbounded. As usual, $C(\overline{\Omega})$ will denote the Banach space consisting of those functions on Ω which are uniformly continuous and bounded, with norm given by

$$\| u \|_C \equiv \sup_{x \in \Omega} | u(x) | .$$

For integer $k \geqslant 0$, $C^k(\overline{\Omega})$ denotes the Banach space of those functions u such that $D^\nu u \in C(\overline{\Omega})$ for $|\nu| \leqslant k$, with norm given by

$$\| u \|_{C^k} = \sum_{|\nu| \leqslant k} | D^\nu u |_C .$$

Obviously, we have the embeddings $C^l(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega})$ for $k < l$; when Ω is bounded these embeddings are compact. For noninteger $\sigma > 0$, say $\sigma = k + \eta$ (k integer, $0 < \eta < 1$), $C^\sigma(\overline{\Omega})$ will denote the Banach space consisting of those functions belonging to $C^k(\overline{\Omega})$ and such that the derivatives $D^\nu u$ of order $|\nu| = k$ satisfy a uniform Hölder condition of exponent η; as usual, the norm in this space is defined as

$$\| u \|_{C^\sigma} = \| u \|_{C^k} + \sum_{|\nu| = k} \{ D^\nu u \}_\eta$$

where

$$\{ v \}_\eta \equiv \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{| v(x) - v(y) |}{| x - y |^\eta} .$$

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We have the obvious embeddings $C^{k+\zeta}(\overline{\Omega}) \hookrightarrow C^{k+\eta}(\overline{\Omega}) \hookrightarrow C^{k}(\overline{\Omega})$ (k integer $\geqslant 0$, $0<\eta<\zeta<1$); they are compact whenever Ω is bounded. We shall also need to use the embeddings $C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k+\eta}(\overline{\Omega})$ (k integer $\geqslant 0$, $0<\eta<1$); when $n>1$, to have these embeddings one needs the domain Ω to satisfy the following condition:

(D) There exists a finite constant L such that any two points $x, y \in \Omega$ can be joined by a polygonal arc entirely located in Ω and of length $d \leqslant L|x-y|$.

In the case where Ω is bounded, condition (D) is automatically true if Ω is a domain of class $C^{1+\alpha}$.

With respect to the notation of function spaces and norms, we shall use the same symbols when referring to functions valued in $\mathbf{R}$ or $\mathbf{R}^N$; the exact meaning will always be clear from the context. Let the letter Y denote a generic function space; in the case of N-component functions we understand

$$u \in Y \equiv u^i \in Y(1 \leqslant i \leqslant N); \qquad \|u\|_Y \equiv \sum_{i=1}^{N} \|u^i\|_Y,$$

where on the right-hand sides, Y stands for the corresponding space of scalar functions.

1.3. *General problem*. In fact, our treatment will apply to problems much more general than (X). Specifically, we shall cover problems of the following general form:

$$\text{(I)}\quad \begin{aligned} &\text{(Ia)} \quad D_t u = -\mathcal{A}(x, D_x)u + f\big(x; u, D_x u, \ldots, D_x^l u, v\big),\\ &\text{(Ib)} \quad D_t v = g\big(x; u, D_x u, \ldots, D_x^l u, v\big),\\ &\text{(Ic)} \quad \mathcal{B}(x, D_x)u\,\big|_{\partial\Omega} = 0,\\ &\text{(Id)} \quad \big[u\,\big|_{\infty} = 0\big], \end{aligned}$$

where u, v are vectors with N and N' components, respectively ($N>0$, $N' \geqslant 0$), functions of $x \in \Omega$ and $t \in \mathbf{R}$, and the square brackets are used to indicate optional conditions. Here, Ω, $\mathcal{A}$, $\mathcal{B}$ are described by the following conditions:

(L1) Ω is a domain in $\mathbf{R}^n$ ($n \geqslant 1$); if $n>1$ we assume that Ω is uniformly of class $C^{1+\alpha}$ for some $\alpha>0$ (see, for example, Ladyzhenskaya et al. [**12**, Chapter IV, §4]), and that it satisfies condition (D) of §1.2.

(L2) $\mathcal{A}$ is given by an $N \times N$ matrix of differential operators of order $\leqslant 2m$. The elements of the matrix $\mathcal{A}$ will be denoted by $\mathcal{A}_{ij}$, and the coefficients of the differential operator $\mathcal{A}_{ij}$ will be denoted by $a_{ij\nu}$ ($0 \leqslant |\nu| \leqslant 2m$), i.e.

$$\mathcal{A}_{ij}(x, D_x) = \sum_{|\nu| \leqslant 2m} a_{ij\nu}(x) D_x^{\nu}.$$

$\mathcal{A}$ is assumed to be a strongly elliptic operator in the sense of Petrovskii and Vishik (see, for example, Ladyzhenskaya et al. [**12**, Chapter VII, §8, Definition 7]), and both the ellipticity and the strong ellipticity are assumed to be uniform with respect to $x \in \Omega$.

(L3) $\mathcal{B}$ is given by an $mN \times N$ matrix of differential operators, i.e. equation (Ic) represents a set of mN boundary conditions $\mathcal{B}_i(x, D_x)u\,|_{\partial\Omega} = 0$ ($1 \leqslant i \leqslant mN$) where the $\mathcal{B}_i$ denote the rows of $\mathcal{B}$. The ith row consists of operators $\mathcal{B}_{ij}$ of order less than or equal to a certain integer m_i, and the numbers m_i are assumed to be in the

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interval $0 \leqslant m_i \leqslant 2m-1$. The coefficients of the operator $\mathcal{B}_{ij}$ will be denoted by $b_{ij\nu}$ $(0 \leqslant |\nu| \leqslant m_i)$, i.e.

$$\mathcal{B}_{ij}(x, D_x) = \sum_{|\nu| \leqslant m_i} b_{ij\nu}(x) D_x^\nu.$$

$\mathcal{B}$ is assumed to satisfy the parabolic complementing condition (with respect to $\mathcal{A}$ and $\partial\Omega$) (see, for example, Solonnikov [**21**, §1] or Ladyzhenskaya et al. [**12**, Chapter VII, §9]).

($\mathrm{L4}_s$) The boundary $\partial\Omega$, and the coefficients $a_{ij\nu}$, $b_{ij\nu}$ satisfy the following smoothness conditions:

$\partial\Omega$ is uniformly of class $C^{s+2m+\alpha}$ (if $n > 1$);

$a_{ij\nu} \in C^{s+\alpha}(\overline{\Omega})$;

$b_{ij\nu} \in C^{s+2m-m_i+\alpha}(\overline{\partial\Omega})$ (if $n > 1$),where s is a nonnegative integer, that we shall fix afterwards, and α is some real number in the interval $0 < \alpha < 1$.

On the other hand, f and g stand for nonlinear functions of v, u, and the spatial derivatives $D_x u$ of order $|\nu| \leqslant l$, where $0 \leqslant l \leqslant 2m-1$; the concrete conditions assumed on f and g will be described in §4. We shall refer to (Ib) as boundary conditions, to (Ic) as conditions at infinity, and to both of them as *accessory* conditions. The condition at infinity is optional (for unbounded Ω), and when included it has to be understood in the following sense:

$$\sup_{x \in \Omega - B_k} |u^i(x)| \to 0 \quad \text{as } k \to \infty \qquad (1 \leqslant i \leqslant N)$$

where B_k denotes the ball of radius k, and k runs over the positive integers.

Although problem (I) considers only homogeneous accessory conditions, in the applications it will frequently come from an analogous problem with nonhomogeneous accessory conditions; namely with

$$\text{(Ic}') \qquad \mathcal{B}(x, D_x) u \,|_{\partial\Omega} = b(x),$$

$$\text{(Id}') \qquad \left[u \,|_\infty = b_\infty \right]$$

substituted for (Ic), (Id). Here, b is an mN-vector that in general could depend on $x \in \partial\Omega$, and b_∞ is a constant N-vector (more generally, the boundary conditions could be different on each connected component of $\partial\Omega$, and the conditions at infinity could be different on every "exit to infinity" of Ω). The reduction of the nonhomogeneous problem to an analogous homogeneous one would involve a mere change of variable $u(\text{new}) = u(\text{old}) - u_b$, where u_b is any sufficiently smooth function on Ω satisfying (Ic′), (Id′).

In the particular case of problem (X),

$$\mathcal{A}(x, D_x) = -\nabla \cdot (K(x) \nabla) \quad \text{and} \quad \mathcal{B}(x, D_x) = Q(\nu(x) \cdot \nabla) + P(x),$$

where

$$Q \equiv \operatorname{diag}\left(0, \overset{(S)}{\ldots}, 0, 1 \overset{(N-S)}{,\ldots,} 1 \right)$$

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and

$$P(x) \equiv \operatorname{diag}\left(\overset{(S)}{1, \ldots, 1}, \overset{(N-S)}{p^{S+1}(x), \ldots, p^N(x)}\right)$$

(therefore, $m = 1$, $m_i = 0$ for $i = 1, 2, \ldots, S$, and $m_i = 1$ for $i = S + 1, \ldots, N$). For this particular case, condition (L2) is ensured if K satisfies the following condition (M2), condition (L3) is satisfied automatically, and condition ($L4_s$) reduces to ($M4_s$) below:

(M2) The matrix K is positive definite, uniformly with respect to x; i.e. there exists a constant $\kappa > 0$ such that

$$(K(x)\xi, \xi) \geqslant \kappa |\xi|^2 \quad (\forall \xi \in \mathbf{R}^N, \forall x \in \Omega);$$

furthermore, there exist $\kappa_1, \kappa_2 > 0$ such that

$$\kappa_1 \leqslant \det K(x) \leqslant \kappa_2 \quad (\forall x \in \Omega).$$

($M4_s$) The boundary $\partial\Omega$ and the coefficients $k_{ij}(x)$, $p^i(x)$ satisfy the following smoothness conditions:

$\partial\Omega$ is uniformly of class $C^{s+2+\alpha}$ (if $n > 1$);

$k_{ij} \in C^{s+1+\alpha}(\overline{\Omega})$;

$p^i \in C^{s+1+\alpha}(\overline{\partial\Omega})$ (if $n > 1$).

1.4. *Some definitions and facts from semigroup theory.* We collect here some definitions and facts that we need to use from semigroup theory. Good explanations of this theory can be found, for example, in Kreĭn [**11**], Pazy [**17**], or Tanabe [**26**].

Given a Banach space E, by a *semigroup* on E we shall understand a family ϕ_t ($t \in \mathbf{R}_+$) of bounded linear operators satisfying the following properties:

(a) $\phi_0 = I$, $\phi_{t+s} = \phi_t\phi_s = \phi_s\phi_t$;

(b) $\phi_t u \to u$ as $t \to 0$ ($\forall u \in E$).

It follows that the operators ϕ_t can always be bounded in the following way:

$$\|\phi_t\| \leqslant Me^{\omega t} \quad (\forall t \geqslant 0),$$

where $\omega \in \mathbf{R}$ and $M \geqslant 1$. The infimum of the ω for which a bound of this type is true is called the *order of growth* of the semigroup; we shall usually denote it by ω_0.

Let us consider a linear evolution equation in a Banach space; i.e. an equation of the form

$$\dot{u} = \mathfrak{A} u \tag{1.1}$$

where the unknown u is a function from (a subset of) $\mathbf{R}_+$ to a Banach space E, $\dot{u}$ denotes its derivative with respect to the time (as a Banach space valued function), and $\mathfrak{A}$ is a linear operator on E which we shall assume has a dense domain $\mathfrak{D}$. Naturally, a solution of (1.1) in a certain interval J means a curve $u\colon J \to E$ such that, for every $t \in J$, $u(t) \in \mathfrak{D}$, u is differentiable at time t, and it satisfies equation (1.1). Following Kreĭn [**11**, Chapter I], we shall say that the Cauchy problem for (1.1) is *uniformly correctly posed* if there exists some $T > 0$ such that, for every $u_0 \in \mathfrak{D}$, (1.1) has a unique solution in $[0, T]$ with initial state u_0 (i.e. satisfying $u(0) = u_0$), and furthermore the solution depends continuously on the initial state in the

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following sense: if $u_{0,n} \to 0$ then the corresponding solutions u_n satisfy $u_n(t) \to 0$ $(0 \leqslant t \leqslant T)$ with the convergence being uniform with respect to t. When the Cauchy problem for (1.1) is uniformly correctly posed, the operators ϕ_t: $u_0 \mapsto u(t)$ can be extended uniquely to continuous operators defined on the whole space E, and so extended, they constitute a semigroup on E.

The *generator* of a semigroup ϕ_t is defined as the linear operator A given by

$$Au \equiv \lim_{t \to 0} \frac{1}{t}(\phi_t u - u)$$

with the domain D consisting of the $u \in E$ for which this limit exists. The generator is always a closed and densely defined operator. We shall often use the exponential notation e^{At} to denote a semigroup whose generator is the operator A. A basic property of the generator is that, for $u \in D$, the curves $t \mapsto \phi_t u$ are solutions of the differential equation

$$\dot{u} = Au \tag{1.2}$$

in any interval of the form $[0, T]$. Furthermore, for fixed $u \in D$, $\phi_t u$ is the only solution of this equation with initial state u. Therefore, the Cauchy problem for equation (1.2) is uniformly correctly posed. If ϕ_t is the semigroup determined by the evolution equation (1.1), where we suppose that the corresponding Cauchy problem is uniformly correctly posed, then the generator A of the semigroup is exactly the closure of the operator $\mathfrak{A}$ (Kreĭn [**11**, Chapter I, Theorem 2.6]).

A semigroup e^{At} is called *differentiable* when $\forall u \in E$, the mapping $t \mapsto e^{At}u$ is differentiable in the open interval $(0, \infty)$; for this it suffices that $e^{At}u \in D$ holds for every $u \in E$ and all $t > 0$, where D denotes the domain of A; in that case, the derivative is necessarily given by $D_t e^{At}u = Ae^{At}u$.

An important class of semigroups are the so-called analytic semigroups. A semigroup e^{At} is *analytic* if and only if it is differentiable and the derivative satisfies a bound of the form

$$\|D_t e^{At}u\|_E \leqslant M_1 t^{-1} e^{\omega t} \|u\|_E \quad (\forall t > 0), \tag{1.3}$$

where $\omega \in \mathbf{R}$ and $M_1 \geqslant 0$.

LEMMA 1.1. *In order that a semigroup e^{At} be analytic it suffices that it be differentiable and that the derivative satisfies a bound of the form*

$$\|D_t e^{At}u\|_E \leqslant Ct^{-1} \|u\|_E \quad (\forall t \in (0, T]) \tag{1.4}$$

for some $T > 0$. $\square$

PROOF. We shall show that (1.4) implies (1.3). This can be seen by using the semigroup property on the semigroup

$$\psi_t \equiv e^{-\omega t} e^{At}, \tag{1.5}$$

where $\omega \in \mathbf{R}$ will be taken large enough so that ψ_t has strictly negative order of growth, i.e. $\exists \xi > 0$ such that

$$\|\psi_t\| \leqslant Me^{-\xi t} \quad (\forall t > 0). \tag{1.6}$$

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Differentiating relation (1.5)

$$\psi_t' u = -\omega \psi_t u + e^{-\omega t} D_t e^{At} u, \tag{1.7}$$

and using (1.4) we obtain that

$$\| \psi_t' u \|_E \leqslant C' t^{-1} \| u \|_E \quad (\forall t \in (0, T]). \tag{1.8}$$

This relation can be extended to an analogous one for $t \in (0, \infty)$ by using the semigroup property; this property implies that $\psi'_{r+s} = \psi_r' \psi_s = \psi_s' \psi_r$, so that

$$(r + s) \psi'_{r+s} u = r \psi_r'(\psi_s u) + s \psi_s'(\psi_r u).$$

From here, application of (1.8) and (1.6) gives

$$\| (r + s) \psi'_{r+s} u \|_E \leqslant C'(M e^{-\xi s} + M e^{-\xi r}) \| u \|_E.$$

Restricting r and s to be $r, s \leqslant T/2$, and taking ω large enough so that $M \exp(-\xi T/2) \leqslant \frac{1}{2}$, we obtain that

$$\| \psi_t' u \|_E \leqslant C' t^{-1} \| u \|_E \quad (\forall t \in (0, 2T]),$$

where the constant C' is the same as in (1.8). By induction it follows that

$$\| \psi_t' u \|_E \leqslant C' t^{-1} \| u \|_E \quad (\forall t \in (0, \infty))$$

and, going back to e^{At} through (1.7),

$$\| D_t e^{At} u \|_E \leqslant M_1 t^{-1} e^{\omega t} \| u \|_E \quad (\forall t \in (0, \infty)). \quad \text{Q.E.D.}$$

As is well known, analytic semigroups can also be characterized in terms of their generator A; for example, given a closed and densely defined operator A, it is the generator of an analytic semigroup if and only if the resolvent set of A includes a half-plane $\operatorname{Re} \lambda \geqslant \omega$ ($\omega \in \mathbf{R}$), where the resolvent operator satisfies an estimate of the form $\| (\lambda I - A)^{-1} \| \leqslant C/(1 + |\lambda - \omega|)$.

