# Semistable Sheaves on Projective Varieties and Their Restriction to Curves 

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## Introduction

Let $X$ be a nonsingular projective variety of dimension $n$ over an algebraically closed field $k$. Let $H$ be a very ample line bundle on $X$. If $V$ is a torsion free coherent sheaf on $X$ we define $\operatorname{deg} V$ to be $c_{1}(V) \cdot c_{1}(H)^{n-1}$ and $\mu(V)=\operatorname{deg} V /$ rk $V$. We call $V$ semistable (resp. stable) if for all proper subsheaves $W$ of $V$ we have $\mu(W) \leqq \mu(V)[$ resp. $\mu(W)<\mu(V)]$ (cf. [7, 14]).

In this paper we prove that if $V$ is semistable on $X$ then its restriction to a general complete intersection curve of sufficiently high degree is semistable (Theorem 6.1).

To give an idea of the proof assume $X$ is a surface and $V$ a vector bundle of rank 2. The restriction of $V$ to a general curve $C_{m}$ of degree $m$ is not semistable if and only if it is not semistable on the generic curve $Y_{m}$ defined over the function field of $\mathbb{P} H^{0}\left(X, H^{m}\right)$. Let $\bar{L}_{m}$ be the line bundle on $Y_{m}$ contradicting the semistability of $V \mid Y_{m}$ (cf. Sects. 4.1 and 4.2). First we show that $\bar{L}_{m}$ extends uniquely to a line bundle $L_{m}$ on $X$ (Proposition 2.1). If we can get $L_{m}$ as a subbundle of $V$ we are through, for then $L_{m}$ would contradict the semistability of $V$. So we would like the restriction map $H^{0}\left(X, \operatorname{Hom}\left(L_{m}, V\right)\right) \rightarrow H^{0}\left(C_{m}, \operatorname{Hom}\left(L_{m}, V\right)\right)$ to be surjective. Now for fixed $L$ it follows from the lemma of Enriques-Severi (Proposition 3.2; [6, Corollary 7.8]) that $H^{0}(X, \operatorname{Hom}(L, V)) \rightarrow H^{0}\left(C_{m}, \operatorname{Hom}(L, V)\right)$ is surjective for large $m$. Therefore it is enough if the $L_{m}$ remain the same line bundle $L$ for infinitely many $m$.

To prove that $L_{m}=L$ we construct a degenerating family of curves $D \xrightarrow{f} S$, $X \times S \supset D \xrightarrow{p} X$, such that the generic fibre is a curve $C_{(m+1)}$ of degree $2^{m+1}$ and the special fibre is a reduced curve with two nonsingular components $C_{(m)}^{i}$ of degree $2^{m}$ (cf. Sect. 5). Let ( $m$ ) denote $2^{m}$. Extending the subbundle $L_{(m+1)} \mid C_{(m+1)}$ to a subsheaf of $p^{*}(V)$ on $D$ and restricting the extension to $C_{(m)}^{i}$ gives a lower bound for the maximal degree of a line subbundle of $V \mid C_{(m)}^{i}$ in terms of that for $V \mid C_{(m+1)}$ (Proposition 4.3). This implies that $\operatorname{deg} L_{m}$ is bounded (Lemma 6.5.1) so that for an infinite subsequence of $m, \operatorname{deg} L_{m}$ is constant. If $\operatorname{deg} L_{(m+r)}=\operatorname{deg} L_{(m)}$ by refining the above argument with the degenerating family one can prove that $L_{(m+r)} \mid C_{(m)}^{i}$
$=L_{(m)} \mid C_{(m)}^{i}$ (Lemma 6.5.2). Therefore $L_{(m+r)}\left|Y_{(m)}=L_{(m)}\right| Y_{(m)}$ so that $L_{(m+r)}=L_{(m)}$ (by Proposition 2.1).

When $X$ is of higher dimension and $V$ is a torsion free sheaf of arbitrary rank the pattern of the proof is the same but the details get a bit more technical.

We have made essential use of an unpublished manuscript of Mumford.

## 1. Families of Complete Intersection Subvarieties

Let $X$ be a projective nonsingular algebraic variety of dimension $n \geqq 2$ over an algebraically closed field $k$. Let $H$ be a given very ample line bundle on $X$ corresponding to a projectively normal embedding $X \subset \mathbb{P}^{N}$.

