Semisimple Varieties of Modal Algebras

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In memory of Wim Blok

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1 Introduction

The main theorem of this paper is that semisimple varieties of modal algebras can be characterized by two properties: they are weakly transitive and cyclic. (The proof offered in [3] contains a fatal mistake.) Both notions are well–known and interesting in their own right. The notion of weak transitivity (going back to Wim Blok) generalizes the notion of transitivity in the right way. Algebraically, it amounts to EDPC. Many results in modal logic do not depend on transitivity, they only depend on weak transitivity. Examples are the existence of a deduction theorem for the global frame consequence, finite equivalentiality, and the existence of many splitting algebras. A variety is cyclic if every operator possesses (or is included in) a converse. Cyclicity generalizes the notion of symmetry.

2 Modal Logic

The set of formulae, Fm^κ (otherwise referred to as **terms** over the given set of variables), is the smallest set such that

1. $\{p_i : i \in \omega\} \subseteq \mathrm{Fm}^\kappa$
2. $\bot \in \mathrm{Fm}^\kappa$.
3. If $\varphi, \chi \in \mathrm{Fm}^\kappa$ then $\neg\varphi, \varphi \wedge \chi \in \mathrm{Fm}^\kappa$.
4. If $\varphi \in \mathrm{Fm}^\kappa$ and $i < \kappa$ also $\Box_i \varphi \in \mathrm{Fm}^\kappa$.

$(\varphi \vee \chi)$ and $(\varphi \to \chi)$ are abbreviations. A subset $L \subseteq \mathrm{Fm}^\kappa$ is a (**normal**) **modal logic** if

1. L contains all tautologies of classical logic.
2. L is closed under substitution.
3. For every $i < \kappa$: $\Box_i(p_0 \to p_1) \to .\Box_i p_0 \to \Box_i p_1 \in L$

4. For every $i < \kappa$, if $\varphi \in L$ then $\Box_i\varphi \in L$.

The smallest normal modal logic in κ operators is called K_κ. In what is to follow we write

$$\Box\varphi := \varphi \wedge \bigwedge_{i<\kappa} \Box_i\varphi \tag{1}$$

This is only meaningful if κ is finite. This is an example of a compound modality, where a **compound modality** is any term $\delta(p)$ over one variable that does not contain $\neg$ or $\bot$. (This is slightly less general than the definition given in [3], but this is inessential here.) Examples are $p \wedge \Box_1(p \wedge \Box_0\Box_2 p)$, non–examples are $\Box_0(p \wedge q)$ and $\Box_1(\Box_1 p \vee \Box_0 p)$. Note that p itself is a compound modality. (There is a slight and harmless confusion here between the formula and the modality that it denotes.) We shall denote compound modalities by $\boxtimes$, $\boxplus$, that make them look like modal operators. For if $\boxtimes$ is a compound modality, then for any normal modal logic L the following holds:

- $L \vdash \boxtimes(p \to q). \to . \boxtimes p \to \boxtimes q$
- If $L \vdash \varphi$ then $L \vdash \boxtimes\varphi$

So, a compound modality has all properties of a modal operator. We say that $\boxtimes$ **contains** $\boxplus$ in L if $L \vdash \boxtimes p \to \boxplus p$.

The methods we are outlining here also work for polyadic operators. The generalization to operators of arity > 1 is straightforward. Basically, it is required that if O is an n–ary operator, then the derived unary operators

$$O_i(p) := O(\top, \ldots, \top, p, \ldots, \top) \tag{2}$$

(where p is placed in ith argument position) are normal operators. Now define

$$[O]p := \bigwedge_{i<n} O_i(p) \tag{3}$$

Then, rather than looking at the signature $\langle O^k : k < \kappa \rangle$, we look in stead at the signature $\langle O^k_{n(k)} : k < \kappa \rangle$, where $n(k)$ is the arity of O^k. This reduces the polyadic signature to a monadic signature.