In connection with semilinear equations like (II), an important role is played by the so-called *fractional power spaces* associated to the semigroup e^{At}. These spaces, which we shall denote as D^α, depend on a continuous parameter α, and for α running from 0 to 1 they form a continuous scale of Banach spaces that goes from $D^0 = E$ to $D^1 = D$, and such that, for $\alpha < \beta$, D^β is densely embedded in D^α. For $\alpha \geqslant 0$, the spaces D^α are defined as the domains of the operators B^α, where $B \equiv \omega I - A$ and ω is a real number bigger that ω_0 (ω_0: the order of growth of the semigroup e^{At}); the norm in D^α is given by

$$\| u \|_{D^\alpha} \equiv \| B^\alpha u \|_E$$

(the norms corresponding to different values of ω turn out to be equivalent). The fractional powers B^α can be defined in the following way: First one defines the negative fractional powers $B^{-\alpha}$ ($\alpha > 0$) by the formula

$$B^{-\alpha} u \equiv \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha - 1} e^{-Bs} u \, ds.$$

Of course, for $\alpha = 0$ we put $B^0 \equiv I$. The operators $B^{-\alpha}$ are bounded, they satisfy $B^{-(\alpha+\beta)} = B^{-\alpha} B^{-\beta}$ ($\forall \alpha, \beta \geqslant 0$), and when α is an integer, say $\alpha = n$, then $B^{-\alpha}$ coincides with B^{-n}, understood in the sense of the nth power of the inverse

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operator of B; furthermore, the operators $B^{-\alpha}$ ($\alpha \geqslant 0$) commute with both e^{-Bt} and $(\lambda I + B)^{-1}$. Then one defines the positive fractional powers B^{α} ($\alpha \geqslant 0$) as the inverses of $B^{-\alpha}$, with domain given by the range of $B^{-\alpha}$:

$$B^{\alpha} \equiv (B^{-\alpha})^{-1}, \quad D(B^{\alpha}) \equiv R(B^{-\alpha}) \quad (\forall \alpha \geqslant 0).$$

From these definitions it follows that the spaces D^{α} ($\alpha \geqslant 0$) satisfy the properties already mentioned. Furthermore, from the commutability of B^{α} with e^{At} and A, it follows easily that the semigroup e^{At} restricts to a semigroup on each of the spaces D^{α} ($\alpha \geqslant 0$), and if the semigroup on E is analytic then the ones on D^{α} ($\alpha \geqslant 0$) have the same character.

When the semigroup e^{At} is analytic, one has the crucial property

$$\| e^{At} \|_{E \to D^{\alpha}} \leqslant M_{\alpha} t^{-\alpha} e^{\omega t} \quad (\forall t > 0).$$

LEMMA 1.2. *Let e^{At} be a semigroup on a Banach space E. In order that $D^{\alpha} \hookrightarrow X$ (X being a Banach space), it suffices that $e^{At}u \in X$ ($\forall u \in E$, $\forall t > 0$) with a bound of the form*

$$\| e^{At} u \|_X \leqslant C t^{-\beta} e^{\omega t} \| u \|_E \quad (\forall t > 0) \tag{1.9}$$

for some $\beta < \alpha$. □

PROOF. Let $B \equiv \hat{\omega} I - A$, where $\hat{\omega} > \omega$. Let $v \in D^{\alpha}$, and let $w \in E$ be such that $v = B^{-\alpha} w$. From the definitions it follows that

$$v = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha - 1} e^{-\hat{\omega} s} e^{As} w \, ds. \tag{1.10}$$

From the hypothesis of the lemma we know that $e^{As} w \in X$ ($\forall s > 0$), and using the bound (1.9) on (1.10) we obtain that

$$\| v \|_X \leqslant \frac{C}{\Gamma(\alpha)} \int_0^{\infty} s^{(\alpha - \beta) - 1} e^{-(\hat{\omega} - \omega) s} \, ds \, \| w \|_E \leqslant C' \| w \|_E,$$

i.e. the integral (1.10) converges in X, and $\| v \|_X \leqslant C' \| v \|_{D^{\alpha}}$. Q.E.D.

LEMMA 1.3. *Let e^{At} be a semigroup on a Banach space E. To have a bound of the form* (1.9), *it suffices to have it on a bounded interval of the form* $(0, T]$; *i.e. to have*

$$\| e^{At} u \|_X \leqslant C t^{-\beta} \| u \|_E \quad (\forall t \in (0, T]) \tag{1.11}$$

for some $T > 0$. □

PROOF. Let ψ_t be defined as in the proof of Lemma 1.1. From (1.11) we obtain that

$$\| \psi_t u \|_X \leqslant C t^{-\beta} \| u \|_E \quad (\forall t \in (0, T]).$$

Consider now the expression $(r + s)^{\beta} \psi_{r+s} u$; using the fact that

$$(r + s)^{\beta} \leqslant K(\beta)(r^{\beta} + s^{\beta}) \qquad \big(K(\beta) = 1 \text{ if } 0 \leqslant \beta \leqslant 1, K(\beta) = 2^{\beta - 1} \text{ if } \beta \geqslant 1\big),$$

we have that

$$\begin{aligned} \|(r + s)^{\beta} \psi_{r+s} u \|_X &\leqslant K(\beta)\big(\| r^{\beta} \psi_{r+s} u \|_X + \| s^{\beta} \psi_{r+s} u \|_X \big) \\ &\leqslant K(\beta) C (\| \psi_s u \|_E + \| \psi_r u \|_E) \\ &\leqslant K(\beta) C (M e^{-\xi s} + M e^{-\xi r}) \| u \|_E. \end{aligned}$$

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Taking $r, s \leqslant T/2$, and ω large enough so that $M\exp(-\xi T/2) \leqslant (2K(\beta))^{-1}$, we obtain that

$$\|\psi_t u\|_X \leqslant C t^{-\beta} \|u\|_E \quad (\forall t \in (0, 2T]).$$

Proceeding by induction, this estimate can be extended to hold $\forall t \in (0, \infty)$, from which follows the estimate (1.9) for e^{At}. Q.E.D.

1.5. *Basic results on abstract evolution equations of semilinear type*. Let us consider the abstract equation

$$\text{(II)} \qquad \dot{u} = Au + F(u).$$

In order to obtain a semiflow on the space X, one needs the following hypotheses:

(H1) The linear operator A is the generator of an analytic semigroup e^{At} on the Banach space E.

(H2) X is a Banach space between D and E, i.e. satisfying $D \hookrightarrow X \hookrightarrow E$, and it is such that the semigroup e^{At} satisfies a property of the form

$$\|e^{At}u\|_X \leqslant C t^{-\alpha} e^{\omega t} \|u\|_E \quad (\forall t > 0, \forall u \in E)$$

for some α in the interval $0 \leqslant \alpha < 1$.

(H3) The semigroup e^{At} restricts to a semigroup on the space X.

(H4) The nonlinear operator F maps X to E, and it is Lipschitz on bounded sets from X to E.

REMARKS. 1. From Lemma 1.2 it follows that hypothesis (H2) is equivalent to the following:

(H2′) X is a Banach space such that $D^\beta \hookrightarrow X \hookrightarrow E$ for some β in the interval $0 \leqslant \beta < 1$.

2. In the situation $X \hookrightarrow E$, to verify hypothesis (H3) it suffices to verify the following two conditions:

(H3a) For every $t \leqslant 0$, there exists an M such that $\|e^{At}\|_{X \to X} \leqslant M$.

(H3b) For every $u \in X$, $\|e^{At}u - u\|_X \to 0$ as $t \to 0$.

3. In particular, both hypotheses (H2) and (H3) are automatically satisfied in the case $X = D^\alpha$ $(0 \leqslant \alpha < 1)$. In that case, properties (H3a) and (H3b) follow from the commutability of B^α with e^{At} ($B \equiv \omega I - A$, ω large enough).

THEOREM 1.1. *Under hypotheses* (H1)–(H4), *the differential equation* (II) *determines a semiflow ϕ_t on X. This semiflow satisfies the following maximal property* (*stronger than property* (d) *of* §1.1):

$$\omega(u) < \infty \Rightarrow \lim_{t \to \omega(u)} \|\phi_t u\|_X = \infty.$$

If the operator A has compact resolvent, then one has also the compactness property:

if $\omega(u) = \infty$ and the orbit $\{\phi_t u \mid 0 \leqslant t < \infty\}$ is bounded, then it is contained in a compact set of X.

If F is of class C^r $(1 \leqslant r \leqslant \infty)$ or analytic from X to E, then the semiflow on X is of the same class. □

Next we state the main result on the stability of a stationary state u_0. For this we shall assume that

(H5) F is of class C^1 from X to E,

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and we shall consider the linearized operator $L \equiv A + DF(u_0)$. This operator will always be the generator of an analytic semigroup on E since the operator $DF(u_0)$ is bounded from X to E, and therefore we have

$$\| DF(u_0) e^{At} u \|_E \leqslant C \| e^{At} u \|_X \leqslant C' \| e^{At} u \|_{D^\alpha} \leqslant C'' t^{-\alpha} e^{\omega t} \| u \|_E;$$

from here, the claim follows by applying Theorems 13.4.1 and 13.4.2 of Hille and Phillips [7]. We shall need to assume also that

(H6) the semigroup e^{Lt} restricts to a differentiable semigroup on X.[2]

In that case, the generator of the semigroup on X must coincide with the restriction of L to X (i.e. an operator with domain $\{u \in X \mid Lu \in X\}$); this operator will be denoted by L_X. In the following, Σ_E, Σ_X denote the spectra of L and L_X, respectively, and σ_E, σ_X denote the quantities $\sup \operatorname{Re} \Sigma_E$ and $\sup \operatorname{Re} \Sigma_X$.

REMARKS. 4. In the particular case $X = D^\alpha$, the hypothesis (H6) is automatically satisfied, and one has also $\Sigma_X = \Sigma_E$ (and therefore $\sigma_X = \sigma_E$).

5. Property $\Sigma_X = \Sigma_E$ will also be true when A has a compact resolvent, which in the applications will occur if Ω is a bounded domain and $N' = 0$.

6. In general, even when Ω is not bounded, in many practical cases it happens that the point spectrum is the same on both E and X, and this is where $\sup \operatorname{Re} \Sigma$ is attained; in this case we also have $\sigma_X = \sigma_E$.

THEOREM 1.2. *Let us consider equation* (II) *under hypotheses* (H1)–(H4). *Let u_0 be a stationary state, and assume that hypotheses* (H5) *and* (H6) *are also satisfied. If $\sigma_X < 0$, then u_0 is asymptotically stable in the semiflow on X. If $\sigma_X > 0$, and for some $p > 1$, $\| F(u_0 + z) - DF(u_0) z \|_E \leqslant \mathcal{O}(\| z \|_X^p)$ as $z \to 0$, then u_0 is unstable.*

We remark that the instability part of this theorem allows the existence of a continuous spectrum crossing the imaginary axis.

2. Linear problem.

2.1. Let us consider the linear problem corresponding to (I), i.e.

$$\text{(IL)} \qquad D_t u = -\mathcal{A}(x, D_x) u, \quad \mathcal{B}(x, D_x) u \,|_{\partial\Omega} = 0, \quad \left[u \,|_\infty = 0 \right].$$

We assume that this problem satisfies conditions (L1)–(L3) and ($L4_s$) of §1.3, where s is a nonnegative integer to be fixed later. Based on the existing theory about the classical solvability of this problem, we shall show that its solutions determine a semigroup on certain closed subspaces of $C^k(\overline{\Omega})$, where k is a nonnegative integer. At least for $0 \leqslant k \leqslant 2m - 1$, these closed subspaces, which we shall denote by $\hat{C}^k(\overline{\Omega})$, are essentially completions of sets of functions satisfying the accessory conditions.

[2]After this paper was written, I have been able to prove that hypothesis (H6) is already implied by (H1)–(H5).

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In general, for every real number $\sigma \geqslant 0$, we shall define $\hat{C}^\sigma(\overline{\Omega})$ as the subspace of $C^\sigma(\overline{\Omega})$ given by

$$\text{(2.1)}\qquad \begin{aligned}\hat{C}^\sigma(\overline{\Omega}) \equiv \{u \in C^\sigma(\overline{\Omega}) \mid {} & \text{for every integer } r \text{ in the} \\ & \text{interval } 0 \leqslant r \leqslant \sigma/2m,\ \mathcal{Q}^r u \text{ satisfies} \\ & \text{all the accessory conditions of order} \leqslant \sigma - 2mr\}.\end{aligned}$$

Another way to say this is the following: let us consider the infinite collection of conditions:

$$\text{(2.2)}\qquad \mathfrak{B}_i(x, D_x)[\mathcal{Q}(x, D_x)]^r u\,|_{\partial\Omega} = 0 \qquad (1 \leqslant i \leqslant mN, 0 \leqslant r < \infty),$$

$$\text{(2.3)}\qquad \left[[\mathcal{Q}(x, D_x)]^r u\,\big|_\infty = 0\ (0 \leqslant r < \infty)\right];$$

then (2.1) amounts to saying

$$\text{(2.4)}\qquad \begin{aligned}\hat{C}^\sigma(\overline{\Omega}) \equiv \{u \in C^\sigma(\overline{\Omega}) \mid u & \text{ satisfies the conditions of} \\ & \text{order} \leqslant \sigma \text{ in (2.2) and (2.3)}\}.\end{aligned}$$

In particular, if the problem does not include any condition at infinity and the orders of the boundary conditions satisfy $m_i > \sigma$ for all i, then $\hat{C}^\sigma(\overline{\Omega}) = C^\sigma(\overline{\Omega})$. In fact, with respect to the possible condition at infinity, if we assume that the coefficients of $\mathcal{Q}$ are bounded (which, in particular, will be true if condition ($L4_0$) is satisfied), then (2.1) or (2.4) is equivalent to simply saying

$$\text{(2.5)}\qquad \begin{aligned}\hat{C}^\sigma(\overline{\Omega}) \equiv \{u \in C^\sigma(\overline{\Omega}) \mid u & \text{ satisfies the conditions of} \\ & \text{order} \leqslant \sigma \text{ in (2.2), and } u\,\big|_\infty = 0\}\end{aligned}$$

or

$$\text{(2.6)}\qquad \begin{aligned}\hat{C}^\sigma(\overline{\Omega}) \equiv \{u \in C^\sigma(\overline{\Omega}) \mid u & \text{ satisfies the conditions of} \\ & \text{order} \leqslant \sigma \text{ in (2.2), and } D^\nu u\,\big|_\infty = 0 \text{ for } 0 \leqslant |\nu| \leqslant \sigma\}.\end{aligned}$$

Obviously, (2.6) $\Rightarrow$ (2.4) $\Rightarrow$ (2.5); the fact that (2.5) $\Rightarrow$ (2.6) follows from the following lemma.

