Jana Zlámalová

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SEMIREGULAR FINITE ELEMENTS IN SOLVING SOME NONLINEAR PROBLEMS*

Jana Zlámalová, Brno

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Abstract. In this paper, under the maximum angle condition, the finite element method is analyzed for nonlinear elliptic variational problem formulated in [4]. In [4] the analysis was done under the minimum angle condition.

Keywords: finite element method, nonlinear elliptic problems, semiregular elements, maximum angle condition, effect of numerical integration, approximation of the boundary

MSC 2000: 65N30

1. Introduction

In this paper we will analyze the finite element method for the nonlinear elliptic variational problem formulated in [4] under the maximum angle condition, whereas in [4] the analysis has been done under the minimum angle condition. We restrict ourselves to the problem in a domain Ω whose boundary $\partial\Omega$ is formed by two circles Γ_1, Γ_2 with the same center S_0 and radii R_1, $R_2 = R_1+\varrho$, where $\varrho \ll R_1$. On one circle the homogeneous Dirichlet boundary condition and on the other the nonhomogeneous Neumann boundary condition are prescribed.

Our assumptions concern only the boundary, the data and the form $a(u,v)$, which is nonlinear in u and linear in v. Our problem is discretized in the way used in practice: 1) the given domain Ω is approximated by a polygonal domain Ω_h; 2) Ω_h is triangulated and, using linear triangular finite elements, a finite dimensional space

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$X_h \subset C(\overline{\Omega}_h) \cap H^1(\Omega_h)$ is constructed; 3) the forms $a(u,v)$, $L(v)$ are computed approximately by means of numerical integration.

The theory presented generalizes the results obtained in [4] and [11]. In [4] the same problem is formulated under the minimum angle condition but on an arbitrary domain with a Lipschitz-continuous boundary. In [11] the finite element method is analyzed for a linear strongly elliptic mixed boundary value problem. In this paper we consider the same domain as in [11] but the problem is nonlinear. We prove the convergence of approximate solutions to the exact solution u under the condition $u \in H^1(\Omega)$.

In [10] the finite element method for a special monotone problem, which has applications in magnetostatics, was analyzed under the maximum angle condition. The results can be considered to be a special case of the present paper.

There are relatively many papers devoted to the analysis of the finite element method of nonlinear problems of elliptic type. Their list can be found, for example, in [2]. However, in all these papers the minimum angle condition is used.

The notation of Sobolev spaces, their norms and seminorms is the same as in [6].

2. Formulation of the problem

2.1. Boundary value problem.

We will consider the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, u, \nabla u) + b_0(\cdot, u, \nabla u) = f(x), \quad x \in \Omega, \tag{1}$$

$$u = 0 \text{ on } \Gamma_1, \tag{2}$$

$$\sum_{i=1}^{2} b_i(\cdot, u, \nabla u) n_i(\Omega) = q \text{ on } \Gamma_2 \tag{3}$$

where Ω is a two-dimensional bounded domain with a boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being circles with radii R_1 and $R_2 = R_1 + \varrho$, respectively. We assume that the circles Γ_1, Γ_2 have the same center S_0 and that

$$R_1 \gg \varrho. \tag{4}$$

Obviously, $\partial\Omega$ is Lipschitz continuous. The symbols $n_i(\Omega)$ $(i = 1, 2)$ denote the components of the unit outward normal to $\partial\Omega$. Further, $f\colon \Omega \to \mathbb{R}^1$, $b_i\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^1$ (i.e., $b_i = b_i(x, \xi) = b_i(\cdot, u, \nabla u)$, where $x = (x_1, x_2) \in \Omega$, $\xi = (\xi_0, \xi_1, \xi_2) = \big(u(x), \nabla u(x)\big) \in \mathbb{R}^3$, $i = 0, 1, 2$) are given functions and $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2)$.

2.2. Weak formulation.

We will use the Lebesgue spaces $L_2(\Omega)$, $L_2(\partial\Omega)$, $L_\infty(\Omega)$ and the Sobolev spaces $H^1(\Omega)$, $H^2(\Omega)$, $W^{1,\infty}(\Omega)$ equipped with their usual norms $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{0,\partial\Omega}$, $\|\cdot\|_{0,\infty,\Omega}$ and $\|\cdot\|_{1,\Omega}$, $\|\cdot\|_{2,\Omega}$, $\|\cdot\|_{1,\infty,\Omega}$, respectively (see [1], [6], [7]). The seminorms in the spaces $H^1(\Omega)$ and $H^2(\Omega)$ will be denoted by $|\cdot|_{1,\Omega}$ and $|\cdot|_{2,\Omega}$, respectively.

Assumptions 2.2.1. Let $\{\Omega_h\}$ $(h \in (0,h_0))$ be a set of polygonal approximations of Ω. Let $\widetilde{\Omega} \subset \mathbb{R}^2$ be a bounded domain such that

$$\widetilde{\Omega} \supset \overline{\Omega} \cup \overline{\Omega}_h, \quad \forall h \in (0,h_0). \tag{5}$$

Let the functions $f\colon \widetilde{\Omega} \to \mathbb{R}^1$, $q\colon \Gamma_2 \to \mathbb{R}^1$ and $b_i\colon \widetilde{\Omega} \times \mathbb{R}^3 \to \mathbb{R}^1$, $i = 0,1,2$ have the following properties:

(A) a) $f \in W^{1,\infty}(\widetilde{\Omega})$,

b) q is piecewise of class C^2 (i.e. $\partial\Omega$ can be divided into a finite number of closed arcs Z_k such that

$$\widetilde{q}_k(t) = q\bigl(\varphi_k(t), \psi_k(t)\bigr), \quad t \in [\alpha_k, \beta_k],$$

is a twice continuously differentiable function on $[\alpha_k, \beta_k]$, where $x_1 = \varphi_k(t)$, $x_2 = \psi_k(t)$, $t \in [\alpha_k, \beta_k]$ is a parametric representation of Z_k with $\varphi_k, \psi_k \in C^2([\alpha_k, \beta_k])$).

(B) The functions $b_i(x,\xi)$ $(x \in \widetilde{\Omega},\ \xi = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3)$ are continuous in $\widetilde{\Omega} \times \mathbb{R}^3$. There exists a constant $C > 0$ such that

$$|b_i(x,\xi)| \leqslant C\Bigl(1 + \sum_{j=0}^{2} |\xi_j|\Bigr) \ \forall x \in \widetilde{\Omega},\ \forall \xi \in \mathbb{R}^3 \quad (i = 0,1,2).$$

(C) The derivatives $(\partial b_i/\partial \xi_j)(x,\xi)$, $(i,j = 0,1,2)$ are continuous and bounded in $\widetilde{\Omega} \times \mathbb{R}^3$:

$$\Bigl|\frac{\partial b_i}{\partial \xi_j}(x,\xi)\Bigr| \leqslant C \ \forall x \in \widetilde{\Omega},\ \forall \xi \in \mathbb{R}^3.$$

(D) The functions b_i satisfy

$$\sum_{i,j=0}^{2} \frac{\partial b_i}{\partial \xi_j}(x,\xi)\eta_i\eta_j \geqslant \alpha \sum_{i=1}^{2} \eta_i^2 \qquad \forall x \in \widetilde{\Omega},\ \forall \xi, \eta \in \mathbb{R}^3$$

where $\alpha > 0$ is a constant independent of x, ξ and η.

(E) The functions $\partial b_i/\partial x_j$ $(i = 0,1,2;\ j = 1,2)$ are continuous in $\widetilde{\Omega} \times \mathbb{R}^3$. There exists a constant $C > 0$ such that

$$\Bigl|\frac{\partial b_i}{\partial x_j}(x,\xi)\Bigr| \leqslant C\Bigl(1 + \sum_{j=0}^{2} |\xi_j|\Bigr) \ \forall x \in \widetilde{\Omega},\ \forall \xi \in \mathbb{R}^3 \quad (i = 0,1,2;\ j = 1,2).$$

Assumptions 2.2.1 are the same as in [4].

A weak solution of problem (1)–(3) is a solution of the following variational problem (which can be obtained from (1)–(3) by means of Green's theorem in a standard way).