A **modal algebra** is an algebra $\langle A, 1, -, \cap, \langle \blacksquare_i : i < \kappa \rangle \rangle$ such that $\langle A, 1, -, \cap \rangle$ is a boolean algebra and for every $i < \kappa$: $\blacksquare_i 1 = 1$ and $\blacksquare_i (a \cap b) = \blacksquare_i a \cap \blacksquare_i b$. An example is the algebra obtained by $\mathrm{Fm}_\kappa / \Theta_L$, where $\varphi \; \Theta_L \; \chi$ iff $L \vdash \varphi \leftrightarrow \chi$. It is more convenient to regard a modal algebra as a pair $\langle \mathbf{A}, \langle \blacksquare_i : i < \kappa \rangle \rangle$, where $\mathbf{A} = \langle A, 1, -, \cap \rangle$ is the underlying boolean algebra. A filter is called **open** if from $a \in F$ follows $\blacksquare_i a \in F$. The map $\gamma \mapsto 1/\gamma$ is a lattice isomorphism between congruences and open filters. If κ is finite, put

$$\blacksquare a := a \cap \bigcap_{i < \kappa} \blacksquare_i a \tag{4}$$

Furthermore, for a polymodal algebra $\mathfrak{A}$, let $\mathfrak{A}^\circ := \langle \mathbf{A}, \blacksquare \rangle$. This is a monomodal algebra. For a variety $\mathcal{V}$ we put $\mathcal{V}^\circ := \{\mathfrak{A}^\circ : \mathfrak{A} \in \mathcal{V}\}$. It is not hard to show the following.

Lemma 1 $\mathrm{Con}\, \mathfrak{A} = \mathrm{Con}\, \mathfrak{A}^\circ$.

This is because a congruence of $\mathfrak{A}$ is a congruence of $\mathfrak{A}^\circ$ and conversely. It does not hold, however, that every subalgebra of $\mathfrak{A}^\circ$ is a subalgebra of $\mathfrak{A}$. So, $\mathcal{V}^\circ$ is not a variety, but $\mathsf{S}\mathcal{V}^\circ$ is, because $(-)^\circ$ commutes with products. The following is reformulation of a theorem found in [4].

Lemma 2 (Rautenberg) $\mathfrak{A}$ *is subdirectly irreducible iff* $\mathfrak{A}^\circ$ *is subdirectly irreducible iff there is a* $c < 1$ *such that for every* $a < 1$ *there is an* $n \in \omega$ *such that*

$$\blacksquare^n a \leq c \tag{5}$$

(Notice that $\blacksquare a \leq a$, by definition.) c is any element that generates the unique smallest filter $\neq \{1\}$. In a simple algebra, this filter is A itself.

Corollary 3 $\mathfrak{A}$ *is simple iff* $\mathfrak{A}^\circ$ *is simple iff for each* $a \in A$ *with* $a < 1$ *there is a* $n \in \omega$ *such that* $\blacksquare^n a = 0$.

A **Kripke–frame** is a structure $\mathfrak{F} = \langle F, \langle \lhd_i : i < \kappa \rangle \rangle$ where $\lhd_i \subseteq F \times F$ for every $i < \kappa$. $\beta : \mathrm{Var} \to \wp(F)$ is a **valuation**. $\langle \mathfrak{F}, \beta, x \rangle \vDash \varphi$ is defined inductively.

$$\langle \mathfrak{F}, \beta, x \rangle \vDash \Box_i \varphi \quad \Leftrightarrow \quad \text{for all } y \rhd_i x : \langle \mathfrak{F}, \beta, y \rangle \vDash \varphi \tag{6}$$

Write $\langle \mathfrak{F}, \beta \rangle \vDash \varphi$ if for all x: $\langle \mathfrak{F}, \beta, x \rangle \vDash \varphi$.

The relations for compound modalities are defined as follows. The relation corresponding to p itself is the identity; furthermore, the relation corresponding to $\Box_i \chi(p)$ is $R(\Box_i) \circ R(\chi)$. Finally, the relation corresponding to $\chi(p) \wedge \chi'(p)$ is $R(\chi) \cup R(\chi')$. In particular, the relation corresponding to $\blacksquare$ is

$$R(\Box) = \Delta \cup \bigcup_{i<\kappa} R(\Box_i) \tag{7}$$

Hence, $R(\Box)$ is always reflexive (even in the monomodal case). It is easy to see that for a compound modality $\boxtimes$:

$$\langle \mathfrak{F}, \beta, x \rangle \vDash \boxtimes \chi \quad \Leftrightarrow \quad \text{for all } y \text{ such that } x \; R(\boxtimes) \; y\text{: } \langle \mathfrak{F}, \beta, y \rangle \vDash \chi \tag{8}$$

3 Weakly Transitive Logics

Consider the rules MP: $\varphi, \varphi \to \chi / \chi$ and Necessitation: $\varphi / \Box_i \varphi$. Let L be a modal logic.

1. The **local consequence relation** of L. $\Delta \vdash_L \varphi$ iff φ can be proved from $\Delta \cup L$ using only Modus Ponens.