Lemma 2.1. *Assuming that the domain Ω is uniformly of class $C^{1+\alpha}$, if $u \in C^\sigma(\overline{\Omega})$ and $u\,|_\infty = 0$, then $D^\nu u\,|_\infty = 0$ for any ν such that $|\nu| \leqslant \sigma$.* □

Proof. Obviously it suffices to prove this for σ an integer, and in particular for $\sigma = 1$; from this case the general result follows by induction. Recall that $u \in C^1(\overline{\Omega})$ means, among other things, that the partial derivatives $D_{x_i}u$ are uniformly continuous on Ω; we shall see that in that situation $D_{x_i}u\,|_\infty \neq 0$ (for some i) would imply $u\,|_\infty \neq 0$, and this will prove the lemma. Relation $D_{x_i}u\,|_\infty \neq 0$ means that

$$\text{(2.7)}\qquad \begin{gathered}\exists \varepsilon > 0 \text{ such that } \forall M > 0\ \exists x \in \Omega \text{ such that} \\ |x| > M \quad\text{and}\quad |D_{x_i}u(x)| > \varepsilon.\end{gathered}$$

On the other hand, the uniform continuity of $D_{x_i}u$ implies that $\exists \delta > 0$ such that

$$\text{(2.8)}\qquad y \in B_\delta(x) \cap \Omega \Rightarrow |D_{x_i}u(x) - D_{x_i}u(y)| < \varepsilon/2,$$

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where $B_\delta(x)$ denotes the ball $B_\delta(x) \equiv \{y \in \mathbf{R}^n \mid |y - x| < \delta\}$. From (2.7) and (2.8) it follows that

$$y \in B_\delta(x) \cap \Omega \Rightarrow |D_{x_i} u(y)| > \varepsilon/2. \tag{2.9}$$

Since the domain Ω is assumed to be uniformly of class $C^{+\alpha}$, it is then ensured that if we take δ less than a certain $\delta_0 > 0$, the set $B_\delta(x) \cap \Omega$ contains a closed segment of length $\theta\delta$ in the direction of the ith axis (θ a fixed real number) (see, for instance, Miranda [**13**, §33]). Let $[z_1, z_2]$ denote this segment; using the mean value theorem and the bound (2.9), we obtain that $|u(z_1) - u(z_2)| > \frac{1}{2}\varepsilon\theta\delta$. From z_1 and z_2, let us choose the point where $|u(\cdot)|$ is larger and call it z; certainly this point will satisfy $|u(z)| > \frac{1}{4}\varepsilon\theta\delta$. Therefore, defining $\varepsilon' \equiv \varepsilon\theta\delta/4$, and $M' \equiv M - \delta$, we have obtained that

$$\exists \varepsilon' > 0 \text{ such that } \forall M' > 0 \, \exists z \in \Omega \text{ such that}$$
$$|z| > M' \quad \text{and} \quad |u(z)| > \varepsilon',$$

which implies that $u|_\infty \neq 0$. Q.E.D.

Obviously, $\hat{C}^\sigma(\overline{\Omega})$ is a closed subspace of $C^\sigma(\overline{\Omega})$, and assuming condition (D) of §1.2, we have the embeddings $\hat{C}^\tau(\overline{\Omega}) \hookrightarrow \hat{C}^\sigma(\overline{\Omega})$ $(\sigma < \tau)$. At least in the most frequently encountered cases it happens that, for k integer and $\sigma > k$, $\hat{C}^\sigma(\overline{\Omega})$ is dense in $\hat{C}^k(\overline{\Omega})$ (see §3). In particular, for k an integer in the interval $0 \leqslant k \leqslant 2m - 1$, the space $\hat{C}^k(\overline{\Omega})$ can be thought of as the completion in $C^k(\overline{\Omega})$ of a set of smooth functions satisfying all the accessory conditions, for example the set $\hat{C}^{2m-1}(\overline{\Omega})$. To better handle the infinite collection of conditions (2.2), it will be useful to rewrite it as

$$\mathfrak{B}_j(x, D_x)u|_{\partial\Omega} = 0 \qquad (1 \leqslant j < \infty), \tag{2.10}$$

where, for $j = i + Nmr$ $(1 \leqslant i \leqslant m,\ r \geqslant 0)$, we define

$$\mathfrak{B}_j(x, D_x) \equiv \mathfrak{B}_i(x, D_x)[\mathfrak{A}(x, D_x)]^r, \qquad m_j \equiv m_i + 2mr.$$

(m_j is the order of the differential operator $\mathfrak{B}_j$). As a last remark concerning the definition of the spaces $\hat{C}^\sigma(\overline{\Omega})$, let us say that the conditions required for a function $u \in C^\sigma(\overline{\Omega})$ to belong to $\hat{C}^\sigma(\overline{\Omega})$ are what in the theory of parabolic equations are called "compatibility conditions of order σ".

For every nonnegative integer k, we shall reformulate problem (IL) as an abstract equation to $\hat{C}^k(\overline{\Omega})$:

$$\dot{u} = \mathfrak{A}_k u. \tag{2.11$_k$}$$

Precisely, we shall define $\mathfrak{A}_k$ as the operator on $\hat{C}^k(\overline{\Omega})$ given by the differential operator $\mathfrak{A}$ with domain $\mathfrak{D}_k \equiv \hat{C}^{k+2m+\alpha}(\overline{\Omega})$ (α a fixed real number in the interval $0 < \alpha < 1$). For this we shall assume that the smoothness condition (L4$_k$) is satisfied (with that value of α); then it is clearly true that $\mathfrak{A}$ takes $\hat{C}^{k+2m+\alpha}(\overline{\Omega})$ to $\hat{C}^k(\overline{\Omega})$. We shall also assume that the domain $\mathfrak{D}_k$ is dense in $\hat{C}^k(\overline{\Omega})$:

(L5$_k$) $\hat{C}^{k+2m+\alpha}(\overline{\Omega})$ is dense in $\hat{C}^k(\overline{\Omega})$.

In §3 this condition is shown to hold in the most frequently encountered cases, in particular those appearing in connection with problem (X). According to what we mentioned at the beginning of §1.4, assuming that $\mathfrak{D}_k$ is dense in $\hat{C}^k(\overline{\Omega})$, to obtain

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the semigroup generating property on $\hat{C}^k(\overline{\Omega})$ it would suffice to show that the Cauchy problem for (2.11_k) is uniformly correctly posed, and in that case, the generator of the semigroup, which we shall denote by A_k, is exactly the closure of the operator $\mathfrak{A}_k$. This is essentially the way we shall obtain our semigroup on the spaces $\hat{C}^k(\overline{\Omega})$. The semigroup on the space $\hat{C}^k(\overline{\Omega})$ will be denoted by $\phi_{k,t}$, or also $e^{A_k t}$, with the abbreviations $\phi_{0,t} \equiv \phi_t$ and $A_0 \equiv A$.

For the moment we shall consider the case where the problem does not include any condition at infinity. The case where it includes such a condition will be taken care of a posteriori by showing that the semigroups obtained preserve the property of functions vanishing at infinity.

2.2. For the general situation that we are considering, a basic work on the classical solvability of problem (IL) is the one by Solonnikov **[21]**. This author obtains the following result of existence and uniqueness:

THEOREM 2.1 (SOLONNIKOV **[21**, §19, THEOREM 4.9]). *Let us consider problem* (IL) *without the condition at infinity and assume conditions* (L1)–(L3) and ($L4_s$), *where s is any nonnegative integer. For any $T > 0$ and every initial state u_0 belonging to $\mathfrak{D}_s \equiv \hat{C}^{s+2m+\alpha}(\overline{\Omega})$, problem* (IL) *with the initial condition $u\big|_{t=0} = u_0$ has a unique solution u in the class $H^{s+2m+\alpha}(Q_T)$ defined below. This solution satisfies an estimate of the form*

$$\|u\|_{H^{s+2m+\alpha}} \leqslant C\|u_0\|_{C^{s+2m+\alpha}}, \tag{2.12}$$

where $\|u\|_{H^{s+2m+\alpha}}$ denotes the norm in the Banach space $H^{s+2m+\alpha}(Q_T)$, and the constant C does not depend on u_0. $\square$

For $\sigma \geqslant 0$, $H^\sigma(Q_T)$ denotes the Banach space consisting of the functions u, defined on $Q_T \equiv \Omega \times (0, T)$, whose derivatives $D_t^\mu D_x^\nu u$ of order $2m\mu + |\nu| \leqslant \sigma$ are uniformly continuous, and such that the following expression, which defines the norm, is finite:

$$\|u\|_{H^\sigma} \equiv [u]_\sigma + \{u\}_\sigma^x + \{u\}_\sigma^t, \tag{2.13}$$

where

$$[u]_\sigma \equiv \sum_{0 \leqslant 2m\mu + |\nu| \leqslant [\sigma]} \sup_{\substack{0<t<T \\ x\in\Omega}} |D_t^\mu D_x^\nu u(x,t)|, \tag{2.14}$$

$$\{u\}_\sigma^x \equiv \sum_{2m\mu+|\nu|=[\sigma]} \sup_{\substack{0<t<T \\ x,y\in\Omega \\ x\neq y}} \frac{|D_t^\mu D_x^\nu u(x,t) - D_t^\mu D_x^\nu u(y,t)|}{|x-y|^{\sigma-[\sigma]}} \tag{2.15}$$

(unless σ is integer, in which case $\{u\}_\sigma^x \equiv 0$),

$$\{u\}_\sigma^t \equiv \sum_{\sigma-2m<2m\mu+|\nu|<\sigma} \sup_{\substack{0<t<s<T \\ x\in\Omega}} \frac{|D_t^\mu D_x^\nu u(x,s) - D_t^\mu D_x^\nu u(x,t)|}{|s-t|^{(\sigma-2m\mu-|\nu|)/2m}}. \tag{2.16}$$

In particular, the functions u belonging to $H^\sigma(Q_T)$ have a unique continuous extension to $\overline{Q}_T \equiv \overline{\Omega} \times [0, T]$, for which the derivatives $D_t^\mu D_x^\nu u$ of order $2m\mu + |\nu| \leqslant \sigma$ exist and are continuous on $\overline{Q}_T$. Precisely, it is in the sense of this extension

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that the u of Theorem 2.1 satisfies the boundary and initial conditions. The spaces $H^\sigma(Q_T)$ are denoted $C_{x,t}^{\sigma,\sigma/2m}(Q_T)$ by Solonnikov [**21**]; the present notation is taken from Belonosov [**1**].

Let us look at the consequences of this result in connection with the question of problem (IL) determining a semigroup on a suitable space. In the following, the solution starting from the initial state w will be the denoted by $\phi_t w$.

From the definition of $\|u\|_{H^{s+2m+\alpha}}$, more exactly the terms of $[u]_{s+2m+\alpha}$ and $\{u\}^x_{s+2m+\alpha}$ corresponding to $\mu = 0$, we see that (2.12) implies that

$$\|\phi_t w\|_{C^{s+2m+\alpha}} \leqslant C\|w\|_{C^{s+2m+\alpha}} \quad \left(\forall t \in [0, T], \forall w \in \hat{C}^{s+2m+\alpha}(\overline{\Omega})\right), \tag{2.17}$$

i.e. the operators ϕ_t are bounded as operators on $\hat{C}^{s+2m+\alpha}(\overline{\Omega})$, and the bound is uniform on intervals of the form $[0, T]$.

On the other hand, Theorem 2.1 is not sufficient to ensure that the mapping $t \mapsto \phi_t w$ be continuous from $[0, T]$ to $\hat{C}^{s+2m+\alpha}(\overline{\Omega})$ (in norm); however, we have the following property:

(2.18) For every $w \in \hat{C}^{s+2m+\alpha}(\overline{\Omega})$, the mapping $t \mapsto \phi_t w$ is continuous from $[0, T]$ to $\hat{C}^{s+2m}(\overline{\Omega})$.

This follows from the boundedness of $[u]_{s+2m+\alpha}$ and $\{u\}^t_{s+2m+\alpha}$, more exactly their terms corresponding to $\mu = 1$ and $\mu = 0$.

In fact, from these estimates we can see that:

(2.19) For every $w \in \hat{C}^{s+2m+\alpha}(\overline{\Omega})$, the mapping $t \mapsto \phi_t w$ is differentiable from the closed interval $[0, T]$ to $\hat{C}^{s}(\overline{\Omega})$.

To see that the differentiability is true in the sense of a Banach space valued function, we have to see that, for $|\nu| \leqslant s$, the difference $D_t D_x^\nu u(t, s) - \frac{1}{h}[D_x^\nu u(t+h, x) - D_x^\nu u(t, x)]$ converges uniformly to zero as $h \to 0$; this can be seen by using the mean value theorem:

$$\begin{aligned} D_t D_x^\nu u(t, x) - \tfrac{1}{h}\left[D_x^\nu u(t+h, x) - D_x^\nu u(t, x)\right] \\ = D_t D_x^\nu u(t, x) - D_t D_x^\nu u(t + \eta(x), x), \end{aligned}$$

where $0 \leqslant \eta(x) \leqslant h$ $(\forall x \in \Omega)$, and then using the boundedness of $[u]_{s+2m+\alpha}$ and $\{u\}^t_{s+2m+\alpha}$ (more exactly their terms corresponding to $\mu = 2$ and $\mu = 1$, respectively); in this way we obtain that, for h small enough,

$$|D_t D_x^\nu u(t, x) - D_t D_x^\nu u(t+\eta(x), x)| \leqslant C|\eta(x)|^{\alpha/2m} \leqslant C h^{\alpha/2m} \quad (\forall x \in \Omega).$$

2.3. In order to have the uniform correctness of the Cauchy problem for equation (2.11_s) in the space $\hat{C}^s(\overline{\Omega})$, it only remains to verify the uniform boundedness of the operators ϕ_t as operators on this space. This property can be obtained by using the known bounds for the Green's matrix of problem (IL). We recall that the Green's matrix $G(t, x; \tau, \xi)$, defined for $0 \leqslant \tau < t \leqslant T$ and $x, \xi \in \Omega$, is such that, given an initial state w, the corresponding solution of problem (IL) is given by

$$[\phi_t w](x) = \int_\Omega G(t, x; 0, \xi) w(\xi)\, d\xi. \tag{2.20}$$

The following theorem collects the basic results giving bounds on $G(t, x; \tau, \xi)$ and its derivatives $D_t D_x G(t, x; \tau, \xi)$ (the norm of a matrix H is denoted here by $|H|$).

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THEOREM 2.2 (ÉĬDEL'MAN AND IVASISHEN [4]; SEE ALSO SOLONNIKOV [22]). *Let us consider problem* (IL) *with conditions* (L1)–(L3) *and* ($L4_s$), *where s is any nonnegative integer. The Green's matrix of this problem has the derivatives $D_t^\mu D_x^\nu G$ for $0 \leq 2m\mu + |\nu| \leq s + 2m$, and these derivatives satisfy the following bounds*:

1. *For* $0 \leq 2m\mu + |\nu| \leq s + 2m$,

$$|D_t^\mu D_x^\nu G(t, x; \tau, \xi)| \leq C(t-\tau)^{-(n+2m\mu+|\nu|)/2m} \exp\left[-c\frac{|x-\xi|^{2m/(2m-1)}}{(t-\tau)^{1/(2m-1)}}\right]. \tag{2.21}$$

2. *For* $2m\mu + |\nu| = s + 2m$, *and* $y \in \Omega$,

$$\begin{aligned} &|D_t^\mu D_x^\nu G(t, x; \tau, \xi) - D_t^\mu D_x^\nu G(t, y; \tau, \xi)| \\ &\qquad \leq C|x-y|^\alpha (t-\tau)^{-(n+s+2m+\alpha)/2m} \exp\left[-c\frac{|z-\xi|^{2m/(2m-1)}}{(t-\tau)^{1/(2m-1)}}\right], \end{aligned} \tag{2.22}$$

where $|z-\xi| = \min(|x-\xi|, |y-\xi|)$.

3. *For* $s + 1 \leq 2m\mu + |\nu| \leq s + 2m$, *and* $t' \in (\tau, t)$,

$$\begin{aligned} &|D_t^\mu D_x^\nu G(t, x; \tau, \xi) - D_t^\mu D_x^\nu G(t', x; \tau, \xi)| \\ &\qquad \leq C(t-t')^{(s+2m+\alpha-2m\mu-|\nu|)/2m}(t'-\tau)^{-(n+s+2m+\alpha)/2m} \\ &\qquad\quad \times \exp\left[-c\frac{|x-\xi|^{2m/(2m-1)}}{(t-\tau)^{1/(2m-1)}}\right]. \quad \square \end{aligned} \tag{2.23}$$

Let us consider the consequences of these bounds when they are applied to (2.20). For this, we note that the integral with respect to ξ of the right-hand sides of (2.21)–(2.23) is bounded, specifically

$$\int_\Omega \exp\left[-c\frac{|x-\xi|^{2m/(2m-1)}}{(t-\tau)^{1/(2m-1)}}\right] d\xi \leq C(t-\tau)^{n/2m}.$$

For the moment we shall consider the consequences of Theorem 2.2 in the case $s = 0$.

