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Semisimple Types in GL_n

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Abstract. This paper is concerned with the smooth representation theory of the general linear group $G = \mathrm{GL}_n(F)$ of a non-Archimedean local field F. The point is the (explicit) construction of a special series of irreducible representations of compact open subgroups, called semisimple types, and the computation of their Hecke algebras. A given semisimple type determines a Bernstein component of the category of smooth representations of G; that component is then the module category for a tensor product of affine Hecke algebras; every component arises this way. Moreover, all Jacquet functors and parabolic induction functors connecting G with its Levi subgroups are described in terms of standard maps between affine Hecke algebras. These properties of semisimple types depend on their special intertwining properties which in turn imply strong bounds on the support of coefficient functions.

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This paper is concerned with the smooth (complex) representation theory of the general linear group $\mathrm{GL}_n(F)$ of a non-Archimedean local field F. As such, it builds on and in a certain sense completes our earlier work [5]. On the other hand, it may be seen as carrying out for GL_n the program, initiated in [9], of analysing the category of smooth representations of a reductive p-adic group via the method of *types*. Let us begin by briefly reviewing this program.

1. Let G be the group of F-points of some connected reductive group defined over F and write $\mathfrak{R}(G)$ for the category of smooth complex representations of G. Let L be an F-Levi subgroup of G and denote by $X(L)$ the group of *unramified quasicharacters* of L, i.e., smooth homomorphisms $L \to \mathbb{C}^\times$ which vanish on all compact subgroups of L. Then given an irreducible supercuspidal representation σ of L, we may, following [1], associate to the pair (L, σ) a full subcategory $\mathfrak{R}^{(L,\sigma)}(G)$ of $\mathfrak{R}(G)$ by decreeing that a smooth representation π of G will be an object in $\mathfrak{R}^{(L,\sigma)}(G)$ if each of its irreducible subquotients appears as a composition

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factor of $\iota_P^G(\sigma \otimes \chi)$, for some $\chi \in X(L)$ and some parabolic subgroup P of G with Levi factor L; here, ι_P^G is the functor of (normalized) parabolic induction. We recall two basic results from [1]:

(1) *Let* (L_i, σ_i), $i = 1, 2$ *be as above. Then*

$$\mathfrak{R}^{(L_1,\sigma_1)}(G) = \mathfrak{R}^{(L_2,\sigma_2)}(G)$$

(as subcategories of $\mathfrak{R}(G)$*) if and only if the pairs* (L_i, σ_i) *are* G*-inertially equivalent: that is, there is a quasicharacter* $\chi \in \mathrm{X}(L_2)$ *so that the pairs* (L_1, σ_1) *and* $(L_2, \sigma_2 \otimes \chi)$ *are* G*-conjugate.*

We denote the G-inertial equivalence class of (L, σ) by $[L, \sigma]_G$ and write $\mathfrak{R}^{[L,\sigma]_G}(G)$ in place of $\mathfrak{R}^{(L,\sigma)}(G)$. We denote the set of G-inertial equivalence classes of pairs (L, σ) by $\mathcal{B}(G)$.

(2) *We have* $\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathcal{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G)$.

Suppose now that K is a compact, open subgroup of G and that (ρ, W) is an irreducible smooth representation of K. Let $(\check\rho, \check W)$ be the representation contragredient to (ρ, W) and let $\mathcal{H}(G, \rho)$ be the space of compactly supported, $\mathrm{End}_{\mathbb{C}}(\check W)$-valued functions f on G which satisfy $f(hxk) = \check\rho(h) f(x) \check\rho(k)$, $x \in G$, $h, k \in K$. Then we may view $\mathcal{H}(G, \rho)$ as an algebra under convolution once we have fixed a Haar measure on G. Let (π, V) be a smooth representation of G, set $V_\rho = \mathrm{Hom}_K(W, V)$ and let $V^\rho = \sum_{\phi\in V_\rho} \phi(W)$; denote by $\mathfrak{R}_\rho(G)$ the full subcategory of $\mathfrak{R}(G)$ whose objects (π, V) have the property that V is generated as G-representation by V^ρ. Then V_ρ is naturally a left $\mathcal{H}(G, \rho)$-module (see Section 2 of [9]), and the map $(\pi, V) \mapsto V_\rho$ induces a functor $\mathrm{M}_\rho\colon \mathfrak{R}_\rho(G) \to \mathcal{H}(G, \rho)$-Mod.

The significance of this construction resides in the following [9] Section 4:

(3) *Let* (K, ρ), (L, σ) *be as above, set* $\mathfrak{s} = [L, \sigma]_G$*, and suppose that* $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}_\rho(G)$*. Then* M_ρ *is an equivalence of categories.*

If $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}_\rho(G)$, then we say that (K, ρ) is an $\mathfrak{s}$-*type*.

(4) *Let* (K, ρ), (L, σ) *be as above and suppose that the categories* $\mathfrak{R}^{\mathfrak{s}}(G)$, $\mathfrak{R}_\rho(G)$ *have the same* $\mathfrak{R}(G)$*-irreducible objects. Then* (K, ρ) *is an* $\mathfrak{s}$*-type.*

2. The immediate aim of the present paper is the construction of an $\mathfrak{s}$-type $(K_{\mathfrak{s}}, \rho_{\mathfrak{s}})$ for every $\mathfrak{s} \in \mathcal{B}(\mathrm{GL}_n(F))$.

Such a construction will only be useful, however, if the algebras $\mathcal{H}(G, \rho_{\mathfrak{s}})$ have well-understood module categories. In the case of the types constructed here, we show that the algebras $\mathcal{H}(G, \rho_{\mathfrak{s}})$ are naturally isomorphic to tensor products of affine Hecke algebras of type A.

It is also desirable to have a module-theoretic interpretation of the functors of normalized parabolic induction and Jacquet restriction. Consideration of this in a general context led us to introduce the notion of a *cover*, which we now briefly review ([9] Section 8).

Let M be a Levi subgroup of G, let K_M be a compact open subgroup of M and ρ_M an irreducible smooth representation of K_M. Then a *G-cover* of (K_M, ρ_M) is a compact open subgroup K of G together with a smooth irreducible representation ρ of K, the pair (K, ρ) satisfying the following properties for each parabolic subgroup Q of G with Levi factor M:

(i) *$K \cap M = K_M$ and $K = K \cap N_\ell \cdot K_M \cdot K \cap N_\mathrm{u}$, where $N_\mathrm{u} = N_\mathrm{u}(Q)$ is the unipotent radical of Q and $N_\ell = N_\ell(Q)$ is that of the opposite to Q relative to M;*

(ii) *$K \cap N_\ell$, $K \cap N_\mathrm{u}$ are contained in the kernel of ρ while $\rho|K_M \cong \rho_M$;*

(iii) *there exists a (K, Q)-positive element z in the center, $\mathcal{Z}(M)$, of M and an invertible element $f_z \in \mathcal{H}(G, \rho)$ with support KzK.*

(An element $z \in \mathcal{Z}(M)$ is said to be *(K, Q)-positive* if the sequences $z^k(K \cap N_\mathrm{u})z^{-k}$, $z^{-k}(K \cap N_\ell)z^k$ tend monotonically to 1 as $k \to \infty$.)

We may now state one of the principal results of [9]. Let $\mathfrak{s} = [L, \sigma]_G \in \mathcal{B}(G)$, and let M be a Levi subgroup of G which contains L. The pair (L, σ) then determines an element $\mathfrak{s}_M = [L, \sigma]_M \in \mathcal{B}(M)$. We have ([9] 8.4):

With notation as above, suppose that (K_M, ρ_M) is an $\mathfrak{s}_M$-type in M and that (K, ρ) is a G-cover of (K_M, ρ_M). Then (K, ρ) is an $\mathfrak{s}$-type in G. Further, there is for each parabolic subgroup Q of G with Levi factor M a unique injective algebra homomorphism

$$\boldsymbol{j}_Q\colon \mathcal{H}(M, \rho_M) \to \mathcal{H}(G, \rho)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{s}_M}(M) & \xrightarrow{\approx} & \mathcal{H}(M, \rho_M)\text{–Mod} \\ {\scriptstyle \iota_Q^G}\big\downarrow & & \big\downarrow{\scriptstyle (j_Q)_*} \\ \mathfrak{R}^{\mathfrak{s}}(G) & \xrightarrow[\approx]{} & \mathcal{H}(G, \rho)\text{–Mod}, \end{array}$$

where $(\boldsymbol{j}_Q)_*$ is the ring-theoretic induction functor given by $\boldsymbol{j}_Q$.

(In fact, the statement in [9] 8.4 refers to the *un-normalized* induction functor; however, only a trivial modification is required to treat the normalized one.)

3. We can now describe the results of this paper. We first rephrase those of [5] in the language of types and covers. So, from here on out, $G = \mathrm{GL}_n(F)$. The book [5] is largely concerned with the case of those $\mathfrak{s} = [L, \sigma]_G \in \mathcal{B}(G)$ for which $L \cong \mathrm{GL}_r(F)^s$, $rs = n$, and $\sigma \cong \pi_0 \otimes \cdots \otimes \pi_0$, for some irreducible supercuspidal representation π_0 of $\mathrm{GL}_r(F)$. One of the main accomplishments of [5] was the explicit construction of a compact open subgroup J of G and an irreducible smooth representation λ of J such that the categories $\mathfrak{R}^{\mathfrak{s}}(G)$, $\mathfrak{R}_\lambda(G)$ have the same irreducible objects. From (4) above, therefore, (J, λ) is an $\mathfrak{s}$-type. Moreover, $\mathcal{H}(G, \lambda)$ is a Hecke algebra of affine type *ibid.* 5.6.6.

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In particular, there exists a type (J_0, λ_0) for the inertial equivalence class $[\mathrm{GL}_r(F), \pi_0]_{\mathrm{GL}_r(F)} \in \mathcal{B}(\mathrm{GL}_r(F))$. If, in the original context, we set $\mathfrak{s}_L = [L, \sigma]_L \in \mathcal{B}(L)$, we obtain an $\mathfrak{s}_L$-type (K_L, τ_L) in L by setting

$$K_L = J_0 \times J_0 \times \cdots \times J_0, \quad \tau_L = \lambda_0 \otimes \lambda_0 \otimes \cdots \otimes \lambda_0.$$

The types (J, λ) are referred to (with some prescience) in [5] as *simple types*. However, they cannot be constructed in the appropriate way as covers, and are not the most convenient choice here. We therefore use [5] Ch. 7, where we constructed a modified version of the simple type (J, λ); this *is* a G-cover of (K_L, τ_L) (and has the same Hecke algebra as (J, λ)). We explain this more fully in Section 1 below.

Now we pass to the case of a general $\mathfrak{s} \in \mathcal{B}(G)$, which is the central concern of this paper. A choice of representative (L, σ) for $\mathfrak{s}$ determines a Levi subgroup M which may be described as the smallest Levi subgroup which contains the G-normalizer (in the obvious sense) of $\mathfrak{s}_L$. The classes $\mathfrak{s}$ considered in [5] may then be characterized by the property that $M = G$. In the general case, the arguments outlined above yield an M-cover (K_M, τ_M) of (K_L, τ_L). The main result of this paper asserts:

(i) *There exists a G-cover (K, τ) of (K_M, τ_M). In particular, (K, τ) is an $\mathfrak{s}$-type in G and a G-cover of (K_L, τ_L).*
(ii) *If Q is a parabolic subgroup of G with Levi factor M, the associated algebra homomorphism $\boldsymbol{j}_Q\colon \mathcal{H}(M, \tau_M) \to \mathcal{H}(G, \tau)$ is an isomorphism and preserves support of functions. In particular, $\mathcal{H}(G, \tau)$ is a tensor product of affine Hecke algebras.*

4. Much of the significance of this result flows from the fact that we construct the types (K, τ) *explicitly*. Their particular form is well-adapted to constructing types in the group SL_n; indeed, it was our earlier work [6], [7] on that group which originally motivated our search for semisimple types in GL_n. A totally different application connects semisimple types with results from [13] to give *explicit formulas* for Plancherel measure and conductors of Rankin-Selberg convolutions [10]. However, such matters must be treated elsewhere.

5. The proof of the above theorem is quite elaborate and involves several new ideas, along with most of the machinery of [5] and some of its elaborations in [4]. The simple types of [5] are built from parahoric subgroups of G *with their standard filtrations*. It is clear from the outset that this framework is inadequate for the construction of semisimple types: a reader familiar with the methods of [5] might consider the problem when, for example, $G = \mathrm{GL}_5$, the Levi subgroup L is $\mathrm{GL}_2 \times \mathrm{GL}_3$, and the associated supercuspidal representations of GL_2, GL_3 are given by totally ramified field extensions of F. This necessitates the introduction of nonstandard filtrations attached to 'lattice sequences', as in Section 2 below. These generalize the standard filtrations attached to lattice *chains*.

Next, it is clear that the difficult part in the construction of a cover is achieving the condition (iii). This, however, can be made to follow (in the right cir-

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cumstances) from intertwining properties of comparatively straightforward representations of certain subgroups of the required K. It is the construction of these subgroups which takes up most of the paper.

Lattice sequences are introduced in Section 2. The key point is that lattice sequences, unlike chains, admit a reasonable definition of direct sum. Using this property, we construct in Section 3 a first family of compact open subgroups with the essential intertwining property. The next step is to extend some of the machinery of [4], [5] concerning simple characters (terminology of [5] Ch. 3) to the context of lattice sequences. This takes up Section 4 and Section 5. The main family of subgroups (with the intertwining property) is then treated in Section 6. Finally, we give the explicit construction of the desired covers in Section 7 and Section 8.

We conclude this Introduction with words of thanks to our colleagues. During the writing of this paper, we have received substantial benefit and encouragement from their comments. In this context, we wish to mention Guy Henniart, Marie-France Vignéras, and, particularly, Corinne Blondel. Her detailed criticisms of an earlier draft were extremely helpful to us.

Notation: The following notation will be standard throughout, and is chosen to be consistent with [5], to which we shall often refer. First, F denotes a non-Archimedean local field with discrete valuation ring $\mathfrak{o}_F$. We write $\mathfrak{p}_F$ for the maximal ideal of $\mathfrak{o}_F$, and $\mathfrak{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ for the (finite) residue field of F.

We write $[x]$ for the integer part of a real number x; thus $[x]$ is the largest integer $\leqslant x$.

If V is a finite-dimensional F-vector space, we adhere to the convention that an $\mathfrak{o}_F$*-lattice in* V is a finitely generated $\mathfrak{o}_F$-submodule of V which *spans* V over F.

We fix a continuous character ψ_F of the additive group of F, with conductor $\mathfrak{p}_F$. If V is as above, and $A = \mathrm{End}_F(V)$, we put $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$. If $a \in A$, we write $\psi_{A,a}$, or just ψ_a, for the function $x \mapsto \psi_A(a(x-1))$, $x \in A$.

Let $[\mathfrak{A}, n, m, \beta]$ be a simple stratum in $\mathrm{End}_F(V)$. (See [5] Section 1.5 for this term.) The symbols $\mathfrak{H}(\beta, \mathfrak{A})$, $\mathfrak{J}(\beta, \mathfrak{A})$, $H^{m+1}(\beta, \mathfrak{A})$, $J(\beta, \mathfrak{A})$ and their variations have the same meanings as in [5] Section 3.1. The simple character set $\mathcal{C}(\mathfrak{A}, m, \beta)$ is as in [5] Section 3.2. In particular, the elements of $\mathcal{C}(\mathfrak{A}, m, \beta)$ are (rather special) abelian characters of the group $H^{m+1}(\beta, \mathfrak{A})$.

It will be convenient to have as standard the following 'block matrix' notation. Let V be a finite-dimensional F-vector space, and write $A = \mathrm{End}_F(V)$. Suppose we have subspaces V^1, V^2 of V such that $V = V^1 \oplus V^2$. Let $\mathbf{1}^i$ denote the projection $V \to V^i$ with kernel V^j, $j \neq i$, and put $A^{ij} = \mathbf{1}^i \cdot A \cdot \mathbf{1}^j \subset A$. We identify $A^{ij} = \mathrm{Hom}_F(V^j, V^i)$. We use the notation

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$$

and sometimes abbreviate $A^{ii} = A^i$. If L is an $\mathfrak{o}_F$-lattice in A, we set $L^{ij} = L \cap A^{ij}$. We use analogous notations when there are more than two factors V^i.

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1. Main Theorem

Throughout this section, V denotes a finite-dimensional F-vector space and $G = \mathrm{Aut}_F(V)$. We use the notation of [9], as outlined in the introduction.

1.1. We give a preliminary version of the main result:

MAIN THEOREM (first version). *Let* $\mathfrak{s} \in \mathcal{B}(G)$. *There exists an* $\mathfrak{s}$*-type in* G.

We shall give a more refined statement after we have examined a sequence of special cases. The final explicit form is Theorem 8.2 below.

1.2. We assume in this paragraph that the inertial equivalence class $\mathfrak{s}$ is *supercuspidal,* i.e., of the form $\mathfrak{s} = [G, \pi]_G$, for an irreducible supercuspidal representation π of G. By [5] (8.4.1), there is a maximal simple type (J, λ) occurring in π, unique up to G-conjugation. By [5] (6.2.3), an irreducible representation π' of G contains λ if and only if $\pi' \cong \pi \otimes \chi \circ \det$, for some *unramified* quasicharacter χ of $F^\times$. This says that (J, λ) is an $\mathfrak{s}$-type.

1.3. We return to the case of a general $\mathfrak{s} \in \mathcal{B}(G)$. Thus $\mathfrak{s} = [L, \sigma]_G$, for some Levi subgroup L of G and some irreducible supercuspidal representation σ of L. There is a decomposition of V as a direct sum of nonzero subspaces

$$V = V^1 \oplus V^2 \oplus \ldots \oplus V^r$$

of which L is the G-stabilizer. Thus

$$L = \prod_{i=1}^{r} \mathrm{Aut}_F(V^i), \qquad \sigma = \bigotimes_{i=1}^{r} \pi_i,$$

where π_i is an irreducible supercuspidal representation of the group $\mathrm{Aut}_F(V^i) \cong \mathrm{GL}_{n_i}(F)$. The class $\mathfrak{s}$ determines the integers n_i up to permutation and the corresponding factors π_i up to unramified twist.

By 1.2, there is a maximal simple type (J_i, λ_i) occurring in π_i, for each i. We set

$$K_L = \prod_{i=1}^{r} J_i, \qquad \tau_L = \bigotimes_{i=1}^{r} \lambda_i.$$

Immediately, we have

PROPOSITION. *Define* $\mathfrak{s}_L = [L, \sigma]_L \in \mathcal{B}(L)$. *The pair* (K_L, τ_L) *is then an* $\mathfrak{s}_L$*-type in* L.

The choice of representative (L, σ) of the inertial equivalence class $\mathfrak{s}$ gives rise to another Levi subgroup of G as follows. Write $\mathfrak{s}_L = [L, \sigma]_L \in \mathcal{B}(L)$. The G-normalizer $\mathrm{N}_G(L)$ of L acts on the set $\mathcal{B}(L)$ by conjugation.

LEMMA. *There is a unique Levi subgroup M of G which contains the* $\mathrm{N}_G(L)$*-stabilizer of* $\mathfrak{s}_L$ *and is minimal for this property.*

This is obvious.

1.4. In this paragraph, we consider the case where the Levi subgroup M defined in 1.3 is G itself.

PROPOSITION. *Suppose, in the situation of* 1.3, *that* $M = G$. *The* $\mathfrak{s}_L$*-type* (K_L, τ_L) *of* 1.3 *then admits a* G*-cover* (K, τ). *In particular,* (K, τ) *is an* $\mathfrak{s}$*-type in* G.

Proof. The hypothesis on M implies that L is isomorphic to $\mathrm{GL}_{n_0}(F)^r$, $rn_0 = n$, and that σ may be taken in the form $\sigma = \pi_0 \otimes \cdots \pi_0$. The $\mathfrak{s}_L$-type (K_L, τ_L) is then necessarily L-conjugate to one of the form $(J_0^r, \lambda_0^{\otimes r})$, where (J_0, λ_0) is a maximal simple type in $\mathrm{GL}_{n_0}(F)$ occurring in π_0, [5] (6.2.4).

From this point, the argument is a re-interpretation of certain results in [5], which have to be further generalized in Section 7 below, so we shall be brief.