Problem 2.2.2. Let us set

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1\}, \tag{6}$$

$$a(u,v) = \int_\Omega \Big(\sum_{i=1}^{2} b_i(\cdot, u, \nabla u) \frac{\partial v}{\partial x_i} + b_0(\cdot, u, \nabla u) v \Big) \,\mathrm{d}x \ \ \forall u, v \in H^1(\Omega), \tag{7}$$

$$L(v) = L^\Omega(v) + L^\Gamma(v) = \int_\Omega v f \,\mathrm{d}x + \int_{\Gamma_2} v q \,\mathrm{d}s. \tag{8}$$

Find $u \in V$ such that

$$a(u,v) = L(v) \ \ \forall v \in V. \tag{9}$$

Lemma 2.2.3. *Let a solution $u \in V$ of Problem* 2.2.2 *satisfy $u \in H^2(\Omega)$. Then relation* (1) *holds almost everywhere in Ω and relation* (3) *holds almost everywhere on Γ_2.*

Proof. The proof is omitted. □

We will solve Problem 2.2.2 approximately by the finite element method. To this end let us approximate Γ_2 by a regular polygon Γ_{2h} with vertices $Q_1, \ldots, Q_n$ such that every segment $Q_i Q_{i+1}$ has no common point with Γ_1. Let the vertices $P_1, \ldots, P_n$ of the polygon Γ_{1h} approximating Γ_1 be obtained in the following way: P_i is the intersection of the segment $S_0 Q_i$ with Γ_1. The symbol Ω_h will denote the polygonal domain with the boundary $\partial\Omega_h$.

We divide each segment $P_i Q_i$ by points $A_1^i, A_2^i, \ldots, A_{m-1}^i$ into m parts of the same length in such a way that we have formally $A_0^i = P_i$, $A_m^i = Q_i$. The points A_j^i are the vertices of quadrilaterals into which the domain Ω_h is divided. We further divide every quadrilateral $A_j^i A_j^{i+1} A_{j+1}^i A_{j+1}^{i+1}$ into two triangles.

We admit to use narrow triangles. This means that we will have

$$\frac{\varrho}{m} \ll h \tag{10}$$

in our considerations, where h is the length of the greatest segment in the partition of Ω_h. The corresponding partition consisting of closed triangles $\overline{T}$ will be denoted by $\mathcal{T}_h$.

2.3. Discrete problem. The discrete problem is now formulated in an almost standard way. (The expression "almost" concerns the approximation of the term $L^\Gamma(v)$.) We define spaces

$$X_h = \{v \in C(\overline{\Omega}_h)\colon\ v\big|_T \text{ is a linear polynomial } \forall T \in \mathcal{T}_h\} \tag{11}$$

and

$$V_h = \{v \in X_h\colon\ v = 0 \text{ on } \Gamma_{1h}\}. \tag{12}$$

We set

$$\widetilde{a}_h(u,v) = \int_{\Omega_h} \Big(\sum_{i=1}^{2} b_i(\cdot,u,\nabla u)\frac{\partial v}{\partial x_i} + b_0(\cdot,u,\nabla u)v\Big)\,\mathrm{d}x \ \ \forall u,v \in H^1(\Omega_h) \tag{13}$$

and

$$\widetilde{L}_h^\Omega(u) = \int_{\Omega_h} uf\,\mathrm{d}x \ \ \forall u \in X_h. \tag{14}$$

To define $\widetilde{L}_h^\Gamma(u)$ is more complicated and we refer to [11].

The symbols $a_h(u,v)$, $L_h^\Omega(u)$ and $L_h^\Gamma(u)$, where $u,v \in X_h$, will denote the approximations of $\widetilde{a}_h(u,v)$, $\widetilde{L}_h^\Omega(u)$ and $\widetilde{L}_h^\Gamma(u)$, respectively, when using numerical integration.

Now we define a finite element discrete problem for the solution of Problem 2.2.2 with the use of numerical integration.

Problem 2.3.1. Find $u_h \in V_h$ such that

$$a_h(u_h,v) = L_h(v) \ \ \forall v \in V_h. \tag{15}$$

3. Discrete Friedrichs' inequality

In order to prove Theorem 4.5 (our abstract error estimate) we must derive discrete Friedrichs' inequality in the case of narrow triangles satisfying the maximum angle condition. Contrary to [9], Lemma 6.1, we prove relation (17) without any restrictive assumption concerning ϱ and h. Such a type of an assumption will appear first in Section 8 (see (76)).

In Section 7 we will analyze the problem where the Dirichlet boundary condition (2) is prescribed on Γ_2 and the Neumann condition (3) on Γ_1. Therefore we will prove discrete Friedrichs' inequality for this case, too.

Notation. We denote

$$\tau_h = \Omega_h - \overline{\Omega}, \quad \omega_h = \Omega - \overline{\Omega}_h. \tag{16}$$

Lemma 3.1. *We have*

$$\|v\|_{1,\Omega_h} \leqslant C|v|_{1,\Omega_h}, \ \forall v \in V_h, \ \forall h < h_0. \tag{17}$$

P r o o f. In the case of the Dirichlet boundary condition (2), the proof is the same as that in [12], Lemma 6.1.a).

In the case of the Dirichlet boundary condition $v = 0$ on Γ_2 we have

$$V_h = \{v \in X_h \colon\ v = 0 \text{ on } \Gamma_{2h}\}$$

and we define the quasinatural extension $\overline{v}$ of $v \in V_h$ by

$$\overline{v} = v \text{ on } \Omega_h, \quad \overline{v} = 0 \text{ on } \omega_h. \tag{18}$$

Friedrichs' inequality gives

$$\|\overline{v}\|^2_{0,\Omega} \leqslant C|\overline{v}|^2_{1,\Omega}. \tag{19}$$

Relations (18) imply

$$|\overline{v}|^2_{1,\Omega} \leqslant |v|^2_{1,\Omega_h}. \tag{20}$$

If we prove

$$\|\overline{v}\|^2_{0,\Omega} \geqslant C\|v\|^2_{0,\Omega_h}, \quad C > 0, \tag{21}$$

then (17) follows from (19)–(21).

Let T be a triangle with vertices P_1, P_2, P_3 where P_1, P_2 lie on Γ_1 and P_3 on the line S_0P_1 in the domain Ω. Let N_1 and N_2 be the midpoints of the sides P_1P_3 and P_2P_3, respectively. Let T^* denote the triangle with the vertices N_1, N_2, P_3. The transformation

$$\begin{aligned} x &= x^*(\xi,\eta) = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta, \\ y &= y^*(\xi,\eta) = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta \end{aligned} \tag{22}$$

maps one-to-one the reference triangle $\overline{T}_0$ with vertices $P_1^*(0,0)$, $P_2^*(1,0)$, $P_3^*(0,1)$ onto the triangle $\overline{T}$. Let $\lambda \subset \Gamma_1$ be the arc which is approximated by the segment

$\lambda_h = P_1P_2$ and let $\mathcal{P}_h$ be the bounded domain with the boundary $\partial\mathcal{P}_h = \lambda \cup \lambda_h$. Then we have

$$\|v\|^2_{0,T-\mathcal{P}_h} \geqslant \|v\|^2_{0,T^*}. \tag{23}$$

Let us assume that we have proved

$$\|v\|^2_{0,T^*} \geqslant \frac{1}{64}\|v\|^2_{0,T}. \tag{24}$$

Then

$$\|v\|^2_{0,T-\mathcal{P}_h} \geqslant \frac{1}{64}\|v\|^2_{0,T}. \tag{25}$$

Hence (21) follows with $C = \frac{1}{64}$.

Now we prove (24). According to the definition, the function $v(x,y)$ is on every triangle $\overline{T}$ such that

$$\widetilde{v}(\xi,\eta) \equiv v\big(x^*(\xi,\eta), y^*(\xi,\eta)\big) = \sum_{i=1}^{3} B_i p_i(\xi,\eta),$$

where

$$p_1(\xi,\eta) = 1-\xi-\eta, \quad p_2(\xi,\eta) = \xi, \quad p_3(\xi,\eta) = \eta$$

and $B_i = v(P_i)$.