In this case, from the bound (2.21) with $\mu = 0$, $|\nu| = 0$, one obtains that

$$\|\phi_t w\|_C \leq C\|w\|_C \quad (\forall t \in [0, T]), \tag{2.24}$$

i.e. the operators ϕ_t are bounded from $C(\overline{\Omega})$ to $C(\overline{\Omega})$. With this, we complete the conditions that are needed to ensure that the Cauchy problem for equation (2.11_0) is uniformly correctly posed in the space $\hat{C}(\overline{\Omega})$: Theorem 2.1 (with $s = 0$) ensures the existence and uniqueness of a solution of (IL) for every initial state $w \in \mathfrak{D}_0 \equiv \hat{C}^{2m+\alpha}(\overline{\Omega})$, and in (2.19) we have seen that this solution is continuously differentiable from $[0, T]$ to $\hat{C}(\overline{\Omega})$; therefore, for $w \in \mathfrak{D}_0$ this solution of (IL) is also a solution of the abstract equation (2.11_0) on $[0, T]$; finally, (2.24) gives the continuous dependence on the initial state uniformly on $[0, T]$. In consequence, assuming that condition ($L5_0$) holds, the operators ϕ_t, initially defined only on $\mathfrak{D}_0$, can be extended uniquely to a semigroup on $\hat{C}(\overline{\Omega})$. This semigroup will be denoted by ϕ_t, or also e^{At}, where A stands for the corresponding generator.

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On the other hand, from the bound (2.21) with $\mu = 1$, $|\nu| = 0$, we obtain that

$$\|D_t\phi_t w\|_C \leqslant Ct^{-1}\|w\|_C \quad (\forall t \in (0, T]), \tag{2.25}$$

where the time derivative is understood as applied to a function valued in the Banach space $C(\bar{\Omega})$. From the same bound (2.21), but now with $\mu = 0$ and $|\nu| \leqslant q$, one obtains an estimate of the following form for every integer q in the interval $0 \leqslant q \leqslant 2m$:

$$\|\phi_t w\|_{C^q} \leqslant Ct^{-q/2m}\|w\|_C \quad (\forall t \in (0, T]). \tag{2.26}$$

An analogous property, obtained from the bounds (2.21) and (2.22) together is

$$\|\phi_t w\|_{C^{2m+\alpha}} \leqslant Ct^{-(2m+\alpha)/2m}\|w\|_C \quad (\forall t \in (0, T]). \tag{2.27}$$

Initially, properties (2.25)–(2.27) are obtained only for $w \in \mathfrak{D}_0$, which is the set of possible initial states for Theorem 2.1; however, as $\mathfrak{D}_0$ is assumed to be dense in $\hat{C}(\bar{\Omega})$, these properties can be extended to the whole semigroup on $\hat{C}(\bar{\Omega})$. This can be seen for example in the following way: according to (2.24)–(2.27), for every $u_0 \in \mathfrak{D}_0$, the corresponding solution of Theorem 2.1 satisfies $u(\cdot, t) \in \mathfrak{D}_0 \equiv \hat{C}^{2m+\alpha}(\bar{\Omega})$ and

(2.28)

$$\sup_{0<t<T}\left[t^{(2m+\alpha)/2m}\|u(\cdot, t)\|_{C^{2m+\alpha}} + t\|D_t u(\cdot, t)\|_C + \sum_{q=0}^{2m} t^{q/2m}\|u(\cdot, t)\|_{C^q}\right] \leqslant C\|u_0\|_C.$$

Therefore, the linear mapping $\mathfrak{F}\colon u_0 \mapsto u$ is bounded from $C(\bar{\Omega})$ to the Banach space

$$\mathfrak{X} \equiv C([0, T], \hat{C}(\bar{\Omega})) \cap C((0, T], \hat{C}^{2m+\alpha}(\bar{\Omega})) \cap C^1((0, T], \hat{C}(\bar{\Omega}))$$

with norm $\|u\|_{\mathfrak{X}}$ defined by the left-hand side of (2.28). As $\mathfrak{D}_0$ is assumed to be dense in $\hat{C}(\bar{\Omega})$, the linear operator $\mathfrak{F}\colon u_0 \mapsto u$, $\mathfrak{D}_0 \to \mathfrak{X}$ can be extended uniquely and with the same bound to the whole space $\hat{C}(\bar{\Omega})$. In this way, the operators ϕ_t are extended to the whole $\hat{C}(\bar{\Omega})$ preserving properties (2.24)–(2.27). Furthermore, by the uniqueness of the extension obtained previously from the uniform correctness of the abstract Cauchy problem, it follows that the extension just obtained must coincide with that one.

As we have seen in Lemma 1.1, property (2.25) is sufficient to establish the analyticity of the semigroup on $\hat{C}(\bar{\Omega})$. On the other hand, from (2.26) and (2.27) we can derive that, with respect to the semigroup obtained on $\hat{C}(\bar{\Omega})$, one has the properties:

$$D^\beta \hookrightarrow \hat{C}^q(\bar{\Omega}), \quad \forall\beta > q/2m \qquad (0 \leqslant q \leqslant 2m), \tag{2.29}$$

$$D^\beta \hookrightarrow \hat{C}^{2m+\alpha}(\bar{\Omega}), \quad \forall\beta > (2m+\alpha)/2m. \tag{2.30}$$

To derive (2.29) and (2.30) from (2.26) and (2.27), we use Lemma 1.3 and 1.2 (the condition $\phi_t w \in \hat{C}^{2m+\alpha}(\bar{\Omega})$ $(\forall t > 0,\ \forall w \in \hat{C}(\bar{\Omega}))$, which is needed to apply to Lemma 1.2, follows from the fact that the mapping $t \mapsto \phi_t w$ belongs to the space $\mathfrak{X}$).

2.4. The bounds on the Green's matrix can also be used to obtain the uniform boundedness of the operators ϕ_t in the C^k norms $(k > 0)$, but in the general case this

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requires one to work at the level of the proof of Theorem 2.1. This work has been done by Belonosov [**1**], who obtains the result stated below; a similar result is obtained also by Solonnikov and Khachatryan [**23**].

THEOREM 2.3 (BELONOSOV [**1**, §4]). *Let us consider problem* (IL) *subject to conditions* (LI)–(L3) *and* ($L4_s$), *where s is any nonnegative integer. Let ρ be any real number in the interval* $0 < \rho < s + 2m + \alpha$. *For any* $T > 0$, *and every initial state u_0 belonging to* $\hat{C}^\rho(\overline{\Omega})$, *problem* (IL) *with the initial condition* $u\big|_{t=0} = u_0$ *has a unique solution in the class* $H_\rho^{s+2m+\alpha}(Q_T)$ *defined below. This solution satisfies a bound of the form*

$$\|u\|_{H_\rho^{s+2m+\alpha}} \leqslant C\|u_0\|_{C^\rho}, \tag{2.31}$$

where $\|u\|_{H_\rho^{s+2m+\alpha}}$ *denotes the norm in the Banach space* $H_\rho^{s+2m+\alpha}(Q_T)$, *and the constant C does not depend on* u_0. □

For $0 \leqslant \rho \leqslant \sigma$, the Banach space $H_\rho^\sigma(Q_T)$ consists of those functions belonging to $H^\rho(Q_T)$ and such that the following expression, which defines the norm in $H_\rho^\sigma(Q_T)$, is finite:

$$\|u\|_{H_\rho^\sigma} \equiv \|u\|_{H^\rho} + [u]_{\rho,\sigma} + \{u\}_{\rho,\sigma}^x + \{u\}_{\rho,\sigma}^t, \tag{2.32}$$

where

$$[u]_{\rho,\sigma} \equiv \sum_{\rho<2m\mu+|\nu|\leqslant[\sigma]} \sup_{\substack{0<t<T\\ x\in\Omega}} \left[t^{(2m\mu+|\nu|-\rho)/2m}|D_t^\mu D_x^\nu u(x,t)|\right], \tag{2.33}$$

$$\{u\}_{\rho,\sigma}^x \equiv \sum_{2m\mu+|\nu|=[\sigma]} \sup_{\substack{0<t<T\\ x,y\in\Omega\\ x\neq y}} \left[t^{(\sigma-\rho)/2m}\frac{|D_t^\mu D_x^\nu u(x,t) - D_t^\mu D_x^\nu u(y,t)|}{|x-y|^{\sigma-[\sigma]}}\right] \tag{2.34}$$

(unless σ is integer, in which case $\{u\}_{\rho,\sigma}^x \equiv 0$),

(2.35)

$$\{u\}_{\rho,\sigma}^t \equiv \sum_{\sigma-2m<2m\mu+|\nu|<\sigma} \sup_{\substack{0<t<s<T\\ x\in\Omega}} \left[t^{(\sigma-\rho)/2m}\frac{|D_t^\mu D_x^\nu u(x,s) - D_t^\mu D_x^\nu u(x,t)|}{|s-t|^{(\sigma-2m\mu-|\nu|)/2m}}\right],$$

and $\|u\|_{H^\rho}$ is defined in (2.13). From the definition it follows that $H^\sigma(Q_T) \hookrightarrow H_\rho^\sigma(Q_T) \hookrightarrow H^\rho(Q_T)$, and, in particular, $H_\sigma^\sigma(Q_T) = H^\sigma(Q_T)$ (equivalent norms). One also has the embeddings $H_\rho^{\sigma'}(Q_T) \hookrightarrow H_\rho^\sigma(Q_T)$ ($\sigma' > \sigma$), and $H_{\rho'}^\sigma(Q_T) \hookrightarrow H_\rho^\sigma(Q_T)$ ($\rho' > \rho$).

We shall use Theorem 2.3 with ρ being an integer $k > 0$. For such k, $\phi_{k,t}$ will denote the mapping $u_0 \mapsto u(\cdot, t)$, $\hat{C}^k(\overline{\Omega}) \to \hat{C}^k(\overline{\Omega})$ determined by Theorem 2.3 with $\rho = k$. It is easily verified that the mappings $\phi_{k,t}$ obtained for different values of s and α are the same; this follows from the embedding $H_k^{s_1+2m+\alpha_1}(Q_T) \hookrightarrow H_k^{s_2+2m+\alpha_2}(Q_T)$ ($s_1 + \alpha_1 > s_2 + \alpha_2$), and the uniqueness of the solution of Theorem 2.3.

From the fact that $H_k^{s+2m+\alpha}(Q_T) \hookrightarrow H^k(Q_T)$, we see that Theorem 2.3 implies the following facts: First, from the definition of $\|u\|_{H^k}$, we see that (2.31) implies that

$$\|\phi_{k,t}w\|_{C^k} \leqslant C\|w\|_{C^k} \quad \left(\forall t \in [0,T], \forall w \in \hat{C}^k(\overline{\Omega})\right). \tag{2.36}$$

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Secondly, $u \in H^k(Q_T)$ implies that the derivatives $D_x^\nu u$ ($|\nu| \leq k$) are uniformly continuous in Q_T, which implies that

(2.37) For every $w \in \hat{C}^k(\overline{\Omega})$, the mapping $t \mapsto \phi_{k,t} w$ is continuous from $[0, T]$ to $\hat{C}^k(\overline{\Omega})$.

Properties (2.36) and (2.37) ensure that the operators $\phi_{k,t}$ ($t \geq 0$) constitute a semigroup on $\hat{C}^k(\overline{\Omega})$.

Assuming that $\hat{C}^{2m+\alpha}(\overline{\Omega})$ is dense in $\hat{C}^1(\overline{\Omega})$ (which, in particular, will hold if we assume ($L5_1$)), then it is easily seen that the semigroups $\phi_{k,t}$ coincide with the restrictions to $\hat{C}^k(\overline{\Omega})$ of the semigroup ϕ_t previously obtained on $\hat{C}(\overline{\Omega})$ (§2.3). Let us consider first the case $k = 1$; then the claim follows from the fact that ϕ_t and $\phi_{1,t}$ coincide on $\hat{C}^{2m+\alpha}(\overline{\Omega})$, which is assumed to be dense in $\hat{C}^1(\overline{\Omega})$: In fact, for $w \in \hat{C}^{2m+\alpha}(\overline{\Omega})$, $\phi_t w$ is the solution of Theorem 2.1 with $s = 0$ and $u_0 = w$; according to this theorem, this solution is unique and belongs to the class $H^{2m+\alpha}(Q_T)$; but this space is embedded in $H_1^{2m+\alpha}(Q_T)$, so that $\phi_t w$ must coincide with the unique solution in $H_1^{2m+\alpha}(Q_T)$ ensured by Theorem 2.3 with $s = 0$, $\rho = 1$, that is, $\phi_{1,t} w$. From here, to see that $\phi_{1,t}$ coincides with ϕ_t on the whole space $\hat{C}^1(\overline{\Omega})$, it suffices to use the denseness of $\hat{C}^{2m+\alpha}(\overline{\Omega})$ in $\hat{C}^1(\overline{\Omega})$, the embedding $\hat{C}^1(\overline{\Omega}) \hookrightarrow \hat{C}(\overline{\Omega})$, and the continuity of both operators $\phi_{1,t}$ and ϕ_t on $\hat{C}^1(\overline{\Omega})$ and $\hat{C}(\overline{\Omega})$, respectively. For $k > 1$, the claim can be proved as follows: for $w \in \hat{C}^k(\overline{\Omega})$, Theorem 2.3 with $\rho = k$ ensures the existence of a unique solution $\phi_{k,t} w$ belonging to $H_k^{s+2m+\alpha}(Q_T)$; but we know that this space is embedded in $H_1^{s+2m+\alpha}(Q_T)$, so that $\phi_{k,t}$ must coincide with the unique solution in $H_1^{s+2m+\alpha}(Q_T)$ ensured by Theorem 2.3 with $\rho = 1$.

Having seen that the semigroups $\phi_{k,t}$ are given by the same operator ϕ_t but restricted to the different spaces $\hat{C}^k(\overline{\Omega})$, we can see that the corresponding generators A_k are also given by the same operator A but restricted to different domain D_k. In fact, from the embedding $\hat{C}^l(\overline{\Omega}) \hookrightarrow \hat{C}^k(\overline{\Omega})$ ($0 \leq k < l$) it follows that

$$\left\| \frac{1}{t}(\phi_t u - u) - A_l u \right\|_{C^l} \to 0 \Rightarrow \left\| \frac{1}{t}(\phi_t u - u) - A_l u \right\|_{C^k} \to 0,$$

which implies that $D_l \subset D_k$ and $\forall u \in D_l$, $A_l u = A_k u$.

On the other hand, assuming ($L4_k$) and ($L5_k$) we can see that the generator A_k of the semigroup $\phi_{k,t}$ coincides with the closure of the operator $\mathfrak{A}_k$. For this it suffices to notice that, having shown that $\phi_{k,t}$ coincides with the restriction to $\hat{C}^k(\overline{\Omega})$ of the semigroup obtained in §§2.2 and 2.3, and assuming conditions ($L4_k$) and ($L5_k$), then we can apply (2.19) with $s = k$, which implies that the Cauchy problem is uniformly correctly posed for the abstract equation (2.11_k) on $\hat{C}^k(\overline{\Omega})$; obviously, the semigroup generated by this uniformly correctly posed problem is none other than $\phi_{k,t}$; from here it follows that the operator $\mathfrak{A}_k$ is closable, its closure being the generator A_k of the semigroup $\phi_{k,t}$.