For a positive integer $m$ let $S_{m}$ denote the projective space of lines in the vector space $H^{\mathrm{o}}\left(X, H^{m}\right)$. For a sequence of positive integers $\mathbf{m}=\left(m_{1}, \ldots, m_{t}\right), 1 \leqq t \leqq n-1$, let $S_{\mathrm{m}}$ be the product $S_{m_{1}} \times \ldots \times S_{m_{t}}$. We have the following diagram
1.1.

where $Z_{\mathrm{m}}$ is the correspondence variety:

$$
Z_{\mathbf{m}}=\left\{\left(x, s_{1}, \ldots, s_{\mathbf{t}}\right) \in X \times S_{\mathbf{m}} \mid s_{i}(x)=0,1 \leqq i \leqq t\right\}
$$

and $p_{\mathrm{m}}$ and $q_{\mathrm{m}}$ are the projections.
1.2. The fibre of $q_{\mathrm{m}}$ over $\left(s_{1}, \ldots, s_{\mathrm{f}}\right) \in S_{\mathrm{m}}$ is embedded in $X$ via $p_{\mathrm{m}}$ as the subscheme of $X$ defined by the ideal generated by $s_{1}, \ldots, s_{t}$ in the homogeneous coordinate ring of $X$. So we always think of the fibres of $q_{\mathrm{m}}$ as subschemes of $X$. The projection $p_{\mathrm{m}}$ is a fibration with the fibre over $x \in X$ being identified by $q_{\mathrm{m}}$ with the product of hyperplanes $H_{1} \times \ldots \times H_{t}, H_{i}=\left\{s \in S_{m_{i}} \mid s(x)=0\right\}$. Therefore $Z_{\mathrm{m}}$ is nonsingular.
1.3. Let $K_{m}$ be the function field of $S_{m}$. Let $Y_{m}$ be the generic fibre of $q_{m}$, given by the fibre product


By Bertini's theorem (cf. [17, Theorem I.6.3]) $Y_{\mathrm{m}}$ is an absolutely irreducible nonsingular variety and there is a nonempty open subset of $S_{m}$ over which the geometric fibres of ' $q_{m}$ are irreducible and nonsingular (cf. EGAIV/3, Theorem 12.2.4 (viii), p. 183 and [6, Theorem 8.18]).
1.4. Definition. We call $Y_{m}$ the generic complete intersection subvariety of type $m$. In particular when $t=n-1$ we call $Y_{m}$ the generic curve of type $m$.
1.4.1. Remark. When a property holds for $q_{\mathrm{m}}^{-1}(s)$ for $s$ in a nonempty open subset of $S_{m}$ we say that it holds for a general s.
1.5. Proposition. Let $S_{\mathrm{m}}^{\prime}=\left\{s \in S_{\mathrm{m}} \mid \operatorname{dim} q_{\mathrm{m}}^{-1}(s)=n-t\right\}$. Let $F$ be a coherent sheaf on $X$. For $s=\left(s_{1}, \ldots, s_{t}\right) \in S_{m}^{\prime}$ let $X_{r}$ be the subscheme of $X$ defined by the ideal $I_{r}$ generated by $s_{1}, \ldots, s_{r}$ for $1 \leqq r \leqq t$ and $X_{0}=X$. Let $0 \rightarrow I_{r} \rightarrow \mathcal{O}_{X_{r-1}} \rightarrow \mathcal{O}_{X_{r}} \rightarrow 0$ be the natural exact sequence. Let $S_{\mathrm{m}}^{\prime \prime}=\left\{s \in S_{\mathrm{m}}^{\prime} \mid 0 \rightarrow I_{r} \otimes F \rightarrow \mathcal{O}_{X_{r-1}} \otimes F\right.$ is exact $\}$. Then
i) $S_{\mathrm{m}}^{\prime}$ is a nonempty open subset of $S_{\mathrm{m}}$ and

$$
q_{\mathrm{m}}^{-1}\left(S_{\mathrm{m}}^{\prime}\right) \text { is flat over } S_{\mathrm{m}}^{\prime}
$$

ii) $\mathrm{S}_{\mathrm{m}}^{\prime \prime}$ is a nonempty open subset of $S_{\mathrm{m}}^{\prime}$ and

$$
p_{\mathbf{m}}^{*} F \text { is flat over } S_{\mathbf{m}}^{\prime \prime} .
$$