2. The **global consequence relation** of L. $\Delta \Vdash_L \varphi$ iff φ can be derived from $\Delta \cup L$ using Modus Ponens and Necessitation.

$\Delta \Vdash_L \varphi$ iff for all modal algebras $\mathfrak{A}$ such that $\mathfrak{A} \vDash L$ and all valuations β: if $\overline{\beta}(\delta) = 1$ for all $\delta \in \Delta$ then $\overline{\beta}(\varphi) = 1$. A monomodal logic L is transitive if it contains $\Box p \to \Box\Box p$. Weak transitivity generalizes this notion as follows.

Definition 4 *L is weakly transitive if there is a compound modality containing all $\Box_i$ in L such that $\boxtimes p \to \boxtimes \boxtimes p$.*

Proposition 5 *A polymodal logic L is weakly transitive iff there is an n such that $L \vdash \Box^n p \to \Box^{n+1} p$.*

(Recall that $\Box$ is reflexive!) Notice that a variety $\mathcal{V}$ of polymodal algebras of finite type is weakly transitive iff there is an n such that

$$\mathcal{V} \vDash \blacksquare^n x = \blacksquare^{n+1} x \tag{9}$$

$\Vdash_L$ has a deduction theorem iff there is a formula $\tau(p,q)$ such that $\chi \Vdash_L \varphi$ iff $\Vdash_L \tau(\chi, \varphi)$. In general, one can choose $\tau(p,q) := \boxtimes(p \to q)$ for some compound modality $\boxtimes$. Hence L is weakly transitive iff $\Vdash_L$ has a deduction theorem. A logic is **equivalential** if there is a set $\Delta(p,q)$ of terms in two variables satisfying

① $\vdash_L \Delta(p,p)$

② $\Delta(p,q) \vdash_L \Delta(q,p)$

③ $\Delta(p,q), \Delta(q,r) \vdash_L \Delta(p,r)$

④ $p, \Delta(p,q) \vdash_L q$

⑤ $\bigcup_{i<\Omega(f)} \Delta(p_i, q_i) \vdash_L \Delta(f(\vec{p}), f(\vec{q}))$

L is **finitely equivalential** if Δ can be chosen finite. Every modal logic is equivalential via $\Delta(p,q) := \{p \leftrightarrow q\}$. K is not finitely eqivalential. The following is from [1].

Theorem 6 (Blok, Pigozzi) *For a modal logic L the following are equivalent:*

- *L is weakly transitive.*
- *$\Vdash_L$ has a deduction theorem.*

- $\vdash_L$ *is finitely equivalential.*
- $\mathsf{Alg}\, L$ *has equationally definable principal congruences.*
- $\mathsf{Alg}\, L$ *has definable principal open filters.*

Given a variety $\mathcal{V}$, an algebra $\mathfrak{A}$ is called **finitely presentable** if there is a finite set E of equations such that

$$\mathfrak{Fr}_{\mathcal{V}}(\mathrm{var}(V))/\Theta(E) \cong \mathfrak{A}$$

where $\Theta(E)$ is the congruence generated by E. (In particular, finite algebras are finitely presentable.) $\mathfrak{A}$ is **splitting** if there is a logic L such that for every logic $L' \supseteq \mathsf{Th}\,\mathcal{V}$: either $L' \subseteq \mathsf{Th}\,\mathfrak{A}$ or $L' \supseteq L$, but not both. The following is from [2].

Theorem 7 *Suppose that $\mathcal{V}$ is weakly transitive. Then every subdirectly irreducible $\mathfrak{A}$ which is finitely presentable is splitting.*

4 Cyclic Varieties

Let $\Box$ be a modal operator. $\blacksquare$ is called a **conjugate** of $\Box$ in L if $p \to \Box\blacklozenge p \in L$. Intuitively, this means that for every x in a Kripke–frame, if $x \lhd y$ then $y \blacktriangleleft x$. $\mathcal{V}$ is called **cyclic** if every modal operator has a conjugate.

Proposition 8 *The following are equivalent.*

① *Every compound modality has a conjugate.*

② *Every basic modality has a conjugate.*

③ $\Box$ *has a conjugate.*

It follows that $\mathcal{V}$ is cyclic iff $\mathcal{V}^\circ$ is cyclic. Obviously, $\mathcal{V}$ is cyclic iff $p \to \Box\Diamond^n p$ for some n. (In which case $\mathcal{V}$ is called n–**cyclic**.)