By using Theorem 2.3 with $\rho = s = k$, we can see that inequalities (2.25)–(2.27) generalize to

(2.38) $$\|D_t \phi_t w\|_{C^k} \leq C t^{-1} \|w\|_{C^k} \quad (\forall t \in (0, T], \forall w \in \hat{C}^k(\overline{\Omega})),$$

(2.39) $$\|\phi_t w\|_{C^{k+q}} \leq C t^{-q/2m} \|w\|_{C^k} \quad (\forall t \in (0, T], \forall w \in \hat{C}^k(\overline{\Omega})),$$

(2.40) $$\|\phi_t w\|_{C^{k+2m+\alpha}} \leq C t^{-(2m+\alpha)/2m} \|w\|_{C^k} \quad (\forall t \in (0, T], \forall w \in \hat{C}^k(\overline{\Omega})),$$

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where q is any integer in the interval $0 \leqslant q \leqslant 2m$. For $k > 0$, these inequalities follow from Theorem 2.3 with $\rho = s = k$ by using the bounds on $[u]_k$, $[u]_{k,k+2m+\alpha}$, $\{u\}^x_{k,k+2m+\alpha}$. By Lemma 1.1, property (2.38) means that the semigroups $\phi_{k,t}$ are analytic. On the other hand, using Lemmas 1.3 and 1.2, (2.39) and (2.40) imply the following properties:

$$D^\beta \hookrightarrow \hat{C}^{k+q}(\overline{\Omega}), \quad \forall \beta > q/2m, \tag{2.41}$$

$$D^\beta \hookrightarrow \hat{C}^{k+2m+\alpha}(\overline{\Omega}), \quad \forall \beta > (2m+\alpha)/2m, \tag{2.42}$$

where q is any integer in the interval $0 \leqslant q \leqslant 2m$, and D^β denotes the fractional power spaces corresponding to the semigroup on $\hat{C}^k(\overline{\Omega})$ (to apply Lemma 1.2 we need to know that $\phi_t w \in \hat{C}^{k+2m+\alpha}(\overline{\Omega})$ ($\forall t > 0$, $\forall w \in \hat{C}^k(\overline{\Omega})$); this follows from the fact that the solution belongs to $H_k^{k+2m+\alpha}(Q_T)$, and that $\mathfrak{B}\mathfrak{A}^r u\,|_{\partial\Omega} = \mathfrak{B}D_t^r u\,|_{\partial\Omega} = D_t^r \mathfrak{B} u\,|_{\partial\Omega} = 0$).

2.5. With this, we have proved the assertions in Theorem 2.4 for the case of no condition at infinity. To take care of the case where there is a condition at infinity of the type $u\,|_\infty = 0$, we only need to verify that the semigroup just obtained preserves the property of vanishing at infinity. This can easily be seen to be true by using the following property of the Green's matrix:

$$\int_{\Omega - B_R(x)} |\, G(t, x; \tau, \xi)\,|\, d\xi \to 0 \quad \text{as } R \to \infty, \tag{2.43}$$

where $B_R(x)$ denotes the open ball with center at x and radius R: $B_R(x) \equiv \{\xi \in R^n \mid |\,\xi - x\,| \leqslant R\}$. This property follows by integration of (2.21) with $\mu = |\,\nu\,| = 0$. From (2.43), the fact that the semigroup preserves the property of vanishing at infinity can be derived in essentially the same way as for the heat equation: Let us consider a sequence of balls $B_R(0)$ with $R \to \infty$, and take x in $\Omega \setminus B_R(0)$, i.e. $|\,x\,| > R$. Splitting the integral (2.20) in the following way:

$$[\phi_t w](x) = \int_{\substack{\xi\in\Omega \\ |\xi - x| \leqslant M}} G(t, x; 0, \xi) w(\xi)\, d\xi + \int_{\substack{\xi\in\Omega \\ |\xi - x| > M}} G(t, x; 0, \xi) w(\xi)\, d\xi,$$

we see that when $R \to \infty$ the ξ in the first integral tend towards infinity (because $|\,\xi - x\,| \leqslant M$ implies $|\,\xi\,| \geqslant |\,x\,| - M \geqslant R - M$), and therefore $|\,w(\xi)\,|$ tends to zero; on the other hand, using (2.43) we see that the second integral will tend to zero if $M \to \infty$. Thus, it will suffice to take, for example, $M = R/2$, and then both integrals will tend to zero as $R \to \infty$.

2.6. The results obtained in the preceding paragraphs can be summed up as follows:

THEOREM 2.4. *Let us consider problem* (IL), *with or without the condition at infinity, and subject to conditions* (L1)–($L5_s$), *where s is a nonnegative integer. This problem determines an analytic semigroup ϕ_t on the space $\hat{C}(\overline{\Omega})$, which restricts to an analytic semigroup $\phi_{k,t}$ on each of the spaces $\hat{C}^k(\overline{\Omega})$ (k integer, $0 \leqslant k \leqslant s$); the corresponding generators A_k are also given by the same operator A but restricted to different domains,*

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and they coincide with the closure of the operators $\mathfrak{A}_k$. *For every such* k, *one has the properties*

$$(2.44)\qquad D^\beta \hookrightarrow \hat{C}^{k+q}(\overline{\Omega}),\quad \forall\beta > q/2m \qquad (q \text{ integer}, 0 \leqslant q \leqslant 2m),$$

$$(2.45)\qquad D^\beta \hookrightarrow \hat{C}^{k+2m+\alpha}(\overline{\Omega}),\quad \forall\beta > (2m+\alpha)/2m,$$

where D^β *denotes the fractional power spaces corresponding to the semigroup on* $\hat{C}^k(\overline{\Omega})$. □

REMARK. Many of the estimates and partial results obtained in the preceding paragraphs can be extended from the norms C^k (k integer) to the norms C^ρ (ρ noninteger). For noninteger ρ, as an abstract operator on $\hat{C}^\rho(\overline{\Omega})$ corresponding to the linear problem (IL), we shall take the operator $\mathfrak{A}_\rho$ given by the differential operator $\mathcal{Q}$ with domain $\mathfrak{D}_\rho$ consisting in the set $\hat{C}^{\rho+2m}(\overline{\Omega})$. One well-known fact that causes problems when considering the case of Hölder spaces is that, for ρ noninteger, the spaces $C^\sigma(\overline{\Omega})$ ($\sigma > \rho$) are not dense in $C^\rho(\overline{\Omega})$. In particular this implies that $\mathfrak{D}_\rho$ is not dense in $\hat{C}^\rho(\overline{\Omega})$. In that situation it appears difficult to obtain a semigroup on $\hat{C}^\rho(\overline{\Omega})$ by extending it from $\mathfrak{D}_\rho$. On the other hand, for $\rho < s + 2m + \alpha$, Theorem 2.3 gives a family of operators $\phi_{\rho,t}$ defined on the whole space $\hat{C}^\rho(\overline{\Omega})$, and satisfying the property

$$(2.46)\qquad \|\phi_{\rho,t}w\|_{C^\rho} \leqslant C\|w\|_{C^\rho} \quad (\forall w \in \hat{C}^\rho(\overline{\Omega})).$$

However, the problem is that we cannot ensure that for every $w \in \hat{C}^\rho(\overline{\Omega})$ the mapping $t \mapsto \phi_{\rho,t}w$ be continuous from $[0,T]$ to $\hat{C}^\rho(\overline{\Omega})$; in general, this mapping can only be ensured to be continuous from $[0,T]$ to $\hat{C}^{[\rho]}(\overline{\Omega})$. All this agrees with the results obtained by von Wähl **[29]** and Kielhöfer **[8, 9]**, who, following the approach based upon the corresponding elliptic theory, consider the question of strongly elliptic operators being generators of semigroups in Hölder spaces, and show that the semigroups obtained are not of class C_0.

2.7. We end this section by proving the following result:

THEOREM 2.5. *Assuming the hypotheses of Theorem* 2.4, *let* k *be a fixed integer in the interval* $0 \leqslant k \leqslant s$, *and let us consider the semigroup on the space* $\hat{C}^k(\overline{\Omega})$. *With respect to this semigroup, properties* (2.43) *and* (2.44) *can be generalized to*

$$(2.47)\qquad D^\beta \hookrightarrow \hat{C}^{k+\eta}(\overline{\Omega}),\quad \forall\beta > \eta/2m \qquad (\eta \text{ real}, 0 \leqslant \eta \leqslant 2m+\alpha).$$

On the other hand, one also has the following reverse property:

$$(2.48)\qquad \hat{C}^{k+\eta}(\overline{\Omega}) \hookrightarrow D^\beta,\quad \forall\beta < \eta/2m \qquad (\eta \text{ real}, 0 \leqslant \eta < 2m+\alpha).$$

The combination of (2.47) *and* (2.48) *says that the scales* D^β *and* $\hat{C}^{k+\eta}(\overline{\Omega})$ *are intimately related to each other*; *using* (2.47) *and* (2.48), *many interesting properties can be transferred from the spaces* D^β *to the more concrete spaces* $\hat{C}^{k+\eta}(\overline{\Omega})$. □

To prove Theorem 2.5 we shall make use of the theory of interpolation of Banach spaces; for the basic notions and results of this theory we refer the reader to Bergh and Löfström **[2]** or Triebel **[28]**. Given a pair of Banach spaces X and Y, $(X,Y)_{\theta,p}$ will denote their real interpolation spaces ($0 < \theta < 1$, $1 \leqslant p \leqslant \infty$), and $[X,Y]_\theta$ will denote their complex interpolation spaces ($0 < \theta < 1$); in our cases, the space Y will always be embedded in X.

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Theorem 2.5 will be proved as a consequence of the following lemma, which has also a certain interest in itself.

LEMMA 2.2. *Let us consider problem* (IL) *subject to conditions* (L1)–(L3) *and* ($L4_s$), *where s is any nonnegative integer*; *as usual*, *k will stand for an integer in the interval* $0 \leqslant k \leqslant s$. *The spaces* $\hat{C}^\sigma(\overline{\Omega})$ *defined in* §2.1 *satisfy the following properties*:

$$\hat{C}^{k+\eta}(\overline{\Omega}) \hookrightarrow \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+2m}(\overline{\Omega})\right)_{\eta/2m,\infty} \qquad (0 < \eta < 2m) \tag{2.49}$$

$$\hat{C}^{k+\eta}(\overline{\Omega}) \hookrightarrow \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+2m+\alpha}(\overline{\Omega})\right)_{\eta/(2m+\alpha),\infty} \qquad (0 < \eta < 2m+\alpha). \tag{2.50}$$

Furthermore, *if* η *is not an integer*, *then the two spaces in* (2.49) *are equal* (*with equivalent norms*). □

PROOF OF LEMMA 2.2. Properties (2.49) and (2.50) follow easily from Theorem 2.3 by using the characterization of the real interpolation spaces by the method of traces (see, for example, Bergh and Löfström [**2**, §3.12]). Having this characterization in mind, properties (2.49) and (2.50) are an almost immediate consequence of the fact that the solutions of Theorem 2.3 satisfy the following estimates (they follow from (2.31)–(2.34) with $s = k$ and $\rho = k + \eta$):

$$t^{(2m-\eta)/2m} \| \phi_t w \|_{C^{k+2m}} \leqslant C \| w \|_{C^{k+\eta}} \quad (\forall t \in (0, T]), \tag{2.51}$$

$$t^{(2m-\eta)/2m} \| D_t \phi_t w \|_{C^k} \leqslant C \| w \|_{C^{k+\eta}} \quad (\forall t \in (0, T]), \tag{2.52}$$

$$t^{(2m+\alpha-\eta)/2m} \| \phi_t w \|_{C^{k+2m+\alpha}} \leqslant C \| w \|_{C^{k+\eta}} \quad (\forall t \in (0, T]). \tag{2.53}$$

If we modify $\phi_t w$ to $v(t) \equiv \xi(t)\phi_t w$, where ξ is a C^∞ function on $(0, \infty)$ such that $\xi(t) = 1$ for $t \leqslant T/2$ and $\xi(t) = 0$ for $t \geqslant T$, then $v(t)$ satisfies a set of bounds analogous to (2.51)–(2.53) but on the whole interval $t \in (0, \infty)$. According to Theorem 2.3, any $w \in \hat{C}^{k+\eta}(\overline{\Omega})$ can be represented as the trace at $t = 0$ of such a function $v(t)$, and furthermore, this function satisfies

$$v(t) \in \hat{C}^{k+2m+\alpha}(\overline{\Omega}) \subset \hat{C}^{k+2m}(\overline{\Omega}) \subset \hat{C}^k(\overline{\Omega}) \quad (\forall t > 0).$$

From these considerations, one obtains relations (2.49) and (2.50) ((2.49) follows from (2.51) and (2.52), and (2.50) follows from (2.52) and (2.53); see, for example, Bergh and Löfström [**2**, §§3.12.2, 3.12.3, and the note in §3.14.12 saying that these results are also true for $p_i = \infty$]).

The equivalence of the two spaces in (2.49) when η is not an integer follows from the fact that an analogous relation is true for the corresponding spaces without the boundary conditions: if the domain Ω is sufficiently smooth, then for any two nonnegative integers k and l,

$$C^{k+\eta}(\overline{\Omega}) = \left(C^k(\overline{\Omega}), C^{k+l}(\overline{\Omega})\right)_{\eta/l,\infty} \qquad (\eta \text{ noninteger}, 0 < \eta < l). \tag{2.54}$$

For $\Omega = \mathbf{R}^n$, this property can be obtained by proceeding along the lines of Triebel [**27**, p. 70], who considers the subspaces of $C^\sigma(\overline{\mathbf{R}^n})$ consisting of functions such that $u\big|_\infty = 0$ (and therefore $D^\nu u\big|_\infty = 0$ for $0 \leqslant |\nu| \leqslant \sigma$, by Lemma 2.1). To obtain analogous relations for a general domain Ω substituted for $\mathbf{R}^n$, it suffices to use a linear operator $\mathfrak{S}$ extending functions from Ω to $\mathbf{R}^n$ and such that $\mathfrak{S}$ be continuous from $C^\sigma(\overline{\Omega})$ to $C^\sigma(\overline{\mathbf{R}^n})$ ($k \leqslant \sigma \leqslant k + l$); for example, for our purposes it suffices to

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use the extension method of Hestenes [6]; this method requires Ω to be uniformly of class C^{k+l}.

From property (2.54) we can obtain the relation

$$(2.55)\quad \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega})\right)_{\eta/l,\infty} \hookrightarrow \hat{C}^{k+\eta}(\overline{\Omega}) \qquad (\eta \text{ noninteger}, 0<\eta<l),$$

where the $\hat{C}^\sigma(\overline{\Omega})$ are the spaces defined in §2.1 in connection with problem (IL). Of course, this relation will imply that the two spaces in (2.49) are equivalent. To establish (2.55), we start from the fact that this relation is true with $C^{k+\eta}(\overline{\Omega})$ (without accessory conditions) substituted for the right-hand side:

$$(2.56)\quad \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega})\right)_{\eta/l,\infty} \hookrightarrow C^{k+\eta}(\overline{\Omega}) \qquad (\eta \text{ noninteger}, 0<\eta<l).$$

This is an immediate consequence of the embeddings $\hat{C}^\sigma(\overline{\Omega}) \hookrightarrow C^\sigma(\overline{\Omega})$ and (2.54). On the other hand, if the set of accessory conditions includes $u\big|_\infty = 0$, it is clear that the elements of $(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega}))_{\eta/l,\infty}$ satisfy this condition, because they belong to $\hat{C}^k(\overline{\Omega})$. Therefore, we only have to verify that these elements also satisfy all the conditions of order $m_j \leqslant k+\eta$ in (2.10). To obtain this, we can proceed as follows: Let ζ be such that $[\eta] < \zeta < \eta$; from (2.56) and the basic results of interpolation theory, we have the following chain of embeddings:

$$(2.57)\quad \begin{aligned} \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega})\right)_{\eta/l,\infty} &\hookrightarrow \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega})\right)_{\zeta/l,p} \\ &\hookrightarrow \left(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega})\right)_{\zeta/l,\infty} \hookrightarrow \hat{C}^{k+\zeta}(\overline{\Omega}). \end{aligned}$$

Here, p is taken $< \infty$, in which case it is known that $\hat{C}^{k+l}(\overline{\Omega})$ is dense in $(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega}))_{\zeta/l,p}$ (see Bergh and Löfström [2, §3.4.2]). Now, for $m_j \leqslant k+\eta$, $\mathfrak{B}_j$ constitutes a continuous linear operator from $C^{k+\zeta}(\overline{\Omega})$ to $C^{k+\zeta-m_j}(\overline{\partial\Omega})$, and by (2.57) it has the same property from $(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+1}(\overline{\Omega}))_{\zeta/l,p}$ to $C^{k+\zeta-m_j}(\overline{\partial\Omega})$; on the other hand, for $m_j \leqslant k+\eta$, the operators $\mathfrak{B}_j$ vanish on $\hat{C}^{k+l}(\overline{\Omega})$ by definition of this space; therefore, by continuity they must vanish on the whole space $(\hat{C}(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega}))_{\zeta/l,p}$. In particular, they vanish on $(\hat{C}^k(\overline{\Omega}), \hat{C}^{k+l}(\overline{\Omega}))_{\eta/l,\infty}$, and this establishes property (2.55) Q.E.D.