The triangle T^* is the image of T_0^* with vertices $N_1^*[0,\frac{1}{2}]$, $N_2^*[\frac{1}{2},\frac{1}{2}]$, $P_3^*[0,1]$ in transformation (22). First we prove that

$$\frac{1}{64}\int_{T_0}\big[\widetilde{v}(\xi,\eta)\big]^2\,\mathrm{d}\xi\,\mathrm{d}\eta \leqslant \int_{T_0^*}\big[\widetilde{v}(\xi,\eta)\big]^2\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{26}$$

Let us express the integrals

$$J_1 = \int_{T_0}\big[\widetilde{v}(\xi,\eta)\big]^2\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1\int_0^{1-\xi}\Big(\sum_{i=1}^{3} B_i p_i(\xi,\eta)\Big)^2\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$J_2 = \int_{T_0^*}\big[\widetilde{v}(\xi,\eta)\big]^2\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^{1/2}\int_{1/2}^{1-\xi}\Big(\sum_{i=1}^{3} B_i p_i(\xi,\eta)\Big)^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$

as quadratic forms of B_1, B_2, B_3. Then

$$\begin{aligned}2304(64J_2 - J_1) = {} & \frac{1}{576}(576B_1 + 288B_2 + 1824B_3)^2\\ & + \frac{1}{432}(432B_2 + 912B_3)^2 + \frac{1664}{3}B_3^2,\end{aligned}$$

from which estimate (26) follows. Estimate (26) gives (24). □

Definition 4.1. We say that the forms $a_h\colon X_h \times X_h \longrightarrow \mathbb{R}^1$, $h \in (0, h_0)$, are uniformly X_h-strongly monotone with respect to the seminorms $|\cdot|_{1,\Omega_h}$, if there exists $\alpha > 0$ such that

$$a_h(v, v-w) - a_h(w, v-w) \geqslant \alpha |v-w|^2_{1,\Omega_h} \quad \forall v, w \in X_h, \ \ \forall h \in (0, h_0). \tag{27}$$

We say that the forms $a_h\colon X_h \times X_h \longrightarrow \mathbb{R}^1$, $h \in (0, h_0)$, are uniformly X_h-Lipschitz continuous, if there exists a constant $C > 0$ such that

$$|a_h(v, z) - a_h(w, z)| \leqslant C \|v-w\|_{1,\Omega_h} \|z\|_{1,\Omega_h} \quad \forall v, w, z \in X_h, \ \ \forall h \in (0, h_0). \tag{28}$$

We say that the forms $\widetilde{a}_h\colon H^1(\Omega_h) \times H^1(\Omega_h) \longrightarrow \mathbb{R}^1$, $h \in (0, h_0)$, are uniformly $H^1(\Omega_h)$-Lipschitz continuous, if there exists a constant $C > 0$ such that

$$|\widetilde{a}_h(v, z) - \widetilde{a}_h(w, z)| \leqslant C \|v-w\|_{1,\Omega_h} \|z\|_{1,\Omega_h} \quad \forall v, w, z \in H^1(\Omega_h), \ \ \forall h \in (0, h_0). \tag{29}$$

Theorem 4.2. *The following implications hold:*

(a) (D) $\Rightarrow$ (27),
(b) (C) $\Rightarrow$ (28),
(c) (C) $\Rightarrow$ (29).

Implications a), b), c) are proved in [4], Theorem 3.1.2.

Remark 4.3. If we combine the discrete form of Friedrichs' inequality (17) with (27), we see that there exists a constant $\alpha > 0$ such that

$$a_h(v, v-w) - a_h(w, v-w) \geqslant \alpha \|v-w\|^2_{1,\Omega_h} \quad \forall v, w \in X_h, \ \ \forall h \in (0, h_0). \tag{30}$$

Lemma 4.4. *Let Γ_0 be the circle with a center S_0 and radius $R_0 = R_1 - \varrho$. Let $\widetilde{\Omega}$ be a bounded domain such that $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_2$. There exists a linear and bounded extension operator $E\colon H^k(\Omega) \to H^k(\widetilde{\Omega})$ such that the constant C appearing in the inequality*

$$\|E(v)\|_{k,\widetilde{\Omega}} \leqslant C\|v\|_{k,\Omega}$$

does not depend on R_1/ϱ. The operator E is also a linear and bounded extension operator from $H^{k-i}(\Omega)$ into $H^{k-i}(\widetilde{\Omega})$ $(1 \leqslant i \leqslant k)$.

Lemma 4.4 is the same as in [12] and follows from the considerations introduced in [8], pp. 20–22.

Theorem 4.5. *Let the forms $a_h(v,w)$ be uniformly X_h-strongly monotone with respect to the seminorms $|\cdot|_{1,\Omega_h}$ (see* (27)*) and uniformly X_h-Lipschitz continuous (see* (28)*). Let the forms $\widetilde{a}_h(v,w)$ be uniformly $H^1(\Omega_h)$-Lipschitz continuous (see* (29)*). Then Problem* 2.3.1 *has a unique solution and for all $h \in (0,h_0)$ we have*

$$\begin{aligned}(31)\quad C^{-1}\|\widetilde{u}-u_h\|_{1,\Omega_h} \leqslant \inf_{v\in V_h}&\Bigg(\|v-\widetilde{u}\|_{1,\Omega_h} + \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|a_h(v,w)-\widetilde{a}_h(v,w)|}{\|w\|_{1,\Omega_h}}\Bigg)\\ &+ \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{\big|\widetilde{L}_h^{\Omega}(w)-L_h^{\Omega}(w)\big|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{\big|\widetilde{L}_h^{\Gamma}(w)-L_h^{\Gamma}(w)\big|}{\|w\|_{1,\Omega_h}}\\ &+ \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{\big|\widetilde{a}_h(\widetilde{u},w)-\widetilde{L}_h(w)\big|}{\|w\|_{1,\Omega_h}},\end{aligned}$$

where C is a positive constant, $u \in H^1(\Omega)$ is the solution of Problem 2.2.2*, $u_h \in V_h$ is the solution of Problem* 2.3.1*, and $\widetilde{u} = E(u)$ with $E\colon H^1(\Omega) \to H^1(\widetilde{\Omega})$.*

P r o o f. The proof follows the same lines as in [9], Theorems 35.4 and 38.5. It should be noted that the use of (30) is essential in this proof. □

5. The interpolation error

Definition 5.1. Let $u \in H^2(\Omega)$. We define $Q_h u \in X_h$ by

$$Q_h u\big|_{\overline{T}\in\mathcal{T}_h} = I_T u = \text{the linear interpolant of } u.$$

Theorem 5.2. *We have*

$$\inf_{v\in V_h} \|v-\widetilde{u}\|_{1,\Omega_h} \leqslant \|Q_h u - \widetilde{u}\|_{1,\Omega_h} \leqslant Ch\|u\|_{1,\Omega}$$

where the constant C is independent of h, u and the triangulation $\mathcal{T}_h$, and $\widetilde{u} \in H^2(\widetilde{\Omega})$.

P r o o f. The proof follows from the definition of $Q_h u$, Lemma 4.4 and the following lemma. □

Lemma 5.3. *Let $u \in H^2(T)$ and let $I_T u$ be the linear polynomial satisfying $(I_T u)(P_i^T) = u(P_i^T)$ $(i = 1,2,3)$ where P_1^T, P_2^T, P_3^T are the vertices of $\overline{T}$. Then*

$$\|u - I_T u\|_{1,T} \leqslant \frac{C}{\sin\gamma_T} h_T \|u\|_{2,T}$$

where γ_T is the maximum angle of T and the constant C does not depend on $\overline{T}$ and u.

P r o o f. For the proof see [5]. □

In this section we estimate the second, third and fourth terms appearing on the right-hand side of (31). These terms express the error of numerical integration. We will use the notation $x = (x_1, x_2)$, $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2$, $\xi = (\xi_1, \xi_2)$ and $\mathrm{d}\xi = \mathrm{d}\xi_1\,\mathrm{d}\xi_2$ in this section. The transformation (22) has the corresponding expression.

In the analysis of the second term from the right-hand side we will use the following two lemmas.

Lemma 6.1. *Let $p \in \mathcal{P}_1(T)$, where T is an arbitrary triangle. Then*

$$\|p\|_{0,\infty,T} \equiv \max_T |p| \leqslant \frac{C}{\sqrt{\operatorname{meas}_2 T}} \|p\|_{0,T} \tag{32}$$

where the constant C does not depend on T and p.