The existence of an $\mathfrak{s}$-type is given by [5] (8.4.3). Indeed, *ibid.* (7.3.12), (7.3.14) say that this type may be taken to be a simple type (J, λ), associated to a simple stratum $[\mathfrak{A}, n, 0, \beta]$. In particular, $J = J(\beta, \mathfrak{A})$. (There is also the possibility that (J, λ) is 'of level zero', i.e., of the form [5] (5.5.10)(b). In this case, the proof is easier, and we omit the details.)

However, (J, λ) is not a cover of (K_L, τ_L), as remarked above. We therefore use [5] Theorem 7.2.17. That result produces a representation λ_P of a subgroup J_P of J attached to a particular parabolic subgroup P of G with Levi component L (this Levi is M in the notation of [5]). The Iwahori decomposition properties (i) and (ii) in the definition of cover (see Introduction) are easy to check for (J_P, λ_P). To verify the third condition, we note that λ_P induces the representation λ of J *loc. cit.* Thus we have an isomorphism $\mathcal{H}(G, \lambda_P) \cong \mathcal{H}(G, \lambda)$ of Hecke algebras. By [5] (7.2.19), this isomorphism has the following property: if $f_P \in \mathcal{H}(G, \lambda_P)$ has support $J_P g J_P$, for some $g \in G$, then its image $f \in \mathcal{H}(G, \lambda)$ has support JgJ. By [5] (5.6.6), any such function f is invertible in $\mathcal{H}(G, \lambda)$, so f_P is also invertible. Thus $(K, \tau) = (J_P, \lambda_P)$ is the cover we seek. (Alternatively, property (iii) of a cover follows in this case from [5] (7.3.2) and [9] 7.14.) □

Continuing in the same situation, we recall ([5] 5.6.6) that the Hecke algebra $\mathcal{H}(G, \lambda_P) = \mathcal{H}(G, \lambda)$ is isomorphic to an *affine Hecke algebra* $\mathcal{H}(G, \lambda) \cong \mathcal{H}(r, q_E^f)$, in the notation of [5]. Here, E denotes the field $F[\beta]$, the integer f is determined by the relation $[F[\beta]{:}\, F]rf = n$, and $q_E = |k_E|$. For further comments on the associated algebra homomorphisms $\mathcal{H}(L, \tau_L) \to \mathcal{H}(G, \tau)$, see [5] 7.6.20: in a certain sense, they depend only on the parameters r and q_E^f.

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1.5. We return to the general case, where the Levi subgroup M defined in Lemma 1.3 is possibly not equal to G. As an immediate consequence of Proposition 1.4, we have:

COROLLARY. *Let* $\mathfrak{s}_M = [L, \sigma]_M \in \mathcal{B}(M)$. *There exists an* $\mathfrak{s}_M$*-type* (K_M, τ_M) *in* M *which is an* M*-cover of* (K_L, τ_L).

Let us make this explicit (as we will need this notation later). The Levi subgroup M is the G-stabilizer of a decomposition $V = W^1 \oplus W^2 \oplus \cdots \oplus W^t$ of V as a direct sum of nonzero subspaces W^j. Set $G_M^j = \mathrm{Aut}_F(W^j)$. We then have $L = \prod L_j$, where $L_j = L \cap G_M^j$ (with the obvious abuse of notation). The type (K_L, τ_L) decomposes as the tensor product of types (K_{L_j}, τ_{L_j}), each of which admits a G_M^j-cover (K_M^j, τ_M^j) as in 1.4. We put

$$K_M = \prod_{j=1}^{t} K_M^j, \qquad \tau_M = \bigotimes_{j=1}^{t} \tau_M^j.$$

We then have

$$\mathcal{H}(M, \tau_M) = \bigotimes_{j=1}^{t} \mathcal{H}(G_M^j, \tau_M^j),$$

and each of the $\mathcal{H}(G_M^j, \tau_M^j)$ is affine.

We now give a more precise statement of our main result.

MAIN THEOREM (second version). *There exists a* G*-cover* (K, τ) *of* (K_M, τ_M). *In particular,* (K, τ) *is an* $\mathfrak{s}$*-type in* G *and is a* G*-cover of* (K_L, τ_L).

The proof will occupy the remainder of the paper. Notice, however, that the second and third assertions follow from the first, via [9] 8.3, 8.5 respectively.

If we choose a parabolic subgroup Q of G with Levi component M, we get a homomorphism $\boldsymbol{j}_Q \colon \mathcal{H}(M, \tau_M) \to \mathcal{H}(G, \tau)$ which realizes the induction functor $\boldsymbol{\iota}_Q^G$, as mentioned in the Introduction. In this case, $\boldsymbol{j}_Q$ is an *algebra isomorphism*, which moreover preserves support of functions [9] 12.1:

$$\mathrm{supp}(\boldsymbol{j}_Q f) = K\,\mathrm{supp}(f)\,K, \quad f \in \mathcal{H}(M, \tau_M).$$

In particular, $\mathcal{H}(G, \tau)$ is a tensor product of affine Hecke algebras.

If we now choose a parabolic subgroup P of G with Levi component L, then $P \cap M$ (resp. $Q = MP$) is a parabolic subgroup of M (resp. G) with Levi component L (resp. M). The algebra homomorphism $\boldsymbol{j}_P \colon \mathcal{H}(L, \tau_L) \to \mathcal{H}(G, \tau)$ which realizes the induction functor $\boldsymbol{\iota}_P^G$ then factors as $\boldsymbol{j}_P = \boldsymbol{j}_Q \circ \boldsymbol{j}_{P\cap M}$, by [9] 8.7. Thus $\boldsymbol{j}_P$ is the composite of a tensor product of standard maps between affine Hecke algebras and an algebra isomorphism. Further, we get a similar factorization of $\boldsymbol{j}_P$ relative to any Levi subgroup of G containing L: this follows from the transitivity property of covers [9] 8.5.

1.6. There is another striking consequence of [9] in the context of Theorem 1.5. Using the same notation as above, we have

COROLLARY. *An element $g \in G$ intertwines the representation τ if and only if it is of the form $g = k_1 m k_2$, where $k_1, k_2 \in K$ and m is an element of M which intertwines τ_M.*

This follows from [9] 12.2.

2. Lattice Sequences

The lattice sequences of the title generalize the lattice chains used extensively in [5]. They give new filtrations of the parahoric subgroups of $\mathrm{GL}_n(F)$, somewhat along the lines of the very general filtrations introduced in [12]. The present section lays out their basic properties, generalizing the early parts of [2], [3].

Throughout, V denotes a finite-dimensional F-vector space, and $A = \mathrm{End}_F(V)$.

2.1. We start with a definition:

DEFINITION. An $\mathfrak{o}_F$-lattice sequence in V is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V such that

(i) $n \geqslant m$ implies $\Lambda(n) \subset \Lambda(m)$;
(ii) there exists $e = e(\Lambda) \in \mathbb{Z}$, $e \geqslant 1$, such that $\Lambda(n+e) = \mathfrak{p}_F \Lambda(n)$, $n \in \mathbb{Z}$.

2.2 *Remarks*. (i) Let Λ be a lattice sequence in V, and define $\mathcal{L}_\Lambda = \{\Lambda(n): n \in \mathbb{Z}\}$. Then $\mathcal{L}_\Lambda$ is a nonempty set of lattices in V which is linearly ordered under inclusion and stable under multiplication by $F^\times$. In other words, it is a *lattice chain* in the sense of [5] (1.1).

(ii) In the opposite direction, suppose we are given a lattice chain $\mathcal{L}$ in V. We can index the elements of $\mathcal{L}$ by $\mathbb{Z}$, $\mathcal{L} = \{L_j: j \in \mathbb{Z}\}$, so that $L_j \supsetneq L_{j+1}$, $j \in \mathbb{Z}$, and define a function Λ by $\Lambda(j) = L_j$, $j \in \mathbb{Z}$. This is a lattice sequence with $\mathcal{L}_\Lambda = \mathcal{L}$ and the additional property that $\Lambda(j) \neq \Lambda(j+1)$ for any j. In other words, an (indexed) lattice chain is the same as a lattice sequence which is *injective* as a function. We call such lattice sequences *strict.*

2.3. Recall that a lattice chain $\mathcal{L}$ in V determines a hereditary $\mathfrak{o}_F$-order in A, which we denote by $\mathfrak{A}(\mathcal{L})$ or $\mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L})$, in the manner of [5]:

$$\mathfrak{A}(\mathcal{L}) = \{x \in A: xL \subset L, L \in \mathcal{L}\}.$$

If Λ is a lattice sequence in V, we define

$$\mathfrak{a}_n = \mathfrak{a}_n(\Lambda) = \{x \in A: x\Lambda(m) \subset \Lambda(m+n), m \in \mathbb{Z}\}, \quad n \in \mathbb{Z}.$$

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Note that, by definition, $\mathfrak{a}_0(\Lambda) = \mathfrak{A}(\mathcal{L}_\Lambda)$. Further, each $\mathfrak{a}_n(\Lambda)$ is an $\mathfrak{o}_F$-lattice in A and also a bimodule over $\mathfrak{a}_0(\Lambda)$. Observe that if Λ is strict, with $\mathcal{L} = \mathcal{L}_\Lambda$, then the definition gives $\mathfrak{a}_n(\Lambda) = \mathfrak{P}^n$, where $\mathfrak{P}$ is the Jacobson radical of the order $\mathfrak{A}(\mathcal{L})$ ([5] (1.1)).

The lattice sequence Λ determines a 'valuation' map $\boldsymbol{\nu}_\Lambda\colon A \to \mathbb{Z}$ by

$$\boldsymbol{\nu}_\Lambda(x) = \max\{n \in \mathbb{Z}\colon x \in \mathfrak{a}_n(\Lambda)\}, \quad x \in A,$$

with the usual understanding that $\boldsymbol{\nu}_\Lambda(0) = \infty$.

PROPOSITION. *Let Λ be a lattice sequence in V. We have:*

(i) $\mathfrak{a}_0(\Lambda) = \mathfrak{A}(\mathcal{L}_\Lambda)$, that is, $\mathfrak{a}_0(\Lambda)$ is the hereditary $\mathfrak{o}_F$-order in A defined by the lattice chain $\mathcal{L}_\Lambda$.
(ii) $\mathfrak{a}_1(\Lambda)$ is the Jacobson radical of $\mathfrak{a}_0(\Lambda)$.
(iii) $\mathfrak{p}_F\mathfrak{a}_k(\Lambda) = \mathfrak{a}_{k+e(\Lambda)}(\Lambda)$, $k \in \mathbb{Z}$.
(iv) $\mathfrak{a}_k(\Lambda)\cdot\mathfrak{a}_l(\Lambda) \subset \mathfrak{a}_{k+l}(\Lambda)$, $k, l \in \mathbb{Z}$.

Proof. Only (ii) requires any comment. Let $\mathcal{L} = \mathcal{L}_\Lambda$. For $L \in \mathcal{L}$, let L' be the largest element of $\mathcal{L}$ such that $L' \subsetneq L$. From the standard theory of lattice chains and hereditary orders, we know that an element $x \in A$ lies in the radical of $\mathfrak{a}_0(\Lambda) = \mathfrak{A}(\mathcal{L})$ if and only if $xL \subset L'$ for all $L \in \mathcal{L}$. On the other hand, if $m \in \mathbb{Z}$ satisfies $\Lambda(m) = L$, then we have $\Lambda(m+1) = L$ or L', the latter case occurring precisely when m is the *largest* integer such that $\Lambda(m) = L$. Thus $x \in A$ lies in $\mathfrak{a}_1(\Lambda)$ if and only if $xL \subset L'$ for all $L \in \mathcal{L}$. This proves (ii). □

2.4. Suppose for the moment that Λ is a strict lattice sequence in V. We may therefore identify Λ with its associated lattice chain $\mathcal{L}$. In this case, the lattice chain $\mathcal{L}$ and the order $\mathfrak{A}(\mathcal{L}) = \mathfrak{a}_0(\Lambda)$ determine each other: the chain $\mathcal{L}$ is simply the set of all $\mathfrak{A}$-lattices in V. This cannot hold for lattice sequences in general. However, weaker versions of many related properties do generalize to this situation, for example:

PROPOSITION. *Let Λ be a lattice sequence in V, and write $e = e(\Lambda)$. The natural map*

$$\frac{\mathfrak{a}_j(\Lambda)}{\mathfrak{a}_{j+1}(\Lambda)} \to \prod_{i=0}^{e-1} \mathrm{Hom}_{k_F}\left(\frac{\Lambda(i)}{\Lambda(i+1)}, \frac{\Lambda(i+j)}{\Lambda(i+j+1)}\right)$$

is an isomorphism, for all $j \in \mathbb{Z}$.

Proof. By periodicity and the definition of $\mathfrak{a}_j$, this map is certainly injective. It is enough therefore to show that $\dim_{k_F}\mathfrak{a}_j/\mathfrak{a}_{j+1} = \Sigma_{i=0}^{e-1} d_i d_{i+j}$ for all $j \in \mathbb{Z}$, where we have written $d_i = \dim_{k_F}(\Lambda(i)/\Lambda(i+1))$.

This injectivity property shows that

$$\dim_{k_F}(\mathfrak{a}_j(\Lambda)/\mathfrak{a}_{j+1}(\Lambda)) \leqslant \sum_{i=0}^{e-1} d_i d_{i+j}, \qquad j \in \mathbb{Z}.$$

This implies

$$\sum_{j=0}^{e-1} \dim_{k_F} \mathfrak{a}_j/\mathfrak{a}_{j+1} \leqslant \sum_{i,j=0}^{e-1} d_i d_{i+j}.$$

The left hand side here is $\dim_{k_F} \mathfrak{a}_0/\mathfrak{a}_e = \dim_{k_F} \mathfrak{a}_0/\mathfrak{p}_F\mathfrak{a}_0 = N^2$, where $N = \dim_F V$. By periodicity, the right hand side reduces to $(\Sigma_{i=0}^{e-1} d_i)^2 = N^2$. It follows that $\dim_{k_F} \mathfrak{a}_j/\mathfrak{a}_{j+1} = \Sigma_{i=0}^{e-1} d_i d_{i+j}$ for $0 \leqslant j \leqslant e-1$. The same equality then holds for all j by periodicity, and the proposition is proved. □

Using this Proposition, it is easy to find examples of lattice sequences Λ with $\mathfrak{a}_j(\Lambda) = \mathfrak{a}_{j+1}(\Lambda)$ for some $j \in \mathbb{Z}$.

2.5. If we have a lattice sequence Λ and an integer t, we can define another lattice sequence $\Lambda{+}t$ by $(\Lambda{+}t)(n) = \Lambda(n+t)$, $n \in \mathbb{Z}$. We refer to $\Lambda{+}t$ as a *translate* of Λ. Of course, the lattice sequences Λ, $\Lambda{+}t$ have many properties in common. In particular, we have $\mathfrak{a}_j(\Lambda{+}t) = \mathfrak{a}_j(\Lambda)$ for all $j \in \mathbb{Z}$. We next show that the converse of this property holds.

PROPOSITION. *Let Λ, Λ' be lattice sequences in V, and suppose that $\mathfrak{a}_j(\Lambda) = \mathfrak{a}_j(\Lambda')$, for all j. There is then an integer t such that $\Lambda = \Lambda'{+}t$.*

Proof. The relation $\mathfrak{a}_0(\Lambda) = \mathfrak{a}_0(\Lambda')$ shows that the associated lattice chains $\mathcal{L}_\Lambda$, $\mathcal{L}_{\Lambda'}$ are equal.

Assume for a contradiction that Λ is not equal to any translate of Λ'. Replacing Λ, Λ' by translates, we can therefore assume we have the following situation: $\Lambda(-1) \neq \Lambda(0) = \Lambda'(0) \neq \Lambda'(-1)$ and, for some nonnegative integer j, $\Lambda(j) = \Lambda(0) = \Lambda'(j) = \Lambda'(j+1)$, while $\Lambda(j+1) \neq \Lambda(j)$. Let $l \geqslant 1$ be the least integer such that $\Lambda'(j) \neq \Lambda'(j+l+1)$. If we number the lattice chain $\mathcal{L}_\Lambda = \{L_k : k \in \mathbb{Z}\}$ so that $L_0 = \Lambda(0)$, we then have $\Lambda(j+1) = \Lambda'(j+l+1) = L_1$. Since $j \geqslant 0$, we have $\mathfrak{a}_{j+1}(\Lambda) = \mathfrak{a}_{j+1}(\Lambda') \subset \mathfrak{P}$, where $\mathfrak{P}$ is the radical of the hereditary order defined by $\mathcal{L}_\Lambda$. We identify

$$\mathfrak{P}/\mathfrak{P}^2 = \prod_{i=-1}^{e(\mathcal{L}_\Lambda)-2} \mathrm{Hom}(L_i/L_{i+1}, L_{i+1}/L_{i+2}),$$

and consider the image of $\mathfrak{a}_{j+l+1}(\Lambda)$ here. By construction, this maps L_{-1}/L_0 trivially to L_0/L_1. However, by (2.4), the image of $\mathfrak{a}_{j+l+1}(\Lambda')$ contains an element mapping L_{-1}/L_0 *nontrivially* to L_0/L_1. Thus we have a contradiction and the

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Proposition follows. □

2.6. If we are given a lattice sequence Λ in V, we can extend Λ to a function on the real line $\mathbb{R}$ by setting $\Lambda(r) = \Lambda(n)$, $r \in \mathbb{R}$, where n is the integer defined by the relation $n - 1 < r \leqslant n$. Equivalently,

$$\Lambda(r) = \Lambda(-[-r]) = \bigcup_{n \geqslant r} \Lambda(n) = \bigcap_{n < r+1} \Lambda(n).$$

We still have the relation $\Lambda(s) \subset \Lambda(r)$, $r, s \in \mathbb{R}$, $r \leqslant s$. It will also be useful to likewise extend the domain of definition of the function $n \mapsto \mathfrak{a}_n(\Lambda)$. In this regard, the following is helpful.

PROPOSITION. *Let Λ be a lattice sequence in V. Let $r \in \mathbb{R}$ and set $n = -[-r]$. For an element $x \in A$, the following are equivalent:*

(i) $x \in \mathfrak{a}_n(\Lambda)$;
(ii) $x\Lambda(s) \subset \Lambda(n+s)$ *for all* $s \in \mathbb{R}$;
(iii) $x\Lambda(m) \subset \Lambda(r+m)$ *for all* $m \in \mathbb{Z}$;
(iv) $x\Lambda(s) \subset \Lambda(r+s)$ *for all* $s \in \mathbb{R}$.

Proof. The implication (i) ⇒ (ii) is easy, and (ii) ⇒ (iv) because $n + s \geqslant r + s$. On the other hand, (iii) is a special case of (iv), so it remains only to show that (iii) implies (i).

To do this, we note that $n - 1 < r \leqslant n$, so $n + m - 1 < r + m \leqslant n + m$ for all integers m. Thus $\Lambda(r + m) = \Lambda(n + m)$ and $x\Lambda(m) \subset \Lambda(n + m)$ for all m, as required. □

For $r \in \mathbb{R}$, we now define $\mathfrak{a}_r(\Lambda)$ to be the set of $x \in A$ which satisfy the conditions of the Proposition; in other words,

$$\mathfrak{a}_r(\Lambda) = \mathfrak{a}_{-[-r]}(\Lambda), \quad r \in \mathbb{R}.$$

As an immediate consequence of this definition, we have

$$\left.\begin{aligned} \mathfrak{p}_F \mathfrak{a}_r(\Lambda) &= \mathfrak{a}_{r+e(\Lambda)}(\Lambda), \\ \mathfrak{a}_r(\Lambda)\mathfrak{a}_s(\Lambda) &\subset \mathfrak{a}_{r+s}(\Lambda), \end{aligned}\right\} \quad r, s \in \mathbb{R}.$$

2.7. The main reason for introducing lattice sequences is that, unlike lattice chains, they admit a natural notion of direct sum. To define this, we first need an operation of ‘scaling’ on lattice sequences. This is the subject of the present section.

