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Nonparametric analysis of doubly truncated data

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Abstract One of the principal goals of the quasar investigations is to study luminosity evolution. A convenient one-parameter model for luminosity says that the expected log luminosity, T^*, increases linearly as $\theta_0 \cdot \log(1+Z^*)$, and $T^*(\theta_0) = T^* - \theta_0 \cdot \log(1+Z^*)$ is independent of Z^*, where Z^* is the redshift of a quasar and θ_0 is the true value of evolution parameter. Due to experimental constraints, the distribution of T^* is doubly truncated to an interval (U^*, V^*) depending on Z^*, i.e., a quadruple (T^*, Z^*, U^*, V^*) is observable only when $U^* \leq T^* \leq V^*$. Under the one-parameter model, $T^*(\theta_0)$ is independent of $(U^*(\theta_0), V^*(\theta_0))$, where $U^*(\theta_0) = U^* - \theta_0 \cdot \log(1+Z^*)$ and $V^*(\theta_0) = V^* - \theta_0 \cdot \log(1+Z^*)$. Under this assumption, the nonparametric maximum likelihood estimate (NPMLE) of the hazard function of $T^*(\theta_0)$ (denoted by $\hat{\mathbf{h}}$) was developed by Efron and Petrosian (J Am Stat Assoc 94:824–834, 1999). In this note, we present an alternative derivation of $\hat{\mathbf{h}}$. Besides, the NPMLE of distribution function of $T^*(\theta_0)$, $\hat{F}$, will be derived through an inverse-probability-weighted (IPW) approach. Based on Theorem 3.1 of Van der Laan (1996), we prove the consistency and asymptotic normality of the NPMLE $\hat{F}$ under certain condition. For testing the null hypothesis $H_{\theta_0} : T^*(\theta_0) = T^* - \theta_0 \cdot \log(1+Z^*)$ is independent of Z^*, (Efron and Petrosian in J Am Stat Assoc 94:824–834, 1999). proposed a truncated version of the Kendall's tau statistic. However, when T^* is exponential distributed, the testing procedure is futile. To circumvent this difficulty, a modified testing procedure is proposed. Simulations show that the proposed test works adequately for moderate sample size.

Keywords Double truncation · Nonparametric MLE · Kendall's tau

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1 Introduction

Doubly truncated failure-time arises if an individual is potentially observed and only if its failure-time falls within a certain interval, unique to that individual. Doubly truncated data play an important role in the statistical analysis of astronomical observations as well as in survival analysis. For example, in a quasar survey (see Efron and Petrosian 1999), one of the principal goals is to study luminosity evolution. Quasars may have been intrinsically brighter in the early universe and evolved toward a dimmer state as time went on. Since larger redshifts (denoted by Z^*) correspond to quasar seen longer ago, a convenient one-parameter model for luminosity says that the expected log luminosity, T^*, increases linearly as $\theta_0 \cdot \log(1 + Z^*)$, and $T^*(\theta_0) = T^* - \theta_0 \cdot \log(1 + Z^*)$ is independent of Z^*, where θ_0 is the true value of evolution parameter. However, apparent magnitude is doubly truncated by the investigator to avoid confusion with nonquasar stellar objects at low end and to eliminate unreliable redshift measurements at the high end. Hence, the distribution of T^* is doubly truncated to an interval (U^*, V^*) depending on Z^*, i.e., a quadruple (T^*, Z^*, U^*, V^*) is observable only when $U^* \le T^* \le V^*$. Under the one-parameter model, $T^*(\theta_0)$ is independent of $(U^*(\theta_0), V^*(\theta_0))$, where $U^*(\theta_0) = U^* - \theta_0 \cdot \log(1 + Z^*)$ and $V^*(\theta_0) = V^* - \theta_0 \cdot \log(1 + Z^*)$. Under this assumption the nonparametric maximum likelihood estimate (NPMLE) of the hazard function of $T^*(\theta_0)$ (denoted by $\hat{\mathbf{h}}$) was developed by Efron and Petrosian (1999). In Sect. 2, we present an alternative derivation of $\hat{\mathbf{h}}$. In Sect. 3, the NPMLEs of the distribution functions of $T^*(\theta_0)$ and $(U^*(\theta_0), V^*(\theta_0))$ will be derived through an inverse-probability-weighted (IPW) approach. Based on Theorem 3.1 of Van der Laan (1996), we will prove that the NPMLE $\hat{F}$ of the distribution function of $T^*(\theta_0)$ is consistent and asymptotically normal under certain condition. In applying the NPMLEs, it is important to test the null hypothesis $H_{\theta_0} : T^*(\theta_0) = T^* - \theta_0 \cdot \log(1 + Z^*)$ is independent of Z^*. Under the one-parameter model, testing the null hypothesis H_{θ_0} amounts to testing if θ_0 is the true value of the evolution parameter. Efron and Petrosian (1999) proposed a truncated version of the Kendall's tau statistic for testing the null hypothesis H_{θ_0}. However, when T^* is exponential distributed, the testing procedure is futile. To circumvent this difficulty, a modified testing procedure is proposed in Sect. 4. Simulations show that the proposed test works adequately for moderate sample size.

2 The NPMLE of hazard function

Without loss of generality, we assume that $\theta_0 = 0$ is the true value of the evolution parameter. Then T^* is independent of (U^*, V^*). Let $f(t)$ and $h(t)$ denote the common density and hazard function of T^*. Let $k(u, v)$ denote the common density function of (U^*, V^*). For any distribution function W denote the left and right endpoints of its support by $a_W = \inf\{t : W(t) > 0\}$ and $b_W = \inf\{t : W(t) = 1\}$, respectively. Let $G(u) = K(u, \infty)$ and $Q(v) = K(\infty, v)$ be the marginal distribution function of U^* and V^*, respectively. Under the assumption that $a_G \le a_F \le a_Q$ and $b_G \le b_F \le b_Q$, F and K are both identifiable (see Woodroofe 1985). Let $(T_1, Z_1, U_1, V_1), \ldots, (T_n, Z_n, U_n, V_n)$ denote the truncated sample. The NPMLE is a discrete distribution putting all of its probability on the observed responses $(U_1, V_1, T_1), \ldots, (U_n, V_n, T_n)$ (see Turnbull 1976). Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a distribution putting probability f_j on T_j $(j = 1, \ldots, n)$. Similarly, let $\mathbf{k} = (k_1, \ldots, k_n)$ be a distribution putting joint probability k_j on (U_j, V_j) $(j = 1, \ldots, n)$. The full nonparametric likelihood based on observed data can be written as

$$L = \prod_{j=1}^{n} \frac{f_j k_j}{\sum_{i=1}^{n} F_i k_i} = \prod_{j=1}^{n} \frac{f_j}{F_j} \times \prod_{j=1}^{n} \frac{F_j k_j}{\sum_{i=1}^{n} F_i k_i} = L_1(\mathbf{f}) \times L_2(\mathbf{f}, \mathbf{k}),$$

where $F_i = \sum_{m=1}^{n} f_m J_{im}$, where $J_{im} = I_{[U_i \le T_m \le V_i]} = 1$ if $U_i \le T_m \le V_i$ and equal to zero otherwise. According to $L_1(\mathbf{f})$, the NPMLE of $\mathbf{f}$ can be obtained by solving the following equation:

$$\frac{1}{\hat{f}_j} = \sum_{i=1}^{n} J_{ij} \frac{1}{\hat{F}_i}, \quad (j = 1, \ldots, n) \tag{1}$$

where $\hat{F}_i = \sum_{m=1}^{n} \hat{f}_m J_{im}$. Efron and Petrosian (1999) showed that the NPMLE $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ has the hazard function $\hat{\mathbf{h}} = (\hat{h}_1, \ldots, \hat{h}_n)$ satisfying

$$\frac{1}{\hat{h}_j} = N_j + \sum_{i=1}^{n} J_{ij} \hat{Q}_i, \quad (j = 1, \ldots, n)$$

where $N_j = \sum_{i=1}^{n} I_{[U_i \le T_j \le T_i]}$, $\hat{Q}_i = \hat{S}(V_i)/\hat{F}_i$, and $\hat{S}(V_i) = \sum_{m=1}^{n} \hat{f}_m I_{[T_m > V_i]}$.