Proof. The proof is the same as the proof of [4], Lemma 2.2.6; it does not depend on the geometry of the triangle and the assertion of Lemma 6.1 holds also for irregular triangles. □

Lemma 6.2. *Let $f \in W^{1,\infty}(T)$ and $p \in \mathcal{P}_1(T)$, where T is an arbitrary triangle. Let the degree of precision d of the formula on the reference triangle T_0*

$$\int_{T_0} F^*(\xi)\,\mathrm{d}\xi \approx \sum_{j=1}^{I} \omega_{T_0,j} F^*(\xi_{T_0,j}), \quad F^* \in C(\overline{T}_0) \tag{33}$$

be $d \geqslant 0$. Then we have

$$\begin{aligned} \Bigl| \int_T fp\,\mathrm{d}x - \operatorname{meas}_2 T \sum_{j=1}^{I} 2\omega_{T_0,j} f(x_{T,j}) p(x_{T,J}) \Bigr| \\ \leqslant C h_T \sqrt{\operatorname{meas}_2 T} \|f\|_{1,\infty,T} \|p\|_{1,T}, \end{aligned} \tag{34}$$

where the constant C does not depend on T and p.

Proof. The assertion (34) is a special case of [1], Theorem 4.1.5 for $k = 1$ and $q = \infty$. Nevertheless, we present the proof which is methodically different and simpler then Ciarlet's proof. We remind that

$$x_{T,j} = \bigl[x_1^*(\xi_{T_0,j}), x_2^*(\xi_{T_0,j})\bigr], \tag{35}$$

where $\xi_{T_0,j} = [\xi_{1_{T_0,j}}, \xi_{2_{T_0,j}}]$ $(j = 1, \ldots, I)$ are the nodal points of formula (33). We say that a quadrature formula on T has degree of precision d if it computes exactly every integral $\int_T v \,\mathrm{d}x$ with $v \in P_2(d)$. J_T is the Jacobian of the transformation (22). As $|J_T| = 2 \operatorname{meas}_2 T$ we obtain from (33), (35) by means of the Change of Variable Theorem

$$\int_T F(x_1, x_2)\,\mathrm{d}x \approx 2 \operatorname{meas}_2 T \sum_{j=1}^{I} \omega_{T_0,j} F(x_{T,j}).$$

We will denote

$$E_T(w) = \int_T w(x)\,\mathrm{d}x - \sum_{j=1}^{I} |J_T| \omega_{T_0,j} w(x_{T,j}), \tag{36}$$

$$E_{T_0}(\varphi) = \int_{T_0} \varphi(\xi)\,\mathrm{d}\xi - \sum_{j=1}^{I} \omega_{T_0,j} \varphi(\xi_{T_0,j}), \tag{37}$$

$$F^*(\xi_1, \xi_2) = F\big((x_1^*(\xi_1, \xi_2), x_2^*(\xi_1, \xi_2)\big).$$

Hence, by means of the Change of Variable Theorem,

$$E_T(F) = E_{T_0}(F^*|J_T|) = |J_T| E_{T_0}(F^*). \tag{38}$$

By (38),

$$|E_T(\omega f)| = |J_T|\,|E_{T_0}(f^* w^*)|. \tag{39}$$

Using (37) we find

$$|E_{T_0}(f^* w^*)| \leqslant \Big(\frac{1}{2} + \sum_{j=1}^{I} |\omega_{T_0,j}|\Big) \|f^* w^*\|_{0,\infty,T_0} \leqslant C \|f^* w^*\|_{1,\infty,T_0}. \tag{40}$$

The assumption of Lemma 6.2 concerning $d = 0$ implies $E_T(F) = 0$ for all $F \in \mathcal{P}_0(T)$. This fact and (38) yield

$$E_{T_0}(F^*) = 0 \;\; \forall F^* \in \mathcal{P}_0(T_0). \tag{41}$$

Relation (40) expresses the boundedness of the functional E_{T_0} on $W^{1,\infty}(T_0)$. This fact, linearity of E_{T_0} and (41) imply, according to the Bramble-Hilbert lemma (see [9], Theorem 9.3),

$$|E_{T_0}(f^* w^*)| \leqslant C |f^* w^*|_{1,\infty,T_0}. \tag{42}$$

The rule on differentiation of a product yields (see also [9], Lemma $\mathcal{P}$.64)

$$|f^* w^*|_{1,\infty,T_0} \leqslant |f^*|_{1,\infty,T_0} |w^*|_{0,\infty,T_0} + |f^*|_{0,\infty,T_0} |w^*|_{1,\infty,T_0}. \tag{43}$$

As estimates [9], (9.5) hold for all triangles—they do not depend on the geometry of the triangle, only on the linearity of the transformation $x_1 = x_1(\xi_1, \xi_2)$, $x_2 = x_2(\xi_1, \xi_2)$—we can write

$$\left|\frac{\partial x_j}{\partial \xi_i}(\xi_1, \xi_2)\right| \leqslant h_T \quad (i, j = 1, 2) \tag{44}$$

and obtain by means of the theorem on differentiation of a composite function

$$|f^*|_{1,\infty,T_0} \leqslant C h_T |f|_{1,\infty,T}. \tag{45}$$

It is obvious that

$$|f^*|_{0,\infty,T_0} = |f|_{0,\infty,T}. \tag{46}$$

By [9], (11.36)

$$|w^*|_{0,\infty,T_0} \leqslant C|w^*|_{0,T_0} \ \ \forall w^* \in \mathcal{P}_1(T_0), \tag{47}$$

$$|w^*|_{1,\infty,T_0} \leqslant C|w^*|_{1,T_0} \ \ \forall w^* \in \mathcal{P}_1(T_0). \tag{48}$$

The Change of Variable Theorem implies

$$\|w^*\|_{0,T_0} = |J_T|^{-1/2} \|w\|_{0,T} \tag{49}$$

and (47) yields

$$|w^*|_{0,\infty,T_0} \leqslant \frac{C}{\sqrt{\operatorname{meas}_2 T}} |w|_{0,T} \ \ \forall w \in \mathcal{P}_1(T). \tag{50}$$

We obtain easily from (44) and (48) by means of the Change of Variable Theorem

$$|w^*|_{1,\infty,T_0} \leqslant \frac{C h_T}{\sqrt{\operatorname{meas}_2 T}} |w|_{1,T} \ \ \forall w \in \mathcal{P}_1(T). \tag{51}$$

Combining (39), (42), (43), (45), (46), (50), (51) we obtain

$$|E_T(wf)| \leqslant C h_T \sqrt{\operatorname{meas}_2 T} \|f\|_{1,\infty,T} \|w\|_{1,T}, \tag{52}$$

which is (34) written in another form. □

Theorem 6.3. *In the case $d \geqslant 0$ we have*

(53) $$|\widetilde{a}_h(v,w) - a_h(v,w)| \leqslant Ch(\|v\|_{1,\Omega_h} + 1)\|w\|_{1,\Omega_h} \ \ \forall v, w \in X_h, \ \ \forall h \in (0, h_0),$$

where the constant C is independent of h, v, w.

P r o o f. For $v, w \in X_h$ we can write

(54) $$\widetilde{a}_h(v,w) - a_h(v,w) = I_1 + I_2,$$

where

$$I_1 = \sum_{T\in\mathcal{T}_h}\sum_{i=1}^{2}\frac{\partial w}{\partial x_i}\Big|_T\Big\{\int_T b_i(\cdot, v, \nabla v\big|_T)\,\mathrm{d}x - \operatorname{meas}_2 T\sum_{j=1}^{k_T} 2\omega_{T_0,j} b_i\big(x_{T,j}, v(x_{T,j}), \nabla v\big|_T\big)\Big\}$$

and

$$I_2 = \sum_{T\in\mathcal{T}_h}\Big\{\int_T b_0(\cdot, v, \nabla v\big|_T) w\,\mathrm{d}x - \operatorname{meas}_2 T\sum_{j=1}^{k_T} 2\omega_{T_0,j} b_0\big(x_{T,j}, v(x_{T,j}), \nabla v\big|_T\big) w(x_{T,j})\Big\}.$$

We can estimate I_1 and I_2 in the same way as in the proof of [4], Theorem 2.2.7. Instead of [4], Lemma 2.2.5 and 2.2.6 we use Lemma 6.2 and Lemma 6.1, respectively. □

Theorem 6.4. *Let the degree of precision of quadrature formulas be $d \geqslant 0$. Then we have*

(55) $$\sup_{\substack{w\in V_h\\ w\neq 0}} \frac{\big|\widetilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)\big|}{\|w\|_{1,\Omega_h}} \leqslant Ch\|f\|_{1,\infty,\widetilde{\Omega}}.$$

The theorem is a consequence of Lemma 6.2.