DEFINITION. Let Λ be a lattice sequence in V and let k be a positive integer. Define a function $k\Lambda$ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V by

$$k\Lambda : m \mapsto \Lambda(m/k), \quad m \in \mathbb{Z}.$$

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PROPOSITION. *Let* Λ *be a lattice sequence in* V *and* $k \in \mathbb{Z}$, $k > 0$. *Then:*

(i) $k\Lambda$ *is a lattice sequence with* $e(k\Lambda) = ke(\Lambda)$;
(ii) *we have* $k\Lambda(r) = \Lambda(r/k)$, *for all* $r \in \mathbb{R}$;
(iii) $\mathfrak{a}_r(k\Lambda) = \mathfrak{a}_{r/k}(\Lambda)$, *for all* $r \in \mathbb{R}$.

Proof. The function $k\Lambda\colon n \mapsto \Lambda(n/k)$ is an order-preserving map from $\mathbb{Z}$ to the set of lattices in V. Moreover,

$$k\Lambda(n) = \begin{cases} \Lambda(n/k) \text{ if } k \text{ divides } n, \\ \Lambda(1+[n/k]) \text{ otherwise}. \end{cases}$$

The first assertion is now clear.

In (ii), we let n be the integer such that $n-1 < r \leqslant n$. Thus $k\Lambda(r) = k\Lambda(n) = \Lambda(n/k)$. We write $n = mk - l$, for integers m, l such that $0 \leqslant l < k$. We then have $m - 1 < r/k \leqslant n/k \leqslant m$, so that $k\Lambda(r) = \Lambda(m) = \Lambda(n/k) = \Lambda(r/k)$, as required.

For (iii), we take $x \in A$. Then $x \in \mathfrak{a}_r(k\Lambda)$ if and only if $x\Lambda(s/k) \subset \Lambda(s/k + r/k)$ for all $s \in \mathbb{R}$. If s is of the form mk with $m \in \mathbb{Z}$, this condition implies $x \in \mathfrak{a}_{r/k}(\Lambda)$ by Proposition 2.6(iii). Thus $\mathfrak{a}_r(k\Lambda) \subset \mathfrak{a}_{r/k}(\Lambda)$. The opposite containment is clear. □

Let $\boldsymbol{\nu}_\Lambda$ be the valuation map attached to Λ, as in 2.3. The Proposition yields:

COROLLARY. *Let* Λ *be a lattice sequence in* V *and let* k *be a positive integer. Then* $\boldsymbol{\nu}_{k\Lambda} = k\boldsymbol{\nu}_\Lambda$.

2.8. We now define the *direct sum* of two lattice sequences. For $i = 1, 2$, let V^i be a finite-dimensional F-vector space and Λ^i a lattice sequence in V^i. We abbreviate $e_i = e(\Lambda^i)$, and set

$$e = \operatorname{lcm}\{e_1, e_2\},$$

$$\Lambda(er) = \Lambda^1(e_1 r) \oplus \Lambda^2(e_2 r), \quad r \in \mathbb{R}.$$

Thus Λ, which we tend to denote

$$\Lambda = \Lambda^1 \oplus \Lambda^2,$$

is an order-preserving function from $\mathbb{R}$ to the set of lattices in $V^1 \oplus V^2$.

PROPOSITION. *We have:*

(i) $\Lambda = \Lambda^1 \oplus \Lambda^2$ *is a lattice sequence in* $V^1 \oplus V^2$ *of period* e;
(ii) $\Lambda = \frac{e}{e_1}\Lambda^1 \oplus \frac{e}{e_2}\Lambda^2$;
(iii) *if* Λ^i *is a lattice sequence in* V^i, *for* $i = 1, 2, 3$, *then*

$$(\Lambda^1 \oplus \Lambda^2) \oplus \Lambda^3 = \Lambda^1 \oplus (\Lambda^2 \oplus \Lambda^3);$$

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(iv) *the transposition isomorphism* $V^1 \oplus V^2 \to V^2 \oplus V^1$ *induces an isomorphism*

$$(\Lambda^1 \oplus \Lambda^2)(r) \cong (\Lambda^2 \oplus \Lambda^1)(r),$$

for all $r \in \mathbb{R}$.

Proof. Straightforward. □

EXAMPLE. Suppose we have two lattice chains $\{L_j\}$, $\{L'_j\}$ with the same period e. We can view these as corresponding to strict lattice sequences Λ, Λ' respectively. The sum $\Lambda \oplus \Lambda'$ is then strict of period e, and the corresponding lattice chain is

$$\cdots \supset L_0 \oplus L'_0 \supset L_1 \oplus L'_1 \supset \cdots \supset L_e \oplus L'_e \supset \cdots.$$

On the other hand, there is an obvious way of sticking these lattice chains together to get a chain of period $2e$, namely

$$\cdots \supset L_0 \oplus L'_0 \supset L_0 \oplus L'_1 \supset L_1 \oplus L'_1 \supset L_1 \oplus L'_2 \supset \cdots$$

This is the lattice chain associated to the strict lattice sequence Λ'' where $\Lambda'' = (2\Lambda - 1) \oplus 2\Lambda'$.

2.9. We continue with the same notation, and set $V = V^1 \oplus V^2$, $A = \mathrm{End}_F(V)$. We use our standard block matrix notation (see 'Notation' above), and start wi th a simple general observation.

LEMMA. *Let* X *be an* $\mathfrak{o}_F$*-lattice in* A. *The following are equivalent:*

(i) $\mathbf{1}^i \cdot X \cdot \mathbf{1}^j \subset X$, *for all* i, j;
(ii) $\mathbf{1}^i \cdot X \cdot \mathbf{1}^j = X \cap A^{ij}$, *for all* i, j;
(iii) $X = \coprod_{i,j} X \cap A^{ij}$.

The proof is straightforward.

PROPOSITION. *For* $i = 1, 2$, *let* V^i *be a finite-dimensional* F*-vector space and let* Λ^i *be a lattice sequence in* V^i *of period* e_i. *Define* e *and* Λ *as in* 2.8, *and use the other notation above. We have* $\mathbf{1}^i \in \mathfrak{a}_0(\Lambda)$, $i = 1, 2$. *Consequently,*

$$\mathfrak{a}_r(\Lambda) \cap A^{ij} = \mathbf{1}^i \cdot \mathfrak{a}_r(\Lambda) \cdot \mathbf{1}^j, \quad 1 \leqslant i, j \leqslant 2,$$

$$\mathfrak{a}_r(\Lambda) = \coprod_{1 \leqslant i, j \leqslant 2} \mathfrak{a}_r(\Lambda) \cap A^{ij},$$

and, further,

$$\mathfrak{a}_r(\Lambda) \cap A^i = \mathfrak{a}_{re_i/e}(\Lambda^i), \quad i = 1, 2,$$

for all $r \in \mathbb{R}$.

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Proof. For the first assertion, the definition gives us $\mathbf{1}^i\Lambda(r) = \Lambda^i(e_i r/e) \subset \Lambda(r)$, so $\mathbf{1}^i \in \mathfrak{a}_0(\Lambda)$. Since $\mathfrak{a}_r(\Lambda)$ is a bimodule over $\mathfrak{a}_0(\Lambda)$, the first two relations are then consequences of the Lemma. The third follows from (2.6). □

2.10. We now take a finite-dimensional F-vector space V, with $A = \mathrm{End}_F(V)$, and form the character $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$ of A, as in our list of general notations. If S is a subset of A, we write

$$S^* = \{y \in A: \psi_A(xy) = 1, x \in S\}.$$

PROPOSITION. *Let V be a finite-dimensional F-vector space and Λ a lattice sequence in V. We then have*

$$\mathfrak{a}_n(\Lambda)^* = \mathfrak{a}_{1-n}(\Lambda), \quad n \in \mathbb{Z}. \tag{2.10.1}$$

Proof. Abbreviate $e = e(\Lambda)$, and let W be an F-vector space of dimension e. Let Λ_W be a *strict* lattice sequence in W of period e. Set $X = V \oplus W$, $\Lambda_X = \Lambda \oplus \Lambda_W$. The lattice sequence Λ_X is then strict of period e. Proposition 2.9 gives us the relation

$$\mathfrak{a}_m(\Lambda) = \mathfrak{a}_m(\Lambda_X) \cap \mathrm{End}_F(V) = \mathbf{1}_V \mathfrak{a}_m(\Lambda_X)\mathbf{1}_V$$

for all $m \in \mathbb{Z}$, where $\mathbf{1}_V$ denotes the obvious projection $X \to V$.

Next, we write $A_X = \mathrm{End}_F(X)$ and use ψ_F to define the analogue of the 'star' operation in A_X:

$$S^\dagger = \{y \in A_X: \psi_{A_X}(xy) = 1, \ x \in S\},$$

where $S \subset A_X$ and $\psi_{A_X} = \psi_F \circ \mathrm{tr}_{A_X/F}$. If S is an $\mathfrak{o}_F$-lattice, we have $S^\dagger \cap A = (\mathbf{1}_V S \mathbf{1}_V)^*$. Since Λ_X is strict, we can use [2] p. 190 to get

$$\mathfrak{a}_n(\Lambda_X)^\dagger = \mathfrak{a}_{1-n}(\Lambda_X), \quad n \in \mathbb{Z}.$$

Thus

$$\begin{aligned} \mathfrak{a}_{1-n}(\Lambda) &= \mathfrak{a}_n(\Lambda_X)^\dagger \cap \mathrm{End}_F(V) \\ &= (\mathbf{1}_V \mathfrak{a}_n(\Lambda_X)\mathbf{1}_V)^* \\ &= \mathfrak{a}_n(\Lambda)^* \end{aligned}$$

as required. □

Remark. The relation (2.10.1) does *not* hold for real indices. Indeed, we have

$$\mathfrak{a}_r(\Lambda)^* = \mathfrak{a}_{-r}(\Lambda), \quad r \in \mathbb{R}, r \notin \mathbb{Z}. \tag{2.10.2}$$

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3. Split Strata

We need to generalize the notion of stratum (as in [5] Chapter 1) to the context of lattice sequences. The main result is Theorem 3.7, together with its Corollary 3.9; this is the first real step in our construction of semisimple types.

3.1. Let V be a finite-dimensional F-vector space and put $A = \mathrm{End}_F(V)$. A *stratum* in A is now a quadruple $[\Lambda, n, s, b]$, where Λ is an $\mathfrak{o}_F$-lattice sequence in V, $n \in \mathbb{Z}$, $s \in \mathbb{R}$ with $s < n$, and $b \in \mathfrak{a}_{-n}(\Lambda)$. Two strata $[\Lambda, n, s, b_i]$ are *equivalent*, denoted $[\Lambda, n, s, b_1] \sim [\Lambda, n, s, b_2]$, if $b_1 \equiv b_2 \pmod{\mathfrak{a}_{-s}(\Lambda)}$. To all intents and purposes therefore, there is no real distinction between $[\Lambda, n, [s], b]$ and $[\Lambda, n, s, b]$, but it will be useful to have the more general grading structure.

A stratum denoted $[\mathfrak{A}, n, m, b]$ in [5] (with $m, n \in \mathbb{Z}$) is the same as what we here call $[\Lambda_{\mathfrak{A}}, n, m, b]$, where $\Lambda_{\mathfrak{A}}$ denotes the strict lattice sequence determined by the hereditary order $\mathfrak{A}$. We shall, however, continue to use the notation of [5] alongside its extension to lattice sequences: this is useful when we are working with strict lattice sequences and wish to emphasize the fact.

3.2. As in the standard case, we can define the *characteristic polynomial* $\varphi_b(X) \in k_F[X]$ of a stratum $[\Lambda, n, s, b]$. To do this, we need to choose a prime element π_F of F, and we write $e = e(\Lambda)$, $g = \gcd(n, e)$. The element $b_0 = \pi_F^{n/g} b^{e/g}$ then lies in $\mathfrak{a}_0(\Lambda)$. Its characteristic polynomial as an endomorphism of V is monic and lies in $\mathfrak{o}_F[X]$; we define $\varphi_b(X)$ as the reduction mod $\mathfrak{p}_F$ of this characteristic polynomial. We observe that $\varphi_b(X)$ depends only on the equivalence class of the stratum $[\Lambda, n, n-1, b]$ or, equivalently, that of $[\Lambda, n, n-\varepsilon, b]$ for any $1 \geqslant \varepsilon > 0$.

We have to remember here the possibility $\mathfrak{a}_{-n}(\Lambda) = \mathfrak{a}_{1-n}(\Lambda)$. If this happens and we have a stratum $[\Lambda, n, n-1, b]$, then the element b_0 above lies in $\mathfrak{a}_1(\Lambda)$ and $\varphi_b(X) = X^N$, where $N = \dim_F V$.

3.3. Let V be a finite-dimensional F-vector space; we write $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Let Λ be a lattice sequence in V, as in Section 2. We write

$$\boldsymbol{u}(\Lambda) = \boldsymbol{u}_0(\Lambda) = \mathfrak{a}_0(\Lambda)^{\times},$$

$$\boldsymbol{u}_r(\Lambda) = 1 + \mathfrak{a}_r(\Lambda), \quad r \in \mathbb{R}, r > 0.$$

If $\mathfrak{A}$ denotes the hereditary order in A defined by the lattice chain $\mathcal{L}_\Lambda$, i.e., $\mathfrak{A} = \mathfrak{a}_0(\Lambda)$, we have $\boldsymbol{u}(\Lambda) = \boldsymbol{U}(\mathfrak{A})$. If $\mathfrak{P} = \mathfrak{a}_1(\Lambda)$ is the radical of $\mathfrak{A}$, we further have $\boldsymbol{u}_r(\Lambda) = \boldsymbol{U}^1(\mathfrak{A}) = 1 + \mathfrak{P}$ when $0 < r \leqslant 1$.

The set $\{\boldsymbol{u}_n(\Lambda): n \in \mathbb{Z}, n \geqslant 1\}$ gives a filtration of the parahoric subgroup $\boldsymbol{U}(\mathfrak{A})$ of G by open normal subgroups; this, however, is not usually the standard filtration of $\boldsymbol{U}(\mathfrak{A})$ by the principal congruence subgroups $\boldsymbol{U}^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$, $n \geqslant 1$.

3.4. Now let $x \in G = \mathrm{Aut}_F(V)$, and abbreviate $\nu = \boldsymbol{\nu}_\Lambda(x)$, $\nu' = \boldsymbol{\nu}_\Lambda(x^{-1})$ (notation of 2.7). For $r \in \mathbb{R}$, we have by definition

$$\Lambda(r) = x^{-1} x \Lambda(r) \subset x^{-1} \Lambda(r + \nu) \subset \Lambda(r + \nu + \nu'),$$

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whence $\nu + \nu' \leqslant 0$. We have $\nu + \nu' = 0$ if and only if the containments above are equalities for all r. We say that x is Λ-*invertible* if it satisfies this condition, i.e., $\boldsymbol{\nu}_\Lambda(x^{-1}) = -\boldsymbol{\nu}_\Lambda(x)$.

PROPOSITION.

(i) *Let $x \in G$. Then x is Λ-invertible if and only if $x\Lambda(r) = \Lambda(r + \boldsymbol{\nu}_\Lambda(x))$, for all $r \in \mathbb{R}$.*

(ii) *If $x \in G$ is Λ-invertible and $\nu = \boldsymbol{\nu}_\Lambda(x)$, we have $x\mathfrak{a}_n(\Lambda) = \mathfrak{a}_n(\Lambda)x = \mathfrak{a}_{n+\nu}(\Lambda)$, for all $n \in \mathbb{Z}$.*

(iii) *The set $\mathfrak{K}(\Lambda)$ of Λ-invertible elements of G is a subgroup of G which normalizes and contains $\boldsymbol{u}(\Lambda)$.*

Proof. Immediate. □

Remarks. (i) The lattice chain $\mathcal{L}_\Lambda$ determined by Λ is certainly stabilized by $\mathfrak{K}(\Lambda)$, in the sense that $gL \in \mathcal{L}_\Lambda$ whenever $g \in \mathfrak{K}(\Lambda)$ and $L \in \mathcal{L}_\Lambda$. Thus $\mathfrak{K}(\Lambda)$ is contained in the G-stabilizer of $\mathcal{L}_\Lambda$. The G-stabilizer of $\mathcal{L}_\Lambda$ is the group

$$\mathfrak{K}(\mathfrak{A}) = \{x \in G : x^{-1}\mathfrak{A}x = \mathfrak{A}\},$$

where $\mathfrak{A}$ is the hereditary order $\mathfrak{a}_0(\Lambda)$ defined by $\mathcal{L}_\Lambda$. (We recall that $\mathfrak{K}(\mathfrak{A})$ is also the G-normalizer of the group $\boldsymbol{U}(\mathfrak{A}) = \boldsymbol{u}(\Lambda)$.) We thus have

$$\mathfrak{K}(\mathfrak{A}) \supset \mathfrak{K}(\Lambda) \supset F^\times \boldsymbol{U}(\mathfrak{A}),$$

but, in general, both containments here may be strict.

(ii) The definitions imply that $\mathfrak{K}(\Lambda)$ normalizes the filtration subgroups $\boldsymbol{u}_r(\Lambda)$ or, equivalently, the lattices $\mathfrak{a}_r(\Lambda)$, $r \geqslant 0$. Proposition 2.5 shows that we can recover the lattice sequence Λ, up to a translation, from the lattice-valued function $n \mapsto \mathfrak{a}_n(\Lambda)$, $n \in \mathbb{Z}$, $n \geqslant 1$, whence it follows that $\mathfrak{K}(\Lambda) = \bigcap_{r\geqslant 0} \mathrm{N}_G(\boldsymbol{u}_r(\Lambda))$, where N denotes normalizer.

3.5. We now return to the extended notion of stratum 3.1.

PROPOSITION. *Let $[\Lambda, n, s, b]$ be a stratum. Then:*

(i) *If some element $b' \in b + \mathfrak{a}_{1-n}(\Lambda)$ is Λ-invertible with $\boldsymbol{\nu}_\Lambda(b') = -n$, then we have* (a) *$\mathfrak{a}_{1-n}(\Lambda) \neq \mathfrak{a}_{-n}(\Lambda)$, and* (b) *$b + \mathfrak{a}_{1-n}(\Lambda) = b\boldsymbol{u}_1(\Lambda)$ and every element of this set is Λ-invertible of valuation $-n$.*

(ii) *The stratum $[\Lambda, n, s, b]$ satisfies the properties in* (i) *if and only if the characteristic polynomial $\varphi_b(X)$ is not divisible by X.*

The proof is immediate. We temporarily call strata satisfying these conditions *nondegenerate.*

3.6. We shall need a notion of *direct sum* for strata, corrresponding to the direct sum operation on lattice sequences given in 2.8.

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For $i = 1, 2$, let V^i be a finite-dimensional F-vector space and $[\Lambda^i, n_i, n_i - 1, b_i]$ a stratum in $\mathrm{End}_F(V^i)$. We can form the lattice sequence $\Lambda = \Lambda^1 \oplus \Lambda^2$ in $V = V^1 \oplus V^2$. Setting $e = e(\Lambda) = \mathrm{lcm}\,(e(\Lambda^1), e(\Lambda^2))$, $n = \max\, n_i e/e(\Lambda^i)$, we get a stratum $[\Lambda, n, n-1, b]$ in $\mathrm{End}_F(V)$, where $b = b_1 \oplus b_2$.

In this situation, it is easy to compute the characteristic polynomial $\varphi_b(X)$ of the stratum $[\Lambda, n, n-1, b]$:

LEMMA. *Assume (by symmetry) that* $n_1/e(\Lambda^1) \geqslant n_2/e(\Lambda^2)$.

(i) *If* $n_1/e(\Lambda^1) = n_2/e(\Lambda^2)$, *we have* $\varphi_b(X) = \varphi_{b_1}(X)\varphi_{b_2}(X)$.
(ii) *If* $n_1/e(\Lambda^1) > n_2/e(\Lambda^2)$, *we have* $\varphi_b(X) = \varphi_{b_1}(X)X^m$, *where* $m = \dim(V^2)$. *In this case, moreover, we have* $[\Lambda, n, n-1, b] \sim [\Lambda, n, n-1, b_1 \oplus 0]$.