Now, we present an alternative derivation of $\hat{\mathbf{h}}$. First, let $p = P(U^* \le T^* \le V^*)$. We consider the distribution function of T_i's:

$$\begin{aligned}\tilde{F}(t) &= P(T_i \le t) = P(T^* \le t | U^* \le T^* \le V^*) \\ &= p^{-1} P(T^* \le t, U^* \le T^* \le V^*) = p^{-1} \int_0^t P(U^* \le u \le V^*) F(\mathrm{d}u).\end{aligned}$$

Hence, $\tilde{f}(t) = \frac{\tilde{F}(\mathrm{d}t)}{\mathrm{d}t} = p^{-1} P(U^* \le t \le V^*) f(t)$. Next, let $\tilde{S}(t) = p^{-1} P(T^* \ge t, U^* \le t \le V^*) = p^{-1} P(U^* \le t \le V^*)\ S(t-)$, where $S(t-) = P(T^* \ge t)$. Thus, $\frac{\tilde{f}(t)}{\tilde{S}(t)} = \frac{f(t)}{S(t-)} = h(t)$. Note that $\tilde{F}(t)$ can be estimated by the empirical function $\tilde{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} I_{[T_i \le t]}$. Therefore, if we can estimate $\tilde{S}(t)$, then an estimator of $h(t)$ can be obtained. Now,

$$\tilde{S}(t) = p^{-1} P(U^* \le t \le T^* \le V^*) + p^{-1} P(U^* \le t \le V^* < T^*) = \tilde{S}_1(t) + \tilde{S}_2(t).$$

Since $\tilde{S}_1(t) = P(U^* \le t \le T^* | U^* \le T^* \le V^*) = P(U_i \le t \le T_i)$, $\tilde{S}_1(t)$ can be estimated by the empirical function $\tilde{S}_{1n}(t) = n^{-1} \sum_{i=1}^{n} I_{[U_i \le t \le T_i]}$. The question left is "how to estimate $\tilde{S}_2(t)$?" To estimate $\tilde{S}_2(t)$, we consider the joint distribution of U_i and V_i:

For $u < b_G$ and $v < b_Q$,

$$\begin{aligned}\tilde{K}(u,v) &= P(U_i \le u, V_i \le v) = P(U^* \le u, V^* \le v | U^* \le T^* \le V^*)\\ &= p^{-1}\int_{a_Q}^{v}\int_{a_G}^{\min(u,y)}[F(y) - F(x-)]k(x,y)\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Hence, the joint density of U_i and V_i is given by $\tilde{k}(u,v) = p^{-1}[F(v) - F(u-)]k(u,v)$, and $k(u,v) = p\tilde{k}(u,v)/[F(v) - F(u-)]$. Now,

$$\begin{aligned}\tilde{S}_2(t) &= p^{-1}\int_{u\le t\le v} P(T > v)k(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_{u\le t\le v}\frac{P(T > v)}{[F(v) - F(u-)]}\tilde{k}(u,v)\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_{u\le t\le v}\frac{1 - F(v)}{[F(v) - F(u-)]}\tilde{k}(u,v)\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Given $F(u)$, $\tilde{S}_2(t)$ can be estimated based on the empirical distribution of $(U_i, V_i)(i = 1,\ldots,n)$, i.e., $n^{-1}\sum_{i=1}^{n} J_i(t)[1 - F(V_i)]/[F(V_i) - F(U_i-)]$, where $J_i(t) = I_{[U_i \le t \le V_i]}$. Hence, an estimator of $h(t)$ is given by

$$\hat{h}(t) = \frac{\tilde{F}_n(\mathrm{d}t)}{\tilde{S}_{1n}(t) + n^{-1}\sum_{i=1}^{n} J_i(t)\hat{Q}_i},$$

where $\tilde{F}_n(\mathrm{d}t) = \tilde{F}_n(t) - \tilde{F}_n(t-)$, $\hat{Q}_i = \hat{S}(V_i)/\hat{F}_i$, and $\hat{S}(V_i) = \sum_{m=1}^{n}\hat{f}_m I_{[T_m > V_i]}$. It follows that $\hat{h}(T_j) = \hat{h}_j$ for $j = 1,\ldots,n$. For left-truncated data, $\tilde{S}_2(t) = 0$, because $V = \infty$, so the estimator is reduced to $\tilde{F}_n(\mathrm{d}t)/\tilde{S}_{1n}(t)$, which is Lynden-Bell (1971) estimator.

By expressing the likelihood in terms of hazard rate, Efron and Petrosian (1999, p. 833), showed that $\hat{h}(T_j)$ is the NPMLE of $h(T_j)$. By induction, the following Lemma provides an alternative proof.

Lemma 1 *For* $j = 1,\ldots,n$,

$$\hat{h}(T_j) = \frac{\hat{f}_j}{\sum_{k=1}^{n}\hat{f}_k I_{[T_k \ge T_j]}}.$$

Proof Let $0 = T_{(0)} < T_{(1)} < T_{(2)} < \cdots < T_{(n)}$ be the ordered observations of T_i's. Define $\tilde{h}(T_{(j)}) = \frac{\hat{F}(T_{(j)}) - \hat{F}(T_{(j-1)})}{1 - \hat{F}(T_{(j-1)})}$ $(j = 1,\ldots,n)$. By Eq. (1), we have

$$\hat{F}(T_{(j)}) - \hat{F}(T_{(j-1)}) = \frac{1}{\sum_{i=1}^{n}\frac{I_{[U_i \le T_{(j)} \le V_i]}}{\hat{F}(V_i) - \hat{F}(U_i-)}}.$$

Hence, we have

$$\tilde{h}(T_{(j)}) = \frac{1}{\sum_{i=1}^{n} I_{[U_i \le T_{(j)} \le V_i]} \left(\frac{1-\hat{F}(T_{(j-1)})}{\hat{F}(V_i)-\hat{F}(U_i-)} \right)}.$$

Now,

$$\hat{h}(T_{(j)}) = \frac{1}{\sum_{i=1}^{n} \left\{ I_{[U_i \le T_{(j)} \le T_i]} + I_{[U_i \le T_{(j)} \le V_i]} \frac{1-\hat{F}(V_i)}{\hat{F}(V_i)-\hat{F}(U_i-)} \right\}}$$

To prove Lemma 2.1, it suffices to show that $\hat{h}(T_{(j)}) = \tilde{h}(T_{(j)})$ for $j = 1, \ldots, n$, i.e.,

$$\sum_{i=1}^{n} I_{[U_i \le T_{(j)} \le V_i]} \frac{1 - \hat{F}(T_{(j-1)})}{\hat{F}(V_i) - \hat{F}(U_i-)}$$
$$= \sum_{i=1}^{n} \left\{ I_{[U_i \le T_{(j)} \le T_i]} + I_{[U_i \le T_{(j)} \le V_i]} \frac{1 - \hat{F}(V_i)}{\hat{F}(V_i) - \hat{F}(U_i-)} \right\}.$$

This is equivalent to the following equation:

$$\sum_{i=1}^{n} I_{[U_i \le T_{(j)} \le T_i]} = \sum_{i=1}^{n} I_{[U_i \le T_{(j)} \le V_i]} \left[\frac{\hat{F}(V_i) - \hat{F}(T_{(j-1)})}{\hat{F}(V_i) - \hat{F}(U_i-)} \right]$$

We use induction on i.

For $j = 1$, $\hat{F}(T_{(j-1)}) = \hat{F}(T_{(0)}) = 0$. For $i = 1, \ldots, n$, when $I_{[U_i \le T_{(1)} \le V_i]} = 1$, one has $\hat{F}(U_i-) = 0$. Since $\sum_{i=1}^{n} I_{[U_i \le T_{(1)} \le T_i]} = \sum_{i=1}^{n} I_{[U_i \le T_{(1)}]} = \sum_{i=1}^{n} I_{[U_i \le T_{(1)} \le V_i]}$, the assertion holds for j=1.

Next, assume the assertion holds for $j = m$, i.e.,

$$\sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le T_i]} = \sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le V_i]} \left[\frac{\hat{F}(V_i) - \hat{F}(T_{(m-1)})}{\hat{F}(V_i) - \hat{F}(U_i-)} \right].$$

We want to show the assertion holds for $j = m + 1$, i.e.,

$$\sum_{i=1}^{n} I_{[U_i \le T_{(m+1)} \le T_i]} = \sum_{i=1}^{n} I_{[U_i \le T_{(m+1)} \le V_i]} \left[\frac{\hat{F}(V_i) - \hat{F}(T_{(m)})}{\hat{F}(V_i) - \hat{F}(U_i-)} \right]. \tag{2}$$

Now, the right-hand side of Eq. (2) is equal to

$$\sum_{i=1}^{n} \left\{ I_{[U_i \le T_{(m)} \le T_{(m+1)} \le V_i]} + I_{[T_{(m)} < U_i \le T_{(m+1)} \le V_i]} \right\} \left[\frac{\hat{F}(V_i) - \hat{F}(T_{(m)})}{\hat{F}(V_i) - \hat{F}(U_i-)} \right]$$

$$= \sum_{i=1}^{n} \left\{ I_{[U_i \le T_{(m)} \le V_i]} - I_{[U_i \le T_{(m)} \le V_i < T_{(m+1)}]} + I_{[T_{(m)} < U_i \le T_{(m+1)} \le V_i]} \right\} \times \frac{\hat{F}(V_i) - \hat{F}(T_{(m)})}{\hat{F}(V_i) - \hat{F}(U_i-)}$$

$$= \underbrace{\sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le V_i]} \frac{\hat{F}(V_i) - \hat{F}(T_{(m-1)})}{\hat{F}(V_i) - \hat{F}(U_i-)}}_{(3)} - \underbrace{\sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le V_i]} \frac{\hat{F}(T_{(m)}) - \hat{F}(T_{(m-1)})}{\hat{F}(V_i) - \hat{F}(U_i-)}}_{(4)}$$

$$- \underbrace{\sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le V_i < T_{(m+1)}]} \frac{\hat{F}(V_i) - \hat{F}(T_{(m)})}{\hat{F}(V_i) - \hat{F}(U_i-)}}_{(5)}$$

$$+ \underbrace{\sum_{i=1}^{n} I_{[T_{(m)} < U_i \le T_{(m+1)} \le V_i]} \frac{\hat{F}(V_i) - \hat{F}(T_{(m)})}{\hat{F}(V_i) - \hat{F}(U_i-)}}_{(6)}$$

By the assumption that the assertion holds for $j = m$, we have
(3) $= \sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le T_i]}$.

