# Semiregular automorphisms of vertex-transitive cubic graphs

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**Abstract**

An old conjecture of Marušič, Jordan and Klin asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Marušič and Scapellato proved this for cubic graphs. For these graphs, we make a stronger conjecture, to the effect that there is a semiregular automorphism of order tending to infinity with $n$. We prove that there is one of order greater than 2.

*Key words:* vertex-transitive graph, semiregular automorphism

A permutation $\sigma$ is *semiregular* if all its cycles have the same length. An old conjecture made independently by Marušič, Jordan and Klin (see the introduction to [3] for details) asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Clearly there is no loss of generality in assuming that the graph is connected. Marušič and Scapellato proved:

**Theorem 1** *A vertex-transitive connected cubic simple graph has a non-trivial semiregular automorphism.*

We need to reproduce the proof since we will use parts of it later.

**PROOF.** We argue by contradiction. Let $G$ be a connected cubic vertex-transitive graph, and suppose that $G$ has no non-trivial semiregular automorphism.

We first observe that, if $\sigma$ is an automorphism of prime order greater than 3, then $\sigma$ is semiregular. For, if $\sigma$ fixes a vertex $v$, then it must fix the three

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neighbours of $v$, and then their neighbours, ad so on; since $G$ is connected, we would find that $\sigma$ is the identity, a contradiction.

So $|\operatorname{Aut}(G)| = 2^x 3^y$ for some $x, y$.

Next we show that $y \neq 0$. For suppose that $y = 0$. Then $\operatorname{Aut}(G)$ is a 2-group, so there is a non-identity element $\sigma$ in its centre; and since $\operatorname{Aut}(G)$ is transitive, $\sigma$ is semiregular, contrary to assumption.

Now it follows that $G$ is arc-transitive, that is, $\operatorname{Aut}(G)$ is transitive on ordered pairs of adjacent vertices. For let $\sigma$ be an automorphism of order 3. Then $\sigma$ fixes a vertex. Arguing as before, there must be a vertex $v$ such that $\sigma$ fixes $v$ and permutes its three neighbours transitively. Since $G$ is vertex-transitive (by assumption), it is thus arc-transitive.

By Burnside's $p^\alpha q^\beta$-theorem, $\operatorname{Aut}(G)$ is soluble. So a minimal normal subgroup $N$ of $\operatorname{Aut}(G)$ is elementary abelian. We split the argument into two cases, according as $N$ is a 3-group or a 2-group.

**Case 1:** $N$ is a 3-group. Since $N$ is abelian, it acts regularly on each or its orbits. We further subdivide into cases as follows. Consider the stabiliser $N_v$ of a vertex $v$.

**Case 1A:** $N_v$ fixes the three neighbours of $v$. Since $N$ is a normal subgroup of $\operatorname{Aut}(G)$, this holds for all vertices $v$. As before, we find that $N_v = 1$, so that $N$ is semiregular, as of course are all its non-identity elements.

**Case 1B:** $N_v$ permutes the three neighbours of $v$ transitively. Then these neighbours lie in the same $N$-orbit, say $O_1$. Let $v$ lie in the orbit $O_2$. Then $N_v$ fixes $O_2$ pointwise (since $N$ acts regularly on $O_2$) but acts semiregularly on $O_1$ as a group of order 3. So $O_1 \neq O_2$. Moreover, all edges from vertices in $O_1$ go to vertices in $O_2$, so $G$ is bipartite, with bipartite blocks $O_1$ and $O_2$. Thus $G$ is the complete bipartite graph $K_{3,3}$. But this graph has a regular automorphism of order 6.

**Case 2:** $N$ is a 2-group. Choose a vertex $v$. Then $\operatorname{Aut}(G)_v$ acts transitively on the three neighbours of $v$, hence as either the symmetric group $S_3$ or the cyclic group $A_3$. So its normal subgroup $N_v$ acts as $S_3$, $A_3$ or the trivial group. Since $N$ is a 2-group, the first two cases are impossible, and $N_v$ fixes the neighbours of $v$. Then the usual argument shows that $N_v$ is trivial, so $N$ is semiregular, and we are done. $\square$

The proof allows the possibility that the semiregular element has order 2 or 3. However, in all the examples known to us, there are semiregular elements with order at least 4. We make the following conjecture:

**Conjecture 2** *There is a function $f$, so that $f(n) \to \infty$ as $n \to \infty$, with the property that a connected vertex-transitive cubic graph on $n$ vertices has a semiregular automorphism of order at least $f(n)$.*

In the rest of this paper, we show that there is always a semiregular automorphism of order at least 3, and end with some examples which give an upper bound to the growth of such a function.