We will be particularly interested in a special case of the foregoing. The following notation will be standard for some time:

NOTATION. *For* $i = 1, 2$, V^i *is a finite-dimensional* F*-vector space and* $V = V^1 \oplus V^2$. *We let* Λ^i *be a lattice sequence in* V^i, *and set* $\Lambda = \Lambda^1 \oplus \Lambda^2$. *We write* $V = V^1 \oplus V^2$ *and* $A = \mathrm{End}_F(V)$. *Let* $b_i \in A^i = \mathrm{End}_F(V^i)$ *satisfy*

(i) $\boldsymbol{\nu}_{\Lambda^1}(b_1) = -n_1 < 0$ *and* b_1 *is* Λ^1*-invertible*;
(ii) *either* $\boldsymbol{\nu}_{\Lambda^2}(b_2) > -n_1 e(\Lambda^2)/e(\Lambda^1)$, *or else all the following conditions hold*:
   (a) $\boldsymbol{\nu}_{\Lambda^2}(b_2) = -n_2 = -n_1 e(\Lambda^2)/e(\Lambda^1)$,
   (b) b_2 *is* Λ^2*-invertible*,
   (c) $\gcd(\varphi_{b_1}, \varphi_{b_2}) = 1$.

We use our standard block matrix notation $A = \bigoplus A^{ij}$, and put

$$b = b_1 \oplus b_2 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

As above, we have $\boldsymbol{\nu}_\Lambda(b) = -n = -n_1 e\,(\Lambda)/e(\Lambda^1) \in \mathbb{Z}$. For this value of n, we define a pair of $\mathfrak{o}_F$-lattices in A:

$$\mathfrak{h}_1 = \begin{pmatrix} (\mathfrak{a}_n)^{11} & (\mathfrak{a}_0)^{12} \\ (\mathfrak{a}_{n+1})^{21} & (\mathfrak{a}_n)^{22} \end{pmatrix}, \quad \mathfrak{h}_2 = \begin{pmatrix} (\mathfrak{a}_{n+1})^{11} & (\mathfrak{a}_0)^{12} \\ (\mathfrak{a}_{n+1})^{21} & (\mathfrak{a}_{n+1})^{22} \end{pmatrix}. \tag{3.6.1}$$

We also put $H_i = 1 + \mathfrak{h}_i$, $i = 1, 2$.

PROPOSITION.

(i) *The sets* H_i, $i = 1, 2$, *are compact open subgroups of* G.
(ii) *The map* $x \mapsto 1 + x$ *induces an isomorphism of* $\mathfrak{h}_1/\mathfrak{h}_2$ *with* H_1/H_2.

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(iii) *We have*

$$\mathfrak{h}_1^* = \begin{pmatrix} (\mathfrak{a}_{1-n})^{11} & (\mathfrak{a}_{-n})^{12} \\ (\mathfrak{a}_1)^{21} & (\mathfrak{a}_{1-n})^{22} \end{pmatrix},$$

$$\mathfrak{h}_2^* = \begin{pmatrix} (\mathfrak{a}_{-n})^{11} & (\mathfrak{a}_{-n})^{12} \\ (\mathfrak{a}_1)^{21} & (\mathfrak{a}_{-n})^{22} \end{pmatrix}.$$

Proof. These are straightforward computations, based on 2.9 and 2.10. □

We take ψ_F, ψ_A as usual, and form the character ψ_b of the group H_1. Thus

$$\psi_b(1+x) = \psi_A(bx), \quad x \in \mathfrak{h}_1. \tag{3.6.2}$$

This is indeed a character of H_1 which is trivial on H_2 by (2.10).

3.7. We use the notation introduced in 3.6. We write $I_G(\psi_b|H_1)$ for the set of $g \in G$ which intertwine the character ψ_b of the group H_1. We also write $G_i = \mathrm{Aut}_F(V^i)$ and $M = G_1 \times G_2$, regarded as a Levi subgroup of $G = \mathrm{Aut}_F(V)$. We also tend to identify, e.g., G_1 with its canonical image $G_1 \times \{1\}$ in M.

THEOREM. *We have* $I_G(\psi_b|H_1) \subset H_1 \cdot M \cdot H_1$.

Proof. The character ψ_b of H_1, and hence also the intertwining set $I(\psi_b) = I_G(\psi_b|H_1)$, depends only on the equivalence class of stratum $[\Lambda, n, n-1, b]$. We can therefore assume that, when we are in case (ii) of Notation 3.6, we actually have $b_2 = 0$. The element b then is Λ-invertible if and only if $b_2 \neq 0$.

We need a sequence of preliminary results.

PROPOSITION. *Under the hypotheses above, let* $c \in \mathfrak{a}_{1-n}(\Lambda) \cap A^{21}$ *and define a map* $\partial_c \colon A^{12} \to A^{12}$ *by* $\partial_c(x) = b_1x - xb_2 + xcx$, $x \in A^{12}$. *Then* $\partial_c(\mathfrak{a}_0(\Lambda) \cap A^{12}) = \mathfrak{a}_{-n}(\Lambda) \cap A^{12}$.

Proof. We need some lemmas.

LEMMA 1. *The map* $\partial = \partial_0$ *maps* $\mathfrak{a}_r(\Lambda) \cap A^{12}$ *onto* $\mathfrak{a}_{r-n} \cap A^{12}$, *for all* r.

Proof. It follows from the Λ^1-invertibility of b_1 that $\mathfrak{a}_{r-n}(\Lambda) \cap A^{12} = b_1(\mathfrak{a}_r(\Lambda) \cap A^{12})$. This gives the result in the case $b_2 = 0$. Suppose then that $b_2 \neq 0$, and consider the map $\delta \colon x \mapsto b_1^{-1}xb_2$. We have

$$b^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} b = \begin{pmatrix} 1 & b_1^{-1}xb_2 \\ 0 & 1, \end{pmatrix}$$

and since b is Λ-invertible, it follows that our map δ takes $\mathfrak{a}_r \cap A^{12}$ to itself. We consider the e_0-th power of this map, where $e_0 = e/\gcd(n, e)$. Since the polynomials φ_{b_i} are relatively prime, no eigenvalue of δ^{e_0} (in some splitting field) is a 1-unit. It

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follows that the map $x \mapsto x - \delta(x)$ has only unit eigenvalues (in the splitting field), and therefore maps $\mathfrak{a}_r(\Lambda) \cap A^{12}$ bijectively to itself. The lemma now follows. □

LEMMA 2. *We have* $\partial_c(\mathfrak{a}_0(\Lambda) \cap A^{12}) + \mathfrak{a}_k(\Lambda) \cap A^{12} = \mathfrak{a}_{-n}(\Lambda) \cap A^{12}$, *for all integers* $k \geqslant -n$.

Proof. Since $c \in \mathfrak{a}_{1-n}(\Lambda) \cap A^{21}$, we have $\partial_c(\mathfrak{a}_0(\Lambda) \cap A^{12}) \subset \mathfrak{a}_{-n}(\Lambda) \cap A^{12}$, and the lemma holds if $k = -n$.

We therefore assume that the assertion holds for some fixed integer $k \geqslant -n$. Take $y \in \mathfrak{a}_{-n}(\Lambda) \cap A^{12}$ and choose $x \in \mathfrak{a}_0(\Lambda) \cap A^{12}$, $z \in \mathfrak{a}_k(\Lambda) \cap A^{12}$ such that $\partial_c(x) + z = y$.

By Lemma 1, there exists $x_1 \in \mathfrak{a}_{k+n}(\Lambda) \cap A^{12}$ such that $\partial_0(x_1) = z$. Also, $x_1 c x_1 \in \mathfrak{a}_{2(k+n)+1-n}(\Lambda) \cap A^{12} \subset \mathfrak{a}_{k+1}(\Lambda) \cap A^{12}$, while the elements xcx_1, $x_1 cx$ both lie in $\mathfrak{a}_{k+1}(\Lambda) \cap A^{12}$. It follows that $\partial_c(x + x_1) - y \in \mathfrak{a}_{k+1}(\Lambda) \cap A^{12}$, and the lemma holds by induction. □

Thus, if $y \in \mathfrak{a}_{-n}(\Lambda) \cap A^{12}$ and $k \geqslant 1$, there exists $y_k \in \mathfrak{a}_0(\Lambda) \cap A^{12}$ such that $\partial_c(y_k) - y \in \mathfrak{a}_k(\Lambda) \cap A^{12}$. The set $\mathfrak{a}_0(\Lambda) \cap A^{12}$ is compact, so $\{y_k\}$ has a convergent subsequence $\{y_{k_i}\}$. The limit, call it y_∞, of this subsequence then satisfies $\partial_c(y_\infty) = y$, and we have proved the Proposition. □

We now prove the Theorem. We write $\mathcal{M}$ for the algebra $A^1 \oplus A^2 \subset A$, so that $M = \mathcal{M}^\times$. We first need:

LEMMA 3. *Let* $x = b + y$, $y \in \mathfrak{h}_1^*$. *There exists* $h \in H_1$ *such that* $hxh^{-1} \in b + \mathfrak{h}_1^* \cap \mathcal{M}$.

Proof. We write x in the form

$$x = \begin{pmatrix} b_1' & a \\ c & b_2' \end{pmatrix}.$$

For an element $u \in A^{12}$, we have

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1' - uc & 0 \\ c & b_2' + cu \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1' & \partial_c(-u) \\ c & b_2' \end{pmatrix}.$$

The element a above lies in $(\mathfrak{a}_{-n})^{12}$, so we can apply the Proposition to get an element $u \in (\mathfrak{a}_0)^{12}$ such that $\partial_c(-u) = a$. The matrix $1 + u$ lies in H_1, so we have reduced to the case where x is lower triangular,

$$x = \begin{pmatrix} b_1' & 0 \\ c & b_2' \end{pmatrix}.$$

LEMMA 4. *The map* $y \mapsto yb_1' - b_2'y$, $y \in A^{21}$, *induces an isomorphism* $(\mathfrak{a}_{n+r})^{21} \cong (\mathfrak{a}_r)^{21}$ *for all* $r \in \mathbb{Z}$.

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Proof. It is enough to show that the map is surjective. The case $b_1' = b_1$, $b_2' = b_2$ is essentially identical to Lemma 1, so we omit the details. In general, we write δ, δ' for the maps $y \mapsto yb_1 - b_2y$, $y \mapsto yb_1' - b_2'y$ respectively. For $y \in (\mathfrak{a}_{n+r})^{21}$, we have $\delta'(y) \equiv \delta(y) \pmod{(\mathfrak{a}_{r+1})^{21}}$, from which it follows that $(\mathfrak{a}_r)^{21} = \delta'(\mathfrak{a}_{n+r})^{21} + (\mathfrak{a}_{r+1})^{21}$. The lemma follows immediately. □

In particular, $y \mapsto yb_1' - b_2'y$ gives a surjection $(\mathfrak{a}_{n+1})^{21} \twoheadrightarrow (\mathfrak{a}_1)^{21}$. We have

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} b_1' & 0 \\ c & b_2' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} = \begin{pmatrix} b_1' & 0 \\ c + yb_1' - b_2'y & b_2' \end{pmatrix}.$$

For suitable choice of $y \in (\mathfrak{a}_{n+1})^{21}$, this matrix is diagonal, while $1 + y \in H_1$. This proves Lemma 3. □

We now prove the theorem. For elementary reasons, an element $g \in G$ intertwines the character ψ_b if and only if $g^{-1}(b + \mathfrak{h}_1^*)g \cap (b + \mathfrak{h}_1^*) \neq \emptyset$. Thus, if g intertwines ψ_b, we have elements $x, y \in \mathfrak{h}_1^*$ such that $g^{-1}(b + x)g = b + y$. Lemma 3 allows us to replace g by h_1gh_2, $h_i \in H_1$, and assume that $x, y \in \mathfrak{h}_1^* \cap \mathcal{M}$. We put

$$b + x = \begin{pmatrix} b_1' & 0 \\ 0 & b_2' \end{pmatrix}, \qquad b + y = \begin{pmatrix} b_1'' & 0 \\ 0 & b_2'' \end{pmatrix},$$

and write out the equation $(b + x)g = g(b + y)$. Comparing (1,2)-entries, we have $b_1'g_{12} = g_{12}b_2''$. The upper triangular analogue of Lemma 4 shows that the map $A^{12} \to A^{12}$ given by $z \mapsto b_1'z - zb_2''$ is injective, so $g_{12} = 0$. Likewise $g_{21} = 0$, whence $g \in M$ as required. □

3.8. We will also need an approximate version of Theorem (3.7). For this, we simplify our block matrix notation (3.6.1) in the obvious way.

COROLLARY. *In the situation of* 3.7, *let* q *be an integer with* $0 \leqslant q \leqslant n$, *and define a group by*

$${}_qH_1 = 1 + \begin{pmatrix} \mathfrak{a}_n & \mathfrak{a}_q \\ \mathfrak{a}_{n+1} & \mathfrak{a}_n \end{pmatrix}.$$

An element $g \in G$ *then intertwines the character* $\psi_b|_q H_1$ *if and only if there exist* $x, y \in (\mathfrak{a}_{n+1-q})^{21}$ *such that* $(1 + x)g(1 + y)$ *intertwines* $\psi_b|H_1$. *In other words,*

$$I_G(\psi_b|_q H_1) \subset \begin{pmatrix} 1 & 0 \\ \mathfrak{a}_{n+1-q} & 1 \end{pmatrix} H_1 M H_1 \begin{pmatrix} 1 & 0 \\ \mathfrak{a}_{n+1-q} & 1 \end{pmatrix}.$$

Proof. It follows from 3.7 Lemma 1 that, under the conditions (3.6), the group $1 + (\mathfrak{a}_{n+1-q})^{21}$ acts transitively (by conjugation) on the set of characters of H_1

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agreeing with ψ_b on the subgroup ${}_qH_1$. Thus, if ξ temporarily denotes the representation of H_1 induced by $\psi_b|_q H_1$, the G-intertwining of ξ is:

$$(1+(\mathfrak{a}_{n+1-q})^{21})I(\psi_b)(1+(\mathfrak{a}_{n+1-q})^{21}).$$

This is the same as the G-intertwining of $\psi_b|_q H_1$, so the result follows. □

3.9. We continue in the situation of 3.7. We now give a corollary of Theorem 3.7, which is one of our major results.

We suppose given, for each i, a subgroup K_i of $\boldsymbol{u}(\Lambda^i)$ containing $H_1 \cap G_i$, and an irreducible representation ϱ_i of K_i whose restriction to $K_i \cap H_1$ is a multiple of ψ_b there. It will also be useful to have the notation

$$P_{\mathrm{u}} = \begin{pmatrix} A^{11} & A^{12} \\ 0 & A^{22} \end{pmatrix} \cap G, \quad N_{\mathrm{u}} = 1 + A^{12},$$

$$P_{\ell} = \begin{pmatrix} A^{11} & 0 \\ A^{21} & A^{22} \end{pmatrix} \cap G, \quad N_{\ell} = 1 + A^{21}.$$

In these circumstances, we have the following. (For the notion of cover, see the Introduction above or [9] (8.1).)

COROLLARY.

(i) *The set* $K = (K_1 \times K_2) \cdot H_1$ *is a group.*
(ii) *There is a unique irreducible representation* ϱ *of* K *which is trivial on* $K \cap N_{\mathrm{u}}$, $K \cap N_{\ell}$, *and whose restriction to* $K_1 \times K_2$ *is equivalent to* $\varrho_1 \otimes \varrho_2$.
(iii) *The pair* (K, ϱ) *is a* G*-cover of* $(K_1 \times K_2, \varrho_1 \otimes \varrho_2)$.

Proof. We have $K_1 \times K_2 \subset \boldsymbol{u}(\Lambda) \cap M$, from which it follows that the group $K_1 \times K_2$ normalizes H_1. Assertion (i) follows, and assertion (ii) is immediate.

To prove (iii), we first have to check that there is an Iwahori decomposition

$$K = K \cap N_{\ell} \cdot K \cap M \cdot K \cdot N_{\mathrm{u}},$$

and this follows from the definition of K. The outer factors here are certainly contained in the kernel of ϱ.

Since P_{u}, P_{ℓ} are the only parabolic subgroups of G with Levi component M, it remains only to verify the following. (As usual, we write $\mathcal{H}(G, \varrho)$ for the convolution algebra of compactly supported ϱ-spherical functions on G.)

LEMMA. *Let* π_F *be some prime element of* F, *and define*

$$\zeta = \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix} \in M.$$

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Let $f \in \mathcal{H}(G, \varrho)$ have support $K\zeta K$. Then f is invertible in $\mathcal{H}(G, \varrho)$ and its inverse is supported on $K\zeta^{-1}K$.

Proof. First we note that the function f is uniquely determined, up to scalar constant factor, by the condition $\operatorname{supp} f = K\zeta K$. We may as well assume that $f(\zeta)$ is the identity map (denote it **1**) on the space underlying the contragredient of ϱ. Likewise, there is a unique function $f' \in \mathcal{H}(G, \varrho)$ with support $K\zeta^{-1}K$ such that $f'(\zeta^{-1}) = \mathbf{1}$. We compute $f' \star f$.

The support of this convolution is contained in

$$K\zeta^{-1}K\zeta K = K \cdot \zeta^{-1}K\zeta \cap N_{\mathrm{u}} \cdot K.$$

Now let $x \in N_{\mathrm{u}}$ intertwine ϱ. By the Iwahori decomposition for K and 3.7, there exist $k_1, k_2 \in H_1 \cap N_{\mathrm{u}}$ such that $k_1 x k_2 \in P_\ell$. We deduce that the intertwining of ϱ in N_{u} is contained in $K \cap N_{\mathrm{u}} = H_1 \cap N_{\mathrm{u}}$. In other words, the support of $f' \star f$ must be contained in K. A simple computation shows that $f' \star f(1_G)$ is of the form $c\mathbf{1}$, for some $c > 0$. Thus f is left invertible in $\mathcal{H}(G, \varrho)$. It follows that f is invertible (cf. [9] proof of (7.14)), with inverse $c^{-1}f'$. □

4. Ps-Characters and Endo-Classes

In this section, we are concerned only with the standard situation relating to lattice chains, rather than lattice sequences. We recall one of the basic concepts of [5], namely that of *simple character* (*ibid.* Chapter 3), and some developments of the idea in [4].

4.1. Let E/F be a finite field extension, and $\beta \in E^\times$ an element such that $E = F[\beta]$. The algebra $A(E) = \operatorname{End}_F(E)$ contains a unique hereditary $\mathfrak{o}_F$-order $\mathfrak{A}(E)$ with the property $\mathfrak{K}(\mathfrak{A}(E)) \supset E^\times$; indeed, $\mathfrak{A}(E)$ is the hereditary order defined by the lattice chain $\{\mathfrak{p}_E^j : j \in \mathbb{Z}\}$. Attached to this is the quantity $k_0(\beta, \mathfrak{A}(E)) \in \mathbb{Z} \cup \{\infty\}$, whose definition we now recall. Write $\mathfrak{P}(E)$ for the radical of $\mathfrak{A}(E)$, and a_β for the map $x \mapsto \beta x - x\beta$, $x \in A(E)$. Then $k_0(\beta, \mathfrak{A}(E))$ is the least integer k such that $\mathfrak{P}(E)^k \cap a_\beta(A) \subset a_\beta(\mathfrak{A}(E))$, with the understanding that $k_0(\beta, \mathfrak{A}(E)) = -\infty$ in the case $E = F$. (See [5] (1.4.11).)

Here, we prefer to use the briefer notation

$$k_0(\beta, \mathfrak{A}(E)) = k_F(\beta).$$

Write ν_E for the standard additive valuation on E. We thus have either $k_F(\beta) = -\infty$ or else $k_F(\beta) \geqslant \nu_E(\beta)$.

More generally, if V is a finite-dimensional E-vector space and $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in $\operatorname{End}_F(V)$ with $E^\times \subset \mathfrak{K}(\mathfrak{A})$, we can define $k_0(\beta, \mathfrak{A})$ in the same way *loc. cit.* We have the relation

$$k_0(\beta, \mathfrak{A}) = k_F(\beta) e(\mathfrak{A}|\mathfrak{o}_F)/e(E|F).$$

4.2. In the language of [4] Section 1, a *simple pair* (k, β) *over* F consists of an integer k and a nonzero element β generating a finite field extension E of F such that

$$-k > \max\{k_F(\beta), \nu_E(\beta)\}.$$

Let (k, β) be a simple pair over F with $k \geqslant 0$. Let V be some finite-dimensional F-vector space, and let $[\mathfrak{A}, n, m, \beta]$ be a simple stratum ([5] (1.5)) in $A = \mathrm{End}_F(V)$, with $m \geqslant 0$. Thus, in particular, $m < -k_0(\beta, \mathfrak{A})$ and the simple character set $\mathcal{C}(\mathfrak{A}, m, \beta)$ of [5] (3.2) is defined. We recall that the elements of $\mathcal{C}(\mathfrak{A}, m, \beta)$ are certain abelian characters of the group $H^{m+1}(\beta, \mathfrak{A})$ defined in [5] (3.1).