By Eq. (1), $\hat{F}(T_{(m)}) - \hat{F}(T_{(m-1)}) = \left(\sum_{i=1}^{n} \frac{I_{[U_i \le T_{(m)} \le V_i]}}{\hat{F}(V_i) - \hat{F}(U_i-)} \right)^{-1}$. Hence, (4) $= 1$
Since $\hat{F}(V_i) = \hat{F}(T_{(m)})$ for $T_{(m)} \le V_i < T_{(m+1)}$, we have (5) $= 0$.
Since $\hat{F}(T_{(m)}) = \hat{F}(U_i-)$ for $T_{(m)} < U_i \le T_{(m+1)}$, we have
(6) $= \sum_{i=1}^{n} I_{[T_{(m)} < U_i \le T_{(m+1)} \le V_i]}$. Thus, the right-hand side of Eq. (2) is equal to

$$\sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le T_i]} - 1 + \sum_{i=1}^{n} I_{[T_{(m)} < U_i \le T_{(m+1)} \le V_i]}. \tag{7}$$

Now, the left-hand side of Eq. (2) can be written as

$$\sum_{i=1}^{n} I_{[U_i \le T_{(m+1)} \le T_i]} = \sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le T_i]} + \sum_{i=1}^{n} I_{[T_{(m)} < U_i \le T_{(m+1)} \le T_i]} - \sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le T_i < T_{(m+1)}]}. \tag{8}$$

By (7) and (8), Eq. (2) is equivalent to

$$\underbrace{\sum_{i=1}^{n} I_{[T_{(m)} < U_i \le T_{(m+1)} \le T_i]}}_{(9)} - \underbrace{\sum_{i=1}^{n} I_{[U_i \le T_{(m)} \le T_i < T_{(m+1)}]}}_{(10)} = \underbrace{\sum_{i=1}^{n} I_{[T_{(m)} < U_i \le T_{(m+1)} \le V_i]}}_{(11)} - 1.$$

Note that $(11) = (9) + \underbrace{\sum_{i=1}^{n} I_{[T_{(m)}<U_i\le T_i<T_{(m+1)}\le V_i]}}_{(12)}$. Since $I_{[T_{(m)}<U_i\le T_i<T_{(m+1)}\le V_i]} = 0$ for all i, we have $(12) = 0$. Furthermore, $(10) = 1$. The proof is completed. □

3 Inverse-probability-weighted (IPW) estimator

In practice, we may be interested in estimating the joint distribution function of U^* and V^*, $K(u, v)$. Note that the full nonparametric likelihood can be written as

$$L = \prod_{j=1}^{n} \frac{k_j}{K_j} \times \prod_{j=1}^{n} \frac{K_j f_j}{\sum_{i=1}^{n} K_i f_i} = L_1(\mathbf{k}) \times L_2(\mathbf{k}, \mathbf{f}),$$

where $K_i = \sum_{m=1}^{n} k_m I_{[U_m\le T_i\le V_m]} = \sum_{m=1}^{n} k_m J_{mi}$. According to $L_1(\mathbf{k})$, the NPMLE of $\mathbf{k} = (\hat{k}_1, \ldots, \hat{k}_n)$ can be obtained by solving the following equation:

$$\frac{1}{\hat{k}_j} = \sum_{i=1}^{n} J_{ji} \frac{1}{\hat{K}_i}, \quad (j = 1, \ldots, n)$$

where $\hat{K}_i = \sum_{m=1}^{n} \hat{k}_m J_{mi}$. It is questionable whether $\hat{\mathbf{f}}$ and $\hat{\mathbf{k}}$ are also the NPMLEs of the full nonparametric likelihood L. Note that $L = L_1(\mathbf{f}) \times L_2(\mathbf{f}, \mathbf{k}) = L_1(\mathbf{k}) \times L_2(\mathbf{k}, \mathbf{f})$, where $L_2(\mathbf{f}, \mathbf{k})$ and $L_2(\mathbf{k}, \mathbf{f})$ are both multinomial likelihood with support points on (U_i, V_i)'s and T_i's, respectively. Thus, as indicated in Wang (1987) $\hat{\mathbf{f}}$ and $\hat{\mathbf{k}}$ would be the NPMLE's of L if

$$\frac{\hat{F}_j \hat{k}_j}{\sum_{i=1}^{n} \hat{F}_i \hat{k}_i} = \frac{1}{n}, \quad \frac{\hat{K}_j \hat{f}_j}{\sum_{i=1}^{n} \hat{K}_i \hat{f}_i} = \frac{1}{n}.$$

In this section, using an inverse-probability-weighted (IPW) approach, we will show that $\hat{\mathbf{f}}$ and $\hat{\mathbf{k}}$ are the NPMLEs of full nonparametric likelihood L.

For random censoring model, Satten and Datta (2001) showed that the Kaplan and Meier (1958) estimator of survival function can be expressed as an IPW average (see Robins 1993 and Robins and Finkelstein 2000). For the univariate random truncation and censoring model, Shen (2003) showed that the truncation NPMLE (see Woodroofe 1985) and the censoring-truncation NPMLE (see Wang 1987) of survival function can also be expressed as IPW averages. For double-truncated data, the following arguments provide the motivation for using IPW estimators.

Note that $k(u, v) = p\tilde{k}(u, v)/[F(v) - F(u-)]$. Thus, when F and p are known, $K(u, v)$ can be estimated by $n^{-1} p \sum_{i=1}^{n} \frac{I_{[U_i\le u, V_i\le v]}}{F(V_i)-F(U_i-)}$. Let $u = \infty$ and $v = \infty$. It follows that when F is known, p can be estimated by $n[\sum_{i=1}^{n} \frac{1}{F(V_i)-F(U_i-)}]^{-1}$ and $K(u, v)$ can be estimated by $[\sum_{i=1}^{n} \frac{1}{F(V_i)-F(U_i-)}]^{-1} \sum_{i=1}^{n} \frac{I_{[U_i\le u, V_i\le v]}}{F(V_i)-F(U_i-)}$. Similarly, $\tilde{f}(t) = p^{-1} P(U^* \le t \le V^*) f(t)$. Hence, when $K(u, v)$ and p is known, $F(t)$ can be estimated by $n^{-1} p \sum_{i=1}^{n} \frac{I_{[T_i \le t]}}{K(T_i,\infty)-K(T_i,T_i)}$. Let $t = \infty$. It follows that when K is known, p can be estimated by $n[\sum_{i=1}^{n} \frac{1}{K(T_i,\infty)-K(T_i,T_i)}]^{-1}$ and $F(t)$ can be estimated by $[\sum_{i=1}^{n} \frac{1}{K(T_i,\infty)-K(T_i,T_i)}]^{-1} \sum_{i=1}^{n} \frac{I_{[T_i \le t]}}{K(T_i,\infty)-K(T_i,T_i)}$. Hence, the IPW estimators of $F(t)$ and $K(u, v)$ can be obtained by simultaneous solving of the following two equations:

$$\hat{F}(t) = \left[\sum_{i=1}^{n} \frac{1}{\hat{K}(T_i,\infty) - \hat{K}(T_i, T_i)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_i \le t]}}{\hat{K}(T_i,\infty) - \hat{K}(T_i, T_i)}, \tag{13}$$

$$\hat{K}(u, v) = \left[\sum_{i=1}^{n} \frac{1}{\hat{F}(V_i) - \hat{F}(U_i-)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[U_i \le u, V_i \le v]}}{\hat{F}(V_i) - \hat{F}(U_i-)}. \tag{14}$$

Theorem 1 *Let* $\hat{F}_{NP}(t) = \sum_{i=1}^{n} \hat{f}_i I_{[T_i \le t]}$ *and* $\hat{K}_{NP}(u, v) = \sum_{i=1}^{n} \hat{k}_i I_{[U_i \le u, V_i \le v]}$ *according to* $L_1(\mathbf{f})$ *and* $L_1(\mathbf{k})$. *Then*

(a) $\hat{F}(t) = \hat{F}_{NP}(t)$ *and* $\hat{K}(u, v) = \hat{K}_{NP}(u, v)$ *are the NPMLEs of* F *and* K*, respectively, and*
(b) $\hat{\mathbf{f}}$ *and* $\hat{\mathbf{k}}$ *are the NPMLEs of full* L.