When considering the line integrals we need also the trace inequalities which are introduced in the following lemma.

Lemma 6.5. *We have*

(56) $$\|v\|_{0,\partial\Omega} \leqslant \frac{C}{\sqrt{\varrho}}\|v\|_{1,\Omega} \ \ \forall v \in H^1(\Omega),$$

(57) $$\|v\|_{0,\partial\Omega_h} \leqslant \frac{C}{\sqrt{\varrho}}\|v\|_{1,\Omega_h} \ \ \forall v \in H^1(\Omega_h)$$

where the constant C does not depend on v, h and ϱ.

The proofs of (56) and (57) are similar to [7], pp. 15–16.

Theorem 6.6. *Let the degree of precision of quadrature formulas be $d = 2$. Then we have*

$$\sup_{\substack{w\in V_h\\ w\neq 0}} \frac{\bigl|\widetilde{L}_h^{\Gamma}(w) - L_h^{\Gamma}(w)\bigr|}{\|w\|_{1,\Omega_h}} \leqslant \frac{C}{\sqrt{\varrho}}h^2 \tag{58}$$

where the constant C does not depend on q, ϱ and h.

The proof can be obtain by combining the ideas of [4] with the proof of [11], Theorem 22.

7. The error of the approximation of the boundary

Notation. Let $w \in X_h$. The symbol $\overline{w}$ is called the natural extension of w and denotes the function $\overline{w}\colon \overline{\Omega}_h \to \mathbb{R}^1$ such that $\overline{w} = w$ on Ω_h and

$$\overline{w}\big|_{\overline{T}^{id}-\overline{T}} = p\big|_{\overline{T}^{id}-\overline{T}}$$

where $p \in \mathcal{P}_1$ satisfies $p\big|_{\overline{T}} = w\big|_{\overline{T}}$. $\overline{T}^{id} \subset \Omega$ is a curved triangle which is approximated by $\overline{T}$. (The symbol $\overline{T}^{id}$ denotes an "ideal triangle".)

Lemma 7.1. *Let $u \in H^2(\Omega)$. Then for $w \in V_h$ we have*

$$\begin{aligned} \bigl|\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)\bigr| \leqslant \bigl|L^{\Gamma}(\overline{w}) - \widetilde{L}_h^{\Gamma}(w)\bigr| &+ \left|\int_{\omega_h} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u)\frac{\partial \overline{w}}{\partial x_i}\,\mathrm{d}x\right| \\ &+ \left|\int_{\omega_h} \sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, u, \nabla u)\overline{w}\,\mathrm{d}x\right| \\ &+ \left|\int_{\tau_h} \left(\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla\widetilde{u}) - b_0(\cdot, \widetilde{u}, \nabla\widetilde{u}) + f\right) w\,\mathrm{d}x\right|, \end{aligned} \tag{59}$$

where $\widetilde{u} = E(u)$ is the extension of u in the sense of Lemma 4.4.

P r o o f. In the proof we use a modification of the trick with the use of Green's theorem introduced in [3], Theorem 3.2.5. By the definitions of $\widetilde{a}_h(\widetilde{u}, w)$ and $\widetilde{L}_h(w)$ we have

$$\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = \int_{\Omega_h} \left(\sum_{i=1}^{2} b_i(\cdot, \widetilde{u}, \nabla\widetilde{u})\frac{\partial w}{\partial x_i} + b_0(\cdot, \widetilde{u}, \nabla\widetilde{u})w\right) \mathrm{d}x - \widetilde{L}_h^{\Omega}(w) - \widetilde{L}_h^{\Gamma}(w).$$

Using Green's theorem and the fact that $w \in V_h$ we obtain

$$\begin{aligned}\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = & \int_{\partial\Omega_h} \sum_{i=1}^{2} w b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) n_i(\Omega_h)\,\mathrm{d}s - \int_{\Omega_h} \sum_{i=1}^{2} w \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla\widetilde{u})\,\mathrm{d}x \\ & + \int_{\Omega_h} b_0(\cdot, \widetilde{u}, \nabla\widetilde{u}) w\,\mathrm{d}x - \int_{\Omega_h} wf\,\mathrm{d}x - \widetilde{L}_h^{\Gamma}(w) \\ = & \int_{\Gamma_{2h}} \sum_{i=1}^{2} b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) n_i(\Omega_h) w\,\mathrm{d}s \\ & - \int_{\Omega_h} \Big(\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla\widetilde{u}) + f \Big) w\,\mathrm{d}x \\ & + \int_{\Omega_h} b_0(\cdot, \widetilde{u}, \nabla\widetilde{u}) w\,\mathrm{d}x - \widetilde{L}_h^{\Gamma}(w).\end{aligned}$$

To the right-hand side let us add zero in the form

$$-\int_{\Gamma_2} \sum_{i=1}^{2} b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) n_i(\Omega) \overline{w}\,\mathrm{d}s + L^{\Gamma}(\overline{w}) = 0.$$

Then

$$\begin{aligned}\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = & \int_{\Gamma_{2h}} \sum_{i=1}^{2} b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) n_i(\Omega_h) w\,\mathrm{d}s - \int_{\Gamma_2} \sum_{i=1}^{2} b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) n_i(\Omega) \overline{w}\,\mathrm{d}s \\ & - \int_{\Omega_h} \Big(\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla\widetilde{u}) - b_0(\cdot, \widetilde{u}, \nabla\widetilde{u}) + f \Big) w\,\mathrm{d}x \\ & - \widetilde{L}_h^{\Gamma}(w) + L^{\Gamma}(\overline{w}).\end{aligned}$$

If we denote $\triangle = \overline{T}^{id} - T$ and use Lemma 2.2.3 then we can write

$$\begin{aligned}\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) = & -\sum_{\triangle \subset \omega_h} \int_{\partial\triangle} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u) n_i(\triangle) \overline{w}\,\mathrm{d}s \\ & - \int_{\tau_h} \Big(\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla\widetilde{u}) - b_0(\cdot, \widetilde{u}, \nabla\widetilde{u}) + f \Big) w\,\mathrm{d}x \\ & - \widetilde{L}_h^{\Gamma}(w) + L^{\Gamma}(\overline{w}).\end{aligned}$$

Transforming the first term on the right-hand side by means of Green's theorem we obtain

$$\int_{\partial\triangle} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u) n_i(\triangle) \overline{w}\,\mathrm{d}s = \int_{\triangle} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u) \frac{\partial \overline{w}}{\partial x_i}\,\mathrm{d}x + \int_{\triangle} \sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, u, \nabla u) \overline{w}\,\mathrm{d}x.$$

These results give (59). □

Lemma 7.2. *Let* (2) *hold. Then*

$$\|v\|_{0,\omega_h} \leqslant Ch(\|v\|_{0,\Gamma_2} + h|v|_{1,\omega_h}) \quad \text{for } v \in H^1(\Omega), \tag{60}$$

and

$$|\overline{w}|_{1,\omega_h} \leqslant Ch\sqrt{\frac{m}{\varrho}}|w|_{1,\Omega_h}, \tag{61}$$

$$\|\overline{w}\|_{0,\omega_h} \leqslant Ch(\|w\|_{0,\Gamma_{2h}} + h|\overline{w}|_{1,\omega_h}) \leqslant Ch\left(\frac{1}{\sqrt{\varrho}} + h^2\sqrt{\frac{m}{\varrho}}\right)\|w\|_{1,\Omega_h}, \tag{62}$$

$$\|w\|_{0,\tau_h} \leqslant Ch(\|w\|_{0,\Gamma_{1h}} + h|w|_{1,\tau_h}) = Ch^2|w|_{1,\tau_h} \tag{63}$$

for $w \in V_h$ *with* $\overline{w}$ *defined in Notation* 7.1.