Proof. i) It follows from (EGAIV/3, Theorem 12.2.4) that $S_{\mathbf{m}}^{\prime}$ is open and by Bertini's theorem it is nonempty. If $\operatorname{dim} q_{\mathrm{m}}^{-1}(s)=n-t$ then its Hilbert polynomial (w.r.t. the given polarisation $H$ on $X$ ) depends only on $m$ as can be seen easily from the cohomology sequence corresponding to the exact sequence $0 \rightarrow I_{r} \rightarrow \mathcal{O}_{X_{r-1}} \rightarrow \mathcal{O}_{X_{r}} \rightarrow 0$ tensored with $H^{l}$, using induction on $r$. Therefore the fibres of $q_{\mathrm{m}}$ over $S_{\mathrm{m}}^{\prime}$ have the same Hilbert polynomial and hence $q_{\mathrm{m}}^{-1}\left(S_{\mathrm{m}}^{\prime}\right)$ is flat over $S_{\mathrm{m}}^{\prime}$ [6, Theorem 9.9].
ii) We use induction on $t$. Assume ii) holds for $t-1$. Let $\mathbf{l}=\left(m_{1}, \ldots, m_{t-1}\right)$ and $T=\left(S_{1}^{\prime \prime} \times S_{m_{\mathrm{t}}}\right) \cap S_{\mathrm{m}}^{\prime} \subset S_{\mathrm{m}}$. Note that $S_{\mathrm{m}}^{\prime \prime} \subset T$ and $T$ is open in $S_{\mathrm{m}}$ by the induction assumption. We have the diagram

where $\pi$ is the projection and $Z_{1}^{\prime}=q_{1}^{-1}\left(S_{\mathrm{m}}^{\prime \prime}\right)$ and $Z_{\mathrm{m}}^{\prime \prime}=q_{\mathrm{m}}^{-1}(T)$. Note that $Z_{\mathrm{m}}^{\prime \prime}$ sits in $\pi^{*}\left(Z_{1}^{\prime}\right)$ as the natural correspondence variety.

By the induction assumption $p_{1}^{*}(F)$ on $Z_{1}^{\prime}$ is flat over $S_{1}^{\prime \prime}$. Therefore $\pi^{*} p_{1}^{*}(F)$ on $\pi^{*}\left(Z_{1}^{\prime}\right)$ is flat over $T$ and moreover, since $T \subset S_{m}^{\prime}, Z_{\mathbf{m}}^{\prime \prime}$ is flat over $T$. In this situation one can deduce from the openness of flatness that $S_{\mathrm{m}}^{\prime \prime}$ is open in $T$ [for example, by taking for the $\mathscr{F}$ ' of Corollary 11.1.2 in EGA IV/3, the sheaf $I \otimes \pi^{*} p_{1}^{*}(F)$ where $I$ is the ideal sheaf of $Z_{\mathrm{m}}^{\prime \prime}$ in $\pi^{*}\left(Z_{1}^{\prime}\right)$ and using the properness of $\left.\pi^{*}\left(Z_{\mathrm{l}}^{\prime}\right) \rightarrow T\right]$. That $S_{\mathrm{m}}^{\prime \prime}$ is not empty follows by noting that for the sequence

$$
0 \rightarrow I_{r} \otimes F \rightarrow \mathscr{O}_{X_{r-1}} \otimes F \rightarrow \mathcal{O}_{X_{r}} \otimes F \rightarrow 0
$$

to be exact it is sufficient that $s_{r}$ is not in any of the associated primes of $\mathcal{O}_{X_{r-1}} \otimes F$ in the homogeneous coordinate ring of $X_{r-1}$.

Once we have the exact sequences

$$
0 \rightarrow I_{r} \otimes F \rightarrow \mathcal{O}_{X_{r-1}} \otimes F \rightarrow \mathcal{O}_{X_{r}} \otimes F \rightarrow 0, \quad 1 \leqq r \leqq t
$$

it follows from the exact cohomology sequences, using induction on $r$, that the Hilbert polynomial of $F$ restricted to $q_{\mathrm{m}}^{-1}(s)$ is independent of $s \in S_{\mathrm{m}}^{\prime \prime}$. Therefore $p_{\mathrm{m}}^{*}(F)$ is flat over $S_{\mathrm{m}}^{\prime \prime}$ (cf. [10, Lectures 7 and 8] and [6, Theorem 9.9]).