It will be convenient to extend our indexing as in earlier sections. For $r \in \mathbb{R}$, $r > 0$, we define

$$H^r(\beta, \mathfrak{A}) = H^{-[-r]}(\beta, \mathfrak{A}), \qquad \mathcal{C}(\mathfrak{A}, r, \beta) = \mathcal{C}(\mathfrak{A}, [r], \beta).$$

For $i = 1, 2$, let V_i be a finite-dimensional vector space over $E = F[\beta]$, and let $[\mathfrak{A}_i, n_i, m_i, \beta]$ be a simple stratum in $\mathrm{End}_F(V_i)$ with $m_i \geqslant 0$. Set $e_i = e(\mathfrak{A}_i|\mathfrak{o}_F)/e(E|F)$. Suppose we have

$$\left[\frac{m_1}{e_1}\right] = \left[\frac{m_2}{e_2}\right] = k.$$

By [5] (3.6.14) there is a canonical bijection

$$\boldsymbol{\tau}_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}\colon \mathcal{C}(\mathfrak{A}_1, m_1, \beta) \xrightarrow{\approx} \mathcal{C}(\mathfrak{A}_2, m_2, \beta). \tag{4.2.1}$$

For any $r \in \mathbb{R}$, we have

$$\left[\frac{[r]}{e_1}\right] = \left[\frac{\left[\frac{re_2}{e_1}\right]}{e_2}\right],$$

so 4.2.1 gives a bijection $\boldsymbol{\tau}_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}\colon \mathcal{C}(\mathfrak{A}_1, r, \beta) \cong \mathcal{C}(\mathfrak{A}_2, re_2/e_1, \beta)$ for any real $r > 0$.

4.3. The bijections of 4.2.1 exhibit a strong coherence property which is most conveniently expressed via a notion from [4] Section 8.

We start with a simple pair (k, β) in which $k \geqslant 0$, and write $E = F[\beta]$. We suppose given a triple $(V, \mathfrak{B}, m)$, where

(i) *V is a finite-dimensional E-vector space,*
(ii) *$\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order in* $\mathrm{End}_E(V)$,

(iii) *m is an integer such that* $[m/e(\mathfrak{B}|\mathfrak{o}_E)] = k$.

To this data, we can attach the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(V)$ defined by the same lattice chain as $\mathfrak{B}$. If we put $n = -\nu_E(\beta)e(\mathfrak{B}|\mathfrak{o}_E)$, the stratum $[\mathfrak{A}, n, m, \beta]$ is simple. Moreover, for any two choices of m satisfying (iii), the simple character sets $\mathcal{C}(\mathfrak{A}, m, \beta)$ are in canonical bijection (4.2.1).

A *ps-character* (attached to the simple pair (k, β), $k \geqslant 0$) is then a triple (Θ, k, β), where Θ is a simple-character-valued function as follows: to each triple $(V, \mathfrak{B}, m)$ as above, Θ attaches a simple character $\Theta(\mathfrak{A}) \in \mathcal{C}(\mathfrak{A}, m, \beta)$. We call this the *realization of* Θ *on* $\mathfrak{A}$ *of level* m. These realizations are subject to the following coherence condition: if we have two realizations $\Theta(\mathfrak{A}_i)$ of (Θ, k, β) on orders $\mathfrak{A}_i$, $i = 1, 2$, they are related by

$$\Theta(\mathfrak{A}_2) = \boldsymbol{\tau}_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}\Theta(\mathfrak{A}_1).$$

Thus, for example, $\Theta(\mathfrak{A}_2)$ is completely determined by $\Theta(\mathfrak{A}_1)$ (and the element β). Given the simple pair (k, β), the ps-character (Θ, k, β) is therefore determined by any one of its realizations.

Now suppose we are given ps-characters (Θ_i, k_i, β_i) over F, $i = 1, 2$. We say they are *endo-equivalent*, denoted

$$(\Theta_1, k_1, \beta_1) \approx (\Theta_2, k_2, \beta_2),$$

if there exists an F-vector space V, hereditary $\mathfrak{o}_F$-orders $\mathfrak{A}_i$ in $\mathrm{End}_F(V)$, and realizations $\Theta_i(\mathfrak{A}_i)$ of the Θ_i *of the same level*, such that $\mathfrak{A}_1 \cong \mathfrak{A}_2$ as $\mathfrak{o}_F$-orders, and such that the simple characters $\Theta_i(\mathfrak{A}_i)$ intertwine (hence, by [5] Theorem 3.5.11, are conjugate) in $\mathrm{Aut}_F(V)$. We then know ([4] (8.7)) that any realizations of the given ps-characters on isomorphic orders in the same endomorphism algebra must intertwine.

In particular, endo-equivalence is an equivalence relation on the set of ps-characters over F. We refer to the equivalence classes as *endo-classes* of simple characters.

Comment. This definition of endo-equivalence is not quite the same as that given in [4]; however, it is easy to see (using [4] (8.3)) that the two definitions are equivalent.

4.4. It is convenient to extend this terminology by admitting a trivial ps-character Θ^0: if $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in some $\mathrm{End}_F(V)$, the realization of Θ^0 on $\mathfrak{A}$ is the trivial character of $\boldsymbol{U}^1(\mathfrak{A})$. This ps-character Θ^0 is not endo-equivalent to any nontrivial Θ.

4.5. Let π be an irreducible supercuspidal representation of the group $G = \mathrm{Aut}_F(V)$. Then π contains a maximal simple type (J, λ), which is uniquely determined up to G-conjugacy ([5] Theorem 8.4.1).

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If π contains the trivial character of $\boldsymbol{U}^1(\mathfrak{A})$, for some hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(V)$, we say that π is *of level zero* and set $\Theta_\pi = \{\Theta^0\}$.

Otherwise, there is a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathrm{End}_F(V)$ such that $J = J(\beta, \mathfrak{A})$ and a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ such that $\lambda | H^1(\beta, \mathfrak{A})$ is a multiple of θ. This character θ determines a ps-character $(\Theta, 0, \beta)$ and hence an endo-class which we denote Θ_π. The uniqueness of (J, λ) implies that the endo-class Θ_π depends only on the equivalence class of π. Indeed, replacing π by $\pi \otimes \chi$, where χ is an unramified quasicharacter, has no effect on Θ_π and so:

PROPOSITION. *The endo-class Θ_π defined above depends only on the inertial equivalence class $[G, \pi]_G \in \mathcal{B}(G)$.*

4.6. We need to recall an 'approximation' property of simple characters. We are given simple strata $[\mathfrak{A}, n, 0, \beta_i]$ in some $A = \mathrm{End}_F(V)$, $i = 1, 2$, and an integer m such that $1 \leqslant m < n$. We further have a simple stratum $[\mathfrak{A}, n, m, \gamma]$, such that $H^{m+1}(\gamma, \mathfrak{A}) = H^{m+1}(\beta_i, \mathfrak{A})$ and $\mathcal{C}(\mathfrak{A}, m, \gamma) = \mathcal{C}(\mathfrak{A}, m, \beta_i)$. (This arises if, for example, $[\mathfrak{A}, n, m, \gamma] \sim [\mathfrak{A}, n, m, \beta_i]$.)

In these circumstances, one knows $H^m(\gamma, \mathfrak{A}) = H^m(\beta_i, \mathfrak{A})$ ([5] (3.1.9) plus [8] 2.11). Thus, if we take characters $\theta_i \in \mathcal{C}(\mathfrak{A}, m-1, \beta_i)$, $\vartheta \in \mathcal{C}(\mathfrak{A}, m-1, \gamma)$ which agree on H^{m+1}, we get elements $c_i \in \mathfrak{P}^{-m}$ (where $\mathfrak{P}$ is the radical of $\mathfrak{A}$) such that $\theta_i = \vartheta\psi_{c_i}$, for each i. We let $\mathfrak{B}_\gamma = \mathfrak{A} \cap \mathrm{End}_{F[\gamma]}(V)$, and we choose a tame corestriction s on A relative to $F[\gamma]/F$ (see [5] Section 1.3). Thus we get strata $[\mathfrak{B}_\gamma, m, m-1, s(c_i)]$ in $\mathrm{End}_{F[\gamma]}(V)$.

PROPOSITION. *For $i = 1, 2$, the stratum $[\mathfrak{B}_\gamma, m, m-1, s(c_i)]$ is either null or equivalent to a simple stratum. The strata $[\mathfrak{B}_\gamma, m, m-1, s(c_i)]$ have the same characteristic polynomials (relative to some prime element of $F[\gamma]$) if and only if the ps-characters defined by the characters θ_i are endo-equivalent.*

This follows from [8] 2.8 and the observation [5] (proof of (2.6.1)) that the conjugacy class of a simple stratum is effectively determined by its characteristic polynomial.

5. Simple Characters for Lattice Sequences

In this section we generalize the notion of simple character to the context of lattice sequences. To do this, we require a substantial technical result (5.2 below) generalizing [5] (7.1.19). First, however, we need to extend some basic concepts from [5] Chapter 3.

5.1. Let E/F be some finite field extension, and let Λ be an $\mathfrak{o}_E$-lattice sequence in a finite-dimensional E-vector space V. Then Λ is also an $\mathfrak{o}_F$-lattice sequence in the F-space V, and the extension of Λ to a function on $\mathbb{R}$, as in 2.6, is independent of the base field. We have the period relation $e_F(\Lambda) = e_E(\Lambda)e(E|F)$. We write $\mathfrak{a}_r(\Lambda)$, $r \in \mathbb{R}$, for the $\mathfrak{o}_F$-lattices in $A = \mathrm{End}_F(V)$ defined in 2.6. Using the same definitions relative to the base field E, we get a sequence of $\mathfrak{o}_E$-lattices $\mathfrak{b}_r(\Lambda)$ in $B = \mathrm{End}_E(V)$ along with the relation $\mathfrak{b}_r(\Lambda) = \mathfrak{a}_r(\Lambda) \cap B$, $r \in \mathbb{R}$.

Suppose we are now given an element $\beta \in E^\times$ with $E = F[\beta]$. If Λ is an E-lattice sequence, we can then define an integer $k_0(\beta, \Lambda)$ by $k_0(\beta, \Lambda) = k_F(\beta)e_E(\Lambda)$.

DEFINITION. Let V be a finite-dimensional F-vector space, and set $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. A stratum $[\Lambda, n, r, \beta]$ in A is simple if

(i) the algebra $E = F[\beta]$ is a field, and Λ is an $\mathfrak{o}_E$-lattice sequence;
(ii) $\boldsymbol{\nu}_\Lambda(\beta) = -n$;
(iii) $r < -k_0(\beta, \Lambda)$.

Note that condition (iii) here is equivalent to $[r] < -k_0(\beta, \Lambda)$, so this definition is consistent with the standard one [5] (1.5) for lattice chains.

5.2. We can now state the central result of the section. We first need a system of notation; this will remain standard for some time.

HYPOTHESES. *Let E/F be a finite field extension. For each integer i, $0 \leqslant i \leqslant t$, let V^i be a finite-dimensional E-vector space and Λ^i an $\mathfrak{o}_E$-lattice sequence in V^i. Write*

$$e_i = e_F(\Lambda^i), \quad 1 \leqslant i \leqslant t, \qquad e = e_F(\Lambda^0).$$

We assume that:

(a) *the lattice sequence Λ^0 is strict*;
(b) *e_i divides e, $1 \leqslant i \leqslant t$.*

We set

$$\begin{aligned}
&V = V^0 \oplus V^1 \oplus \cdots \oplus V^t,\\
&\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^t,\\
&A = \mathrm{End}_F(V), \quad G = \mathrm{Aut}_F(V),\\
&\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}_\Lambda), \quad \mathfrak{P} = \mathrm{rad}\,\mathfrak{A},\\
&B = \mathrm{End}_E(V),\\
&\mathfrak{B} = \mathfrak{A} \cap B, \quad \mathfrak{Q} = \mathfrak{P} \cap B.
\end{aligned} \tag{5.2.1}$$

Thus Λ is a strict $\mathfrak{o}_E$-lattice sequence in V whose $\mathfrak{o}_F$-period is e. (We note here that the definition of direct sum of lattice sequences is independent of the choice of base field.)

In this situation, we can use our standard block notation,

$$A^{ij} = \mathrm{Hom}_F(V^j, V^i), \quad 0 \leqslant i, j \leqslant t.$$

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Thus we get a parabolic subgroup P_u of G by $P_\mathrm{u} = G \cap (\coprod_{i \leq j} A^{ij})$ with Levi component $M = \prod_{i=0}^{t} \operatorname{Aut} FV^i$ and unipotent radical $N_\mathrm{u} = 1 + \coprod_{i<j} A^{ij}$. We write $P_\ell = MN_\ell$ for the opposite of P_u relative to the Levi factor M.

PROPOSITION. *Suppose we have an element β such that $E = F[\beta]$ and integers $n > m \geqslant 0$ such that $[\mathfrak{A}, n, m, \beta]$ is a simple stratum in A. Let $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$.*

(i) *The groups $H^{m+1} = H^{m+1}(\beta, \mathfrak{A})$ and $J^{m+1} = J^{m+1}(\beta, \mathfrak{A})$ then have Iwahori decomposition relative to the pair (M, P_u):*

$$H^{m+1} = H^{m+1} \cap N_\ell \cdot H^{m+1} \cap M \cdot H^{m+1} \cap N_\mathrm{u},$$
$$J^{m+1} = J^{m+1} \cap N_\ell \cdot J^{m+1} \cap M \cdot J^{m+1} \cap N_\mathrm{u}.$$

(ii) *The characters $\theta|(H^{m+1} \cap N_\mathrm{u})$, $\theta|(H^{m+1} \cap N_\ell)$ are both trivial.*

(iii) *We have $H^{m+1}(\beta, \mathfrak{A}) \cap \operatorname{Aut}_F(V^0) = H^{m+1}(\beta, \mathfrak{A}^0)$, where $\mathfrak{A}^0 = \mathfrak{A}(\Lambda^0)$. Moreover, the character $\theta^0 = \theta|H^{m+1}(\beta, \mathfrak{A}^0)$ lies in $\mathcal{C}(\mathfrak{A}^0, m, \beta)$ and indeed $\theta^0 = \boldsymbol{\tau}_{\mathfrak{A}, \mathfrak{A}^0, \beta}(\theta)$.*

Proof. We have $H^{m+1} = 1 + \mathfrak{H}^{m+1}$, where $\mathfrak{H}^{m+1} = \mathfrak{H}^{m+1}(\beta, \mathfrak{A})$, in the notation of [5]. The lattice $\mathfrak{H}^{m+1}$ is a $(\mathfrak{B}, \mathfrak{B})$-bimodule (see [5] (3.1.9)) and the canonical projection $V \to V^i$ (i.e., the one with kernel $\coprod_{j \neq i} V^j$) lies in $\mathfrak{B}$, as in (2.9). Thus $\mathfrak{H}^{m+1}$ is the direct sum of its intersections with the A^{ij}, and the Iwahori decomposition follows immediately. The proof for J^{m+1} is identical.

The proof of the second assertion requires some technical preparation, which we give in the next section.

5.3. We generalize the notion of '(W, E)-decomposition' from [5] (1.2). For this purpose, we fix a finite field extension E/F and a finite-dimensional E-vector space V. Consistent with (5.2.1), we put $B = \operatorname{End}_E(V)$, $A = \operatorname{End}_F(V)$.

We also write $A(E)$ for the F-endomorphism ring of E (viewed as F-vector space) and $\mathfrak{A}(E)$ for the unique hereditary $\mathfrak{o}_F$-order in $A(E)$ which is normalized by $E^\times$; the order $\mathfrak{A}(E)$ is attached to the lattice chain $\{\mathfrak{p}_E^k : k \in \mathbb{Z}\}$ in E.

Let W be some F-subspace of V such that the canonical map $E \otimes_F W \to V$ is an isomorphism. This induces an algebra isomorphism $A(E) \otimes_F \operatorname{End}_F(W) \cong A$, and so gives V the structure of left $A(E)$-module.

We observe that $A(E)$ is, in a natural way, a right E-space. The trivial observations $A(E) \otimes_E E = A(E)$, $E \otimes_F \operatorname{End}_F(W) = B$ combine with the algebra isomorphism above to give an isomorphism

$$A(E) \otimes_E B \cong A \tag{5.3.1}$$

of $(A(E), B)$-bimodules.

Next, let Λ be an $\mathfrak{o}_E$-lattice sequence in V. We can view this as a $\mathfrak{o}_F$-lattice sequence in V, and define the lattices $\mathfrak{a}_r(\Lambda)$ in A, $\mathfrak{b}_r(\Lambda)$ in B as in 5.1.

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LEMMA. *Suppose that the vector space W has an F-basis $\{w_1, w_2, \dots, w_m\}$ with the following property: for $j \in \mathbb{Z}$, there are integers $j(k)$, $1 \leqslant k \leqslant m$, such that*

$$\Lambda(j) = \mathfrak{p}_E^{j(1)} w_1 \oplus \mathfrak{p}_E^{j(2)} w_2 \oplus \cdots \oplus \mathfrak{p}_E^{j(m)} w_m.$$

The isomorphism (5.3.1) *then restricts to an isomorphism* $\mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{b}_r(\Lambda) \cong \mathfrak{a}_r(\Lambda)$, *for every* $r \in \mathbb{R}$.

Proof. Suppose first that the lattice sequence Λ is strict. The assertion then strengthens [5] (1.2.10), but the same proof applies. In the general case, let $\mathfrak{A} = \mathfrak{A}(\Lambda)$, and $\mathfrak{B} = \mathfrak{A} \cap B$. By the first case, we have $\mathfrak{A}(E) \otimes \mathfrak{B} = \mathfrak{A}$. On the other hand, we certainly have $\mathfrak{A}(E) \otimes \mathfrak{b}_r \subset \mathfrak{a}_r$. The result then follows on comparing indices via Proposition 2.4. □

If the subspace W of V satisfies $E \otimes W \cong V$ and the conditions of the lemma, we say that it is in *general position relative to* Λ *over* E. When this holds, we have $\mathfrak{K}(\mathfrak{A}(E)) \otimes 1 \subset \mathfrak{K}(\Lambda)$.

We now take an element β such that $E = F[\beta]$ and a simple stratum $[\Lambda, n, r, \beta]$ in A.

COROLLARY. *Let $n > s > r$, $s \in \mathbb{R}$. There exists a simple stratum $[\Lambda, n, s, \gamma]$ in A which is equivalent to $[\Lambda, n, s, \beta]$. Moreover, given a (W, E)-decomposition $A = A(E) \otimes B$ with W in general position relative to Λ over E, we may choose $\gamma \in \mathfrak{K}(\mathfrak{A}(E)) \otimes 1$.*

Proof. The argument is exactly the same as that given on p. 66 of [5], in the proof of *ibid.* Theorem (2.4.1). □

We now complete the proof of Proposition 5.2. For each i, we choose an F-subspace W^i of V^i in general position relative to Λ^i over E. The space $\sum_i W^i = W \subset V$ is then in general position relative to Λ over E. Using this W to decompose A as $A(E) \otimes B$, the canonical projections $V \to V^i$ then commute with the implied action of $A(E)$ on V. Given this, we simply imitate the proof of [5] (7.1.19) to obtain the result. □

5.4. We continue in the same situation as 5.2, *but we assume for the moment that* $t = 1$. We now define a filtered lattice $\mathfrak{H}(\beta, \Lambda^1)$ in A^{11} by

$$\mathfrak{H}^r(\beta, \Lambda^1) = \mathfrak{H}^{re/e_1}(\beta, \mathfrak{A}) \cap A^{11}. \tag{5.4.1}$$

(Observe that this is consistent with 2.9). We can do the same thing with $\mathfrak{J}$, to obtain groups

$$H^r(\beta, \Lambda^1) = 1 + \mathfrak{H}^r(\beta, \Lambda^1), \qquad J^r(\beta, \Lambda^1) = 1 + \mathfrak{J}^r(\beta, \Lambda^1),$$

for real $r > 0$. We also get a group $J(\beta, \Lambda^1) = \mathfrak{J}(\beta, \Lambda^1) \cap \boldsymbol{u}(\Lambda^1)$. We remark that the lattice $\mathfrak{H}^0(\beta, \Lambda^1)$ is a ring, even an $\mathfrak{o}_F$-order, and the $\mathfrak{H}^r(\beta, \Lambda^1)$ are ideals of $\mathfrak{H}(\beta, \Lambda^1)$: this follows from the corresponding property for strict lattice sequences.