Proposition 9 *If $\mathcal{V}$ is cyclic then every finite subdirectly irreducible algebra is simple.*

This need not hold for infinite algebras. Take $\mathfrak{F} := \langle \mathbb{Z}, \lhd \rangle$, where $x \lhd y$ iff $|x - y| = 1$. Then the algebra $\mathfrak{Z}$ of finite and cofinite subsets is subdirectly irreducible. The subset of cofinite sets forms an open filter. It is not hard to see that $\mathrm{Con}\,\mathfrak{Z} \cong 3$. The variety generated by $\mathfrak{Z}$ is cyclic.

Theorem 10 *Suppose $\mathcal{V}$ is weakly transitive and cyclic. Then every subdirectly irreducible algebra is simple.*

5 Discriminator Varieties

A **discriminator term** for $\mathfrak{A}$ is a term $t(x, y, z)$ such that for all elements a, b, c: $t(a, b, c) = c$ if $a = b$ and a otherwise. An algebra with a discriminator term is simple. A variety is **discriminator** if it is generated from algebras with the same discriminator term. Put $u(a) := -t(1, a, 0)$. Then $u(a) = 1$ if $a = 1$ and $u(a) = 0$ else.

Proposition 11 *A modal algebra has a discriminator term if there is a term such u that $u(a) = 0$ iff $a < 1$, and $u(1) = 1$.*

Such a u is often called a **unary discriminator**. Let $\mathcal{V}$ be m–transitive and cyclic. Then $\tau(p) := \Box^m p$ is a (unary) discriminator term for every subdirectly irreducible member. Conversely, suppose that $\mathcal{V}$ is discriminator. Then if $\mathcal{V}$ is not weakly transitive, for every n there is $\mathfrak{A}_n$ and $a_n < 1$ such that $\blacksquare^n a_n > \blacksquare^{n+1} a_n$. Put n the modal degree of u. Then $u(a) \leftrightarrow u(1) > 0$. Contradiction.

Theorem 12 *($\kappa < \omega$.) $\mathcal{V}$ is discriminator iff it is weakly transitive and cyclic.*

The condition $\kappa < \omega$ is essential. For suppose that κ is infinite. Then let $\mathfrak{H}_i = \langle \mathbf{H}, \langle \blacksquare_i : i < \kappa \rangle \rangle$ be the following algebra. $\mathbf{H}$ is the 4–element boolean algebra. $\blacksquare_j$ is the diagonal if $j \neq i$ and the full relation otherwise. Now, let $\mathfrak{P}$ be an ultraproduct of the $\mathfrak{H}_i$ over a nonprincipal ultrafilter. Then $\vec{x} \lhd_i \vec{y}$ iff for a set F of U: for all $j \in F$: $\vec{x}(j) \lhd_i \vec{y}(j)$. This is easily seen to be equivalent to $\vec{x} = \vec{y}$ modulo U. Hence, $\mathfrak{P}$ is directly decomposable, but the $\mathfrak{H}_i$ are simple. In fact, it is isomorphic to the four element algebra in which $x = \blacksquare_i x$ for all x. Now, the variety generated by the $\mathfrak{H}_i$ is cyclic and weakly transitive. But it is not a discriminator variety, as the example shows. By Jónsson's theorem, the subdirectly irreducible algebras are in $\mathsf{HS}(\{\mathfrak{H}_i : i < \kappa\} \cup \{\mathfrak{P}\})$. They all have at most four elements. Every two element algebra is simple, hence there is only one nonsimple algebra in this class: $\mathfrak{P}$, which is not subdirectly irreducible. Therefore the variety is semisimple.

6 Semisimple Varieties of Finite Type

A variety is **semisimple** if every subdirectly irreducible member is simple.

Theorem 13 *($\kappa < \omega$.) The following are equivalent:*

① *$\mathcal{V}$ is semisimple.*

② *$\mathcal{V}$ is discriminator.*

③ *$\mathcal{V}$ is weakly transitive and cyclic.*

We have seen ② ⇔ ③. Also ② ⇒ ① is generally true. ③ ⇒ ① is Theorem 10. The really hard part is to show that ① implies weak transitivity. For suppose the variety n–transitive but not cyclic. Then there is a s.i. $\mathfrak{A}$ and a such that $a \not\leq \blacksquare\blacklozenge^n a$. So, $\blacklozenge^n a \neq 1$. Its complement $-a$ therefore generates an open filter $F = \{x : x \geq \blacksquare^n - a\} \neq A$. Contradiction.