P r o o f. For the proof see [11]. □

Lemma 7.3. *Let* $u \in H^2(\Omega)$. *Then*

$$\left|\int_{\omega_h}\sum_{i=1}^{2}\frac{\partial b_i}{\partial x_i}(\cdot,u,\nabla u)\overline{w}\,\mathrm{d}x\right| \leqslant Ch^2\left(\frac{1}{\sqrt{\varrho}}+h^2\sqrt{\frac{m}{\varrho}}\right)\left(1+\frac{1}{\sqrt{\varrho}}\|u\|_{2,\Omega}\right)\|w\|_{1,\Omega_h}. \tag{64}$$

P r o o f. We have

$$\left|\int_{\omega_h}\sum_{i=1}^{2}\frac{\partial b_i}{\partial x_i}(\cdot,u,\nabla u)\overline{w}\,\mathrm{d}x\right| \leqslant \sum_{i=1}^{2}\sqrt{\int_{\omega_h}\left(\frac{\partial b_i}{\partial x_i}\right)^2(\cdot,u,\nabla u)\,\mathrm{d}x}\;\|\overline{w}\|_{0,\omega_h}. \tag{65}$$

By assumption (E) we have

$$\left|\frac{\partial b_i}{\partial x_i}(\cdot,u,\nabla u)\right| \leqslant C\left(1+|u|+\left|\frac{\partial u}{\partial x_1}\right|+\left|\frac{\partial u}{\partial x_2}\right|\right), \quad i=0,1,2.$$

Due to the inequality

$$\sqrt{\operatorname{meas}_2\omega_h} \leqslant Ch, \tag{66}$$

this relation yields

$$\begin{aligned}\sqrt{\int_{\omega_h}\left(\frac{\partial b_i}{\partial x_i}\right)^2(\cdot,u,\nabla u)\,\mathrm{d}x} &\leqslant \sqrt{\int_{\omega_h} C\left(1+|u|+\left|\frac{\partial u}{\partial x_1}\right|+\left|\frac{\partial u}{\partial x_2}\right|\right)^2\,\mathrm{d}x}\\ &\leqslant C(\sqrt{\operatorname{meas}_2\omega_h}+\|u\|_{0,\omega_h}+|u|_{1,\omega_h})\\ &\leqslant C(h+\|u\|_{1,\omega_h}).\end{aligned} \tag{67}$$

As $u \in H^2(\Omega)$, by (56) and (60) we obtain

$$\|u\|_{1,\omega_h} \leqslant C\frac{h}{\sqrt{\varrho}}\|u\|_{2,\Omega}. \tag{68}$$

Combining (65) with (62) and (67), (68) we easily derive (64). □

Lemma 7.4. *Let $u \in H^2(\Omega)$. Then*

$$\left|\int_{\omega_h} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u) \frac{\partial \overline{w}}{\partial x_i} \,\mathrm{d}x\right| \leqslant Ch^2 \sqrt{\frac{m}{\varrho}} \left(1 + \frac{1}{\sqrt{\varrho}} \|u\|_{2,\Omega}\right) \|w\|_{1,\Omega_h}, \quad w \in V_h. \tag{69}$$

If in addition

$$u \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \tag{70}$$

then

$$\left|\int_{\omega_h} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u) \frac{\partial \overline{w}}{\partial x_i} \,\mathrm{d}x\right| \leqslant Ch^2 \sqrt{\frac{m}{\varrho}} (1 + |u|_{1,\infty,\Omega}) \|w\|_{1,\Omega_h}, \quad w \in V_h. \tag{71}$$

P r o o f. We have

$$\left|\int_{\omega_h} \sum_{i=1}^{2} b_i(\cdot, u, \nabla u) \frac{\partial \overline{w}}{\partial x_i} \,\mathrm{d}x\right| \leqslant \sum_{i=1}^{2} \sqrt{\int_{\omega_h} b_i^2(\cdot, u, \nabla u) \,\mathrm{d}x} |\overline{w}|_{1,\omega_h}. \tag{72}$$

If we use assumption (B) we obtain

$$\sqrt{\int_{\omega_h} b_i^2(\cdot, u, \nabla u) \,\mathrm{d}x} \leqslant C(h + \|u\|_{1,\omega_h}). \tag{73}$$

This result together with (68), (72) and (61) implies (69).

Assumption (70) and inequality (66) give

$$\|u\|_{1,\omega_h} \leqslant Ch \|u\|_{1,\infty,\Omega}. \tag{74}$$

From this and the preceding part of the proof we obtain (71). □

Lemma 7.5. *For $w \in V_h$ and $u \in H^2(\Omega)$ we have*

$$\left|\int_{\tau_h} \left(\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla \widetilde{u}) - b_0(\cdot, \widetilde{u}, \nabla \widetilde{u}) + f\right) w \,\mathrm{d}x\right| \leqslant Ch^2 \big(\|\widetilde{A}\widetilde{u}\|_{0,\widetilde{\Omega}} + \|f\|_{0,\widetilde{\Omega}}\big) \|w\|_{1,\Omega_h}$$

where

$$\widetilde{A}\widetilde{u} := -\sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}(\cdot, \widetilde{u}, \nabla \widetilde{u}) + b_0(\cdot, \widetilde{u} \nabla \widetilde{u}).$$

P r o o f. Owing to the assumption that $w \in V_h$ the assertion follows from estimate (63). □

Lemma 7.6. *We have*

$$\big|L^{\Gamma}(\overline{w}) - \widetilde{L}_h^{\Gamma} w\big| \leqslant Ch^2\sqrt{\frac{m}{\varrho}}\|q\|_{0,\Gamma_2}\|w\|_{1,\Omega_h} \qquad w \in V_h. \tag{75}$$

P r o o f. For the proof see [11], Lemma 29. □

8. The final estimate

In this section we use the assumption

$$C_1 h^2 \leqslant \frac{\varrho}{m}, \quad C_1 > 0. \tag{76}$$

The preceding results then yield the following theorem:

Theorem 8.1. *Let $u \in H^2(\Omega)$, $f \in W^{1,\infty}(\widetilde{\Omega})$. Let assumption* (76) *and the assumptions concerning the degree of precision of the quadrature formulas (see Theorems* 6.3, 6.4 *and* 6.6*) be satisfied. Then*

$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \leqslant \frac{C}{\sqrt{\varrho}}\,h \tag{77}$$

where the constant C does not depend on ϱ, m, h and the triangulation $\mathcal{T}_h$.

If in addition condition (70) *is satisfied then*

$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \leqslant Ch \tag{78}$$

where again the constant C does not depend on ϱ, m, h and the triangulation $\mathcal{T}_h$.

9. The case of opposite boundary conditions

We will analyze the boundary value problem for equation (1) with boundary conditions opposite to conditions (2) and (3):

$$u = 0 \quad \text{on } \Gamma_2, \tag{79}$$

$$\sum_{i=1}^{2} b_i(\cdot, u, \nabla u) n_i(\Omega) = q \quad \text{on } \Gamma_1. \tag{80}$$

Problem 2.3.1 and all results up to relation (16) remain without changes except for Lemma 2.2.3, where (3) is replaced by (80), and

$$V_h = \{v \in X_h \colon v = 0 \text{ on } \Gamma_{2h}\} \tag{81}$$

is substituted for relation (12).

The natural extension $\overline{w} \colon \overline{\Omega}_h \cup \overline{\Omega} \to \mathbb{R}^1$ of w is now defined by

$$\overline{w} = w \text{ on } \overline{\Omega}_h, \quad \overline{w} = 0 \text{ on } \omega_h.$$

We will use again assumption (76). Lemma 7.1 is replaced by the following lemma:

Lemma 9.1. *For $w \in V_h$ we have*

$$\begin{aligned} \left|\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)\right| \leqslant \left|L^\Gamma(\overline{w}) - \widetilde{L}_h^\Gamma(w)\right| + \left|\int_{\tau_h} \sum_{i=1}^2 b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) \frac{\partial w}{\partial x_i} \, \mathrm{d}x\right| \\ + \left|\int_{\tau_h} b_0(\cdot, \widetilde{u}, \nabla\widetilde{u}) w \, \mathrm{d}x\right| + \left|\int_{\tau_h} f w \, \mathrm{d}x\right|. \end{aligned} \tag{82}$$

P r o o f. The proof is a simple modification of the proof of Lemma 7.1. The changes are small: Γ_2 and Γ_{2h} are replaced by Γ_1 and Γ_{1h}, respectively, and ω_h by τ_h. □

Now we estimate the terms appearing on the right-hand side of (82).