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LEMMA. *In the situation of* 5.2, *suppose that* $t = 1$ *and that the lattice sequence* Λ^1 *is strict. Let* $\mathfrak{A}^1$ *denote the hereditary order defined by* Λ^1. *We have:*

(i) $\mathfrak{H}^r(\beta, \mathfrak{A}) \cap A^{11} = \mathfrak{H}^{re_1/e}(\beta, \mathfrak{A}^1)$;

(ii) *for* $\theta \in \mathcal{C}(\mathfrak{A}, -(1+[-r]), \beta)$, *the character* $\theta^1 = \theta | H^{re_1/e}(\beta, \mathfrak{A}^1)$ *lies in* $\mathcal{C}(\mathfrak{A}^1, -(1+[-re_1/e]), \beta)$ *and* $\theta^1 = \boldsymbol{\tau}_{\mathfrak{A},\mathfrak{A}^1,\beta}\theta$.

Proof. We recall that Λ^0 is strict of period e, and that e_1 divides e. The method of [5] (7.2) (see especially (7.2.18)) reduces us to the case where $e = e_1$, and that case is treated in *ibid.* (3.6.14). □

PROPOSITION. *The definitions above of* $\mathfrak{H}^r(\beta, \Lambda^1)$, $\mathfrak{J}^r(\beta, \Lambda^1)$ *depend only on* β *and the lattice sequence* Λ^1. *In particular, they are independent of the choice of the strict lattice sequence* Λ^0.

Proof. Suppose first that the lattice chain Λ^1 is strict. Let $\mathfrak{A}^1$ denote the hereditary $\mathfrak{o}_F$-order defined by the lattice chain $\mathcal{L}_{\Lambda^1}$. The group $\mathfrak{H}^r(\beta, \Lambda^1)$, as defined by 5.4.1, is the same as $\mathfrak{H}^r(\beta, \mathfrak{A}^1)$ for all r, by the lemma.

Reverting to the general situation, suppose we have another strict $\mathfrak{o}_E$-lattice sequence $\bar{\Lambda}^0$ with $\mathfrak{o}_F$-period $\bar{e}$ divisible by e_1. We can define $\mathfrak{H}^r(\beta, \Lambda^1)$ relative to $\bar{\Lambda}^0$. We have to show that these two definitions coincide. To do this, we choose a strict $\mathfrak{o}_E$-lattice sequence $\tilde{\Lambda}^0$ of $\mathfrak{o}_F$-period $\tilde{e}$ divisible by both e and $\bar{e}$. We consider the strict $\mathfrak{o}_E$-lattice chain $\Lambda' = \tilde{\Lambda}^0 \oplus \bar{\Lambda}^0 \oplus \Lambda^0$. We can form $\Lambda'' = \Lambda' \oplus \Lambda^1$, and write $\mathfrak{A}''$ for the associated hereditary $\mathfrak{o}_F$-order. Our two definitions of $\mathfrak{H}^r(\beta, \Lambda^1)$ are then both equal to the intersection of $\mathfrak{H}^{r\tilde{e}/e_1}(\beta, \mathfrak{A}'')$ with $\mathrm{End}_F(V^1)$. □

5.5. In the situation of 5.2, we consider the characters $\theta | H^{m+1}(\beta, \mathfrak{A}) \cap \mathrm{Aut}_F(V^i)$, $0 \leqslant i \leqslant t$, where $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$. Again it is only the case $t = 1$ which immediately interests us. Let us assume for the moment that *the lattice chain* Λ^1 *is strict*. We then have $H^{m+1}(\beta, \mathfrak{A}) \cap \mathrm{Aut}_F(V^1) = H^{m_1+1}(\beta, \mathfrak{A}^1)$, where $\mathfrak{A}^1 = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}_{\Lambda^1})$, and m_1 is the integer $[me_1/e]$. Lemma 5.4 shows that the character $\theta^1 = \theta | H^{m_1+1}(\beta, \mathfrak{A}^1)$ lies in $\mathcal{C}(\mathfrak{A}^1, m_1, \beta)$ and that $\theta^1 = \boldsymbol{\tau}_{\mathfrak{A},\mathfrak{A}^1,\beta}(\theta)$.

We can perform this same construction without the hypothesis that Λ^1 is strict. To express its invariance properties, it is convenient to use the notion of *ps-character* recalled in Section 4. In those terms, the same argument used to prove Proposition 5.4 gives us:

PROPOSITION. *Use the notation of* 5.2, *and set* $t = 1$. *Let* $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$, *and let* θ^1 *denote the restriction of* θ *to the group*

$$H^{(m+1)e_1/e}(\beta, \Lambda^1) = H^{m+1}(\beta, \mathfrak{A}) \cap \mathrm{Aut}_F(V^1).$$

The character θ^1 *then depends only on the ps-character defined by* (θ, m, β), *and not on the choice of* Λ^0.

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If (Θ, k, β) denotes the ps-character defined by (θ, m, β), we can think of θ^1 as the *realization of* Θ *on* Λ^1, and use the notation $\theta^1 = \Theta(\Lambda^1)$.

The process $\theta \mapsto \theta | H^{re/e_1}(\beta, \mathfrak{A}) \cap \mathrm{Aut}_F(V^1)$ defines a set of characters of the group $H^r(\beta, \Lambda^1)$. In the case where r is not integral, we denote this set by $\mathcal{C}(\Lambda^1, r, \beta)$. We extend to integer parameters by

$$\mathcal{C}(\Lambda^1, r, \beta) = \mathcal{C}(\Lambda^1, [r], \beta), \quad r \in \mathbb{R} \setminus \mathbb{Z}.$$

This is consistent with the conventions of Section 4 for the standard case. Our construction above then gives us a map

$$\mathcal{C}(\mathfrak{A}, r, \beta) \to \mathcal{C}(\Lambda^1, re_1/e, \beta), \quad r \in \mathbb{R},$$

which is surjective by definition.

5.6. We conclude by recalling and extending some more notation from [5]. Suppose, for the moment, we have a simple stratum $[\mathfrak{A}, n, m, \beta]$ in some $A = \mathrm{End}_F(V)$, where n, m are integers with $m \geqslant 0$. Write $E = F[\beta]$, $B = \mathrm{End}_E(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$. If $\mathfrak{P}$ denotes the Jacobson radical of $\mathfrak{A}$, then $\mathfrak{Q} = \mathfrak{P} \cap B$ is the radical of $\mathfrak{B}$.

We write a_β for the adjoint map $x \mapsto \beta x - x\beta$, $x \in A$. We set

$$\mathfrak{N}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} \colon a_\beta(x) \in \mathfrak{P}^k\}, \quad k \in \mathbb{Z}.$$

We put $r = -k_0(\beta, \mathfrak{A})$ and then set $\mathfrak{N}(\beta, \mathfrak{A}) = \mathfrak{N}_{-r}(\beta, \mathfrak{A})$. This is a $(\mathfrak{B}, \mathfrak{B})$-bimodule, and an $\mathfrak{o}_F$-lattice in A.

If we take $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$, the set $I_G(\theta)$ of $g \in G = \mathrm{Aut}_F(V)$ which intertwine θ is ([5] (3.3.2)) $I_G(\theta) = (1 + \mathfrak{M}_m) \cdot B^\times \cdot (1 + \mathfrak{M}_m)$, where we have abbreviated

$$\mathfrak{M}_m = \mathfrak{M}_m(\beta, \mathfrak{A}) = \mathfrak{Q}^{r-m}\mathfrak{N}(\beta, \mathfrak{A}) + \mathfrak{J}^{\left[\frac{r+1}{2}\right]}(\beta, \mathfrak{A}).$$

We extend this notation to a simple stratum $[\Lambda, n, m, \beta]$ attached to a lattice sequence Λ in V. We choose a simple stratum $[\Lambda^0, n, m, \beta]$, for a strict lattice sequence Λ^0 in a space V^0, with $e(\Lambda^0) = e(\Lambda)$; we form the strict lattice sequence $\tilde{\Lambda} = \Lambda^0 \oplus \Lambda$ and the simple stratum $[\tilde{\Lambda}, n, m, \beta]$ in $V^0 \oplus V$. Let $\tilde{\mathfrak{A}}$ be the hereditary $\mathfrak{o}_F$-order defined by $\tilde{\Lambda}$ and $\tilde{\mathfrak{B}}$ the corresponding $\mathfrak{o}_E$-order. The lattice $\mathfrak{M}_m(\beta, \tilde{\mathfrak{A}})$ is then a $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}})$-bimodule and so its projection to A equals the intersection $\mathfrak{M}_m(\beta, \tilde{\mathfrak{A}}) \cap A$. We then define $\mathfrak{M}_m(\beta, \Lambda)$ to be this projection. As a direct consequence of the corresponding property for strict lattice sequences, we get:

PROPOSITION. *Let* $[\Lambda, n, m, \beta]$ *be a simple stratum, for nonnegative integers* m, n. *Set* $\mathfrak{a} = \mathfrak{a}_0(\Lambda)$, $\mathfrak{b} = \mathfrak{a} \cap B$, *in the notation above. Let* $\theta \in \mathcal{C}(\Lambda, m, \beta)$. *An element* $x \in \boldsymbol{u}(\mathfrak{a})$ *normalizes the pair* $(H^{m+1}(\beta, \Lambda), \theta)$ *if and only if* $x \in \boldsymbol{u}(\mathfrak{b})(1 + \mathfrak{M}_m(\beta, \Lambda))$.

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6. Relatively Split Strata

We seek a generalization of Corollary 3.9 to a certain class of extensions of simple characters attached to lattice sequences. These are sharper versions of certain of the 'split types' of [5] Chapter 8. The main definitions come in 6.1, and the main result is Corollary 6.6, which gives the most important step in the proof of the Main Theorem 1.5.

6.1. For $i = 1, 2$, we are given a simple stratum $[\Lambda^i, n_i, 0, \beta]$ in $A^i = \mathrm{End}_F(V^i)$. We form the lattice sequence $\Lambda = \Lambda^1 \oplus \Lambda^2$ in $V = V^1 \oplus V^2$; we thus obtain a simple stratum $[\Lambda, n, 0, \beta]$ in $A = \mathrm{End}_F(V)$. We fix an integer m of the form $m = m_1 e(\Lambda)/e(\Lambda^1)$, $m_1 \in \mathbb{Z}$, $0 < m < -k_0(\beta, \Lambda)$. We write

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$$

in our standard block matrix notation.

Let us abbreviate $\mathfrak{h}^q = \mathfrak{H}^q(\beta, \Lambda)$, $\mathfrak{b} = \mathfrak{b}_0(\Lambda)$, $\mathfrak{M}_m = \mathfrak{M}_m(\beta, \Lambda)$ (in the notation of 5.6). We consider the lattice

$$\mathfrak{k} = \begin{pmatrix} \mathfrak{h}^m & \mathfrak{b} + \mathfrak{M}_m \\ \mathfrak{h}^{m+1} & \mathfrak{h}^m \end{pmatrix}. \tag{6.1.1}$$

More precisely, $\mathfrak{k}$ is the direct sum of the intersections $\mathfrak{k}^{ij} = \mathfrak{k} \cap A^{ij}$, with $\mathfrak{k}^{11} = (\mathfrak{h}^m)^{11}$ etc. In particular, the diagonal blocks are respectively $\mathfrak{H}^{m_1}(\beta, \Lambda^1)$, and $\mathfrak{H}^{me(\Lambda^2)/e(\Lambda)}(\beta, \Lambda^2)$.

The group $\boldsymbol{u}(\mathfrak{b})(1 + \mathfrak{M}_m)$ normalizes $H^m = H^m(\beta, \Lambda)$ (as follows from [5] (3.1.9)) and so the set

$$K = 1 + \mathfrak{k} \tag{6.1.2}$$

is a compact open subgroup of $G = \mathrm{Aut}_F(V)$.

PROPOSITION. *Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and choose $\tilde{\theta} \in \mathcal{C}(\Lambda, m-1, \beta)$ which extends θ. There exists a unique character ϑ of K which agrees with $\tilde{\theta}$ on the group $K \cap H^m(\beta, \Lambda)$ and is trivial on $1 + (\mathfrak{b} + \mathfrak{M}_m)^{12}$.*

Proof. The uniqueness assertion is clear. To prove existence, it is enough to treat the case where Λ^1 is a strict lattice sequence of period divisible by $e(\Lambda^2)$: to achieve this, we replace Λ^1 by $\Lambda^0 \oplus \Lambda^1$, where Λ^0 is a suitable strict lattice sequence. This done, the assertion follows readily from the commutator calculations in [5] Section 3.2. □

6.2. We continue with the notation of 6.1. We now define a character $\xi = \vartheta \psi_b$ of K, for a special kind of element b as follows:

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NOTATION. *Let s be a tame corestriction on A relative to the field extension $F[\beta]/F$. Let $b_i \in \mathfrak{a}_{-m}(\Lambda) \cap A^{ii}$, and set $\delta_i = s(b_i) \in \mathfrak{b}_{-m}(\Lambda) \cap A^{ii}$, $i = 1, 2$. We assume that the pair (δ_1, δ_2) satisfies the conditions of* 3.6 *relative to the base field $E = F[\beta]$. We put*

$$b = b_1 \oplus b_2 \in \mathfrak{a}_{-m}(\Lambda), \qquad \delta = s(b) = \delta_1 \oplus \delta_2 \in \mathfrak{b}_{-m}(\Lambda).$$

Note that the restriction of s to A^{ii} is again a tame corestriction on $A^{ii} = \mathrm{End}_F(V^i)$ relative to $F[\beta]/F$.

We now form the character $\xi = \vartheta\psi_b$ of the group K.

THEOREM. *Let $x \in A^{12}$ be such that $1 + x$ intertwines the character ξ of K. Then $x \in \mathfrak{k} \cap A^{12}$.*

6.3. We first observe that it is enough to prove Theorem 6.2 under the assumption that Λ^1 is a strict lattice sequence of period divisible by $e(\Lambda^2)$: to achieve this, we replace Λ^1 by $\Lambda^0 \oplus \Lambda^1$ and b_1 by $b_0 \oplus b_1$, where Λ^0 is strict of period divisible by both $e(\Lambda^i)$ and b_0 is chosen so that the pair $(b_0 \oplus b_1, b_2)$ still satisfies the conditions in 6.2. If the result holds in this case, it holds in the original one, simply by intersecting with A.

We henceforward assume, therefore, that Λ^1 is strict of period divisible by $e(\Lambda^2)$. In particular, Λ is a strict lattice sequence of the same period as Λ^1. We therefore use the standard notation of [5], and write $\mathfrak{A}$ for the hereditary $\mathfrak{o}_F$-order $\mathfrak{a}_0(\Lambda)$. We also set $E = F[\beta]$, $B = \mathrm{End}_E(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$. We denote the radicals of $\mathfrak{A}$, $\mathfrak{B}$ by $\mathfrak{P}$, $\mathfrak{Q}$. We set $r = -k_0(\beta, \mathfrak{A})$.

Using this notation, we recall some results from [5]. Let $0 \leqslant q < r$, and let $\Theta \in \mathcal{C}(\mathfrak{A}, q, \beta)$. According to [5] (3.3.2), the G-intertwining of the character Θ is given by

$$I_G(\Theta) = (1 + \mathfrak{M}_q) \cdot B^\times \cdot (1 + \mathfrak{M}_q),$$

where

$$\mathfrak{M}_q = \mathfrak{Q}^{r-q}\mathfrak{N} + \mathfrak{j}, \qquad \mathfrak{N} = \mathfrak{N}_{-r}(\beta, \mathfrak{A}), \qquad \mathfrak{j} = \mathfrak{J}^{\left[\frac{r+1}{2}\right]}(\beta, \mathfrak{A}).$$

We use the standard abbreviation $\mathfrak{H}^t = \mathfrak{H}^t(\beta, \mathfrak{A})$. We observe that, in the case $q \leqslant \left[\frac{r}{2}\right]$, we have $\mathfrak{M}_q = \mathfrak{j}$ [5] (3.1.10).

6.3.1. *Let q, k, ℓ be positive integers satisfying*

$$\left[\frac{r}{2}\right] \leqslant q < r, \qquad k + \ell \geqslant q + 1, \qquad k + 2\ell \geqslant r + 1.$$

Let $x \in \mathfrak{Q}^k\mathfrak{N} + \mathfrak{j}$, $y \in \mathfrak{Q}^\ell\mathfrak{N} + \mathfrak{j}$ and $\Theta \in \mathcal{C}(\mathfrak{A}, q, \beta)$. The commutator $[1+x, 1+y]$ then lies in H^{q+1} and $\Theta[1 + x, 1 + y] = \psi_{(1+x)^{-1}\beta(1+x)-\beta}(1 + y)$.

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Proof. This combines [5] (3.2.6), (3.2.8) and (3.2.12). □

6.3.2. *Let t, q be positive integers satisfying*

$$t < r, \qquad \left[\frac{t}{2}\right] \leqslant q < t,$$

and let $\Theta \in \mathfrak{C}(\mathfrak{A}, q, \beta)$. Let $g \in G$ intertwine the restriction of Θ to H^{t+1}. Then, for $h \in g^{-1}H^{q+1}g \cap H^{q+1}$, we have $\Theta^g(h) = \Theta(h)\psi_{g^{-1}\beta g-\beta}(h)$.

Proof. This is the second assertion of [5] (3.3.9). □

We also need a collection of exact sequences, derived from [5] Section 3.1. Recall that we have chosen (in our definition of δ above) a tame corestriction s on A relative to E/F. We use 'star' to denote duals with respect to ψ_A, as before. We also use M to denote the group $\mathrm{Aut}_F(V^1) \times \mathrm{Aut}_F(V^2)$.

LEMMA. *Let $0 \leqslant q < r$. The following sequences are exact:*

$$0 \longrightarrow \mathfrak{Q}^{r-q} \longrightarrow \mathfrak{M}_q \xrightarrow{a_\beta} (\mathfrak{H}^{q+1})^* \xrightarrow{s} \mathfrak{Q}^{-q} \longrightarrow 0;$$

$$0 \longrightarrow \mathfrak{Q}^{q+1} \longrightarrow \mathfrak{H}^{q+1} \xrightarrow{a_\beta} \mathfrak{M}_q^* \xrightarrow{s} \mathfrak{Q}^{q+1-r} \longrightarrow 0;$$

$$0 \longrightarrow \mathfrak{Q}^{\left[\frac{r+1}{2}\right]} \longrightarrow \mathfrak{H}^{\left[\frac{r+1}{2}\right]} \xrightarrow{a_\beta} \mathfrak{j}^* \xrightarrow{s} \mathfrak{Q}^{1-\left[\frac{r+1}{2}\right]} \longrightarrow 0.$$

If $0 \to \mathfrak{l}_1 \to \mathfrak{l}_2 \to \mathfrak{l}_3 \to \mathfrak{l}_4 \to 0$ *denotes any of these sequences, the sequence*

$$0 \to t^{-1}\mathfrak{l}_1 t + \mathfrak{l}_1 \to t^{-1}\mathfrak{l}_2 t + \mathfrak{l}_2 \to t^{-1}\mathfrak{l}_3 t + \mathfrak{l}_3 \to t^{-1}\mathfrak{l}_4 t + \mathfrak{l}_4 \to 0$$

is exact, for any $t \in B^\times$.

For any choice of $\mathfrak{l}_k$, we have $\mathfrak{l}_k = \coprod_{i,j} \mathfrak{l}_k \cap A^{ij}$. We write $\mathfrak{l}_k^{ij} = \mathfrak{l}_k \cap A^{ij}$. If $t \in B^\times \cap M$, we further have

$$(t^{-1}\mathfrak{l}_k t + \mathfrak{l}_k) \cap A^{ij} = t^{-1}\mathfrak{l}_k^{ij} t + \mathfrak{l}_k^{ij},$$

and the sequence

$$0 \to t^{-1}\mathfrak{l}_1^{ij} t + \mathfrak{l}_1^{ij} \to t^{-1}\mathfrak{l}_2^{ij} t + \mathfrak{l}_2^{ij} \to t^{-1}\mathfrak{l}_3^{ij} t + \mathfrak{l}_3^{ij} \to t^{-1}\mathfrak{l}_4^{ij} t + \mathfrak{l}_4^{ij} \to 0$$

is exact, for all choices of i, j and $(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4)$.