Now we shall show that for modal algebras of finite type semisimplicity implies weakly transitivity. The problem can be simplified at the outset by looking at $\mathcal{V}^\circ$ (or $\mathsf{S}\mathcal{V}^\circ$ for that matter). So, from now on we assume that $\mathcal{V}$ is a variety of monomodal algebras satisfying $x \leq \blacklozenge x$. Then $\mathcal{V}$ is n–transitive iff $\mathcal{V} \models \blacklozenge^{n+1}x = \blacklozenge^n x$. In what is to follow we assume that $\mathcal{V}$ is a nontrivial semisimple variety of modal algebras of finite type.

Let $\mathfrak{A}$ be an algebra in $\mathcal{V}$ such that for a nonzero element $a \in A$ we have $\blacklozenge^n a < 1$ for every $n \in \omega$. Such an algebra obviously exists: for instance the free algebra $\mathfrak{Fr}_{\mathcal{V}}(x)$ on one generator x must be such, as otherwise $\mathcal{V}$ would satisfy $\blacklozenge^n x = 1$ for some $n \in \omega$, and therefore also $\blacklozenge^n 0 = 1$, and thus $0 = 1$, forcing $\mathcal{V}$ to be trivial, contrary to the initial assumption. Put $\alpha := \mathrm{Cg}^{\mathfrak{A}}(a, 0)$. By assumption on $\mathfrak{A}$, $0 < \alpha < 1$. As α is principal, there must be a congruence β with $\beta \prec \alpha$, that is, β is a lower cover of α.

Lemma 14 *For every congruence β with $\beta \prec \alpha$ there is an $m \in \omega$ such that*

(i) $\blacklozenge^{m+1}a \equiv_\beta \blacklozenge^m a$, *and*

(ii) $-\blacklozenge^m a \equiv_\beta \blacklozenge - \blacklozenge^m a$.

Proof. Consider the set $\Gamma := \{\theta \in \mathrm{Con}\,\mathfrak{A} : \theta \geq \beta,\ \theta \not\geq \alpha\}$. Observe that if $\Gamma = \{\beta\}$, then $\mathfrak{A}/\beta$ is si but not simple. So there is a $\theta \in \Gamma$ such that $\theta \neq \beta$. By congruence distributivity, $\gamma := \bigvee \Gamma$ is a member of Γ. Therefore, $\mathfrak{A}/\gamma$ is si hence simple. From this and congruence permutability it follows that $\alpha \circ \gamma = 1$. Thus, $(0, 1) \in \alpha \circ \gamma$, and there must be a $c \in A$ with $(0, c) \in \alpha$ and $(c, 1) \in \gamma$; hence also $(-c, 0) \in \gamma$. Now, $(0, c) \in \alpha$ iff for some $m \in \omega$ we have $\blacklozenge^m a \geq c$. Thus, $-\blacklozenge^m a \leq -c$ and therefore $(-\blacklozenge^m a, 0) \in \gamma$. We can then assume $c = \blacklozenge^m a$. By definition we have $\alpha \cap \gamma = \beta$, i.e., $0/\alpha \cap 0/\gamma = 0/\beta$. Now, to prove (i), take $\blacklozenge^{m+1}a \cap -\blacklozenge^m a$. This belongs to $0/\alpha \cap 0/\gamma = 0/\beta$ and thus we obtain $\blacklozenge^{m+1}a \equiv_\beta \blacklozenge^m a$. Then, for (ii), take $\blacklozenge^m a \cap \blacklozenge - \blacklozenge^m a$. This again belongs to $0/\alpha \cap 0/\gamma = 0/\beta$; therefore $-\blacklozenge^m a \equiv_\beta \blacklozenge - \blacklozenge^m a$. ⊣

Now, consider the following condition on $\mathcal{V}$.

($\star$) For every $k \in \omega$ there are $r, \ell \in \omega$ such that $\mathcal{V} \models x \leq \blacklozenge^\ell \blacksquare^k \blacklozenge^r x$.

Notice that ($\star$) is a weakened form of cyclicity.