Proof For $j = 1, \ldots, n$ let $\hat{f}_{wj} = [\sum_{i=1}^{n} \frac{1}{\hat{K}_i}]^{-1} \frac{1}{\hat{K}_j}$ and $\hat{k}_{wj} = [\sum_{i=1}^{n} \frac{1}{\hat{F}_i}]^{-1} \frac{1}{\hat{F}_j}$. It follows that

$$\frac{\hat{F}_j \hat{k}_{wj}}{\sum_{i=1}^{n} \hat{F}_i \hat{k}_{wi}} = \frac{1}{n}, \quad \frac{\hat{K}_j \hat{f}_{wj}}{\sum_{i=1}^{n} \hat{K}_i \hat{f}_{wi}} = \frac{1}{n}.$$

Note that under the constraint $\sum_{j=1}^{n} \hat{f}_j = 1$, Efron and Petrosian (1999) obtain the NPMLE $\hat{\mathbf{f}}$ by solving the equation

$$\hat{f}_j = \left[\sum_{i=1}^{n} J_{ij} \frac{1}{\sum_{m=1}^{n} \hat{f}_m J_{im}}\right]^{-1}. \tag{15}$$

It can be verified that for $j = 1, \ldots, n$, $\hat{f}_{wj} = [\sum_{i=1}^{n} J_{ij} \frac{1}{\sum_{m=1}^{n} \hat{f}_{wm} J_{im}}]^{-1}$. Similarly, under the constraint $\sum_{j=1}^{n} \hat{k}_j = 1$, $\hat{\mathbf{k}}$ is uniquely determined by solving the following equation: $\hat{k}_j = [\sum_{i=1}^{n} J_{ji} \frac{1}{\sum_{m=1}^{n} \hat{k}_m J_{mi}}]^{-1}$. It can be verified that for $j = 1, \ldots, n$, $\hat{k}_{wj} = [\sum_{i=1}^{n} J_{ji} \frac{1}{\sum_{m=1}^{n} \hat{k}_{wm} J_{mi}}]^{-1}$. Since $\sum_{j=1}^{n} \hat{f}_{wj} = \sum_{j=1}^{n} \hat{k}_{wj} = 1$, it follows that $\hat{f}_j = \hat{f}_{wj}$ and $\hat{k}_j = \hat{k}_{wj}$ for all $j = 1, \ldots, n$. Thus, $\hat{\mathbf{f}}$ and $\hat{\mathbf{k}}$ are the NPMLEs of full nonparametric likelihood L, and the NPMLEs can be obtained by simultaneously solving the following two equations:

$$\hat{f}_j = \left[\sum_{i=1}^{n} \frac{1}{\hat{K}_i}\right]^{-1} \frac{1}{\hat{K}_j}, \quad (j = 1, \ldots, n) \tag{16}$$

$$\hat{k}_j = \left[\sum_{i=1}^{n} \frac{1}{\hat{F}_i}\right]^{-1} \frac{1}{\hat{F}_j}. \quad (j = 1, \ldots, n) \tag{17}$$

Note that $\hat{K}_i = \hat{K}(T_i, \infty) - \hat{K}(T_i, T_i)$ and $\hat{F}_i = \hat{F}(V_i) - \hat{F}(U_i)$. It follows from (16) and (17) that $\hat{F}_{NP}(t) = \hat{F}(t)$ and $\hat{K}_{NP}(u, v) = \hat{K}(u, v)$. This completes the proof of Theorem 3.1.

Efron and Petrosian (1999) solve for the MLE $\hat{\mathbf{f}}$ by beginning with any initial estimate of $\mathbf{f}$ and then iterating between $\sum_{m=1}^{n} \hat{f}_m J_{im}$ and Eq. (15). They also propose an alternative algorithm by beginning with $\hat{h}_j = 1/N_j$ and iterating between $\hat{f}_j = \hat{G}_j - \hat{G}_{j+1}$, where $\hat{G}_j = \exp\{\sum_{i<j} \log(1 - \hat{h}_i)\}$, and $\hat{h}_j = \{N_j + \sum_{i=1}^{n} J_{ij} \hat{Q}_i\}^{-1}$. The IPW approach differs from the method of Efron and Petrosian (1999) in simultaneous estimating $\mathbf{f}$ and $\mathbf{k}$. In practice, we can use $\hat{\mathbf{f}}^{(0)} = (1/n, \ldots, 1/n)$ as the initial estimator of $\mathbf{f}$ and then the first-step estimator of $\mathbf{k}$, $\hat{\mathbf{k}}^{(1)}$, can be obtained by (16). Based on (16), the first-step estimator of $\mathbf{f}$, $\hat{\mathbf{f}}^{(1)}$, can be obtained. Repeat steps (16) and (17) until the solution is stable.

An alternative proof of Theroem 1 can be obtained by considering the estimation of p. Since $I_{[U_j \le T_i, V_j \le T_i]} = I_{[V_j \le T_i]}$, we have

$$n\left[\sum_{i=1}^{n} \frac{1}{\hat{K}(T_i, \infty) - \hat{K}(T_i, T_i)}\right]^{-1} = n\left[\sum_{i=1}^{n} \frac{1}{\hat{K}(T_i, \infty) - \hat{K}(\infty, T_i)}\right]^{-1}.$$

Note that $\int \hat{\bar{F}}(x-)\hat{K}(\mathrm{d}x, \infty) = \int \hat{K}(x, \infty)\hat{F}(\mathrm{d}x)$, and $\int \hat{\bar{F}}(x-)\hat{K}(\infty, \mathrm{d}x) = \int \hat{K}(\infty, x)\hat{F}(\mathrm{d}x)$, where $\hat{\bar{F}}(x-) = 1 - \hat{F}(x-)$. Hence,

$$\int \hat{\bar{F}}(x-)[\hat{K}(\mathrm{d}x, \infty) - \hat{K}(\infty, \mathrm{d}x)] = \int [\hat{K}(x, \infty) - \hat{K}(\infty, x)]\hat{F}(\mathrm{d}x),$$

and

$$n\left[\sum_{i=1}^{n} \frac{1}{\hat{K}(T_i, \infty) - \hat{K}(\infty, T_i)}\right]^{-1} = n\left[\sum_{i=1}^{n} \frac{1}{\hat{F}(V_i) - \hat{F}(U_i-)}\right]^{-1}.$$

Theorem 1 follows from (13) and (14). □

Let $(D[0, \tau], ||\cdot||_\infty, \mathcal{B})$ be the space of cadlag functions as defined in Neuhaus (1971), i.e., real valued functions which are right-continuous with left-hand limits, endowed with the supremum-norm and the Borel-sigma-algebra. Next, we will prove the consistency of $\hat{F}$ on $[0, \tau]$. The proof of the following theorem is based on Theorem 3.1 of Van der Laan (1996).

Theorem 2 *Let $[a_F, \tau] \in [0, \infty)$ be such that $F(v) - F(u-) > \delta > 0$ for $[u, v] \in [a_F, \tau]$. Moreover, assume that (a) $\int_{a_F}^{\tau} F(\mathrm{d}x)/[K(x, \infty) - K(x, x)] < \infty$ and (b) $[K(\mathrm{d}x, \infty) - K(\mathrm{d}x, \mathrm{d}x)]/F(\mathrm{d}x)$ is uniformly bounded on $[a_F, \tau]$. Then $\hat{F}$ is uniformly consistent on $[a_F, \tau]$.*

Proof Since $n[\sum_{i=1}^{n} \frac{1}{\hat{K}(T_i,\infty)-\hat{K}(T_i,T_i)}]^{-1} = n[\sum_{i=1}^{n} \frac{1}{\hat{F}(V_i)-\hat{F}(U_i-)}]^{-1}$, it follows that the estimator $\hat{F}(t)$ is equivalent to the solution of $U(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) = 0$, where

$$U(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) = \hat{F}(t) - \int_{a_F}^{t} \frac{\tilde{F}_n(\mathrm{d}x)}{\int_x^{\infty}\int_0^{x} \frac{\tilde{K}_n(\mathrm{d}u\,\mathrm{d}v)}{\hat{F}(v)-\hat{F}(u-)}}, \tag{18}$$

where $\tilde{K}_n$ is the empirical distribution function of $\tilde{K}$.

Note that in Eq. (18), $\int_x^{\infty}\int_0^{x} \frac{\tilde{K}_n(\mathrm{d}u\,\mathrm{d}v)}{\hat{F}(v)-\hat{F}(u-)}$ converges to zero if $x \to a_G$ or $x \to b_Q$, where a_G and b_Q are the left and right support of U^* and V^*, respectively. Therefore, we will have to control this singularity with the assumption (a), which is like the assumption $\int_{a_F}^{\tau} F(\mathrm{d}x)/G(x) < \infty$ for left-truncated data (see Woodroofe 1985). Note that assumption (a) holds, i.e., $\int_{a_F}^{\tau} F(\mathrm{d}x)/[K(x, \infty) - K(x, x)] < \infty$, if $a_G < a_F$ and $b_F <= b_Q$. When $a_G = a_F$, assumption (a) is not easily justified in practice. However, it is the minimal condition under which our proof works.

Let

$$H(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) = \int_0^{\infty}\int_0^{v} \frac{\hat{F}(v \vee t) - \hat{F}((u-) \vee t)}{\hat{F}(v) - \hat{F}(u-)} \tilde{K}_n(\mathrm{d}u,\ \mathrm{d}v) - \tilde{F}_n(t).$$

It follows that $U(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) = 0$ is equivalent to $H(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) = 0$.

By the argument of Van der Laan (1996, p. 122), $\hat{F}$ has a convergent subsequence which converges uniformly to a F_{∞}, which has the same support as F. Let $\hat{F}_{n(k)}$ be this convergent subsequence. Note that for $[u, v] \in [a_F, \tau]$, $\hat{F}_{n(k)}(v) - F_{n(k)}(u-)$ is uniformly bounded away from zero for n large enough and $\hat{F}(v) - \hat{F}(u-)$ is of uniformly bounded sectional variation. Thus, by Lemma 3.2 and 3.3 of Van der Laan (1996, p. 121), we have

$$\int_0^{\infty}\int_0^{v} \frac{\hat{F}_{n(k)}((v \vee t)) - \hat{F}_{n(k)}((u-) \vee t)}{\hat{F}_{n(k)}(v) - \hat{F}_{n(k)}(u-)} [\tilde{K}_{n(k)}(\mathrm{d}u,\ \mathrm{d}v)] \le C||\tilde{K}_{n(k)} - \tilde{K}||_{\infty}$$

for a $C < \infty$. Empirical process theory and the uniform consistency of $F_{n(k)}$ to F_{∞} imply that $H(F_{\infty}, \tilde{F}, \tilde{K})(t) = 0$ for all t. It remains to show that $H(F, \tilde{F}_0, \tilde{K}_0)(t) = 0$ implies $F = F_0$. It follows that $F_{\infty} = F_0$ due to $F(v) - F(u-) > \delta > 0$. Hence, $\hat{F}$ is uniformly consistent on $[a_F, \tau]$.