Proof. The exactness of the first sequence is [5] (3.1.16). That of the third is given by *ibid.* (3.1.22). The second sequence is simply the dual of the first: the canonical inclusion $B \to A$ and the tame corestriction $s\colon A \to B$ are mutually dual, while the dual of $a_\beta\colon A \to A$ is simply $a_{-\beta}$.

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Now let us assume $(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) = (\mathfrak{Q}^{r-q}, \mathfrak{M}_q, (\mathfrak{H}^{q+1})^*, \mathfrak{Q}^{-q})$. The sequence

$$0 \to t^{-1}\mathfrak{l}_1 t + \mathfrak{l}_1 \to t^{-1}\mathfrak{l}_2 t + \mathfrak{l}_2 \to t^{-1}\mathfrak{l}_3 t + \mathfrak{l}_3 \to t^{-1}\mathfrak{l}_4 t + \mathfrak{l}_4 \to 0$$

is certainly a complex, and exactness at the ends is clear. To get exactness at the second place, we have to show that

$$(t^{-1}\mathfrak{M}_q t + \mathfrak{M}_q) \cap B = t^{-1}\mathfrak{Q}^{r-q} t + \mathfrak{Q}^{r-q}.$$

We have containment in the direction $\supset$ here. On the other hand, $\mathfrak{M}_q \subset \mathfrak{P}^{r-q}$, and

$$(t^{-1}\mathfrak{P}^{r-q} t + \mathfrak{P}^{r-q}) \cap B = t^{-1}\mathfrak{Q}^{r-q} t + \mathfrak{Q}^{r-q}$$

by [5] (1.3.16).

It remains, in this case, to establish exactness at the third place. Dualizing, this is equivalent to showing that

$$t^{-1}\mathfrak{Q}^{q+1} t \cap \mathfrak{Q}^{q+1} \to t^{-1}\mathfrak{H}^{q+1} t \cap \mathfrak{H}^{q+1} \xrightarrow{a_\beta} t^{-1}\mathfrak{M}_q^* t \cap \mathfrak{M}_q^*$$

is exact. This is equivalent to the relation

$$t^{-1}\mathfrak{H}^{q+1} t \cap \mathfrak{H}^{q+1} \cap B = t^{-1}\mathfrak{Q}^{q+1} t \cap \mathfrak{Q}^{q+1},$$

which follows easily, as before.

A simple ‘Snake Lemma’ argument now shows that the sequence

$$0 \to t^{-1}\mathfrak{l}_1 t \cap \mathfrak{l}_1 \to t^{-1}\mathfrak{l}_2 t \cap \mathfrak{l}_2 \to t^{-1}\mathfrak{l}_3 t \cap \mathfrak{l}_3 \to t^{-1}\mathfrak{l}_4 t \cap \mathfrak{l}_4 \to 0$$

is exact, where the $\mathfrak{l}_i$ are as in the first case above. Dualizing, this gives the exactness of the sequence in the second case

$$(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4) = (\mathfrak{Q}^{q+1}, \mathfrak{H}^{q+1}, \mathfrak{M}_q^*, \mathfrak{Q}^{q+1-r}).$$

The final case is similar to the first one, so we omit the details.

To prove the remaining assertions, it is enough to observe that all the $\mathfrak{l}_k$ are $\mathfrak{B}$-bimodules, and the canonical projections $V \to V^i$ both lie in $\mathfrak{B}$, cf. 2.9. □

6.4. We now prove Theorem 6.2 in the case where $m \geqslant \left[\frac{r}{2}\right]$. Let us write ξ_0 for the restriction of ξ to the group $K \cap H^m$. Thus $\xi_0 = \tilde{\theta}\psi_b$ on this group.

LEMMA. *Let $g \in G$ intertwine the character ξ_0. Then g lies in $\Gamma \cdot B^\times \cap M \cdot \Gamma$, where Γ denotes the group*

$$\Gamma = \begin{pmatrix} 1 + \mathfrak{Q} + \mathfrak{M}_m & \mathfrak{B} + \mathfrak{M}_m \\ \mathfrak{Q} + \mathfrak{M}_{m-1} & 1 + \mathfrak{Q} + \mathfrak{M}_m \end{pmatrix}.$$

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Proof. It will be convenient to have the notation

$$P_{\mathrm{u}} = M \cdot 1 + A^{12}, \quad N_{\mathrm{u}} = 1 + A^{12},$$

$$P_{\ell} = M \cdot 1 + A^{21}, \quad N_{\ell} = 1 + A^{21},$$

and also the abbreviation $L = K \cap H^m$. Our group Γ has Iwahori decomposition with respect to (P_{u}, M), and the subgroup $\Gamma_{\mathrm{u}} = \Gamma \cap P_{\mathrm{u}}$ normalizes the pair (L, ξ_0).

Any $g \in G$ which intertwines ξ_0 on L must also intertwine the restriction of ξ_0 to the group $H^{m+1} = H^{m+1}(\beta, \mathfrak{A})$. This restriction lies in $\mathcal{C}(\mathfrak{A}, m, \beta)$, so $g \in (1+\mathfrak{M}_m)B^\times(1+\mathfrak{M}_m)$. Absorbing factors into Γ_{u}, we may assume g has the form $(1+x)t(1+y)$, with $t \in B^\times$ and $x, y \in \mathfrak{M}_m \cap A^{21}$. The elements $1+x, 1+y$ normalize L and fix the character ψ_b on L. On the other hand, the commutator relation 6.3.2 gives us $\vartheta^{1+x}|L = \vartheta\psi_{a_\beta(x)}|L$ and likewise with $(1-y) = (1+y)^{-1}$ in place of $1+x$. By definition, our element t intertwines the characters ξ^{1+x}, ξ^{1-y} of L, and so intertwines their restriction to $B^\times \cap L$. The factors $\psi_{a_\beta(x)}, \psi_{a_\beta(-y)}$ restrict trivially here. Also by definition, there exists $\tilde\theta \in \mathcal{C}(\mathfrak{A}, m-1, \beta)$ such that $\vartheta|L = \tilde\theta|L$. By the definition [5] (3.2.3) of simple character, the restriction $\tilde\theta|H^m \cap B^\times$ factors through the determinant map $\det_B\colon B^\times \to E^\times$, so the same applies to ϑ on $B^\times \cap L$. In particular, it is intertwined by every element of $B^\times$. We deduce that t intertwines the character $\psi_b|B^\times \cap L$.

However, we can recognize the character $\psi_b|B^\times \cap L$ in different terms. Let s be the tame corestriction on A used in the definition of δ. There is then a character ψ_B of B, of the form $\psi_E \circ \mathrm{tr}_{B/E}$, such that $\psi_a|B = \psi_{B,s(a)}$ for any $a \in A$. In particular, we have $\psi_b|B^\times \cap L = \psi_{B,\delta}|B^\times \cap L$. Corollary (3.8) now shows that

$$t \in \begin{pmatrix} 1 & 0 \\ \mathfrak{Q} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathfrak{B} \\ 0 & 1 \end{pmatrix} \cdot B^\times \cap M \cdot \begin{pmatrix} 1 & 0 \\ \mathfrak{Q} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathfrak{B} \\ 0 & 1 \end{pmatrix}.$$

We can absorb the upper triangular factors into Γ and $B^\times \cap M$, and we may now take $g = (1+x)t(1+y)$, with $t \in B^\times \cap M$, $x, y \in (\mathfrak{Q} + \mathfrak{M}_m) \cap A^{21}$. Our adjustments do not alter the hypothesis that g intertwines ξ_0.

In particular, this element g intertwines the restriction of ξ to the group

$$K' = 1 + \begin{pmatrix} \mathfrak{H}^{m+1} & \mathfrak{H}^m \\ \mathfrak{H}^{m+1} & \mathfrak{H}^{m+1} \end{pmatrix}.$$

The factor ψ_b is trivial on this group, so g intertwines $\vartheta|K'$. This last is the restriction of $\tilde\theta \in \mathcal{C}(\mathfrak{A}, m-1, \beta)$ as above. We now apply 6.3.1 to $\tilde\theta$ to show that, as characters of $K'^g \cap K'$, we have $\vartheta^g = \vartheta\psi_{g^{-1}\beta g - \beta}$. We next use this relation to show that

$$g = (1+x')t(1+y'), \quad x', y' \in (\mathfrak{Q} + \mathfrak{M}_{m-1}) \cap A^{21}.$$

To do this, let us write $K' = 1+\mathfrak{k}'$. Since, by hypothesis, the element g intertwines ϑ on K', the relation above says that g intertwines the coset $\beta + \mathfrak{k}'^*$, in the sense that $g^{-1}(\beta + \mathfrak{k}'^*)g \cap (\beta + \mathfrak{k}'^*) \neq \emptyset$. We have, for example,

$$(1+y)(\beta + \mathfrak{k}'^*)(1+y)^{-1} = \beta - a_\beta(y) + \mathfrak{k}'^*,$$

so this formal intertwining condition amounts to

$$t^{-1}a_\beta(x)t + a_\beta(y) \equiv 0 \pmod{t^{-1}\mathfrak{k}'^* t + \mathfrak{k}'^*}.$$

The lattice $\mathfrak{k}'$ decomposes as the direct sum of its intersections with the blocks A^{ij}, so the same applies to $\mathfrak{k}'^*$. Since $t \in M$, in particular, we have

$$t^{-1}\mathfrak{k}'^* t + \mathfrak{k}'^* = \coprod (t^{-1}\mathfrak{k}'^* t + \mathfrak{k}'^*) \cap A^{ij} = \coprod (t^{-1}\mathfrak{k}'^* t \cap A^{ij}) + (\mathfrak{k}'^* \cap A^{ij}),$$

just as in Lemma 6.3. We can therefore examine this congruence relation 'block by block'. Since $x, y \in A^{21}$, it is automatically satisfied in all blocks except A^{21}. It is therefore equivalent to

$$t^{-1}a_\beta(x)t + a_\beta(y) \equiv 0 \pmod{(t^{-1}\mathfrak{h}^* t + \mathfrak{h}^*) \cap A^{21}},$$

where we have abbreviated $\mathfrak{h} = \mathfrak{H}^m$. We can now use 6.3 Lemma 3 to produce elements $x', y' \in \mathfrak{M}_{m-1} \cap A^{21}$ such that $a_\beta(t^{-1}x't + y') = a_\beta(t^{-1}xt + y)$. In other words, $t^{-1}x't + y' - t^{-1}xt - y$ lies in the (2, 1) block of

$$(t^{-1}(\mathfrak{Q} + \mathfrak{M}_m)t + \mathfrak{Q} + \mathfrak{M}_m) \cap B = t^{-1}\mathfrak{Q}t + \mathfrak{Q}.$$

In all, there exist elements $x'', y'' \in (\mathfrak{Q} + \mathfrak{M}_{m-1}) \cap A^{21}$ such that $g = (1+x'')\, t(1+y'')$, as required. This completes the proof of the Lemma. □

Let us now deduce 6.2 from the Lemma, in the present case $m \geqslant \left[\frac{r}{2}\right]$. Let $g \in N_{\mathrm{u}}$ intertwine ξ. We thus have $g = \gamma t \gamma'$, for some $t \in B^\times \cap M$ and $\gamma, \gamma' \in \Gamma$. The group Γ has Iwahori decomposition $\Gamma = \Gamma \cap N_{\mathrm{u}} \cdot \Gamma \cap M \cdot \Gamma \cap N_\ell$. We accordingly write $\gamma = \gamma_{\mathrm{u}}\, \gamma_M \gamma_\ell$, $\gamma' = \gamma'_\ell\, \gamma'_M \gamma'_{\mathrm{u}}$, in the obvious notation. (We get this second decomposition by applying the Iwahori decomposition to γ'^{-1}.) We thus get $\gamma_{\mathrm{u}}^{-1} g\, \gamma_{\mathrm{u}}'^{-1} = \gamma_\ell t'\, \gamma'_\ell$, for some $t' \in M$. The first of these elements lies in N_{u}, and the second in P_ℓ. We deduce that $\gamma_{\mathrm{u}}^{-1} g \gamma_{\mathrm{u}}'^{-1} = 1$, whence $g \in N_{\mathrm{u}} \cap \Gamma = N_{\mathrm{u}} \cap K$, as required.

6.5. We now prove Theorem 6.2 in the remaining case, where $1 \leqslant m \leqslant \left[\frac{r}{2}\right]$. Here we have $\mathfrak{M}_m = \mathfrak{j}$ and the group K is given by

$$K = 1 + \begin{pmatrix} \mathfrak{H}^m & \mathfrak{B} + \mathfrak{j} \\ \mathfrak{H}^{m+1} & \mathfrak{H}^m \end{pmatrix}.$$

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We define the group Γ exactly as in 6.4.

Let $g \in G$ intertwine ξ. The group Γ_{u} normalizes ξ, so we may as well take $g = (1+x)t(1+y)$, with $t \in B^\times$ and $x, y \in \mathfrak{j} \cap A^{21}$. Consider the restriction of ξ to the group

$$K' = 1 + \begin{pmatrix} \mathfrak{H}^m & \mathfrak{H}^1 \\ \mathfrak{H}^{m+1} & \mathfrak{H}^m \end{pmatrix}.$$

Here, the character ϑ agrees with a character $\Theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. The character Θ is invariant under conjugation by $1+\mathfrak{j}$. We deduce that t intertwines the character $\psi_{B,\delta}$ on the group

$$K' \cap B^\times = 1 + \begin{pmatrix} \mathfrak{Q}^m & \mathfrak{Q} \\ \mathfrak{Q}^{m+1} & \mathfrak{Q}^m \end{pmatrix}.$$

Therefore, by (3.8), we have

$$t \in \begin{pmatrix} 1+\mathfrak{Q} & \mathfrak{B} \\ \mathfrak{Q} & 1+\mathfrak{Q} \end{pmatrix} \cdot B^\times \cap M \cdot \begin{pmatrix} 1+\mathfrak{Q} & \mathfrak{B} \\ \mathfrak{Q} & 1+\mathfrak{Q} \end{pmatrix}.$$

Therefore $g \in \Gamma \cdot B^\times \cap M \cdot \Gamma$, and the Theorem follows in this case, just as before.

This completes the proof of 6.2. □

6.6. We now give what will be the crucial consequence of Theorem 6.2. We remain in the same situation, but it will now be convenient to abbreviate $G^i = \mathrm{Aut}_F(V^i)$, so that $M = G^1 \times G^2$. In particular, K is the group defined by 6.1.2.

NOTATION. *For $i = 1, 2$, we suppose given*

(i) *an open subgroup K_i of $\boldsymbol{u}(\Lambda^i)$ containing and normalizing the group $H^m(\beta, \Lambda) \cap G^i$;*
(ii) *an irreducible smooth representation ϱ_i of K_i whose restriction to $K_i \cap K$ is a multiple of $\xi | K_i \cap K$.*

Under these conditions, we have

COROLLARY.

(i) *The set $(K_1 \times K_2) \cdot K = \tilde{K}$ is a group.*
(ii) *There is a unique irreducible smooth representation ϱ of $\tilde{K}$ which is trivial on $K \cap N_{\mathrm{u}}$, $K \cap N_\ell$, and whose restriction to $K_1 \times K_2 = \tilde{K} \cap M$ is equivalent to $\varrho_1 \otimes \varrho_2$.*
(iii) *The pair $(\tilde{K}, \varrho)$ is a G-cover of the pair $(\tilde{K} \cap M, \varrho_1 \otimes \varrho_2)$.*

Proof. The group K_1 is contained in $\boldsymbol{u}(\Lambda^1)$ and normalizes $H^m \cap G^1$. It therefore normalizes $\mathfrak{h}^{m+1} \cap G^1 = \boldsymbol{u}_{m+1}(\Lambda) \cap H^m \cap G^1$; it also fixes the restriction of ξ to this

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group. The restriction of ξ to this group lies in $\mathcal{C}(\Lambda^1, m_1, \beta)$. By 5.7, the normalizer of such a character is the intersection with G_1 of the normalizer $\boldsymbol{u}(\mathfrak{b})(1+\mathfrak{M}_m)$ of our original character $\theta \in \mathcal{C}(\Lambda, m, \beta)$. It follows that K_1 normalizes both $K \cap N_{\mathrm{u}}$ and $K \cap N_\ell$. The argument for K_2 is the same, and this proves (i). Assertion (ii) is now immediate.

Since P_{u}, P_ℓ are the only parabolic subgroups of G with Levi component M, the Iwahori decomposition properties required by (iii) are immediate. Let ζ denote the element

$$\zeta = \begin{pmatrix} \pi_F & 0 \\ 0 & 1 \end{pmatrix} \in M.$$

Here, π_F is some prime element of F. There is a unique function $f \in \mathcal{H}(G, \varrho)$ supported on $\tilde{K}\zeta\tilde{K}$ whose value at ζ is the identity transformation $\mathbf{1}$ (of the space underlying the contragredient of ϱ). To complete the proof of (iii), it is enough to show that f is invertible in $\mathcal{H}(G, \varrho)$ and that its inverse is supported on $\tilde{K}\zeta^{-1}\tilde{K}$. To do this, we let $f' \in \mathcal{H}(G, \varrho)$ have support $\tilde{K}\zeta^{-1}\tilde{K}$ with $f'(\zeta^{-1}) = \mathbf{1} = f(\zeta)$. The convolution $f' \star f$ has support contained in

$$\tilde{K}\zeta^{-1}\tilde{K}\zeta\tilde{K} = \tilde{K} \cdot \zeta^{-1}\tilde{K} \cap N_{\mathrm{u}}\zeta \cdot \tilde{K}.$$

However, by 6.2, the intertwining of ξ, and hence of ϱ, in N_{u} is precisely $K \cap N_{\mathrm{u}} = \tilde{K} \cap N_{\mathrm{u}}$ (cf. the proof of 3.9). We deduce that the support of $f' \star f$ is contained in $\tilde{K}$. An easy direct computation shows that $f' \star f(1_G) = c\mathbf{1}$, for some positive constant c.

We deduce that f is left-invertible in $\mathcal{H}(G, \varrho)$; it follows easily (cf. the proof of [9] (7.14)) that f is in fact invertible, and its inverse is therefore $c^{-1}f'$. □

7. The Homogeneous Case

In this section, we prove the Main Theorem 1.5 in a special case; this will serve as the first step in the general inductive argument of Section 8.

We use the notation introduced in Section 1. In particular, L is the G-stabilizer of a decomposition $V = \coprod_{1\leqslant i\leqslant r} V^i$ of V as a direct sum of nonzero subspaces V^i, and we set $G^i = \mathrm{Aut}_F(V^i)$. Thus we can write $\sigma = \bigotimes_{1\leqslant i\leqslant r} \pi_i$, for an irreducible supercuspidal representation π_i of G^i.

The representation π_i determines an endo-class $\Theta_i = \Theta_{\pi_i}$ of simple characters as in 4.5. For the remainder of this section, we assume that $\Theta_1 = \Theta_2 = \cdots = \Theta_r = \Theta$.

7.1. We return to the $\mathfrak{s}_L$-type $(K_L, \tau_L) = (\prod J_j, \bigotimes \lambda_j)$ of 1.3, and give a more detailed description of the maximal simple types (J_j, λ_j). (We are just summarizing the definition [5] (5.5.10).) Our hypothesis on the Θ_j says that we may

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take the maximal simple type (J_j, λ_j) in G^j in the following form. There is an element β and, for each j, a simple stratum $[\mathfrak{A}_j, n_j, 0, \beta]$ in $\mathrm{End}_F(V^j)$ such that $J_j = J(\beta, \mathfrak{A}_j)$; moreover, the restriction of λ_j to $H^1(\beta, \mathfrak{A}_j)$ is a multiple of some $\theta_j \in \mathcal{C}(\mathfrak{A}_j, 0, \beta)$, and the θ_j are all realizations of the same ps-character $(\Theta, 0, \beta)$ (of endo-class Θ).

Let us abbreviate $E = F[\beta]$. The $\mathfrak{o}_E$-order $\mathfrak{B}_j = \mathfrak{A}_j \cap \mathrm{End}_E(V^j)$ is maximal, since (J_j, λ_j) is a maximal simple type. We have $J(\beta, \mathfrak{A}_j)/J^1(\beta, \mathfrak{A}_j) = \boldsymbol{U}(\mathfrak{B}_j)/\boldsymbol{U}^1(\mathfrak{B}_j) \cong \mathrm{GL}_{f_j}(k_E)$, for some integer f_j. In particular, the orders $\mathfrak{A}_j$ all have the same $\mathfrak{o}_F$-period, namely $e(E|F)$.