Lemma 9.2. *Let $u \in H^2(\Omega)$ and let assumption (76) be satisfied. Then*

$$\left|\int_{\tau_h} \sum_{i=1}^2 b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) \frac{\partial w}{\partial x_i} \, \mathrm{d}x\right| \leqslant Ch \left(1 + \frac{1}{\sqrt{\varrho}} \|u\|_{2,\Omega}\right) \|w\|_{1,\Omega_h} \quad w \in V_h. \tag{83}$$

If in addition

$$\widetilde{u} \in W^{1,\infty}(\widetilde{\Omega}) \tag{84}$$

then

$$\left|\int_{\tau_h} \sum_{i=1}^2 b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) \frac{\partial w}{\partial x_i} \, \mathrm{d}x\right| \leqslant Ch\big(1 + \|\widetilde{u}\|_{1,\infty,\widetilde{\Omega}}\big) \|w\|_{1,\Omega_h} \quad w \in V_h. \tag{85}$$

P r o o f. We have

$$\left|\int_{\tau_h} \sum_{i=1}^2 b_i(\cdot, \widetilde{u}, \nabla\widetilde{u}) \frac{\partial w}{\partial x_i} \, \mathrm{d}x\right| \leqslant \sum_{i=1}^2 \sqrt{\int_{\tau_h} b_i^2(\cdot, \widetilde{u}, \nabla\widetilde{u}) \, \mathrm{d}x} |w|_{1,\tau_h}.$$

By assumption (B) we have

$$\sqrt{\int_{\tau_h} b_i^2(\cdot,\widetilde{u},\nabla\widetilde{u})\,\mathrm{d}x} \leqslant C(h+\|\widetilde{u}\|_{1,\tau_h}). \tag{86}$$

If we use a relation analogous to (60) with τ_h instead of ω_h, by the trace inequality (56) and Lemma 4.4 we obtain

$$\|\widetilde{u}\|_{1,\tau_h} \leqslant C\frac{h}{\sqrt{\varrho}}\|u\|_{2,\Omega}. \tag{87}$$

These results together with a relation analogous to (61) give (83).

Assumption (84) implies that

$$\|\widetilde{u}\|_{1,\tau_h} \leqslant Ch\|\widetilde{u}\|_{1,\infty,\widetilde{\Omega}}. \tag{88}$$

From here we obtain (85). □

Lemma 9.3. *Let $f\in W^{1,\infty}(\widetilde{\Omega})$. Then*

$$\left|\int_{\tau_h} fw\,\mathrm{d}x\right| \leqslant Ch\|f\|_{0,\infty,\widetilde{\Omega}}\|w\|_{1,\Omega_h} \quad w\in V_h. \tag{89}$$

P r o o f. The assertion follows from $\|f\|_{0,\tau_h} \leqslant Ch\|f\|_{0,\infty,\widetilde{\Omega}}$. □

Lemma 9.4. *Let $u\in H^2(\Omega)$ and let assumption (76) be satisfied. Then for $w\in V_h$ we have*

$$\left|\int_{\tau_h} b_0(\cdot,\widetilde{u},\nabla\widetilde{u})w\,\mathrm{d}x\right| \leqslant C\frac{h^2}{\sqrt{\varrho}}\left(1+\frac{1}{\sqrt{\varrho}}\|u\|_{2,\Omega}\right)\|w\|_{1,\Omega_h}. \tag{90}$$

If in addition

$$\widetilde{u}\in W^{1,\infty}(\widetilde{\Omega}) \tag{91}$$

then

$$\left|\int_{\tau_h} b_0(\cdot,\widetilde{u},\nabla\widetilde{u})w\,\mathrm{d}x\right| \leqslant C\frac{h^2}{\sqrt{\varrho}}(1+\|\widetilde{u}\|_{1,\infty,\widetilde{\Omega}})\|w\|_{1,\Omega_h}. \tag{92}$$

P r o o f. We have

$$\left|\int_{\tau_h} b_0(\cdot,\widetilde{u},\nabla\widetilde{u})w\,\mathrm{d}x\right| \leqslant \sqrt{\int_{\tau_h} b_0^2(\cdot,\widetilde{u},\nabla\widetilde{u})\,\mathrm{d}x}|w|_{0,\tau_h}. \tag{93}$$

If we use assumption (B) we obtain

$$\sqrt{\int_{\tau_h} b_0^2(\cdot, \widetilde{u}, \nabla \widetilde{u}) \, \mathrm{d}x} \leqslant C(h + \|\widetilde{u}\|_{1,\tau_h}).$$

By (63), trace inequality (57) and a relation analogous to (61) we obtain

$$\|w\|_{0,\tau_h} \leqslant C \frac{h}{\sqrt{\varrho}} \|w\|_{1,\Omega_h}.$$

This result together with Lemma 4.4, (87) and (93) gives (90).

If we use assumption (91) we obtain inequality (88). From this we obtain (92). □

In the case of (79) and (80) the preceding results yield the following final theorem:

Theorem 9.5. *Let the assumptions of Theorem* 8.1 *be satisfied except for the additional assumption* (70) *which is replaced by* (84). *Then estimates* (77) *and* (78) *are again valid.*

10. General convergence theorem

In this section we will assume that $u \in H^1(\Omega)$ only and we will prove the convergence under a stronger assumption than (76), namely

(94) $$C_1 h^{2-\delta} \leqslant \frac{\varrho}{m} \leqslant C_2 h^{2-\delta},$$

where

(95) $$0 < \delta < 1$$

is a given number, which can be arbitrary small, and C_1, C_2 are positive constants. The lack of reguarality of $u \in H^1(\Omega)$ is usually a consequence of the fact that the Dirichlet condition is prescribed only on a part of Γ_1 or Γ_2 (and the Neumann condition is considered on the rest of Γ_1 or Γ_2).

The first term on the right-hand side of (31) is estimated by [12], Lemma 5.11 and Theorem 5.1:

Theorem 10.1. *We have*

(96) $$\lim_{h \to 0} \Big(\inf_{v \in V_h} \|v - \widetilde{u}\|_{1,\Omega_h} \Big) = 0.$$

The estimate of the second term can be obtained in the same way as in [9], Theorem 38.7.:

Theorem 10.2. *For all* $h \in (0, h_0)$ *we have*

$$\inf_{v\in V_h} \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|a_h(v,w) - \widetilde{a}_h(v,w)|}{\|w\|_{1,\Omega_h}} \leqslant Ch(1 + \|u\|_{1,\Omega}). \tag{97}$$

The third and fourth terms appearing on the right-hand side of (31) are estimated in Theorems 6.4 and 6.6. The remaining part of this section is devoted to estimating the fifth term on the right-hand side of (31).

Notation.

a) The symbol M_h denotes the set of ideal triangles $T^{id} \in \mathcal{T}_h^{id}$ lying along the part of $\partial\Omega$ where the homogeneous Dirichlet condition is prescribed.

b) The function $\widehat{w} \in H^1(\Omega)$ is said to be associated with a given function $w \in X_h$ if

(i) $\widehat{w} \in C(\overline{\Omega})$;

(ii) $\widehat{w}(P_i) = w(P_i)$ at all nodal points P_i of $\mathcal{T}_h$;

(iii) $\widehat{w}$ is linear on each triangle $\overline{T} \in (\mathcal{T}_h \cap \mathcal{T}_h^{id})$ and on each triangle $\overline{T}^{id} \in \mathcal{T}_h^{id}$ such that $\overline{T}^{id} \notin M_h$;

(iv) if $\overline{T}^{id} \in M_h$ then $\widehat{w}\big|_{T^{id}} = \widetilde{w}\big|_{T^{id}}$, where $\widetilde{w}$ is the simplest Zlámal's ideal triangular C^0-element uniquely determined by the values $w(P_1^T)$, $w(P_2^T)$, $w(P_3^T)$, P_1^T, P_2^T, P_3^T being the local notation of the vertices of T. (See [13] and also [9], p. 257.)