Notice that $H(F, \tilde{F}_0, \tilde{K}_0)(t) = 0$ is equivalent to $U(F, \tilde{F}_0, \tilde{K}_0)(t) = 0$. Since $\tilde{K}_0(\mathrm{d}u, \mathrm{d}v) = p^{-1}[F_0(v) - F_0(u-)]K(\mathrm{d}u, \mathrm{d}v)$ and $\tilde{F}_0(\mathrm{d}x) = p^{-1}[K_0(x, \infty) - K_0(x, x)]F_0(\mathrm{d}x)$, we have

$$0 = U(F, \tilde{F}_0, \tilde{K}_0)(t) - U(F_0, \tilde{F}_0, \tilde{K}_0)(t) = F(t) - F_0(t) \\ + p^{-1} \int_0^t \int_x^\infty \int_0^x \frac{[F(v) - F(u-)] - [F_0(v) - F_0(u-)]}{F(v) - F(u-)} [K_0(\mathrm{d}u, \mathrm{d}v)] \frac{F_0(\mathrm{d}x)}{a_S(x)}, \tag{19}$$

where $a_S(x) = \int_x^\infty \int_0^x \frac{1}{F(v)-F(u-)}[\tilde{K}_0(\mathrm{d}u, \mathrm{d}v)]$. Consider the linear operator $I - A_{F,0} : (D[0, \tau], || \cdot ||_\infty) \to (D[0, \tau], || \cdot ||_\infty)$ defined by:

$$(I - A_{F,0})(h)(t) = h(t) + p^{-1} \int_0^t \int_x^\infty \int_0^x \frac{h(v) - h(u-)}{F(v) - F(u-)} [K_0(\mathrm{d}u, \mathrm{d}v)] \frac{F_0(\mathrm{d}x)}{a_S(x)}.$$

Equation (19) tells us that $(I - A_{F,0})(F - F_0)(t) = 0$. If we can prove that the linear operator $I - A_{F,0}$ is 1-1, then $F = F_0$ follows. The operator can be proved to be invertible in the same way as was in Sect. 3.4 of Van der Laan (1996, p. 125). We simply replace in his proof $\Lambda_1(\mathrm{d}u, \mathrm{d}v)$ by $K_0(\mathrm{d}u, \mathrm{d}v)/[F(v) - F(u-)]$ and $\Lambda_2(\mathrm{d}x)$ by $F_0(\mathrm{d}x)/a_S(x)$. Let $\Lambda_1^*(x) = \int_x^\infty \int_0^x \Lambda_1(\mathrm{d}u\, \mathrm{d}v)$. In the proof in Sect. 3.4 of Van der Laan (1996), it is necessary that $\Lambda_1^*(\mathrm{d}x)/\Lambda_2(\mathrm{d}x)$ is uniformly bounded on $[a_F, \tau]$ and that $\int_{a_F}^\tau \Lambda_2(\mathrm{d}x) < \infty$ (i.e., $\int_{a_F}^\tau F_0(\mathrm{d}x)/a_S(x) < \infty$). Because $F(v) - F(u-) > \delta > 0$ on $[u, v] \in [a_F, \tau]$, $\Lambda_1^*(\mathrm{d}x)/\Lambda_2(\mathrm{d}x)$ is uniformly bounded if the assumption (b) holds, i.e., $\frac{K_0(\mathrm{d}x,\infty)-K_0(\mathrm{d}x,\mathrm{d}x)}{F_0(\mathrm{d}x)}$ is uniformly bounded on $[a_F, \tau]$. Note that the assumption (b) holds for many commonly used distribution functions $F(x)$ and $K(x, y)$. Similarly, because

$$a(x) = \int_x^\infty \int_0^x \frac{1}{F(v) - F(u-)} [\tilde{K}_0(\mathrm{d}u, \mathrm{d}v)] \\ = p^{-1} \int_x^\infty \int_0^x \frac{F_0(v) - F_0(u-)}{F(v) - F(u-)} [K_0(\mathrm{d}u, \mathrm{d}v)] \geq p^{-1}\delta[K_0(x, \infty) - K_0(x, x)],$$

we only need $\int_{a_F}^\tau F_0(\mathrm{d}x)/[K_0(x, \infty) - K_0(x, x)] < \infty$. The proof is completed. □

Next, we investigate the weak convergence of $\hat{F}$.

Theorem 3 *Let $W(x) = K(x, \infty) - K(x, x)$, $\tilde{W}(x) = \tilde{K}(x, \infty) - \tilde{K}(x, x)$ and $\tilde{W}_n(x) = \tilde{K}_n(x, \infty) - \tilde{K}_n(x, x)$. Under the assumptions (a) and (b) of Theorem 2, we assume that (c) the class of functions $\mathcal{F}$, where $\mathcal{F}$ consists of functions with envelop $1/W(s)$ is a $\tilde{F}(s)$-Donsker class, and*

$$(d) \int_u^v \frac{\tilde{F}_n(\mathrm{d}x)}{W(x)\tilde{W}_n(x)} \leq M(u, v)$$

with probability tending to 1, where $M(\cdot, \cdot)$ is such that the class of functions with envelope $M(\cdot, \cdot)$ is $\tilde{K}(u, v)$-Donsker.

Then $\sqrt{n}(\hat{F}(t) - F(t))$ is asymptotically normal for every $t \in [a_F, \tau]$.

Proof Let $a_n(s) = \int_s^\infty \int_0^s \frac{1}{\hat{F}(v)-\hat{F}(u-)} \tilde{K}(\mathrm{d}u, \mathrm{d}v)$, $a_n^n(s) = \int_s^\infty \int_0^s \frac{1}{\hat{F}(v)-\hat{F}(u-)} \tilde{K}_n$ $(\mathrm{d}u, \mathrm{d}v)$. and $a(s) = \int_x^\infty \int_0^x \frac{1}{F(v)-F(u-)} \tilde{K}(\mathrm{d}u, \mathrm{d}v)$. Define the empirical processes

$h_{\tilde{F}_n}(x) = \sqrt{n}(\tilde{F}_n(x) - \tilde{F}(x))$ and $h_{\tilde{K}_n}(u, v) = \sqrt{n}(\tilde{K}_n(u, v) - \tilde{K}(u, v))$. We know that $h_{\tilde{F}_n} \xrightarrow{d} h_{\tilde{F}}$ and $h_{\tilde{K}_n} \xrightarrow{d} h_{\tilde{K}}$ (jointly) in $(D[0, \tau], ||\cdot||_\infty, \mathcal{B})$ for two Gaussian processes $h_{\tilde{F}}$ and $h_{\tilde{K}}$. By telescoping it follows that we have

$$\sqrt{n}(U(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) - U(\hat{F}, \tilde{F}, \tilde{K})(t)) \\ = \int_{a_F}^{t} \frac{h_{\tilde{F}_n}(\mathrm{d}s)}{a_n(s)} - \int_{a_F}^{t} \frac{1}{a_n^n(s)a_n(s)} \int_s^\infty \int_0^s \frac{1}{F(v) - F(u-)} h_{\tilde{K}_n}(\mathrm{d}u\,\mathrm{d}v)\tilde{F}_n(\mathrm{d}s). \tag{20}$$

First, both $1/a_n(s)$ and $1/a(s)$ fall in a class of functions with envelope $1/W(s)$ with probability tending to 1. Under the assumption (c), both $1/a_n(s)$ and $1/a(s)$ fall in a $\tilde{F}$-Donsker class. Since

$$|1/a_n(s) - 1/a(s)| \le M||\hat{F} - F||_\infty 1/W(s),$$

under the assumption (a) of Theorem 2, we have $\int_{a_F}^t [1/a_n(s) - 1/a(s)]^2 \tilde{F}(\mathrm{d}s) \to 0$. Hence, $\int_{a_F}^t (1/a_n(s) - 1/a(s))h_{\tilde{F}_n}(\mathrm{d}s)$ converges to zero in probability. Since $\int_{a_F}^t 1/a(s)h_{\tilde{F}_n}(\mathrm{d}s)$ converges weakly, the weak convergence of the first term in (20) holds.