Proof of Theorem 2.5. To obtain Theorem 2.5 we shall use the real interpolation spaces between $E \equiv \hat{C}^k(\overline{\Omega})$ and $D \equiv D_k$, the domain of the generator of the semigroup $\phi_{k,t}$. The spaces $(E, D)_{\theta,p}$ are known to be related to the spaces D^β by the following relations:

$$(2.58)\qquad (E, D)_{\beta,1} \hookrightarrow D^\beta \hookrightarrow (E, D)_{\beta,\infty} \qquad (0<\beta<1)$$

(see, for example, Komatsu [10]).

Let us first consider property (2.47). By Theorem 2.4 we know that it holds for η integer, and also for $2m < \eta < 2m+\alpha$ (redefine $\alpha(\text{new}) \equiv \eta - 2m$ and use (2.45)); our objective is to extend this property to noninteger values of η in the interval $0 < \eta < 2m$. Given any such value of η, let q_1 and q_2 be two integers such that $0 \leqslant q_1 < \eta < q_2 \leqslant 2m$, and let $\theta \equiv (\eta - q_1)/(q_2 - q_1)$; from relations (2.44) applied to these values of q, and Lemma 2.2, we can derive that

$$(2.59)\qquad (D^{\beta_1}, D^{\beta_2})_{\theta,\infty} \hookrightarrow \left(\hat{C}^{k+q_1}(\overline{\Omega}), \hat{C}^{k+q_2}(\overline{\Omega})\right)_{\theta,\infty} = \hat{C}^{k+\eta}(\overline{\Omega}),$$

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where β_1 and β_2 are any pair of numbers such that $\beta_1 > q_1/2m$ and $\beta_2 > q_2/2m$. Therefore, we only need to see that, for any $\beta > \eta/2m$, D^β is embedded in some space of the type appearing at the left-hand side of (2.59). This can be obtained from (2.58) by using certain properties about the relation between the real and complex interpolation spaces (see, for example, Bergh and Löfström [**2**, §§4.7.1, 4.7.2]); using these properties we obtain the following chain of embeddings:

$$\begin{aligned} D^\beta &\hookrightarrow (E, D)_{\beta,\infty} \hookrightarrow (E, D)_{\bar\beta,1} = \left[(E, D)_{\beta_1,1}(E, D)_{\beta_2,1}\right]_\theta \\ &\hookrightarrow [D^{\beta_1}, D^{\beta_2}]_\theta \hookrightarrow (D^{\beta_1}, D^{\beta_2})_{\theta,\infty}, \end{aligned}$$

where $\bar\beta \equiv (1-\theta)\beta_1 + \theta\beta_2$, and β is any number strictly larger than $\bar\beta$. It is easily verified that, for any $\beta > \eta/2m$, we can find $\beta_1 > q_1/2m$ and $\beta_2 > q_2/2m$ such that $\beta > \bar\beta$. This establishes property (2.47).

Let us now consider property (2.48). The only property of this type that we have at hand to use as a starting point is the embedding

$$\hat C^{k+2m+\alpha}(\bar\Omega) \hookrightarrow D. \tag{2.60}$$

This follows from the fact that the generator of the semigroup on $\hat C^k(\bar\Omega)$ is the closure of the operator $\mathfrak{A}_k$, whose domain is $\mathfrak{D}_k \equiv \hat C^{k+2m+\alpha}(\bar\Omega)$. From (2.60), property (2.48) can be obtained by the following chain of embeddings:

$$\begin{aligned} \hat C^{k+\eta}(\bar\Omega) &\hookrightarrow \left(\hat C^k(\bar\Omega), \hat C^{k+2m+\alpha}(\bar\Omega)\right)_{\eta/(2m+\alpha),\infty} \\ &\hookrightarrow (E, D)_{\eta/(2m+\alpha),\infty} \hookrightarrow (E, D)_{\beta,1} \hookrightarrow D^\beta, \end{aligned}$$

where we are using Lemma 2.2 and relation (2.60), and β is any number strictly smaller than $\eta/(2m+\alpha)$. From this, (2.48) follows by taking α sufficiently small. Q.E.D.

3. Denseness of $\hat C^\sigma(\bar\Omega)$ in $\hat C^k(\bar\Omega)$ (k integer, $\sigma > k$). The objective of this section is to verify the validity of properties like $(\mathrm{L5}_k)$ at least for the most frequently encountered cases; in general, we shall study the denseness of $\hat C^\sigma(\bar\Omega)$ in $\hat C^k(\bar\Omega)$ for k integer and $\sigma > k$.

This property is certainly true for the corresponding spaces without boundary conditions: When Ω is a bounded domain, the denseness of $C^\infty(\bar\Omega)$ in $C^k(\bar\Omega)$ follows from the n-dimensional version of Weierstrass' theorem. When $\Omega = \mathbf{R}^n$, this property can be obtained by using the Sobolev-Friedrichs averaging method. For general unbounded domains, one can take the approximating functions in the form

$$u_n = \mathfrak{R}\Phi_{\varepsilon_n}\mathfrak{E}u, \tag{3.1}$$

where $\mathfrak{E}$ denotes an extension operator from $C^k(\bar\Omega)$ to $C^k(\overline{\mathbf{R}^n})$, $\mathfrak{R}$ is the restriction from $\mathbf{R}^n$ to Ω, Φ_ε are the Sobolev-Friedrichs mollifier operators, and $\varepsilon_n \to 0$. Note that, although Ω is not necessarily bounded, $u_n \to u$ in the uniform C^k topology, because $\mathfrak{E}u \in C^k(\overline{\mathbf{R}^n})$ means that u is uniformly continuous together with all its partial derivatives of order $\leqslant k$. The extension $\mathfrak{E}u \in C^k(\overline{\mathbf{R}^n})$ can be obtained by several methods; for our purposes it suffices to use the method of Hestenes [**6**], which can be used whenever Ω is uniformly of class C^k.

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To obtain similar properties for the spaces $\hat{C}^k(\overline{\Omega})$, the idea will be to modify the functions u_n in a neighborhood of $\partial\Omega \cup \{\infty\}$ in order to make them satisfy the accessory conditions, and show that this modification can be made arbitrarily small in the C^k norm. In what follows we show how to do this for scalar-valued functions (i.e. $N = 1$) assuming that the system of boundary operators is what we call uniformly normal. As usual, the term *normal* means that the boundary operators $\mathfrak{B}_i$ $(1 \leqslant i \leqslant m)$ satisfy the following conditions: (a) the orders m_i $(1 \leqslant i \leqslant m)$ are different from each other; (b) for every $x \in \partial\Omega$ and $1 \leqslant i \leqslant m$, we have $\beta_i(x) \equiv \mathfrak{B}_i^0(x, \nu(x)) \neq 0$, where $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at the point x, and $\mathfrak{B}_i^0$ denotes the principal part of $\mathfrak{B}_i$, i.e. $\mathfrak{B}_i^0(x, \xi) \equiv \sum_{|\alpha| = m_i} b_{i\alpha}(x)\xi^\alpha$. By *uniformly normal* we mean that in fact the quantities $\beta_i(x)$ are bounded away from zero by a constant independent of $x \in \partial\Omega$. Of course, when $\partial\Omega$ is bounded, then any normal system is uniformly normal. When $m > 1$ we shall always assume that the indexing of the boundary operators is such that $0 \leqslant m_1 < \cdots < m_m \leqslant 2m - 1$.

3.1. *One-dimensional case*. Let us begin by considering the one-dimensional case; for the sake of definiteness we shall take $\Omega = (0, +\infty)$, but it will be evident that the arguments extend to any other type of interval. For the moment, we shall consider the case where the problem does not include any condition at infinity. In the present one-dimensional situation, the $\mathfrak{B}_j$ $(1 \leqslant j \leqslant \infty)$ of (2.10) are ordinary differential operators of orders m_j satisfying $0 \leqslant m_j < m_2 < \cdots$. Without loss of generality we can assume that the coefficient of the leading term is 1, i.e.

$$\mathfrak{B}_j = D^{m_j} + \sum_{q=0}^{m_j - 1} b_{jq} D^q.$$

Assume for example that $k < m_1 \leqslant \sigma < m_2$; then our problem is the following: given a function $u \in C^k(\overline{\Omega})$, to approximate it by a sequence v_n of functions belonging to $C^\sigma(\overline{\Omega})$ and satisfying the boundary condition $\mathfrak{B}_1 v(0) = 0$. For this, the idea will be to take $v_n = u_0 + \phi_n$, where u_n is any approximating sequence of functions belonging to $C^\sigma(\overline{\Omega})$, for example any sequence of the type (3.1), and ϕ_n are functions also belonging to $C^\sigma(\overline{\Omega})$ and satisfying the following conditions:

$$\mathfrak{B}_1\phi_n(0) = -\mathfrak{B}_1 u_n(0), \qquad \|\phi_n\|_{C^k} \to 0 \quad \text{as } n \to \infty.$$

If we have $k < m_1 < m_2 \leqslant \sigma < m_3$, then we will look for v_n in the form $v_n = u_n + \phi_n^1 + \phi_n^2$, where u_n is the same as before, and ϕ_n^1, ϕ_n^2 satisfy the following set of conditions:

$$\begin{aligned}
&\mathfrak{B}_1\phi_n^1(0) = -\mathfrak{B}_1 u_n(0),\\
&\mathfrak{B}_1\phi_n^2(0) = 0,\\
&\mathfrak{B}_2\phi_n^2(0) = -\mathfrak{B}_2(u_n + \phi_n^1)(0),\\
&\|\phi_n^1\|_{C^k},\ \|\phi_n^2\|_{C^k} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Looking at examples like these, we see that it would be interesting to have a family $\theta_\varepsilon^{m,\sigma}$ of real functions defined in $\Omega = (0, +\infty)$ and satisfying the following properties:

(3.2) $$\theta_\varepsilon^{m,\sigma} \in C^\sigma(\overline{\Omega}), \qquad \operatorname{supp} \theta_\varepsilon^{m,\sigma} = [0, \varepsilon],$$

(3.3) $$D^j\theta_\varepsilon^{m,\sigma}(0) = 0 \quad (\forall j < m),$$

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(3.4) $$D^m\theta_\varepsilon^{m,\sigma}(0) = 1,$$

(3.5) $$\|\theta_\varepsilon^{m,\sigma}\|_{C^k} \to 0 \quad \text{as } \varepsilon \to 0 \quad (\forall k < m),$$

where m is a nonnegative integer and $\sigma \geqslant m$. A little searching shows that, for example, we can use the functions given in the following lemma.

LEMMA 3.1. *The functions* $\theta_\varepsilon^{m,\sigma}$: $[0, +\infty) \to \mathbf{R}$ *given by*

$$\theta_\varepsilon^{m,\sigma}(x) \equiv \begin{cases} (x^m/m!)[1 - x/\varepsilon]^\sigma & \text{if } x \in [0, \varepsilon], \\ 0 & \text{if } x > \varepsilon \end{cases}$$

(m *nonnegative integer*, $\sigma \geqslant m$) *satisfy the preceding properties* (3.2)–(3.5). *In fact, property* (3.5) *is satisfied in the following way*:

(3.6) $$\|\theta_\varepsilon^{m,\sigma}\|_{C^\rho} \leqslant C\varepsilon^{m-\rho} \qquad (0 \leqslant \rho < \sigma),$$

where ρ *denotes any real number in the indicated interval, and the constant* C *depends on* m, σ, ρ, *but not on* ε.

PROOF. Applying Leibniz' rule, the successive derivatives $D^p\theta_\varepsilon^{m,\sigma}$ $(0 \leqslant p \leqslant \sigma)$ are given by

$$D^p\theta_\varepsilon^{m,\sigma}(x) = \sum_{q=0}^{\min(m,p)} \binom{p}{q} \frac{x^{m-q}}{(m-q)!} \prod_{j=0}^{p-q-1} (\sigma - j)\left[1 - \frac{x}{\varepsilon}\right]^{\sigma-(p-q)} \left[-\frac{1}{\varepsilon}\right]^{p-q}$$

from which we easily obtain properties (3.3), (3.4), (3.6). On the other hand, to see that $\theta_\varepsilon^{m,\sigma} \in C^\sigma(\overline{\Omega})$ $(\Omega = (0, +\infty))$, it suffices to verify that $D^p\theta_\varepsilon^{m,\sigma}(\varepsilon) = 0$ $(\forall p \leqslant [\sigma])$, and for noninteger σ, that $D^{[\sigma]}\theta_\varepsilon^{m,\sigma}$ satisfies a Hölder condition with exponent $\sigma - [\sigma]$; both properties are also clear from the preceding formula. Q.E.D.

We can easily see that these functions can indeed be used to solve our problem. For example, in the case $k < m_1 \leqslant \sigma < m_2$, it suffices to take the functions ϕ_n in the form $\phi_n = a_n\theta_{\varepsilon_n}^{m_1,\sigma}$, where $a_n = -\mathfrak{B}_1 u_n(0)$, and ε_n will be a certain sequence tending to zero. From properties (3.3) and (3.4) it follows that $\mathfrak{B}_1 v_n(0) = 0$, and from (3.6) we obtain that

$$\|v_n - u\|_{C^k} \leqslant \|u_n - u\|_{C^k} + \|\phi_n\|_{C^k} \leqslant \|u_n - u\|_{C^k} + K|a_n|\varepsilon_n^{m_1-k}.$$

Therefore, it will suffice to take the sequence ε_n tending to zero rapidly enough so that $|a_n|\varepsilon_n^{m_1-k} \to 0$ as $n \to \infty$. On the other hand, using property (3.3), we can also solve the case where there are several boundary conditions involved. For example, when $k < m_1 < m_2 \leqslant \sigma < m_3$, it suffices to take $\phi_n^1 = a_n\theta_{\varepsilon_n}^{m_1,\sigma}$, $\phi_n^2 = b_n\theta_{\eta_n}^{m_2,\sigma}$, where $a_n = -\mathfrak{B}_1 u_n(0)$, $b_n = -\mathfrak{B}_2(u_n + \phi_n^1)(0)$, $\varepsilon_n = o(|a_n|^{k-m_1})$, $\eta_n = o(|b_n|^{k-m_2})$. Using the same strategy we can handle any situation of the type $k < m_1 < \cdots < m_j \leqslant \sigma < m_{j+1}$.

In fact, a similar strategy also solves the case where some of the m_i's are less than or equal to k. For example, consider the case $m_1 \leqslant k < m_2 \leqslant \sigma < m_3$; if we take v_n as in the last example, the problem is that, for $m_1 \leqslant k$, $\|\theta_\varepsilon^{m_1,\sigma}\|_{C^k}$ does not now tend to zero as $\varepsilon \to 0$. Nevertheless for $m_1 \leqslant k$, $u \in \hat{C}^k(\overline{\Omega})$ implies that it satisfies the condition $\mathfrak{B}_1 u(0) = 0$, and therefore, the approximating sequence u_n satisfies $|a_n| = |\mathfrak{B}_1 u_n(0)| \to 0$ as $n \to \infty$; thus, considering that $\|\phi_n^1\|_{C^k} \leqslant K|a_n|\varepsilon_n^{m_1-k}$, we can still

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make $\|\phi_n^1\|_{C^k}$ tend to zero by taking as ε_n any sequence tending to zero slowly enough to have $|a_n|\varepsilon_n^{m_1-k} \to 0$.

In the case where the problem also contains the condition at infinity $u|_\infty = 0$, if we take u_n in the form (3.1), then the problem is automatically solved, because $u|_\infty = 0$ implies the same property for u_n (and all its derivatives); on the other hand, this property extends automatically to $v_n = u_n + \phi_n$, since ϕ_n has compact support.