Lemma 15 *All p–cyclic varieties of modal algebras satisfy ($\star$) for $r = kp$ and $\ell = 0$.*

Proof. p–cyclic varieties satisfy $x \leq \blacksquare\blacklozenge^p x$. We will show that this forces $x \leq \blacksquare^k\blacklozenge^{kp}x$, for every $k \in \omega$. For $k = 0, 1$ this is trivial. Suppose it holds for k.

$$
\begin{aligned}
\blacksquare^{k+1}\blacklozenge^{(k+1)p}x &= \blacksquare(\blacksquare^k\blacklozenge^{kp}(\blacklozenge^p x)) \\
&= \blacksquare(\blacksquare^k\blacklozenge^{kp}(\blacklozenge^p x)) \\
&\geq \blacksquare\blacklozenge^p x \\
&\geq x
\end{aligned}
\tag{10}
$$

Thus, $x \leq \blacklozenge^0\blacksquare^{k+1}\blacklozenge^{(k+1)p}x$, as required. ⊣

Suppose now that $\mathcal{V}$ falsifies ($\star$). Then there is a $k \in \omega$ such that for all $r, \ell \in \omega$ our variety $\mathcal{V}$ falsifies $x \leq \blacklozenge^\ell\blacksquare^k\blacklozenge^r x$. Let K be the smallest such k; note that for all $k' \geq K$ the variety $\mathcal{V}$ also falsifies $x \leq \blacklozenge^\ell\blacksquare^{k'}\blacklozenge^r x$.

In $\mathfrak{Fr}_\mathcal{V}(x)$, the algebra freely generated by x, we have $x \not\leq \blacklozenge^\ell\blacksquare^K\blacklozenge^r x$, for all $r, \ell \in \omega$. For each $r \in \omega$ define $\theta_r := \mathrm{Cg}^{\mathfrak{Fr}_\mathcal{V}(x)}(\blacksquare^K\blacklozenge^r x, 0)$. Since $\blacksquare^K\blacklozenge^r x \leq \blacksquare^K\blacklozenge^{r+1}x$, the family of congruences $\{\theta_r : r \in \omega\}$ forms an increasing chain. Put $\alpha := \mathrm{Cg}^{\mathfrak{Fr}_\mathcal{V}(x)}(x, 0)$ and $\Theta := \bigvee_{r\in\omega}\theta_r$.

Lemma 16 $0 < \Theta < \alpha$.

Proof. If $\Theta = 0$, then, since the reasoning takes place in the one–generated free algebra in $\mathcal{V}$, we get that $\mathcal{V} \models \blacksquare^K\blacklozenge^r x = 0$, for all $r \in \omega$. In particular, substituting 1 for x we obtain $\mathcal{V} \models 1 = \blacksquare^K\blacklozenge^r 1 = 0$. This means $\mathcal{V}$ is trivial, contradicting the initial assumptions.

That $\Theta \leq \alpha$ is clear from the definitions. Since α is principal, $\Theta = \alpha$ iff there is a finite set $S \subseteq \{\theta_r : r \in \omega\}$ with $\bigvee S = \alpha$. However, as $\{\theta_r : r \in \omega\}$ is an increasing chain there is an $r \in \omega$ such that $\theta_r = \alpha$. Now, both θ_r and α are principal, and thus $\theta_r = \alpha$ iff there is an $\ell \in \omega$ with $x \leq \blacklozenge^\ell\blacksquare^K\blacklozenge^r x$. As we reason in the free algebra, it means that $\mathcal{V}$ satisfies $x \leq \blacklozenge^\ell\blacksquare^K\blacklozenge^r x$. This contradicts our assumption that $\mathcal{V}$ falsifies ($\star$). ⊣

Lemma 17 *All semisimple varieties of modal algebras satisfy* ($\star$).

Proof. Since our fixed variety $\mathcal{V}$ has been chosen arbitrarily, it suffices to prove that it satisfies $(\star)$. We proceed by *reductio*. Suppose $\mathcal{V}$ falsifies $(\star)$ and let K be the smallest number witnessing that, precisely as in Lemma 16. In view of that lemma, we can choose a congruence β such that $\Theta \leq \beta \prec \alpha$. Then, by Lemma 14, we obtain $\blacklozenge^m x \equiv_\beta \blacksquare\blacklozenge^m x$, for some $m \in \omega$. Since $\theta_m \leq \Theta \leq \beta$, this gives us $\blacklozenge^m x \equiv_\beta \blacksquare^K\blacklozenge^m x \equiv_\beta 0$. Therefore, $x \equiv_\beta 0$ and thus $\beta \geq \alpha$ contradicting the choice of β. ⊣

From now on we assume that $\mathcal{V}$ satisfies $(\star)$. Define a function $r : \omega \to \omega$, by taking $r(i)$ to be the smallest number such that there exists an $\ell \in \omega$ with $\mathcal{V} \models \blacklozenge^\ell\blacksquare^i\blacklozenge^{r(i)}x \leq x$.