By Fubini's theorem, we can rewrite the second term of (20) as

$$\int_{a_F}^\infty \int_0^{\min(v,t)} \frac{1}{F(v) - F(u-)} q_n(u, v) h_{\tilde{K}_n}(\mathrm{d}u,\ \mathrm{d}v),$$

where $q_n(u, v) = \int_u^v 1/[a_n^n(s)a(s)]\tilde{F}_n(\mathrm{d}s)$. Under the assumption (d), we have

$$q_n(u, v) \le \int_u^v \frac{\tilde{F}_n(\mathrm{d}s)}{W(s)\tilde{W}_n(s)} \le M(u, v)$$

with probability tending to 1, and both $q_n(u, v)$ and $q(u, v) = \int_u^v 1/a^2(s)\tilde{F}(\mathrm{d}s)$ fall with probability tending to 1 in a $\tilde{K}(u, v)$-Donsker class. Since $\sup_{u > a_G+\epsilon, v \le \tau} |q_n(u, v) - q(u, v)| \to 0$, $(q_n(u, v) - q(u, v))^2 \le M^2(u, v)$ and $\int_0^\tau \int_0^\tau M^2(u, v) \tilde{K}(\mathrm{d}u\,\mathrm{d}v) < \infty$, it follows that $\int_0^\tau \int_0^\tau (q_n(u, v) - q(u, v))^2 \tilde{K}(\mathrm{d}u\,\mathrm{d}v)$ converges to zero in probability. Hence, the weak convergence of the second term in (20) holds. This proves that

$$Z_n(t) \equiv \sqrt{n}(U(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) - U(\hat{F}, \tilde{F}, \tilde{K})(t)) \xrightarrow{d} Z(t) \\ \equiv \int_0^t \frac{h_{\tilde{F}}(\mathrm{d}s)}{a(s)} - \int_0^t \frac{1}{a^2(s)} \int_s^\infty \int_0^s \frac{1}{F(v) - F(u-)} h_{\tilde{K}}(\mathrm{d}u, \mathrm{d}v)\tilde{F}(\mathrm{d}s),$$

where $a(s) = \int_x^\infty \int_0^x \frac{1}{F(v) - F(u-)} [\tilde{K}(\mathrm{d}u,\ \mathrm{d}v)]$. Since $U(\hat{F}, \tilde{F}_n, \tilde{K}_n)(t) = U(F, \tilde{F}, \tilde{K})(t) = 0$, this implies that $\sqrt{n}(U(\hat{F}, \tilde{F}, \tilde{K})(t) - U(F, \tilde{F}, \tilde{K})(t)) \equiv -Z_n(t)$ is asymptotically normal with mean zero. Since F appears in $U(F, \tilde{F}, \tilde{K})$ only as a function it is straightforwardly verified (see Sect. 3.3 of van der Laan) that $F \to U(F, \tilde{F}, \tilde{K})$ is Fréchet-differentiable for any sequence $\hat{F}$ s.t. $||\hat{F} - F||_\infty \to 0$ we have

$$\frac{1}{||\hat{F} - F||_\infty}(U(\hat{F}, \tilde{F}, \tilde{K}) - U(F, \tilde{F}, \tilde{K}) - d_1U(F, \tilde{F}, \tilde{K})(\hat{F} - F)) \to 0$$

with respect to the supnorm, where $d_1U(F, \tilde{F}, \tilde{K})$ is a linear mapping. By the usual kind of argument (see, e.g., Van der Vaart 1995) for M-estimators it follows now that

$$d_1U(F, \tilde{F}, \tilde{K})(\sqrt{n}(\hat{F}(t) - F(t)) = -Z_n(t) + o_p(1).$$

Under the assumptions (a) and (b) of Theorem 2, we can prove that $d_1U(F, \tilde{F}, \tilde{K})$ has a bounded inverse in the same way as was in Sect. 3.4 of Van der Laan (1996); then

$$\sqrt{n}(\hat{F}(t) - F(t)) = -d_1U(F, \tilde{F}, \tilde{K})^{-1}(Z_n(t)) + o_p(1).$$

Hence, the weak convergence of Z_n implies, by the continuous mapping theorem, weak convergence of $\sqrt{n}(F_n - F)$. The proof is completed. □

A simulation study was conducted to investigate the performances of the $\hat{F}$ and $\hat{K}$. The T^*'s are i.i.d. Weibull distributed with scale parameter $\lambda_f = 1$ and shape parameters $\delta_f = 1.0, 4.0$. The U^*'s are i.i.d. exponential distributed with scale parameters $\lambda_g = 1.0, 4.0$. The V^*'s are i.i.d. exponential distributed with scale parameters $\lambda_q = 0.25, 1.0$. The T^*, U^* and V^* are independent of one another. The goal is to estimate the distribution function of T^*: $F(t_{0.5}) = 0.5$, and the joint distribution function of U^* and V^*: $K(u_{0.7}, v_{0.8}) = 0.56$, where t_p, u_p and v_p denote the p^{th} percentile of T^*, U^* and V^*, respectively. The sample sizes are chosen as 50, 100 and 200. The replication is 1000 times. Table 1 shows the biases and standard deviations (denoted by std.) of the two estimators $\hat{F}(t_{0.5})$ and $\hat{K}(u_{0.7}, v_{0.8})$. Table 1 also lists the simulated truncation probability $1 - p$.

Table 1 shows that the bias and standard deviation of the estimator $\hat{F}$ are smaller than those of $\hat{K}$. The biases and standard deviations of both estimators decrease as sample size increases. Given n, the bias and standard deviation of $\hat{F}$ increases as $1-p$ increases. When truncation is severe (i.e., $1 - p = 0.833$) and sample size is small (i.e., $n = 50$), the standard deviations of $\hat{F}$ and $\hat{K}$ are large.

4 Testing the null hypothesis H_{θ_0}

4.1 The test proposed by Efron and Petrosian (1999)

Define $T_i(\theta) = T_i - \theta \cdot \log(1 + Z_i)$, $U_i(\theta) = U_i - \theta \cdot \log(1 + Z_i)$ and $V_i(\theta) = V_i - \theta \cdot \log(1 + Z_i)$. For testing the null hypothesis $H_{\theta_0}: T^*(\theta_0) = T^* - \theta_0 \cdot \log(1 + Z^*)$ is independent of Z^*, define a pair of indices (i, j) as a comparable pair if $U_j(\theta_0) \le T_i(\theta_0) \le V_j(\theta_0)$ and $U_i(\theta_0) \le T_j(\theta_0) \le V_i(\theta_0)$. Efron and Petrosian (1999) proposed

Table 1 Simulation results for biases and std of the estimators $\hat{F}$ and $\hat{K}$

λ_g	λ_q	δ_f	n	$1-p$	$\hat{F}(t_{0.5})$		$\hat{K}(u_{0.7}, v_{0.8})$	
					bias	std	bias	std.
0.25	1.0	1.0	50	0.666	0.0079	0.1245	0.0799	0.1409
0.25	1.0	1.0	100	0.666	−0.0031	0.0976	0.0550	0.1132
0.25	1.0	1.0	200	0.666	−0.0025	0.0618	0.0456	0.0811
0.25	1.0	4.0	50	0.605	0.0115	0.0791	−0.0213	0.1824
0.25	1.0	4.0	100	0.605	0.0031	0.0551	−0.0083	0.1570
0.25	1.0	4.0	200	0.605	−0.0004	0.0399	0.0149	0.1280
0.25	4.0	1.0	50	0.391	−0.0058	0.0940	0.0272	0.1055
0.25	4.0	1.0	100	0.391	−0.0030	0.0682	0.0123	0.0751
0.25	4.0	1.0	200	0.391	−0.0007	0.0544	0.0096	0.0590
0.25	4.0	4.0	50	0.240	0.0012	0.0727	−0.0039	0.1096
0.25	4.0	4.0	100	0.240	0.0014	0.0499	−0.0013	0.0932
0.25	4.0	4.0	200	0.240	0.0000	0.0376	−0.0008	0.0785
1.00	1.0	1.0	50	0.833	−0.0110	0.1482	0.0788	0.1917
1.00	1.0	1.0	100	0.833	0.0078	0.1031	0.0673	0.1495
1.00	1.0	1.0	200	0.833	0.0066	0.0699	0.0626	0.1033
1.00	1.0	4.0	50	0.768	−0.0004	0.0822	0.1014	0.1732
1.00	1.0	4.0	100	0.768	−0.0002	0.0572	0.1009	0.1380
1.00	1.0	4.0	200	0.768	−0.0001	0.0395	0.0852	0.1211
1.00	4.0	1.0	50	0.645	−0.0082	0.1239	0.0487	0.1461
1.00	4.0	1.0	100	0.645	−0.0009	0.0866	0.0476	0.1311
1.00	4.0	1.0	200	0.645	0.0001	0.0685	0.0420	0.0843
1.00	4.0	4.0	50	0.539	0.0045	0.0748	0.1070	0.1560
1.00	4.0	4.0	100	0.539	−0.0006	0.0538	0.0964	0.1327
1.00	4.0	4.0	200	0.539	0.0007	0.0374	0.0891	0.1209

a truncated version of the Kendall's tau statistics as

$$\hat{\tau}_n = \sum_{(i,j)\in\mathcal{C}} sgn[(T_i(\theta_0) - T_j(\theta_0))(Z_i - Z_j)],$$

where $\mathcal{C}$ =set of comparable pairs.