The representation λ_j is given as follows. First, there is a unique irreducible representation η_j of $J^1(\beta, \mathfrak{A}_j)$ whose restriction to $H^1(\beta, \mathfrak{A}_j)$ is a multiple of θ_j. The representation η_j extends to a representation κ_j of J_j which is intertwined by every element of G^j which intertwines η_j, i.e., $I_{G^j}(\kappa_j) = J_j \cdot \mathrm{Aut}_E(V^j) \cdot J_j$. (In the language of [5] Section 5.2, κ is a 'β-extension of η_j'.) We then have $\lambda_j = \kappa_j \otimes \rho_j$, where ρ_j is the inflation of an irreducible cuspidal representation of $\mathrm{GL}_{f_j}(k_E)$.

7.2. It is now our task to assemble the (J_j, λ_j) of 7.1 into the desired $\mathfrak{s}$-type. First we need some more notation. Let P_{u} be the parabolic subgroup of G stabilizing the flag $\{0\} \subset V^1 \subset V^1 \oplus V^2 \subset \ldots$. Thus P_{u} has Levi component L; we write N_{u} for its unipotent radical and $P_\ell = LN_\ell$ for its opposite relative to L.

We write $\mathcal{L}^j = \{L^j_k : k \in \mathbb{Z}\}$ for the lattice chain attached to $\mathfrak{A}_j$; we assemble these into a lattice chain $\mathcal{L} = \{L_k : k \in \mathbb{Z}\}$ in V, of $\mathfrak{o}_F$-period $re(E|F)$, according to the rule:

$$L_0 = L_0^1 \oplus L_0^2 \oplus \cdots \oplus L_0^r,$$

$$L_1 = L_0^1 \oplus L_0^2 \oplus \cdots \oplus L_0^{r-1} \oplus L_1^r,$$

$$L_2 = L_0^1 \oplus L_0^2 \oplus \cdots \oplus L_0^{r-2} \oplus L_1^{r-1} \oplus L_1^r,$$

and so on. This defines a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $\mathrm{End}_F(V)$; we put $\mathfrak{B} = \mathfrak{A} \cap B$. We now follow the procedures of [5] (7.1), (7.2). The set

$$K = H^1(\beta, \mathfrak{A}) \cap N_\ell \cdot J(\beta, \mathfrak{A}) \cap P_{\mathrm{u}}$$

is then a group containing $H^1(\beta, \mathfrak{A})$. It admits an irreducible representation κ with the following properties:

7.2.1.

(i) *the restriction of* κ *to* $H^1(\beta, \mathfrak{A})$ *is a multiple of* $\theta = \Theta(\mathfrak{A})$;
(ii) κ *is trivial on* $K \cap N_\ell$, $K \cap N_{\mathrm{u}}$;
(iii) $K \cap L = K_L$ *and* $\kappa | K_L$ *is of the form* $\kappa'_1 \otimes \cdots \otimes \kappa'_r$, *for some* β*-extension* κ'_j *of* η_j.

We can choose the decomposition $\lambda_j = \kappa_j \otimes \rho_j$ above so that $\kappa_j = \kappa'_j$ for all j; we assume this has been done.

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The quotient $K/K \cap J^1(\beta, \mathfrak{A})$ is canonically isomorphic to the product of the $J_j/J^1(\beta, \mathfrak{A}_j) \cong \mathrm{GL}_{f_j}(k_E)$; we can therefore inflate $\rho_1 \otimes \cdots \otimes \rho_r$ to a representation ρ of K and form $\tau = \kappa \otimes \rho$.

THEOREM. *The pair (K, τ) is a G-cover of (K_L, τ_L), and hence an $\mathfrak{s}$-type in G. It is, moreover, a G-cover of the pair (K_M, τ_M) of* 1.5.

Proof. The second assertion of the theorem implies the first, by [9] (8.5).

In the case where the π_j are all inertially equivalent, i.e., the case $M = G$ of 1.4, the construction above yields the pair (K, τ) of 1.4. In the general case, we have $K \cap M = K_M$ and $\tau | K_M = \tau_M$. The Iwahori decomposition properties of (K, τ) relative to M are easy to establish; to prove the theorem, it is enough therefore, by [9] (7.2), to show that the G-intertwining of the representation τ is contained in KMK.

To do this, we can use the technique of [5] (5.3) to show that $I_G(\tau) = K I_H(\rho) K$, where $H = \mathrm{Aut}_E(V)$ and we view ρ as an irreducible representation of $\boldsymbol{U}(\mathfrak{B}) = K \cap H$. A pleasant exercise along the lines of [5] (5.5.5) (or an appeal to the very general result [11] 7.12) shows that $I_H(\rho) \subset \boldsymbol{U}(\mathfrak{B}) \cdot M \cap H \cdot \boldsymbol{U}(\mathfrak{B})$, and the proof is complete. □

8. The General Construction

8.1. We retain the notation of Section 1 as used in Section 7. In particular, M is the G-stabilizer of the decomposition $V = \coprod_{1 \leqslant i \leqslant t} W^i$. To each W^i, we can associate an endo-class of simple characters, namely Θ_j for any j such that $V^j \subset W^i$. Let $\Theta(1), \Theta(2), \ldots, \Theta(q)$ be the distinct endo-classes arising here. For each k, let $\bar{W}^k$ be the sum of those W^j whose associated endo-class Θ_{π_j} is $\Theta(k)$. Write $\bar{G}^j = \mathrm{Aut}_F \bar{W}^j$, $\bar{M} = \prod_j \bar{G}^j$. We choose a ps-character $(\Theta_i, 0, \beta_i)$ of endo-class $\Theta(i)$.

Theorem 7.2 gives us an $\bar{M}$-cover $(K_{\bar{M}}, \tau_{\bar{M}})$ of (K_L, τ_L) with the following properties:

8.1.1.

(i) $K_{\bar{M}} = \prod_i K_i$, *for subgroups K_i of $\bar{G}^i$ of the following form: there is a simple stratum $[\mathfrak{A}_i, n_i, 0, \beta_i]$ in $\mathrm{End}_F(\bar{W}^i)$ such that*

$$H^1(\beta_i, \mathfrak{A}_i) \subset K_i \subset J(\beta_i, \mathfrak{A}_i).$$

(ii) *We have $\tau_{\bar{M}} = \bigotimes_i \tau_i$, for an irreducible representation τ_i of K_i whose restriction to $H^1(\beta_i, \mathfrak{A}_i)$ is a multiple of $\Theta_i(\mathfrak{A}_i)$.*

(iii) *$(K_{\bar{M}}, \tau_{\bar{M}})$ is an $\bar{M}$-cover of (K_M, τ_M).*

8.2. We now construct a G-cover (K, τ) of $(K_{\bar{M}}, \tau_{\bar{M}})$. This will provide the cover of (K_M, τ_M) announced in 1.5. We will work inductively on the integer q of 8.1,

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noting that, in the case $q = 1$, we have $\bar{M} = G$, a case which has been dealt with in Section 7.

In order to state our inductive hypothesis, we need to introduce a new concept, that of a *common approximation* to a collection of simple characters. We use the same data as in 8.1: we are given F-vector spaces $\bar{W}^1, \ldots, \bar{W}^q$ and, for each j, a simple stratum $[\mathfrak{A}_j, n_j, 0, \beta_j]$ in $\mathrm{End}_F(\bar{W}^j)$, with $\mathfrak{A}_j$ corresponding to a strict lattice sequence Λ^j of period e_j. We are also given a simple character $\theta^j = \Theta_j(\mathfrak{A}_j) \in \mathcal{C}(\mathfrak{A}_j, 0, \beta_j)$, $1 \leqslant j \leqslant q$. We form the lattice sequence $\Lambda = \Lambda^1 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^q$ in $V = \coprod \bar{W}^j$ (see 2.8), and write

$$e = \mathrm{lcm}(e_j), \qquad n = \max(n_j e/e_j).$$

Suppose we are given an integer m with $0 < m < n$. The character $\theta^j | H^1(\beta_j) \cap \boldsymbol{u}_{m+1}(\Lambda)$ determines an endo-class $\Theta^m(j)$. Let us consider the case where these $\Theta^m(j)$ are all the same. We can then find a simple stratum $[\Lambda, n, 0, \gamma]$ and a simple character $\theta \in \mathcal{C}(\Lambda, 0, \gamma)$ with the following properties. First, all of the $\bar{W}^j$ are $F[\gamma]$-subspaces of V. We decompose the group $H^1(\gamma, \Lambda)$ and the character ϑ in the manner of Proposition 5.2, relative to the Levi subgroup $\bar{M}$. We then demand the properties:

$$H^{m+1}(\gamma, \Lambda) \cap \bar{G}^j = H^{(m+1)e_j/e}(\beta_j, \Lambda^j),$$

$$\vartheta | H^{(m+1)e_j/e}(\beta_j, \Lambda^j) = \theta^j | H^{(m+1)e_j/e}(\beta_j, \Lambda^j),$$

for all j. Under these circumstances, we say that $([\Lambda, n, 0, \gamma], \vartheta, m)$ is a *common approximation to the system* (θ^j) *of level* m.

We note that the character ϑ is trivial on $H^{m+1}(\gamma, \Lambda) \cap \bar{N}$, where $\bar{N}$ is the unipotent radical of any parabolic subgroup with Levi component $\bar{M}$.

We remark that the system (θ^j) admits a common approximation of level zero if and only if the θ^j are equal (or, rather, endo-equivalent). In our present situation 8.1, this amounts to $q = 1$.

In general, suppose we have common approximations $([\Lambda, n, 0, \gamma_i], \vartheta_i, m)$, $i = 1, 2$, with $m < n$. We then have $H^{m+1}(\gamma_1, \Lambda) = H^{m+1}(\gamma_2, \Lambda)$, and the characters ϑ_i coincide on this group.

We now give the most precise statement of our main result.

MAIN THEOREM (final version). *There exists a G-cover (K, τ) of $(K_{\bar{M}}, \tau_{\bar{M}})$ with the following properties:*

(i) $\boldsymbol{u}_{n+1}(\Lambda) \subset K \subset \boldsymbol{u}(\Lambda)$.
(ii) *Suppose that $(\theta^1, \theta^2, \ldots, \theta^q)$ admits a common approximation $([\Lambda, n, 0, \gamma], \vartheta, m)$, $m < n$. Then K contains and normalizes the group $H^{m+1}(\gamma, \Lambda) \cdot H^m(\gamma, \Lambda) \cap \bar{M}$. The restriction of τ to $H^{m+1}(\gamma, \Lambda)$ is a multiple of ϑ and its restriction to $H^m(\gamma, \Lambda) \cap \bar{M}$ is a multiple of $\theta^1 \otimes \cdots \otimes \theta^q$.*

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In particular, (K, τ) *is an* $\mathfrak{s}$*-type, and also a cover of the pair* (K_M, τ_M) *of* 1.5.

If P *is any parabolic subgroup of* G *with Levi component* M*, the associated algebra homomorphism* $\boldsymbol{j}_P : \mathcal{H}(M, \tau_M) \to \mathcal{H}(G, \tau)$ *is an isomorphism which preserves support of functions:* $\mathrm{supp}(j_P(f)) = K\,\mathrm{supp}(f)K$, $f \in \mathcal{H}(M, \tau_M)$*. In particular, the* G*-interwining of the representation* τ *is given by* $I_G(\tau) = K \cdot I_M(\tau_M) \cdot K$*.*

We give the proof in the next paragraph. Note, however, that the final assertion follows from the earlier ones and [9] 8.5, 12.1. Also, in the case $q = 1$, the cover constructed in 7.2 satisfies these extra conditions. In other words, the first case $q = 1$ of the induction has been done.

8.3. We now assume that $q > 1$. As we have already observed, the system (θ^j) does not admit a common approximation to level 0. We deal first with the case where (θ^j) does not admit a common approximation to level $n-1$. We need to describe this situation in more detail:

LEMMA. *Let* $(\theta^1, \theta^2, \ldots, \theta^q)$ *be as above. For each* j*, let* $[\mathfrak{A}_j, n_j, n_j - 1, \alpha_j]$ *be a simple stratum equivalent to* $[\mathfrak{A}_j, n_j, n_j - 1, \beta_j]$*, let* $\varphi_j(X) \in k_F[X]$ *be its characteristic polynomial, and let* $f_j(X)$ *be the unique irreducible factor of* $\varphi_j(X)$*. The following conditions are equivalent:*

(i) (θ^j) *admits a common approximation of level* $n-1$;

(ii) $n_1/e_1 = n_2/e_2 = \cdots = n_q/e_q$ *and* $f_1(X) = f_2(X) = \cdots = f_q(X)$.

Proof. This simply says that the endo-equivalence class of the simple character $\psi_{\alpha_j} \in \mathcal{C}(\mathfrak{A}_j, n_j - 1, \beta_j)$ is determined by the 'normalized level' n_j/e_j and the polynomial $f_j(X)$: see [5] (2.6.1). □

Under our present hypothesis, we can renumber the θ^j to achieve the following situation:

8.3.1. *There is an index* j_0, $1 \leqslant j_0 < q$*, such that either:*

(i) $n_1/e_1 = n_2/e_2 = \cdots = n_{j_0}/e_{j_0} > n_{j_0+1}/e_{j_0+1} \geqslant \cdots \geqslant n_q/e_q$*, or else*

(ii) *all* n_j/e_j *are equal,* $f_1(X) = \cdots = f_{j_0}(X)$*, while* $f_k(X) \neq f_1(X)$ *for any* $k > j_0$.

We now let $Y_1 = \bar{W}^1 \oplus \cdots \oplus \bar{W}^{j_0}$, $Y_2 = \bar{W}^{j_0+1} \oplus \cdots \oplus \bar{W}^q$; we set $G_i' = \mathrm{Aut}_F(Y_i)$. Let us set $L_i = K_{\bar{M}} \cap G_i'$, so that $K_{\bar{M}} = L_1 \times L_2$; the representation $\tau_{\bar{M}}$ likewise decomposes as $\varrho_1 \otimes \varrho_2$, for an irreducible representation ϱ_i of L_i. By inductive hypothesis, there exists a G_i'-cover (K_i', ϱ_i') of (L_i, ϱ_i) satisfying the requirements of the theorem.

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Let us make these conditions explicit. First, let $\Lambda_1' = \Lambda^1 \oplus \cdots \oplus \Lambda^{j_0}$, let e_1' be the period of Λ_1', and define an integer n_1' by $n_1'/e_1' = n_1/e_1$. We have (by inductive hypothesis)

$$\boldsymbol{u}_{n_1'+1}(\Lambda_1') \cdot \boldsymbol{u}_n(\Lambda) \cap \bar{M} \cap G_1' \subset K_1' \subset \boldsymbol{u}(\Lambda_1'),$$

and the restriction of ϱ_1' to the first group here is a multiple of ψ_{b_1}, for a Λ_1'-invertible element b_1 of level n_1'. The characteristic polynomial $\varphi_1(X)$ of the stratum $[\Lambda_1', n_1', n_1'-1, b_1]$ is a product of powers of the $f_j(X)$ for $j \leqslant j_0$. In the other component, we form the direct sum Λ_2' of the Λ^j for $j > j_0$ and denote the period of Λ_2' by e_2'. We define the integer n_2' by $n_2'/e_2' = \max_{j>j_0} n_j/e_j$. We have $n_2'/e_2' \leqslant n_1'/e_1'$. If this inequality is strict, $\boldsymbol{u}_{n_2'+1}(\Lambda_2')$ contains $\boldsymbol{u}_n(\Lambda) \cap G_2'$; we have $K_2' \subset \boldsymbol{u}(\Lambda_2')$ and ϱ_2' is trivial on $\boldsymbol{u}_n(\Lambda) \cap G_2'$. If, on the other hand, we have equality, the situation is analogous to that in the first component, but the characteristic polynomial $\varphi_2(X)$ is relatively prime to $\varphi_1(X)$. We let P_{u}, P_ℓ be the parabolic subgroups of G with Levi component $G_1' \times G_2'$. We denote their unipotent radicals by N_{u}, N_ℓ. We define the group K by

$$K = \boldsymbol{u}_{n+1}(\Lambda) \cap N_\ell \cdot K_1' \times K_2' \cdot \boldsymbol{u}(\Lambda) \cap N_{\mathrm{u}},$$

and we extend $\varrho_1' \otimes \varrho_2'$ to a representation τ of K which is trivial on the unipotent factors. This has the required covering properties by Corollary 3.9. The group K certainly lies between $\boldsymbol{u}(\Lambda)$ and $\boldsymbol{u}_{n+1}(\Lambda)$, so it satisfies the first part of the inductive hypothesis. The second part is empty here, so we have produced the required cover (K, τ) in this case.

8.4. We now prove the Main Theorem in the case where the system (θ^j) does admit a common approximation $([\Lambda, n, 0, \gamma], \vartheta)$ to level m, with $0 < m < n$. We choose this approximation so as to minimize m. As in 8.3, there are two cases. First, we know that some me_j/e is an integer: otherwise, we would have $H^m(\gamma, \Lambda) = H^{m+1}(\gamma, \Lambda)$ and this would contradict the minimality of m.

Suppose, for a fixed j, that me_j/e is an integer, m_j say. We then have

$$H^m(\gamma, \Lambda) \cap \bar{G}^j = H^{m_j}(\gamma, \mathfrak{A}_j) = H^{m_j}(\beta_j, \mathfrak{A}_j),$$

$$H^{m+1}(\gamma, \Lambda) \cap \bar{G}^j = H^{m_j+1}(\gamma, \mathfrak{A}_j).$$

Comparing the characters θ^j, ϑ on the first of these groups, we get an element c_j such that $\theta^j = \vartheta\psi_{c_j}$. Write $B_\gamma = \mathrm{End}_F[\gamma](V)$ and fix a tame corestriction s_γ on A relative to $F[\gamma]/F$. The stratum $[\mathfrak{A}_j \cap B_\gamma, m_j, m_j-1, s_\gamma(c_j)]$ is then either null or simple (4.7). Its characteristic polynomial (relative to $F[\gamma]$) is a power of an irreducible polynomial $f_j(X) \in k_{F[\gamma]}[X]$ (and, possibly, $f_j(X) = X$). In the null case, we can adjust ϑ by conjugation to get $\vartheta = \theta^j$ on H^{m_j}. In the cases where $me_j/e \notin \mathbb{Z}$, we set $f_j(X) = X$, $c_j = 0$. With this convention, we see that not all

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$f_j(X)$ can be equal to X: otherwise, we could contradict the minimality of m. This gives us:

LEMMA. *There exists j_0, $1 \leqslant j_0 < q$ such that one of the following holds:*

(i) *all me_j/e are integral, $f_1(X) = f_2(X) = \cdots = f_{j_0}(X) \neq X$, while $f_k(X) \neq f_1(X)$ for any $k > j_0$;*

(ii) *for all $j \leqslant j_0$, the quantity me_j/e is integral and $f_j(X) \neq X$, while $f_k(X) = X$ for all $k > j_0$.*

We use this index j_0 to define spaces Y_i, groups G'_i, lattice sequences Λ'_i, etc., $i = 1, 2$, as in 8.3. By inductive hypothesis, we have covers (K'_i, ϱ'_i) satisfying the conditions of the theorem. We define the group K by

$$K = H^{m+1}(\gamma, \Lambda) \cap N_\ell \cdot K'_1 \times K'_2 \cdot \boldsymbol{u}(\mathfrak{a}(\Lambda) \cap B_\gamma)(1 + \mathfrak{M}_m(\gamma, \Lambda)) \cap N_{\mathrm{u}},$$

where $\boldsymbol{u}(\mathfrak{a}(\Lambda) \cap B_\gamma)\ (1 + \mathfrak{M}_m(\gamma, \Lambda))$ is the $\boldsymbol{u}(\Lambda)$-normalizer of $(H^{m+1}(\gamma, \Lambda), \vartheta)$, as in 5.6. We extend $\varrho'_1 \otimes \varrho'_2$ to a representation τ of K by making it trivial on the unipotent factors. The pair (K, τ) has all the required properties, by Corollary 6.6.

This completes the proof of the Main Theorem. □

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