Lemma 18 *The function r is nondecreasing.*

Proof. Suppose the contrary. Then, for a certain $i \in \omega$, we have $r(i) > r(i+1)$. By definition of r there is an $\ell \in \omega$ such that

$$x \leq \blacklozenge^\ell\blacksquare^{i+1}\blacklozenge^{r(i+1)}x \leq \blacklozenge^\ell\blacksquare^{i}\blacklozenge^{r(i+1)}x \tag{11}$$

This is a contradiction, since $r(i)$ is by definition the smallest number for which a suitable ℓ exists, yet $r(i+1)$ is strictly smaller. ⊣

Now let us define another function $\ell : \omega \to \omega$ by taking $\ell(i)$ to be the smallest number such that $\mathcal{V} \models \blacklozenge^{\ell(i)}\blacksquare^i\blacklozenge^{r(i)}x \leq x$. Thus, ℓ depends on i via $r(i)$.

Lemma 19 *Let $\mathcal{V}$ be semisimple. If $\mathcal{V}$ is not weakly transitive then for each $i \in \omega$ there is a simple algebra $\mathfrak{A}_i$ in $\mathcal{V}$ and an element $a_i \in A_i$ such that $\blacklozenge^{r(i)}a_i < 1$ but $\blacklozenge^{r(i)+1}a_i = 1$.*

Proof. Suppose otherwise. Then there is an $i \in \omega$ such that for each simple algebra $\mathfrak{A} \in \mathcal{V}$ and each element $a \in A$ we have: $\blacklozenge^{r(i)+1}a = 1$ implies $\blacklozenge^{r(i)}a = 1$. By Corollary 3 this gives $\blacklozenge^{r(i)}a = 1$ if $a > 0$. This in turn forces $\mathcal{V}$ to satisfy $\blacklozenge^{n+1}x = \blacklozenge^n x$ for $n = r(i)$, contrary to the assumption. ⊣

Now suppose $\mathcal{V}$ is not weakly transitive and let $\mathfrak{A}_i$ and $a_i \in A_i$ be as in the lemma above, for each $i \in \omega$. Then put $b_i := -\blacklozenge^{r(i)}a_i$ and fix a $k \in \omega$.

Lemma 20 *For every $i \geq k$ we have: $\blacklozenge^k b_i < 1$ and $\blacklozenge^{\ell(k)+r(k)+1} - \blacklozenge^k b_i = 1$.*

Proof. To prove the first statement of the claim, suppose the contrary, and note that $\blacklozenge^k b_i = 1$ forces $\blacklozenge^i b_i = 1$, since $i \geq k$. By the definition of b_i, we have $\blacklozenge^i - \blacklozenge^{r(i)} a_i = 1$, thus $\blacksquare^i \blacklozenge^{r(i)} a_i = 0$. Therefore, $\blacklozenge^\ell \blacksquare^i \blacklozenge^{r(i)} a_i = 0$, for any $\ell \in \omega$. This falsifies $\blacklozenge^{\ell(i)} \blacksquare^i \blacklozenge^{r(i)} a_i \geq a_i$.

For the second part, applying the definition of b_i we get

$$\begin{aligned} \blacklozenge^{\ell(k)+r(k)+1} - \blacklozenge^k b_i &= \blacklozenge^{\ell(k)+r(k)+1} - \blacklozenge^k - \blacklozenge^{r(i)} a_i \\ &= \blacklozenge^{\ell(k)+r(k)+1} \blacksquare^k \blacklozenge^{r(i)} a_i \\ &= \blacklozenge^{\ell(k)+r(k)+1} \blacksquare^k \blacklozenge^{r(k)} (\blacklozenge^{r(i)-r(k)} a_i) \end{aligned} \tag{12}$$

Then, from the inequality $\blacklozenge^{\ell(k)} \blacksquare^k \blacklozenge^{r(k)} x \geq x$, which holds in $\mathcal{V}$, we get $\blacklozenge^{\ell(k)+r(k)+1} \blacksquare^k \blacklozenge^{r(k)} x \geq \blacklozenge^{r(k)+1} x$, by applying $\blacklozenge$ repeatedly $r(k)+1$ times to both sides. Now a substitution yields:

$$\blacklozenge^{\ell(k)+r(k)+1} \blacksquare^k \blacklozenge^{r(k)} (\blacklozenge^{r(i)-r(k)} a_i) \geq \blacklozenge^{r(k)+1} (\blacklozenge^{r(i)-r(k)} a_i) \tag{13}$$

Rewriting both sides back, we obtain

$$\blacklozenge^{\ell(k)+r(k)+1} \blacksquare^k \blacklozenge^{r(i)} a_i \geq \blacklozenge^{r(i)+1} a_i = 1 \tag{14}$$

This is as required. ⊣

Now take the ultraproduct $\mathfrak{B} := \prod_{i \in \omega} \mathfrak{A}_i / U$, over a nonprincipal ultrafilter U on ω. Put $b := \langle b_i : i \in \omega \rangle / U$.