Let $T_{(1)}(\theta_0) < T_{(2)}(\theta_0) < \cdots < T_{(n)}(\theta_0)$ be the ordered observation of $T_i(\theta_0)$'s and $(Z_{(i)}, U_{(i)}(\theta_0), V_{(i)}(\theta_0))$ be the concomitant of $T_{(i)}(\theta_0)$. For each i, define the set $\mathcal{R}_i = \{j | U_{(j)}(\theta_0) \le T_{(i)}(\theta_0) \le T_{(j)}(\theta_0) \le V_{(i)}(\theta_0)\}$ and $\mathcal{R}_i^* = \{j | U_{(i)}(\theta_0) \le T_{(j)}(\theta_0) \le T_{(i)}(\theta_0) \le V_{(j)}(\theta_0)\}$. Note that $\hat{\tau}_n = \sum_{i=1}^{n-1} S_i = \sum_{i=2}^{n} S_i^*$, where $S_i = \sum_{j=i+1}^{n} sgn(Z_{(j)} - Z_{(i)}) I_{[j\in\mathcal{R}_i]}$ and $S_i^* = \sum_{j=1}^{i-1} sgn(Z_{(i)} - Z_{(j)}) I_{[j\in\mathcal{R}_i^*]}$. Note that either $j \in \mathcal{R}_i$ or $j \in \mathcal{R}_i^*$ implies that $U_{(i)}(\theta_0) \le T_{(j)}(\theta_0) \le V_{(i)}(\theta_0)$ and $U_{(j)}(\theta_0) \le T_{(i)}(\theta_0) \le V_{(j)}(\theta_0)$. Let $r_i = \text{card}(\mathcal{R}_i) = \sum_{j=1}^{n} I_{[j\in\mathcal{R}_i]}$ and $r_i^* = \text{card}(\mathcal{R}_i^*) = \sum_{j=1}^{n} I_{[j\in\mathcal{R}_i^*]}$. For left-truncated data (i.e., $V_{(i)}(\theta_0) = \infty$), under H_{θ_0}, the conditional distribution of S_i, given $r_i > 0$ is uniform with probability mass function

$P(S_i = j|H_{\theta_0}, r_i) = 1/r_i$ for $j = r_i - 1, r_i - 3, \ldots, -r_i + 3, -r_i + 1$. Similarly, given $r_i^* > 0$, S_i^* also has a discrete uniform distribution. Tsai (1990) proved that conditionally on r_i $(i = 1, \ldots, n)$, $S_1, \ldots, S_n$ are mutually independent. Hence, for left-truncated data the conditional variance of $\hat{\tau}_n, \sigma_{\hat{\tau}_n^2} = \frac{1}{3}\sum_{i=1}^n (r_i^2 - 1)$. However, for doubly truncated data, S_i does not have a discrete uniform distribution and $S_1 \ldots, S_n$ are not mutually independent. Hence, there is no convenient formula for $\sigma^2_{\hat{\tau}_n^2}$. One way to circumvent this difficulty is to use a bootstrap approximation. Based on $\hat{\tau}_n$, Efron and Petrosian (1999) suggest a bootstrap test as follows:

(i) Randomly generate B samples of size n, with replacement from the observed data.
(ii) Based on the b^{th} bootstrap sample, compute the test statistic $\hat{\tau}_n^{*b}$ using the same procedure in computing $\hat{\tau}_n$.
(iii) Let $\hat{\sigma}^2_{\hat{\tau}_n} = \frac{1}{B-1}\sum_{b=1}^B (\hat{\tau}_n^{*b} - \bar{\hat{\tau}}_n^*)^2$, where $\bar{\hat{\tau}}_n^* = B^{-1}\sum_{b=1}^B \hat{\tau}_n^{*b}$.
(iv) Based on normal approximation, a bootstrap test is given by $\hat{T}_E = \hat{\tau}_n/\sqrt{\hat{\sigma}^2_{\hat{\tau}_n}}$. Note that, the asymptotic normality of $\hat{T}_E$-test has not been generally verified. The situation is more complicated than that of Tsai (1990), where the asymptotic normality of $\hat{\tau}_n/[\frac{1}{3}\sum_{i=1}^n (r_i^2 - 1)]^{\frac{1}{2}}$ holds for left-truncated data. Besides, as discussed by Efron and Petrosian (1999), when T^* is exponential distributed, the testing procedure is futile. Next, we explain why the $\hat{T}_E$-test fails in the exponential case.

A pair of $(T_1^* - \theta \cdot \log(1 + Z_1^*), T_2^* - \theta \cdot \log(1 + Z_2^*))$ is observable and comparable only if

$$\max\left\{U_1^*, U_2^* - \theta \cdot \log\left(\frac{1 + Z_2^*}{1 + Z_1^*}\right)\right\} \le T_1^* \le \min\left\{V_1^*, V_2^* - \theta \cdot \log\left(\frac{1 + Z_2^*}{1 + Z_1^*}\right)\right\}$$

and

$$\max\left\{U_2^*, U_1^* - \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right)\right\} \le T_2^* \le \min\left\{V_2^*, V_1^* - \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right)\right\}. \quad (21)$$

Given $Z_1^* > Z_2^*$ and θ is sufficiently large, (21) is reduced to

$$U_2^* - \theta \cdot \log\left(\frac{1+Z_2^*}{1+Z_1^*}\right) \le T_1^* \le V_1^*, \quad \text{and} \quad U_2^* \le T_2^* \le V_1^* - \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right). \quad (22)$$

Similarly, given $Z_1^* < Z_2^*$ and θ is sufficiently large, (21) is reduced to

$$U_1^* \le T_1^* \le V_2^* - \theta \cdot \log\left(\frac{1+Z_2^*}{1+Z_1^*}\right), \quad \text{and} \quad U_1^* - \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right) \le T_2^* \le V_2^*. \quad (23)$$

Let $x = \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right)$. Then

$$P(T_1^* - \theta \cdot \log(1 + Z_1^*) - [T_2^* - \theta \cdot \log(1 + Z_2^*)] > 0|(22))$$

$$= P(T_1^* > T_2^* + x | U_2^* + x \le T_1^* \le V_1^*,\ U_2^* \le T_2^* \le V_1^* - x), \qquad (24)$$

and

$$\begin{aligned} &P(T_1^* - \theta \cdot \log(1 + Z_1^*) - [T_2^* - \theta \cdot \log(1 + Z_2^*)] > 0 | (23)) \\ &\quad = P(T_1^* - T_2^* > -x | U_2^* \le T_1^* \le V_2^* - x,\ U_2^* + x \le T_2^* \le V_2^*) \\ &\quad = P(T_{1x}^* > T_{2x}^* + x | U_2^* + x \le T_{1x}^* \le V_2^*,\ U_2^* \le T_{2x}^* \le V_2^* - x), \qquad (25) \end{aligned}$$

where $T_{1x}^* = T_1^* + x$ and $T_{2x}^* = T_2^* - x$. Because of the exponential's memoryless property, it follows that (24) is equal to (25) and the testing procedure is futile.

4.2 An alternative test

In this section, we propose a modified Kendall's tau test for testing the null hypothesis H_{θ_0}. The test is valid in the exponential case. First, let $\tilde{\tau}_n = \sum_{i=2}^n \tilde{S}_i^*$, where $\tilde{S}_i^* = \sum_{j=1}^{i-1} sgn(Z_{(i)} - Z_{(j)}) I_{[j \in \tilde{\mathcal{R}}_i]}$, where $\tilde{\mathcal{R}}_i = \{j | j \in \mathcal{R}_i^*, U_{(j)}(\theta_0) \le T_i^{\min}(\theta_0)\}$, and $T_i^{\min}(\theta_0) = \min_{j \in \mathcal{R}_i^*} T_{(j)}(\theta_0)$. Note that given i, every pair of indices (j, k) in $\tilde{\mathcal{R}}_i$ is comparable, i.e., $U_{(j)}(\theta_0) \le T_{(k)}(\theta_0) \le V_{(j)}(\theta_0)$ and $U_{(k)}(\theta_0) \le T_{(j)}(\theta_0) \le V_{(k)}(\theta_0)$. Conditional on $\tilde{r}_i = \text{card}(\tilde{\mathcal{R}}_i)$, let $Z_1^i < \cdots < Z_{\tilde{r}_i}^i$ be the observations in $\tilde{\mathcal{R}}_i$ and $Z_{(1)}^i < \cdots < Z_{(\tilde{r}_i)}^i$ be the ordered values of Z_j^i's in $\tilde{\mathcal{R}}_i$. Under H_{θ_0}, given $\tilde{r}_i$, the conditional probability $P(Z_j^i = Z_{(k)}^i | H_{\theta_0}, \tilde{r}_i) = 1/\tilde{r}_i$ for $k = 1, \ldots, \tilde{r}_i$. Hence, the conditional distribution of $\tilde{S}_i^*$, given $\tilde{r}_i > 0$ is uniform with probability mass function $P(\tilde{S}_i^* = j | H_{\theta_0}, \tilde{r}_i) = 1/\tilde{r}_i$ for $j = \tilde{r}_i - 1, \tilde{r}_i - 3, \ldots, -\tilde{r}_i + 3, -\tilde{r}_i + 1$. These implies $E(\tilde{S}_i^* | H_{\theta_0}) = 0$ and $Var(\tilde{S}_i^* | H_{\theta_0}) = \frac{1}{3}(\tilde{r}_i^2 - 1)$. Hence, $E(\tilde{\tau}_n | H_{\theta_0}, \tilde{r}_1, \ldots, \tilde{r}_n) = 0$. However, since $\tilde{S}_1^* \ldots, \tilde{S}_n^*$ are not mutually independent, $Var(\tilde{\tau}_n | H_{\theta_0}, \tilde{r}_1, \ldots, \tilde{r}_n) \ne \frac{1}{3} \sum_{i=1}^n (\tilde{r}_i^2 - 1)$. We can use a bootstrap approximation as described in Sect. 4.1. Let $\hat{\sigma}_{\tilde{\tau}_n}^2$ denote the bootstrap variance estimator for $\tilde{\tau}_n$. Based on normal approximation of $\tilde{T}_n = \tilde{\tau}_n / \sqrt{\hat{\sigma}_{\tilde{\tau}_n}^2}$, we can construct a bootstrap test (denoted by $\tilde{T}_n$-test).