## 2. Picard Group of the Generic Curve

For any scheme $S$ we denote by $\operatorname{Pic}(S)$ the (abstract) group of invertible sheaves on $S$. We then have the following proposition (cf. [16]).
2.1. Proposition. Let $\operatorname{dim} X=n \geqq 2$. For $\mathbf{m}=\left(m_{1}, \ldots, m_{t}\right), 1 \leqq t \leqq n-1$ with each $m_{i} \geqq 3$ the natural map $\operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(Y_{\mathrm{m}}\right)$, induced by $Y_{\mathrm{m}} \xrightarrow{\varphi_{m}} Z_{\mathrm{m}} \xrightarrow{p_{m}} X$ (cf. 1.3), is a bijection.
2.1.1. Remark. In fact one can show that if $\operatorname{dim} Y_{m} \geqq 2$, then $\operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(Y_{m}\right)$ is bijective for all m with $m_{i} \geqq 1$ and if $\operatorname{dim} Y_{m}=1$ then $\operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(Y_{m}\right)$ is bijective if just one of the $m_{i}$ 's is $\geqq 3$ (see Remark 2.1 .4 below). If $X$ is a surface then $\operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(Y_{m}\right)$ need not be injective for $m=1$ as shown by the example of the quadric $\mathbb{P}^{1} \times \mathbb{P}^{1} \subset \mathbb{P}^{3}$.

Proof of Proposition 2.1. First we prove the surjectivity of the map. Any subscheme of $Y_{m}$ (defined over $K_{m}$ ) can be extended to an open set $q_{m}^{-1}(U), U$ open in $S_{\mathrm{m}}$ (by inverting the finitely many elements which occur in the denominator in a set of generators of the ideal defining the subscheme). Therefore if $L \in \operatorname{Pic}\left(Y_{m}\right)$ corresponds to the divisor $D$ in $Y_{m}$ we can extend the divisor to an open subset of $Z_{\mathrm{m}}$ and hence to the whole of $Z_{\mathrm{m}}$. Thus $L$ can be extended to a line bundle on $Z_{\mathrm{m}}$ i.e. $\operatorname{Pic}\left(Z_{m}\right) \rightarrow \operatorname{Pic}\left(Y_{m}\right)$ is surjective. Now we claim that

$$
\operatorname{Pic}\left(Z_{\mathbf{m}}\right)=p_{\mathbf{m}}^{*}(\operatorname{Pic}(X)) \oplus q_{\mathbf{m}}^{*}\left(\operatorname{Pic}\left(S_{\mathbf{m}}\right)\right)
$$

Since $q_{\mathbf{m}}^{*}\left(\operatorname{Pic}\left(S_{\mathbf{m}}\right)\right)$ is in the kernel of $\operatorname{Pic}\left(Z_{m}\right) \rightarrow \operatorname{Pic}\left(Y_{m}\right)$ it would then follow that $\operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(Y_{m}\right)$ is surjective. To prove the above direct sum decomposition we have only to note that since the fibres of $p_{\mathbf{m}}$ are products of projective spaces embedded in $S_{\mathrm{m}}$ by $q_{\mathrm{m}}$ (Sect. 1.2), given a line bundle $L$ on $Z_{\mathrm{m}}$ we can find a unique line bundle $M$ on $S_{\mathrm{m}}$ such that $L \otimes q_{\mathrm{m}}^{*}(M)$ is trivial on one and hence all the fibres of $p_{\mathrm{m}}$ so that it comes from $X$ (cf. Lemma 2.1.2 below).

For proving injectivity we need the following lemmas.
2.1.2. Lemma. Let $q: Z \rightarrow S$ be a proper flat morphism of irreducible varieties with fibres integral. Let $L \in \operatorname{Pic}(Z)$. Then the following are equivalent:
a) $L$ is trivial on the generic. fibre of $q$.
b) $L$ is trivial on all geometric fibres of $q$ over a nonempty open subset of $S$.
c) $L$ is trivial on all the geometric fibres of $q$.
d) $L \approx q^{*}(M), M \in \operatorname{Pic}(S)$,

Proof. This is a consequence of semicontinuity and the remark that a line bundle $L$ on an integral complete scheme $F$ is trivial if and only if $H^{0}(F, L) \neq 0 \neq H^{0}\left(F, L^{-1}\right)$ (see [11, Corollary 6]).

### 2.1.3. Lemma. Let $\operatorname{dim} X \geqq 2$.

i) For any point $P \in X$ and $m \geqq 3$ the rational map given by the linear system $V=\left\{s \in H^{0}\left(X, H^{m}\right) \mid s=0\right.$ passes through $P$ and is singular at $\left.P\right\}$ gives an isomorphism of $X-P$ onto its image.
ii) For a nonempty open set of points $s \in V$ the divisor $s=0$ is integral (and is singular at $P$ ).
iii) Let $A=\left\{s \in S_{m} \mid q_{m}^{-1}(s)\right.$ is not integral $\}$. Then $A$ is a closed set and if $m \geqq 3, A$ has codimension $\geqq 2$ in $S_{m}$, as does $q_{m}^{-1}(A)$ in $Z_{m}$.