Lemma 21 *For every $k \in \omega$ we have $\blacklozenge^k b < 1$ and $\blacklozenge^{\ell(k)+r(k)+1} - \blacklozenge^k b = 1$.*

Proof. For any fixed $k \in \omega$, it follows by Lemma 20 that the i for which $\blacklozenge^k b_i < 1$ and $\blacklozenge^{\ell(k)+r(k)+1} - \blacklozenge^k b_i = 1$ form a cofinite set; namely of all these with $i \geq k$. The claim then follows by properties of ultraproducts. ⊣

Lemma 22 *If $\mathcal{V}$ satisfies $(\star)$ it is weakly transitive.*

Proof. We will show that assuming otherwise leads to a contradiction. Suppose $\mathcal{V}$ falsifies $\blacklozenge^{n+1} x = \blacklozenge^n x$ for all $n \in \omega$. Given that, the preceding three lemmas show how to produce an algebra $\mathfrak{B} \in \mathcal{V}$ and an element $b \in B$ which for all $k \in \omega$ satisfies $\blacklozenge^k b < 1$ and $\blacklozenge^{\ell(k)+r(k)+1} - \blacklozenge^k b = 1$. Then, putting $\alpha := \mathrm{Cg}^{\mathfrak{B}}(b, 0)$, and taking β as in Lemma 14, we get

$$-\blacklozenge^m b \equiv_\beta \blacklozenge - \blacklozenge^m b \equiv_\beta \blacklozenge^{\ell(m)+r(m)+1} - \blacklozenge^m b = 1 \tag{15}$$

Thus, $\blacklozenge^m b \equiv_\beta 0$ and therefore $b \equiv_\beta 0$. It follows that $\beta \geq \alpha$, contradicting the choice of β as a lower cover of α. ⊣

By Lemma 17 all semisimple varieties satisfy $(\star)$.

Theorem 23 *($\kappa < \omega$.) All semisimple varieties of modal algebras are weakly transitive.* ⊣

It follows that semisimple varieties have a deduction theorem, that every finitely presentable subdirectly irreducible algebra is splitting. This has useful applications in tense logic. By definition, tense logics are cyclic. $\mathsf{K4.2}t$ also is 2–transitive. It follows that its variety is semisimple. A fortiori, tense logics of linear transitive structures define semisimple varieties. By contrast, the variety of $\mathsf{K4}t$–algebras is not weakly transitive and therefore not semisimple.

Let us note some corollary of our result.

Corollary 24 *A semisimple variety $\mathcal{V}$ of modal algebras (BAOs) of finite type is canonical iff $\mathcal{V}$ is closed under canonical extensions of its simple members.*

Proof. $\mathcal{V}$ is discriminator, by our result. Thus, every algebra is $\mathcal{V}$ is a Boolean product of simple algebras. The canonical extension of a Boolean product is (isomorphic to) the product of canonical extensions of the factors (Gehrke). Hence, if canonical extensions of simple members of $\mathcal{V}$ belong to $\mathcal{V}$, the same holds for every algebra in $\mathcal{V}$. ■

This is however also a special case of the following

Theorem 25 *Let $\mathcal{V}$ be weakly transitive. Then $\mathcal{V}$ is canonical iff $\mathcal{V}$ is closed under canonical extensions of its subdirectly irreducible members.*

Proof. Every algebra in $\mathcal{V}$ is a subdirect product of si algebras. Every product can be rendered as a Boolean product of ultraproducts of the factors of the original product (folklore, or maybe Jonsson). Thus, every algebra in $\mathcal{V}$ is a Boolean product of ultraproducts of si algebras. By EDPC, ultraproducts of si algebras are si themselves. So, their canonical extensions are in $\mathcal{V}$. Then by Gehrke's theorem, and the fact that canonical extensions commute with subalgebras, the conclusion follows. ∎

7 Conclusion

We have shown that weak transitivity and cyclicity together characterize semisimplicity. Weak transitivity and cyclicity are important notions in modal logic, which basically generalize the notions of transitivity and symmetry.

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