**Theorem 3** *Let $G$ be a connected vertex-transitive cubic graph. Then $G$ has a semiregular automorphism of order greater than 2.*

The proof depends on the following group-theoretic lemma.

**Lemma 4** *If $P$ is a 2-group which is not elementary abelian, and $Q$ a core-free subgroup of $P$, then there is an element of $P$ of order 4 whose square lies in no conjugate of $Q$.*

**PROOF.** Recall that a subgroup $Q$ of $P$ is *core-free* if $Q$ contains no non-trivial normal subgroup of $P$. Note that, if $Q$ is core-free, then $Q \cap \zeta(P) = 1$, where $\zeta(P)$ is the centre of $P$.

Our proof is by induction on $|P|$. Let $P$ be a minimal counterexample. Let $Q$ be a core-free subgroup of $P$ such that

$$\mathcal{P}^2 = \{g^2 | g \in G\} \subseteq \bigcup_{g \in P} Q^g.$$

If the exponent of $\zeta(P)$ is at least 4, then the centre of $P$ contains a square, and the proposition is clear, so assume $\zeta(P)$ is elementary abelian.

Let $Z$ be a central subgroup of order 2 and consider $P/Z$ and $QZ/Z$. If $P/Z$ is elementary abelian then $1 \neq g^2 \in Z$ for some $g \in P$. In particular, as $Q$ is core-free, $g^2 \notin Q$. So, we may as well assume that $P/Z$ is not elementary abelian.

Now, $|P/Z| < |P|$ and $\{g^2 Z | g \in P\} \subseteq \bigcup_{g \in P} (QZ/Z)^g$. So, by the induction hypothesis, $QZ/Z$ is not core-free in $P/Z$.

Set $N = \bigcap_{g \in P} (QZ)^g$. Now, $Z < N \leq QZ$, hence $NQ = ZQ$. In particular $\bigcap_{g \in P} (QN)^g = N$. Therefore $QN/N$ is a core-free subgroup of $P/N$. Moreover $\{g^2 N | g \in P\} \subseteq \bigcup_{g \in P} (QN/N)^g$. So, by the induction hypothesis, $P/N$ is elementary abelian, so that $\Phi(P) \subseteq N$, where $\Phi(P)$ is the Frattini subgroup of $P$.

Now $NQ = QZ$, $|Z| = 2$ and $Z < N$; so $N \cap Q$ has index 2 in $N$. Furthermore, $Q \cap N$ is core-free, therefore $N$ is a subdirect product of copies of the cyclic group of order 2, and hence is a normal elementary abelian 2-subgroup of $P$. Since $\Phi(P) \subseteq N$, this implies that $\Phi(P)$ is elementary abelian.

Let $Z_1$ be another subgroup of $\zeta(P)$ of order 2. Applying the same argument, we get $\Phi(P) \subseteq QZ_1$ as well as $\Phi(P) \subseteq QZ$. If $QZ \neq QZ_1$ then $Q = QZ \cap QZ_1$. Therefore $Q$ is a normal subgroup of $P$, a contradiction. Therefore $QZ = QZ_1$. This proves that $QZ = Q\zeta(P)$ (since $\zeta(P)$ is elementary abelian). In particular, as $Q$ is core-free, $\zeta(P)$ has order 2, $P' = \zeta(P)$ and $P$ has nilpotency class 2.

Let $g, h$ be elements of $P$, $1 = [g,h]^2 = [g^2, h]$. Therefore $g^2$ lies in the centre of $P$. Thence $\mathcal{P}^2 \subseteq \zeta(P)$. So, either $\mathcal{P}^2 = 1$ or $\zeta(P)$ contains a square. In the former case $P$ is elementary abelian, a contradiction. In the latter case $P$ is not a counterexample. This concludes the proof. □

This is equivalent to the following result about permutation groups:

**Corollary 5** *Let $P$ be a transitive 2-group which is not elementary abelian. Then $P$ contains a semiregular element of order 4.* □

This follows immediately from the lemma, on taking $Q$ to be the stabiliser of a point in $P$. (An element of order 4 is semiregular if and only if its square has no fixed points.)