The following lemma can be obtained in the same way as in [4], Theorem 3.3.10:

Lemma 10.3. *For all* $w \in V_h$ *we have*

$$\begin{aligned} &\big|\widetilde{L}_h(w) - \widetilde{a}_h(\widetilde{u}, w)\big| \\ &\quad\leqslant \big|\widetilde{L}_h^{\Gamma}(w) - L^{\Gamma}(\overline{w})\big| \\ &\qquad + \sum_{T^{id}\in M_h} \left| \int_{T^{id}} \left[\sum_{i=0}^{2} b_i(\cdot, u, \nabla u) \frac{\partial(\widehat{w} - w)}{\partial x_i} - f(\widehat{w} - w) \right] \mathrm{d}x \right| \\ &\qquad + \left| \int_{\omega_h} \left[\sum_{i=0}^{2} b_i(\cdot, u, \nabla u) \frac{\partial \overline{w}}{\partial x_i} - f\overline{w} \right] \mathrm{d}x \right| \\ &\qquad + \left| \int_{\tau_h} \left[\sum_{i=0}^{2} b_i(\cdot, \widetilde{u}, \nabla \widetilde{u}) \frac{\partial w}{\partial x_i} - fw \right] \mathrm{d}x \right|. \end{aligned} \tag{98}$$

The following theorem is a generalization of [12], Theorem 5.10.

Theorem 10.4. *We have*

$$\left|\widetilde{L}_h(w) - \widetilde{a}_h(\widetilde{u}, w)\right| \leqslant Ch^{\delta/2}\|w\|_{1,\Omega_h} \quad \forall w \in V_h, \tag{99}$$

where the constant C does not depend on h and w.

P r o o f. Let us denote by the symbol $\Gamma_{j,D}$ for $j = 1, 2$ the part of Γ_j on which the homogeneous Dirichlet boundary condition is prescribed. Let B_h^j be the union of triangles of $\mathcal{T}_h$ lying along $\Gamma_{j,D}$.
Let us denote the terms appearing on the right-hand side of (98) by $D_1, \ldots, D_4$. By Lemma 7.7 and assumption (94) we have

$$D_1 \leqslant Ch\|q\|_{0,\partial\Omega\setminus(\Gamma_{1,D}\cup\Gamma_{2,D})}\|w\|_{1,\Omega_h}. \tag{100}$$

Now we estimate D_2. By the Cauchy inequality and assumption (B) we have

$$\begin{aligned} D_2 \leqslant\ & C(1 + \|u\|_{1,B_h^1\setminus\tau_h} + \|u\|_{1,B_h^2\cup\omega_h} + \|f\|_{0,B_h^1\setminus\tau_h} + \|f\|_{0,B_h^2\cup\omega_h}) \\ & \times \Big(\sum_{T^{id}\in M_h} \|\widehat{w} - \overline{w}\|_{1,T^{id}}^2\Big)^{1/2}. \end{aligned} \tag{101}$$

By [12], Theorem 5.5 we have

$$\|u - u_I\|_{0,T^{id}} \leqslant Ch^2\|u\|_{2,T^{id}}, \qquad |u - u_I|_{1,T^{id}} \leqslant Ch^{\delta}\|u\|_{2,T^{id}}$$

where u_I is the simplest ideal triangular finite C^0-element interpolating $u \in H^2(T^{id})$. Setting here $u = \overline{w}$ and thus $u_I = \widehat{w}$ we find

$$\sum_{T^{id}\in M_h} \|\widehat{w} - \overline{w}\|_{1,T^{id}}^2 \leqslant Ch^{2\delta} \sum_{T^{id}\in M_h} \|\overline{w}\|_{1,T^{id}}^2 \leqslant Ch^{2\delta}(\|w\|_{1,\Omega_h}^2 + \|\overline{w}\|_{1,\omega_h}^2). \tag{102}$$

By (61), (62) and (94) we obtain

$$\|\overline{w}\|_{1,\omega_h}^2 \leqslant Ch^{\delta}\|w\|_{1,\Omega_h}^2 \tag{103}$$

and

$$\sum_{T^{id}\in M_h} \|\widehat{w} - \overline{w}\|_{1,T^{id}}^2 \leqslant Ch^{2\delta}\|w\|_{1,\Omega_h}^2. \tag{104}$$

According to (101),

$$D_2 \leqslant Ch^{\delta}\|w\|_{1,\Omega_h}. \tag{105}$$

As to the estimate of D_3 we use the Cauchy inequality and assumption (B) and obtain

$$D_3 \leqslant C\big(1 + \|\widetilde{u}\|_{1,\widetilde{\Omega}} + \|f\|_{0,\infty,\widetilde{\Omega}}\sqrt{\operatorname{meas}_2 \omega_h}\big)|\overline{w}|_{1,\omega_h}. \tag{106}$$

Due to (103), (106) we find that

$$D_3 \leqslant Ch^{\delta/2}\|w\|_{1,\Omega_h}. \tag{107}$$

Similarly,

$$D_4 \leqslant Ch^{\delta/2}\|w\|_{1,\Omega_h}. \tag{108}$$

Relations (100), (105), (107), (108) together with Lemma 10.3 yield estimate (99). □

Thus, using the preceding results we obtain

Theorem 10.5. *Let assumptions* (A)–(E) *as well as the assumptions concerning the degrees of precision of quadrature formulas on a triangle and its side (see Theorems* 6.4 *and* 6.6*) be satisfied. Then*

$$\lim_{h\to 0} \|\widetilde{u} - u_h\|_{1,\Omega_h} = 0, \tag{109}$$

where u_h *is the solution of Problem* 2.3.1, $u \in H^1(\Omega)$ *is the solution of Problem* 2.2.2, *and* $\widetilde{u} = E(u) \in H^1(\widetilde{\Omega})$ *is its extension in the sense of Lemma* 4.4 *with* $k = 1$.

References

[1] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. North-Holland Publishing Company, Amsterdam, 1978.
[2] *M. Feistauer, K. Najzar*: Finite element approximation of a problem with a nonlinear Newton boundary condition. Numer. Math. *78* (1998), 403–425.
[3] *M. Feistauer, V. Sobotíková*: Finite element approximation of nonlinear elliptic problems with discontinuous coefficients. RAIRO Modél. Math. Anal. Numér. *24* (1990), 457–500.
[4] *M. Feistauer, A. Ženíšek*: Finite element solution of nonlinear elliptic problems. Numer. Math. *50* (1987), 451–475.
[5] *M. Křížek*: On semiregular families of triangulations and linear interpolation. Appl. Math. *36* (1991), 223–232.

[6] *A. Kufner, O. John and S. Fučík*: Function Spaces. Academia, Praha, 1977.
[7] *J. Nečas*: Les Métodes Directes en Théorie des Equations Elliptiques. Academia-Masson, Prague-Paris, 1967.
[8] *L. A. Oganesian, L. A Rukhovec*: Variational-Difference Methods for the Solution of Elliptic Problems. Izd. Akad. Nauk ArSSR, Jerevan, 1979. (In Russian.)
[9] *A. Ženíšek*: Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations. Academic Press, London, 1990.
[10] *A. Ženíšek*: The maximum angle condition in the finite element method for monotone problems with applications in magnetostatics. Numer. Math. *71* (1995), 399–417.
[11] *A. Ženíšek*: Finite element variational crimes in the case of semiregular elements. Appl. Math. *41* (1996), 367–398.
[12] *A. Ženíšek*: The use of semiregular finite elements. In: Proceedings of EQUADIFF, Conference on Differential Equations and Their Applications (R. P. Agarwal, F. Neuman and J. Vosmanský, eds.). Masaryk University, Brno & Electronic Publishing House, Stony Brook, New York, 1998, pp. 201–251.
[13] *M. Zlámal*: Curved elements in the finite element method I. SIAM J. Numer. Anal. *10* (1973), 229–240.

Author's address: *Jana Zlámalová*, Department of Mathematics, Technical University Brno, Technická 2, 616 69 Brno, Czech Republic, e-mail: `zlamalova@um.fme.vutbr.cz`.