Note that given i, the cardinal of $\tilde{\mathcal{R}}_i$, $\tilde{r}_i$, depends on the cardinal of $\mathcal{R}_i^*$ (i.e. r_i^*). Since for each $j \in \mathcal{R}_i^*$ $(j = 1, \ldots, i-1)$, $U_{(i)}(\theta) \le T_{(j)}(\theta) \le T_{(i)}(\theta)$, this may make the probability $P(U_{(j)}(\theta) \le T_i^{\min}(\theta))$ large enough such that $\tilde{r}_i$, is not too small.

Next, we explain why the $\tilde{T}_n$-test is valid in the exponential case. Given $\tilde{r}_2 = 1$,

$$\begin{aligned} &P(\tilde{S}_2 = 1 | \tilde{r}_2 = 1) \\ &\quad = P(Z_{(2)} > Z_{(1)} | T_{(2)}(\theta_0) > T_{(1)}(\theta_0), T_{(2)}(\theta_0) \le V_{(1)}(\theta_0), T_{(1)}(\theta_0) \ge U_{(2)}(\theta_0)) \\ &\quad = 2P(Z_2^* > Z_1^* | (26)), \end{aligned}$$

where (26) denotes the following conditions:

$$\max\left\{U_1^*, U_2^* - \theta \cdot \log\left(\frac{1 + Z_2^*}{1 + Z_1^*}\right)\right\} \le T_1^* \le \min\left\{V_1^*, T_2^* - \theta \cdot \log\left(\frac{1 + Z_2^*}{1 + Z_1^*}\right)\right\}$$

and

$$\max\left\{U_2^*, T_1^* - \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right)\right\} \leq T_2^* \leq \min\left\{V_2^*, V_1^* - \theta \cdot \log\left(\frac{1+Z_1^*}{1+Z_2^*}\right)\right\}.$$

When θ_0 is sufficiently large, it is easy to see that $P(Z_2^* > Z_1^*|(26)) < P(Z_2^* < Z_1^*|(26))$, i.e., $P(\tilde{S}_2^* = 1|\tilde{r}_2 = 1) < P(\tilde{S}_2^* = -1|\tilde{r}_2 = 1)$. Similar arguments hold for $\tilde{S}_i^*(i = 2, \ldots, n)$ and the $\tilde{T}_n$-test is valid in the exponential case.

To investigate the performances of the $\tilde{T}_n$ and $\hat{T}_E$-test, we conduct the following simulation study. The distribution for Z^*'s are exponential with scale parameter equal to 1.0. The distribution for U^*'s are exponential with scale parameter $\lambda_g = 0.25, 1.0$. The distribution for V^*'s are exponential with scale parameter $\lambda_q = 1.0, 4.0$. The distribution for T^*'s are Weibull with scale parameter $\lambda_f = 1.0$ and shape parameter $\delta_f = 1.0$ and 4.0. The variables T^*, V^*, U^* and Z^*'s are mutually independent. The significance level, α, is set at 0.05 and θ_0 is chosen as 0.0, 0.25 and 0.50. The sample sizes n are chosen as 50 and 100. The replication is 1000 times. For bootstrap replications, B is chosen as 500. Table 2 shows the truncation probability $1-p$, the estimated size and power of the two tests. Table 2 also shows the average of $\tilde{r}_i$'s, (denoted by $\tilde{r} = n^{-1}\sum_{i=1}^n \tilde{r}_i$) and the average of r_i^*'s, (denoted by $r^* = n^{-1}\sum_{i=1}^n r_i^*$).

Table 2 shows that when $\theta_0 = 0$ and $n = 100$, the empirical sizes of both tests are close to the nominal level 0.05 for most of cases considered. Except for the exponential case (i.e., $\delta_f = 1.0$), the power of the $\hat{T}_E$-test is larger than that of $\tilde{T}_n$-test. For the exponential case, the $\hat{T}_E$-test fails. However, the power of the $\tilde{T}_n$-test increases in n and θ_0.

5 Discussion

For doubly truncated data, we have established the equivalence between the IPW estimator and the NPMLE. Based on Theorem 3.1 of van der Laan, we show that the NPMLE is consistent under certain conditions. Simulation study indicates that the NPMLE works satisfactorily for moderate sample size. When T^* is exponential distributed, the alternative test $\tilde{T}_n$ proposed in this note is practically useful for testing if θ_0 is the true value of the evolution parameter. Based on (13), the IPW estimator can be extended to nonidentical truncation models. When the bivariate distribution of (U^*, V^*) is parameterized as K_θ, we can obtain an estimator $\hat{\theta}_n$ by maximizing the conditional likelihood of (U^*, V^*)'s, given T's (see Wang 1989 and Qin and Wang 2001). Based on $K_{\hat{\theta}_n}$, a semiparametric IPW estimator of F is given by

$$F_{\hat{\theta}_n}(t) = \left[\sum_{i=1}^n \frac{1}{K_{\hat{\theta}_n}(T_i, \infty) - K_{\hat{\theta}_n}(T_i, T_i)}\right]^{-1} \sum_{i=1}^n \frac{I_{[T_i \leq t]}}{K_{\hat{\theta}_n}(T_i, \infty) - K_{\hat{\theta}_n}(T_i, T_i)}.$$

The large-sample properties of the estimator $F_{\hat{\theta}_n}$ can be established using the arguments similar to those of Wang (1989).

Table 2 Estimated size and power of the $\tilde{T}_n$-test and $\hat{T}_E$-test

λ_g	λ_q	δ_f	n	$1-p$	$\tilde{r}$	r^*	test	$\theta_0 = 0.0$	$\theta_0 = 0.25$	$\theta_0 = 0.50$
0.25	1.0	1.00	50	0.666	9.3	13.5	$\tilde{T}_n$	0.045	0.558	0.926
0.25	1.0	1.00	50	0.666	9.3	13.5	$\hat{T}_E$	0.055	0.061	0.060
0.25	1.0	1.00	100	0.666	15.5	25.5	$\tilde{T}_n$	0.063	0.974	1.000
0.25	1.0	1.00	100	0.666	15.5	25.5	$\hat{T}_E$	0.058	0.066	0.062
0.25	1.0	4.00	50	0.605	8.9	12.1	$\tilde{T}_n$	0.058	0.523	0.840
0.25	1.0	4.00	50	0.605	8.9	12.1	$\hat{T}_E$	0.061	0.745	0.985
0.25	1.0	4.00	100	0.605	15.0	23.1	$\tilde{T}_n$	0.055	0.658	0.743
0.25	1.0	4.00	100	0.605	15.0	23.1	$\hat{T}_E$	0.052	0.962	1.000
0.25	4.0	1.00	50	0.391	11.7	17.6	$\tilde{T}_n$	0.038	0.618	0.775
0.25	4.0	1.00	50	0.391	11.7	17.6	$\hat{T}_E$	0.054	0.039	0.052
0.25	4.0	1.00	100	0.391	20.4	34.3	$\tilde{T}_n$	0.053	0.992	1.000
0.25	4.0	1.00	100	0.391	20.4	34.3	$\hat{T}_E$	0.052	0.033	0.040
0.25	4.0	4.00	50	0.240	11.7	15.6	$\tilde{T}_n$	0.058	0.369	0.746
0.25	4.0	4.00	50	0.240	11.7	15.6	$\hat{T}_E$	0.046	0.510	0.829
0.25	4.0	4.00	100	0.240	20.7	30.7	$\tilde{T}_n$	0.050	0.726	0.985
0.25	4.0	4.00	100	0.240	20.7	30.7	$\hat{T}_E$	0.056	0.828	0.993
1.00	1.0	1.00	50	0.833	7.2	11.0	$\tilde{T}_n$	0.030	0.248	0.587
1.00	1.0	1.00	50	0.833	7.2	11.0	$\hat{T}_E$	0.062	0.028	0.044
1.00	1.0	1.00	100	0.833	12.4	20.6	$\tilde{T}_n$	0.039	0.705	0.962
1.00	1.0	1.00	100	0.833	12.4	20.6	$\hat{T}_E$	0.053	0.030	0.047
1.00	1.0	4.00	50	0.768	6.9	9.8	$\tilde{T}_n$	0.045	0.545	0.940
1.00	1.0	4.00	50	0.768	6.9	9.8	$\hat{T}_E$	0.053	0.601	0.986
1.00	1.0	4.00	100	0.768	12.0	19.0	$\tilde{T}_n$	0.045	0.752	0.957
1.00	1.0	4.00	100	0.768	12.0	19.0	$\hat{T}_E$	0.047	0.946	1.000
1.00	4.0	1.00	50	0.645	8.9	14.0	$\tilde{T}_n$	0.042	0.427	0.610
1.00	4.0	1.00	50	0.645	8.9	14.0	$\hat{T}_E$	0.045	0.047	0.082
1.00	4.0	1.00	100	0.645	16.0	27.7	$\tilde{T}_n$	0.038	0.585	0.749
1.00	4.0	1.00	100	0.645	16.0	27.7	$\hat{T}_E$	0.040	0.048	0.087
1.00	4.0	4.00	50	0.539	9.0	13.3	$\tilde{T}_n$	0.038	0.395	0.720
1.00	4.0	4.00	50	0.539	9.0	13.3	$\hat{T}_E$	0.065	0.582	0.837
1.00	4.0	4.00	100	0.539	15.6	25.5	$\tilde{T}_n$	0.056	0.614	0.858
1.00	4.0	4.00	100	0.539	15.6	25.5	$\hat{T}_E$	0.059	0.661	0.893

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