Now we can prove the Theorem. Let $G$ be a vertex-transitive cubic graph which has no semiregular automorphism of order greater than 2. As in the proof of Theorem 1, $\mathrm{Aut}(G)$ has order divisible by the primes 2 and 3 only.

Suppose that $P = \mathrm{Aut}(G)$ is a 2-group. By the above Corollary, it is elementary abelian and regular, and $G$ is a Cayley graph for $P$. Since $G$ is a cubic graph, $P$ is generated by three elements. Thus $P$ has order 4 or 8, and $G$ is $K_4$ or the cube; but each of these graphs has a semiregular automorphism of order 4. So 3 must divide $|\mathrm{Aut}(G)|$.

If 3 does not divide the order of the vertex stabiliser, then an element of order 3 is semiregular. (Note that in this case we cannot construct a semiregular automorphism of order greater than 3; but such an automorphism will exist unless the exponent of a Sylow 3-subgroup of $G$ is 3.)

So we may assume that there is an automorphism of order 3 fixing a vertex and permuting its three neighbours transitively. Since $G$ is vertex-transitive, it is arc-transitive.

Let $v$ be any vertex, and $N$ a minimal normal subgroup of $G$ (which we may assume is an elementary abelian 2-group). We separate three cases, according to the behaviour of the neighbours of $v$.

**Case 1:** The neighbours of $v$ are in the same $N$-orbit as $v$. In this case, $N$ is transitive, so $G$ is a Cayley graph, which is dealt with by the same argument as before.

**Case 2:** The neighbours of $v$ are all in a single $N$-orbit which doesn't contain $v$. In this case, as before, there are just two $N$-orbits and $G$ is bipartite; we find easily that it is the 3-cube.

**Case 3:** The neighbours of $v$ are all in different $N$-orbits. In this case, the edges between two orbits (if any) form a 1-factor; the graph obtained by shrinking each $N$-orbit to a single vertex and each such 1-factor to a single edge is a cubic vertex-transitive graph, so has a semiregular automorphism of order greater than 2, by induction. This lifts to a semiregular automorphism group of $G$ which is not an elementary abelian 2-group, and hence contains an element of order greater than 2. □

**Remark** In an earlier version, we asked whether the following stronger version of the Lemma is true:

> If $P$ is a 2-group which is not elementary abelian, then some non-identity element of the centre of $P$ is a square.

It was pointed out to us by Alexander Hulpke and Andreas Caranti that this is not the case: there are counterexamples of order 128, for example, group number 36 in the list of small groups in GAP [2].

We conclude with an example to show that, if our conjecture is true, the function $f(n)$ cannot grow faster than $n^{1/3}$.

Let $p$ be a prime congruent to $\pm 1$ mod 16, and let $G$ be the group $\mathrm{PSL}(2, p)$. Then $G$ has a maximal subgroup $H$ isomorphic to $S_4$. This subgroup contains a dihedral subgroup $K$ of order 8, with $N_G(K)$ a dihedral group of order 16. (See Burnside [1] for the subgroups of $G$.) Then $G$, acting on the cosets of $H$, preserves the orbital graph corresponding to the double coset $HxH$, where

$x \in N_G(K) \setminus K$. Since $H \cap x^{-1}Hx = K$, and $|H : K| = 3$, this orbital graph is cubic and 1-transitive. Since $|H| = 3 \cdot 2^3$, it is 4-transitive. Now it follows that $G$ is the full automorphism group. For Tutte's Theorem [4] shows that the full automorphism group has at most twice the order of $G$. So it contains $G$ as a normal subgroup of index at most 2. If it is larger than $G$, it would be $\mathrm{PGL}(2, p)$. But this group does not contain a subgroup isomorphic to $S_4 \times C_2$.

Now the largest order of an element of $G$ is $p$, and the number of vertices of the graph is $(p^3 - p)/48$.

## References


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