With this, we have proved the result stated in §3.3 below in the case of a one-dimensional domain.

3.2. *Many-dimensional case.* Let us now consider the case of Ω being a domain in $\mathbf{R}^n$ ($n \geqslant 2$). As in the one-dimensional case, we shall begin by considering the case where the problem does not include the condition at infinity. For the moment, we shall restrict ourselves to boundary operators including only normal derivatives D_ν^k; more precisely, the boundary operators $\mathcal{B}_i$ ($1 \leqslant i \leqslant m$) are assumed to be of the form

$$\mathcal{B}_i = D_\nu^{m_i} + \text{lower order terms in } D_\nu. \tag{3.7}$$

Provisionally, we shall also restrict σ by $\sigma < 2m + m_1$; in that situation, from the infinite set of conditions (2.2), the only ones that matter are those with $r = 0$.

In that case, the problem can be solved on the basis of the one-dimensional case by proceeding as follows: Let us assume $\partial\Omega$ uniformly of class $C^{1+\alpha}$, and let $\Gamma_{-\delta}$ denote the region adjacent to the boundary of Ω, and defined by

$$\Gamma_{-\delta} \equiv \{ y - \alpha\nu(y) \mid y \in \partial\Omega, 0 < \alpha < \delta \}$$

where $\nu(y)$ denotes the outward normal to $\partial\Omega$ at the point y, and δ is a positive real number small enough so that (a) $\Gamma_{-\delta} \subset \Omega$, and (b) for every $x \in \Gamma_{-\delta}$ there exists a unique $y \in \partial\Omega$ such that $x = y - \alpha\nu(y)$ for some $a \in (0, \delta)$. The maximal number δ satisfying these conditions will be denoted by $\bar{\delta}$. As usually defined, to be uniformly of class $C^{1+\alpha}$ implies that $\bar{\delta} > 0$. For every $x \in \Gamma_{-\delta}$, we shall denote by x^* and ξ those unique elements of $\partial\Omega$ and $(0, \delta)$ for which $x = x^* - \xi\nu(x^*)$. To treat our problem, the idea will be to apply the one-dimensional procedure on every ray normal to $\partial\Omega$, so that our perturbations will have support in $\overline{\Gamma}_{-\delta}$.

For example, in the case $k < m_1 \leqslant \sigma < m_2$, given a function $u \in \hat{C}^k(\overline{\Omega})$, to approximate it by functions v_n belonging to $\hat{C}^\sigma(\overline{\Omega})$, it suffices to take $v_n = u_n + \phi_n$, where u_n is an approximating sequence of smooth functions given by (3.1), and ϕ_n are functions of the form

$$\phi_n(x) \equiv \begin{cases} -\mathcal{B}_1 u_n(x^*)\theta_{\varepsilon_n}^{m_1,\sigma}(\xi) & \text{if } x \in \Gamma_{-\delta}, \\ 0 & \text{if } x \in \Omega \setminus \Gamma_{-\delta}, \end{cases}$$

where $\varepsilon_n = o(\sup|\mathcal{B}_1 u_n(y)|)$. In order that the functions ϕ_n belong to $C^\sigma(\overline{\Omega})$ it suffices that the u_n belong to $C^{\sigma+m_1}(\overline{\Omega})$ and that $\partial\Omega$ be of class C^σ, because then the mappings $x \mapsto x^*$ and $x \mapsto \xi$ are of class C^σ. From the way they are constructed, the functions ϕ_n will satisfy

$$\mathcal{B}_1\phi_n|_{\partial\Omega} = D_\nu^{m_1}\phi_n|_{\partial\Omega} = -\mathcal{B}_1 u_n|_{\partial\Omega}, \qquad \|\phi_n\|_{C^k} \to 0 \quad \text{as } n \to \infty,$$

which imply that $v_n \in \hat{C}^\sigma(\overline{\Omega})$ and $\|v_n - u\|_{C^k} \to 0$.

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Following the idea of the one-dimensional case, we can solve similarly the case involving different boundary conditions of the form (3.7), and also the case where some of the m_i's are less than or equal to k. In general, for $k < m_j \leqslant \sigma < m_{j+1}$ ($j \leqslant m$), the approximating sequence v_n is obtained in the form $v_n = u_n + \phi_n^1 + \cdots + \phi_n^j$, where the functions ϕ_n^i ($1 \leqslant i \leqslant j$) are of the form

$$\phi_n^i(x) \equiv \begin{cases} a_n^i(x^*)\theta_{\varepsilon_n}^{m_i,\sigma}(\xi) & \text{if } x \in \Gamma_{-\delta}, \\ 0 & \text{if } x \in \Omega \setminus \Gamma_{-\delta}, \end{cases}$$

with $a_n^i(x^*) = -\mathfrak{B}_i[u_n + \phi_n^1 + \cdots + \phi_n^{j-1}](x^*)$, and the ε_n are chosen suitably.

Now we can see that in fact the preceding method can be applied to boundary conditions of more general form than (3.7). The reason for this is that, for $m_i > 0$, the functions ϕ_n^i satisfy $D_\nu^q \phi_n^i|_{\partial\Omega} = 0$ $(0 \leqslant q < m_i)$; this implies that all the tangential derivatives of the functions $D_\nu^q \phi_n^i$ $(0 \leqslant q < m_i)$ are zero, and therefore we see that, for any boundary operator of order m_i (not necessarily of the form (3.7)),

$$\mathfrak{B}_i \phi_n^i|_{\partial\Omega} = \beta_i(x) D_\nu^{m_i} \phi_n^i|_{\partial\Omega},$$

where $\beta_i(x) \equiv \mathfrak{B}_i^0(x, \nu(x))$ (ν: unit outward normal, $\mathfrak{B}_i^0$: principal part of $\mathfrak{B}_i$). Therefore, to treat the general case it will suffice to divide the coefficients $a_n^i(x^*)$ by $\beta_i(x^*)$. In order that one can still choose the sequences ε_n suitably, one needs that the quantitites $\beta_i(y)$ $(y \in \partial\Omega)$ stay bounded away from zero and from infinity; the first part is assured if the system of boundary operators is assumed to be uniformly normal; the second part will be true if we assume that the coefficients $b_{i\alpha}$ belong to $C(\overline{\partial\Omega})$ (in particular, if we assume condition (L4$_0$)). This reasoning also extends to show that we can drop the restriction $\sigma < 2m + m_1$, whenever we assume condition (L4$_s$) with $s + \alpha \geqslant \sigma - 2m$.

Finally, the result also extends to the case where the problem contains the condition at infinity $u|_\infty = 0$. This can be seen by essentially the same argument that we used in the one-dimensional case, with the only difference that when $\partial\Omega$ is not compact the perturbations ϕ_n^i do not have compact support, but they are seen to tend to zero as $|x| \to \infty$ as a consequence of the functions u_n having this property together with all their partial derivatives (this follows from (3.1)).

3.3. The preceding reasoning gives a proof of the theorem stated below in the case of scalar-valued functions. It can easily be seen that the result extends to the case of functions with several components assuming that the boundary conditions consist of a uniformly normal system for each of these components. In particular, this covers the case of problem (X). Although we shall not enter into details, it is clear that the result must remain true for systems of boundary conditions more general than diagonal ones, in which case one should consider some generalization of the condition of uniform normality.

THEOREM 3.1. *Let us consider problem* (IL) *with or without the condition at infinity and subject to conditions* (L1)–(L3) *and* (L4$_s$) (*s is a nonnegative integer*); *assume also that the boundary conditions consist of a uniformly normal system for each one of the components. For every integer k in the interval* $0 \leqslant k \leqslant s + 2m$, *the space* $\hat{C}^{s+2m+\alpha}(\overline{\Omega})$ *is dense in* $\hat{C}^k(\overline{\Omega})$. *If the domain Ω is one dimensional, then* $\hat{C}^\sigma(\overline{\Omega})$ *is dense in* $\hat{C}^k(\overline{\Omega})$ *for every nonnegative integer k and any $\sigma > k$.* □

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4. Nonlinear problem.

4.1. In this last section we apply the abstract results of §1.5 to the nonlinear problem (I) of §1.3. To avoid repetition, we shall consider directly the general case where N' is not necessarily zero. We shall use the following notations:

$$\text{(4.1)} \qquad w \equiv (u, v), \qquad h \equiv (f, g),$$

$$Tw \equiv \text{vector consisting of } D^\nu u^i \ (0 \leqslant |\nu| \leqslant l,\ 1 \leqslant i \leqslant N)$$
$$\text{plus the } v^j \ (1 \leqslant j \leqslant N').$$

The dimension of this vector will be denoted by M (specifically, $M = [(n+l)!/(n!l!)]N + N'$); accordingly, f, g, h will be considered as functions of $x \in \Omega$ and a vector $W \in \mathbf{R}^M$; the capital letter W will always denote a generic vector in $\mathbf{R}^M$.

For E, we shall take spaces of the type $\hat{C}^k(\overline{\Omega}) \times C^k(\overline{\Omega})$, where k is a nonnegative integer, and we define $\|w\|_{X\times Y} \equiv \|u\|_X + \|v\|_Y$. The linear operator A will be the diagonal operator (A_k, O), where A_k denotes the generator of the semigroup determined on $\hat{C}^k(\overline{\Omega})$ by the linear problem (IL) (assuming conditions (L1)–(L3), ($\mathrm{L4}_k$) and ($\mathrm{L5}_k$)), and O is the zero operator on $C^k(\overline{\Omega})$, i.e. $Aw \equiv A_k u$. Note that the zero operator is the generator of the analytic semigroup given by the identity ($\phi_t = I$, $t \geqslant 0$). Since A has this diagonal structure, it is clear that it is the generator of an analytic semigroup on $\hat{C}^k(\overline{\Omega}) \times C^k(\overline{\Omega})$. This gives hypothesis (H1) of §1.5; using Theorem 2.4 we can see that hypotheses (H2) and (H3) will be fulfilled if we take $X = \hat{C}^{k+q}(\overline{\Omega}) \times C^k(\overline{\Omega})$ with q being an integer in the interval $0 \leqslant q \leqslant 2m-1$ (for this we must assume conditions ($\mathrm{L4}_{k+q}$) and ($\mathrm{L5}_{k+q}$)). When $N' = 0$ and Ω is bounded, then the operator $A = A_k$ has compact resolvent; this follows from the fact that $D \hookrightarrow \hat{C}^{k+2m-1}(\overline{\Omega})$ (Theorem 2.4), and the compactness of the embedding $\hat{C}^{k+2m-1}(\overline{\Omega}) \hookrightarrow \hat{C}^k(\overline{\Omega})$. Note, however, that this property is not true when $N' > 0$, even if Ω is bounded; this reflects the fact that the domain of O is the whole space $C^k(\overline{\Omega})$.

Let us now consider the nonlinear operator F; we define it as

$$[F(w)](x) \equiv h(x, Tw(x)) \quad (\forall x \in \Omega),$$

where we are using notation (4.1). Let us consider hypotheses (H4) and (H5), and in general the smoothness of F. Obviously, in order for F to take $X = \hat{C}^{k+q}(\overline{\Omega}) \times C^k(\overline{\Omega})$ to $E = \hat{C}^k(\overline{\Omega}) \times C^k(\overline{\Omega})$, we shall need to take $q \geqslant l$. To reduce the requirements on F to conditions on $h = (f, g)$, note that the general case $q \geqslant l \geqslant 0$ reduces to the case $q = l = 0$ by decomposing F in the following way:

$$\begin{array}{ccccc} C^{k+q}(\overline{\Omega}) \times C^k(\overline{\Omega}) & & C^k(\overline{\Omega}) & & C^k(\overline{\Omega}) \\ w & \underset{T}{\mapsto} & Tw \equiv W & \underset{\hat{F}}{\mapsto} & h(x, W(x)) \end{array}$$

Since T is a bounded linear mapping (because $q \geqslant l$), it is clear that the conditions on F will reduce to analogous conditions on $\hat{F}$. Similarly, the case $k \geqslant 0$ reduces to the case $k = 0$ by substituting for h a new vector consisting of the $(\partial/\partial x)^\nu h^i(x, W(x))$ $(0 \leqslant |\nu| \leqslant k,\ 1 \leqslant i \leqslant N + N')$.

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Using these considerations, it can be seen that, for F to take $X = \hat{C}^{k+q}(\bar{\Omega}) \times C^k(\bar{\Omega})$ to $E = \hat{C}^k(\bar{\Omega}) \times C^k(\bar{\Omega})$ and to satisfy the condition of being Lipschitz on bounded sets, it suffices that the function $h = (f, g)$ satisfies the following conditions:

($N_{k,p,0}$) (a) The function $h(x, W)$ is of class C^k (jointly in x and W), and for $|\alpha| + |\beta| \leqslant k$, $D_x^\alpha D_W^\beta h(\cdot, W) \in C(\bar{\Omega})$ $(\forall W \in R^M)$.

(b) The partial derivatives $D_x^\alpha D_W^\beta h(x, W)$ $(|\alpha| + |\beta| \leqslant k)$ are locally Lipschitz with respect to $W \in R^M$, with Lipschitz constants independent of $x \in \Omega$.

(c) If $u \in \hat{C}^{k+q}(\bar{\Omega})$, i.e. it satisfies the accessory conditions of order $\leqslant k + q$, then $f(x, Tw(x))$ satisfies those of order $\leqslant k$ (for any $v \in C^k(\bar{\Omega})$).

In general, in order that F be of class C^r $(1 \leqslant r \leqslant \infty)$ from $X = \hat{C}^{k+q}(\bar{\Omega}) \times C^k(\bar{\Omega})$ to $E = \hat{C}^k(\bar{\Omega}) \times C^k(\bar{\Omega})$, the following condition must be substituted for the preceding one:

($N_{k,q,r}$) (a) The function $h(x, W)$ has the partial derivatives $D_x^\alpha D_W^\beta h$ for $|\alpha| \leqslant k$ and $|\alpha| + |\beta| \leqslant k + r$, and they satisfy $D_x^\alpha D_W^\beta h(\cdot, W) \in C(\bar{\Omega})$ $(\forall W \in R^M)$.

(b) The partial derivatives $D_x^\alpha D_W^\beta h(x, W)$ $(|\alpha| \leqslant k, |\alpha| + |\beta| \leqslant k + r)$ are locally Hölder continuous with respect to $W \in R^M$, with Hölder constants independent of $x \in \Omega$ (this condition is required only if $r < \infty$).

(c) For any $s \leqslant r$, if $w, \psi_1, \ldots, \psi_s \in \hat{C}^{k+q}(\bar{\Omega}) \times C^k(\bar{\Omega})$, then

$$D_W^s f(x, Tw(x)) T\psi_1(x) \cdots T\psi_s(x)$$

satisfies all the accessory conditions of order $\leqslant k$.

In this case, the Fréchet derivatives $D^s F(w_0)$ $(0 \leqslant s \leqslant r)$ are given by

$$[D^s F(w_0)\psi_1 \cdots \psi_s](x) = D_W^s h(x, Tw_0(x)) T\psi_1(x) \cdots T\psi_s(x)$$

$(\forall x \in \Omega)$, where we are using notation (4.1); in particular, $DF(w_0)$ is given by a linear differential operator of order l, which we shall denote by $\mathcal{F}_{w_0}(x, D_x)$.

Finally, let us assume ($N_{k,p,1}$), and consider hypothesis (H6). For the moment we consider the case $N' = 0$. Let $L_k \equiv A + DF(u_0)$ considered as an operator on $E = \hat{C}^k(\bar{\Omega})$; this operator is the generator of an analytic semigroup on E, call it ϕ_t. Let us define $\mathfrak{L}_i$ $(i = k, k + q)$ as the operator on $\hat{C}^i(\bar{\Omega})$ given by the differential operator $\mathcal{Q} + \mathcal{F}_{u_0}$ with domain $\mathfrak{D}_i = \hat{C}^{i+2m+\alpha}(\bar{\Omega})$; it is clear that L_k is the closure of $\mathfrak{L}_k$. On the other hand, the equation $\dot{u} = \mathfrak{L}_{k+q} u$ is seen to determine an analytic semigroup ψ_t on $X = \hat{C}^{k+q}(\bar{\Omega})$; obviously, this semigroup must coincide with the restriction of the semigroup ϕ_t, and the generator of ψ_t, which is the closure of $\mathfrak{L}_{k+q}$, must coincide with the restriction of L_k to X, i.e. an operator with domain $\{u \in X \mid L_k u \in X\}$. If $N' > 0$, we shall restrict ourselves to the case $q = l = 0$, in which case hypothesis (H6) is obviously satisfied (since $X = E$), and $A + DF(w_0)$ is easily seen to be the closure of the operator $\bar{\mathcal{Q}} + \mathcal{F}_{w_0}$ $(\bar{\mathcal{Q}} \equiv \operatorname{diag}(\mathcal{Q}, O))$ with domain $\hat{C}^{k+2m+\alpha}(\bar{\Omega}) \times C^k(\bar{\Omega})$.

Applying Theorems 1.1 and 1.2 as indicated by the preceding arguments, we obtain the following results:

THEOREM 4.1. *Let us consider problem* (I) *subject to conditions* (L1)–(L3), ($L4_{k+q}$), ($L5_{k+q}$) *and* ($N_{k,q,r}$), *where k, q, r are nonnegative integers, $l \leqslant q \leqslant 2m - 1$, and r can also be ∞. In that situation, problem* (I) *determines a semiflow of class C^r on the*

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space $\hat{C}^{k+q}(\overline{\Omega}) \times C^k(\overline{\Omega})$. *This semiflow* ϕ_t *satisfies the strong maximal property*

$$\omega(w) < \infty \Rightarrow \lim_{t \to \omega(w)} \|\phi_t w\|_{C^{k+q} \times C^k} = \infty. \tag{4.2}$$

If $N' = 0$ *and the domain* Ω *is bounded, one also has the compactness property*

(4.3) *if* $\omega(u) = \infty$ *and the orbit* $\{\phi_t u \mid 0 \leqslant t < \infty\}$ *is bounded in* $\hat{C}^{k+q}(\overline{\Omega})$, *then it is contained in a compact set of* $\hat{C}^{k+q}(\overline{\Omega})$. $\square$

REMARK. When the spatial domain Ω is not bounded, the fact that the compactness property (4.3) is not assured reflects the possibility of traveling waves or similar phenomena. In the case $N' > 0$, the lack of the compactness property reflects the possibility that the variable v (or some of its derivatives $D_x^\nu v$ of order $|\nu| \leqslant k$) tends to a discontinuous function of $x \in \Omega$ as $t \to \infty$.

THEOREM 4.2. *Let us assume the hypotheses of Theorem* 4.1 *with* $r = 1$, *and in the case* $N' > 0$ *assume also that* $q = l = 0$. *Given a stationary state* $w_0 = (u_0, v_0)$, *let* L_k *be the linearized operator* $A + DF(w_0)$ *as an operator on* $E = \hat{C}^k(\overline{\Omega}) \times C^k(\overline{\Omega})$, *and let* L_{k+q} *be its restriction to* $X = \hat{C}^{k+q}(\overline{\Omega}) \times C^k(\overline{\Omega})$; *i.e. the domain of* L_{k+q} *is the set* $\{w \in X \mid L_k w \in X\}$. *Finally, let* $\sigma_i \equiv \sup \operatorname{Re} \Sigma(L_i)$ $(i = k, k+q)$. *If* $\sigma_{k+q} > 0$, *then the stationary state* w_0 *is unstable in the semiflow on* $X = \hat{C}^{k+q}(\overline{\Omega}) \times C^k(\overline{\Omega})$; *if* $\sigma_{k+q} < 0$, *then it is asymptotically stable in the same semiflow. For* $i = k, k+q$, *the operator* L_i *can also be characterized as the closure of the operator* $\mathfrak{L}_i$ *on* $\hat{C}^i(\overline{\Omega}) \times C^k(\overline{\Omega})$ *given by the differential operator* $\overline{\mathcal{A}} + \mathfrak{F}_{w_0}$ $(\overline{\mathcal{A}} \equiv \operatorname{diag}(\mathcal{A}, O))$ *with domain* $\mathfrak{D}_i \equiv \hat{C}^{i+2m+\alpha}(\overline{\Omega}) \times C^k(\overline{\Omega})$. $\square$

REMARK. When $N' = 0$ and Ω is bounded, the spectra of L_k and L_{k+q} consist only of eigenvalues; furthermore, the eigenfunctions of L_k belong to $D(L_k) \subset \hat{C}^{k+2m-1}(\overline{\Omega}) \subset X$, which implies that they are also eigenfunctions of L_{k+q}. This implies that in this case the spectra of L_k and L_{k+q}, and in particular the numbers σ_k and σ_{k+q}, are the same.

4.2. *Application to problem* (X). Let us apply these results to the particular case of problem (X) (with homogeneous boundary conditions, i.e. $b^i(x) = 0$ for $1 \leqslant i \leqslant N$). For the sake of simplicity, we shall restrict ourselves to the following situation: (a) the functions f and g do not depend on $x \in \Omega$; (b) the boundary conditions are of the same type for all the components of u, and they are either of Dirichlet type ($S = N$), or of Neumann type ($S = 0$, $p^i = 0$ for $1 \leqslant i \leqslant N$); (c) in the Dirichlet case we leave the condition at infinity as optional, but in the Neumann case we consider the problem without the condition at infinity. We shall also restrict ourselves to taking $k = 0$ and $r = 0, 1$.

As we have seen in 1.3, for the problem (X), conditions (L1)–(L3) and (L4_q) reduce to (L1), (M2) and (M4_q). On the other hand, Theorem 3.1 guarantees that these conditions imply (L5_q). Finally, for the situation described above, conditions ($\mathrm{N}_{0,q,0}$) and ($\mathrm{N}_{0,q,1}$) ($q = 0, 1$) reduce to the following ones:

(P_0) The functions $f(u, p, v)$ and $g(u, p, v)$ are locally Lipschitz with respect to u, p, v (p is a generic vector in $\mathbf{R}^{nN}$ corresponding to ∇u). In the Dirichlet case, it is also required that $f(0, p, v) = 0$ ($\forall p \in \mathbf{R}^{nN}, \forall v \in \mathbf{R}^{N'}$).

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(P_1) The functions $f(u, p, v)$ and $g(u, p, v)$ are locally of class $C^{1+\alpha}$ with respect to u, p, v. In the Dirichlet case, it is also required that $f(0, p, v) = 0$, $D_v f(0, p, v) = 0$, $D_p f(0, p, v) = 0$ ($\forall p \in \mathbf{R}^{nN}$, $\forall v \in \mathbf{R}^{N'}$).

Therefore, we can state the following corollaries of Theorems 4.1 and 4.2.

COROLLARY 4.1. *Consider problem* (X) *with the restrictions and described above, and assume conditions* (L1), (M2), ($M4_q$) *and* (P_r), *where* $q \in \{0, 1\}$ *if* $l = 0$, $q = 1$ *if* $l = 1$, *and* $r \in \{0, 1\}$ (l *is* 1 *or* 0 *depending on whether the functions* f *and* g *depend on* $p = \nabla u$). *Under these conditions, problem* (X) *determines a semiflow of class* C^r *on the space* $\hat{C}^q(\overline{\Omega}) \times C(\overline{\Omega})$, *where* $\hat{C}^q(\overline{\Omega})$ *is a closed subspace of* $C^q(\overline{\Omega})$ *given as follows: in the Dirichlet case and for* $q = 0, 1$,

$\hat{C}^q(\overline{\Omega})$ = *subspace of* $C^q(\overline{\Omega})$ *consisting of the functions* u *that satisfy the accessory conditions; in the Neumann case,*

$\hat{C}^0(\overline{\Omega}) = C(\overline{\Omega})$;

$\hat{C}^1(\overline{\Omega})$ = *subspace of* $C^1(\overline{\Omega})$ *consisting of the functions* u *that satisfy the boundary conditions. This semiflow satisfies the strong maximal property* (4.2), *and if* $N' = 0$ *and* Ω *is bounded, it also satisfies the compactness property* (4.3). □

COROLLARY 4.2. *Assume the hypotheses of Corollary* 4.1 *with* $r = 1$, *and in the case* $N' > 0$ *assume also that* $q = l = 0$ (*i.e.* f *and* g *do not depend on* $p = \nabla u$). *Then the conclusions of Theorem* 4.2 *hold for the semiflow on* $\hat{C}^q(\overline{\Omega}) \times C(\overline{\Omega})$ *corresponding to problem* (X). □

REMARK. For problems of type (X), several results are known which provide information about the qualitative structure of the semiflows on the spaces $\hat{C}(\overline{\Omega})$ and $\hat{C}^1(\overline{\Omega})$ (for the sake of simplicity, we consider $N' = 0$). In particular, we note the existence of criteria to obtain bounded positively-invariant regions in these spaces, and the application of topological methods like Conley's index (see, for example, Smoller [**18**, Chapter 14, 23]). In connection with the application of Conley's index, it is interesting to have the following property, which can easily be obtained by making use of relation (2.48): Consider problem (X) with $N' = 0$, and assume the hypotheses of Corollary 4.1; assume also that one has a bounded positively-invariant region in $\hat{C}^1(\overline{\Omega})$, which we shall call Σ. Then the solutions with initial state in Σ satisfy an estimate of the form

$$\|u(t) - u(s)\|_C \leqslant C\,|t - s|^\gamma \quad (\forall t, s \geqslant 0), \tag{4.4}$$

where γ is any real number in the interval $0 \leqslant \gamma < \frac{1}{2}$, and the constant C can be taken independently of $u_0 \in \Sigma$. The main point is that the estimate holds uniformly up to time zero; of course, for a general u_0 this can only be obtained at the price of using a weaker norm than the one in $\hat{C}^1(\overline{\Omega})$. The way to obtain property (4.4) is the following: We consider the problem in the abstract setting of §1.5 with $E = \hat{C}(\overline{\Omega})$ and $X = \hat{C}^1(\overline{\Omega})$; without loss of generality we can assume that the order of growth of the semigroup e^{At} is negative (otherwise it would suffice to redefine $A(\text{new}) \equiv A - \omega I$ and $F(\text{new}) \equiv F + \omega I$ with ω large enough). By using the integral equation

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equivalent to (II) (see, for example, Henry **[5]**) we have that

$$(4.5)\qquad u(t)-u(s)=(e^{A(t-s)}-I)u(s)+\int_s^t e^{A(t-\tau)}F(u(\tau))\,d\tau \qquad (0\leqslant s\leqslant t<\infty).$$

The first term in the right-hand side of (4.5) can be estimated in the following way:

$$\begin{aligned}\|(e^{A(t-s)}-I)u(s)\|_E &\leqslant C(t-s)^\gamma\|u(s)\|_{D^\gamma}\\ &\leqslant C'(t-s)^\gamma\|u(s)\|_{C^1}\qquad (0\leqslant\gamma<\tfrac{1}{2}),\end{aligned}$$

where the first step uses a well-known property (see, for example, Henry **[5**, Theorem 1.4.3**]**), and the second one is based on property (2.48). On the other hand, the norm of the expression under the integral sign is easily bounded by a constant, because of the fact that $u(\tau)$ stays in the bounded set Σ, and the F takes bounded sets of $\hat{C}^1(\overline{\Omega})$ to bounded sets of $\hat{C}(\overline{\Omega})$. Therefore, we obtain that

$$\|u(t)-u(s)\|_C\leqslant C'(t-s)^\gamma\|u(s)\|_{C^1}+C''(t-s)\qquad(0\leqslant s\leqslant t<\infty),$$

which establishes property (4.4) for $(t-s)$ small (we use again that $u(s)$ belongs to the bounded set Σ). Of course, for $(t-s)$ large, property (4.4) is a trivial consequence of the fact that the solution stays in Σ.

References

1. V. S. Belonosov, *Estimates of solutions of parabolic systems in weighted Hölder classes and some of their applications*, Mat. Sb. **110** (1979), 163–188 = Math. USSR Sb. **38** (1979), 151–173.

2. J. Bergh and J. Löfström, *Interpolation spaces*, Springer, Berlin, 1976.

3. N. P. Bhatia and O. Hájek, *Local semi-dynamical systems*, Lecture Notes in Math., vol. 90, Springer-Verlag, Berlin and New York, 1969.

4. S. D. Éĭdel'man and S. D. Ivasishen, *Investigation of the Green matrix for a homogeneous parabolic boundary value problem*, Trudy Moskov. Mat. Obšč. **23** 179–234 = Trans. Moscow Math. Soc. **23** (1970), 179–242.

5. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin and New York, 1981.

6. M. R. Hestenes, *Extension of the range of a differentiable function*, Duke Math. J. **8** (1941), 183–192.

7. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

8. H. Kielhöfer, *Halbgruppen und semilineares Anfangs-Randwertprobleme*, Manuscripta Math. **12** (1974), 121–152.

9. ______, *Existenz und Regularität von Lösungen semilinearer parabolischer Anfangs-Randwertprobleme*, Math. Z. **142** (1975), 131–160.

10. H. Komatsu, *Fractional powers of operators*, Pacific J. Math. **19** (1967), 285–346.

11. S. G. Kreĭn, *Linear differential equations in Banach space*, "Nauka", Moskow, 1963; Transl. Math. Monographs, vol. 28, Amer. Math. Soc., Providence, R. I., 1971.

12. O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and quasi-linear equations of parabolic type*, "Nauka", Moskow, 1967; Transl. Math. Monographs, vol. 23, Amer. Math. Soc.,Providence, R. I., 1968.

13. C. Miranda, *Partial differential equations of elliptic type*, Springer, Berlin, 1970.

14. X. Mora, Ph. D. Dissertation, Univ. Autònoma de Barcelona, Fac. Ciències, Sec. Matemàtiques, 1982.

15. J. D. Murray, *Lectures on nonlinear differential equation models in biology*, Clarendon Press, Oxford, 1977.

16. G. Nicolis and I. Prigogine, *Self-organization in nonequilibrium systems*, Wiley, New York, 1977.

17. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Dept. of Math., Univ. of Maryland, Lecture Notes no. 10, 1974.

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18. J. A. Smoller, *Shock waves and reaction-diffusion equations*, Springer, New York, 1982.

19. P. E. Sobolevskiĭ, *Estimates of the Green function for second-order parabolic partial differential equations*, Dokl. Akad. Nauk SSSR **138** (1961), 313–316 = Soviet Math. Dokl. **2** (1961), 617–620.

20. ______, *Green's function for arbitrary (in particular, integral) powers of elliptic operators*, Dokl. Akad. Nauk SSSR **142** (1961), 804–807 = Soviet Math. Dokl. **3** (1961), 183–187.

21. V. A. Solonnikov, *On boundary value problems for linear parabolic systems of differential equations of general form*, Trudy Mat. Inst. Steklov. **83** (1965), 3–163 = Proc. Steklov Inst. Math. **83** (1965), 1–184.

22. ______, *Green matrices for parabolic boundary value problems*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. **14** (1969), 256–287 = Sem. Math. V. A. Steklov Math. Inst. Leningrad **14** (1969), 132–150.

23. V. A. Solonnikov and A. G. Khachatryan, *Estimates for solutions of parabolic initial-boundary value problems in weighted Hölder norms*, Trudy Mat. Inst. Steklov. **147** (1980), 147–155 = Proc. Steklov. Math. Inst. **147** (1980), 153–162.

24. H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators*, Trans. Amer. Math. Soc. **199** (1974), 141–162.

25. ______, *Generation of analytic semigroups by strongly elliptic operators under general boundary conditions*, Trans. Amer. Soc. **259** (1980), 299–310.

26. H. Tanabe, *Evolution equations*, Iwanami Shoten, Tokyo, 1975; English transl., Pitman, London, 1979.

27. H. Triebel, *A remark on embedding theorems for Banach spaces of distributions*, Ark. Mat. **11** (1973), 65–74.

28. ______, *Interpolation theory, function spaces, differential operators*, Akademie Verlag, Berlin; North-Holland, Amsterdam, 1978.

29. W. von Wähl, *Gebrochene Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen hölderstetiger Funktionen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **11** (1972), 231–258. See also Manuscripta Math. **11** (1974), 199–201.

30. F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1980), 79–102.

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