Proof. i) It is easy to see that if i) holds for $X$ and $H$ then it holds for $Y, H / Y$ for any nonsingular subvariety $Y$ of $X$. Therefore to prove i) we can assume $X=\mathbb{P}^{N}$ and $H=\mathcal{O}(1)$. Let $P \neq Q \in \mathbb{P}^{N}$. We can find a linear form $l$ such that $l(P)=0$ and $l(Q) \neq 0$. Then $l^{m} \in V$ so that $P$ is the only base point of $V$. If $Q \neq R$ are two points of $\mathbb{P}^{N}$ different from $P($ and $v$ a tangent at $Q)$ we can find a nonsingular quadric $f=0$ passing through $P$ and $Q$ and not passing through $R$ (or not having $v$ as a tangent at $Q$ ). Choose a linear form $l$ such that $l(P)=0$ and $l(Q) \neq 0 \neq l(R)$ [or $l(v) \neq 0]$. Then $l^{m-2} \cdot f \in V$ showing that $V$ separates points and infinitesmally separates points. This proves i).
ii) This follows from Bertini's theorem as in [17, Theorem I.6.3] supplemented in characteristic $p$ by [17, Proposition I.6.4] whose condition for $p^{e}=1$ is satisfied because of $i$ ).
iii) That $A$ is a closed set follows from [EGA IV/3 Theorem 12.2 .4 (viii)]. From [6, Proof of Theorem 8.18] the closed set $B=\left\{s \in S_{m} \mid q_{m}^{-1}(s)\right.$ has a singularity is irreducible and is not the whole of $S_{m}$. Now $A \subset B$, and hence to show that $\operatorname{codim} A \geqq 2$ it is enough to show that for $m \geqq 3 A \neq B$, i.e. there is at least one point in $S_{m}$ such that $q_{m}^{-1}(s)$ is integral and singular. But that is ii). Since $q_{m}$ is equidimensional codim $q_{m}^{-1}(A)$ is then $\geqq 2$.

Now we return to the proof of the injectivity of $\operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(Y_{\mathbf{m}}\right)$. We use induction on $t$. First assume that $t=1$ so that $\mathbf{m}$ is the single integer $m \geqq 3$. Let $L \in \operatorname{Pic}(X)$ be such that its image $\varphi_{m}^{*} p_{m}^{*} L$ in $\operatorname{Pic}\left(Y_{m}\right)$ is trivial. Then by Lemmas 2.1.2 and 2.1.3iii), on an open subset of $Z_{m}$ whose complement has codimension $\geqq 2$, $p_{m}^{*}(L)$ is isomorphic to $q_{m}^{*}(M), M \in \operatorname{Pic}\left(S_{m}\right)$. But then $p_{m}^{*}(L) \approx q_{m}^{*}(M)$ on the whole of $Z_{m}$. As we have seen $\operatorname{Pic}\left(Z_{m}\right)=p_{m}^{*}(\operatorname{Pic}(X)) \oplus q_{m}^{*}\left(\operatorname{Pic}\left(S_{m}\right)\right)$. Therefore $p_{m}^{*}(L) \approx q_{m}^{*}(M)$ implies $p_{m}^{*} L$ is trivial on $Z_{m}$ and hence $L$ is trivial (since $p_{m}^{*}$ is injective).

Now let $\mathbf{m}=\left(m_{1}, \ldots, m_{t}\right)$. Assume injectivity for $\mathbf{l}=\left(m_{1}, \ldots, m_{t-1}\right)$. Let $L \in \operatorname{Pic}(X)$ be such that $\varphi_{\mathrm{m}}^{*} p_{\mathrm{m}}^{*}(L)$ is trivial on $Y_{\mathrm{m}}$. By Lemma 2.1.2a) $\Rightarrow \mathrm{c}$ ) it follows that $L$ is trivial on $q_{\mathrm{m}}^{-1}(s)$ whenever $q_{\mathrm{m}}^{-1}(s)$ is smooth. For a general $s^{\prime} \in S_{1}, L$ being trivial on all the smooth $q_{\mathrm{m}}^{-1}(s)$ contained in $q_{1}^{-1}\left(s^{\prime}\right)$, by Lemma 2.1 .2 b$) \Rightarrow$ a) and the above case $t=1$ applied to the smooth variety $q_{1}^{-1}\left(s^{\prime}\right)$ it follows that $L$ is trivial on $q_{1}^{-1}\left(s^{\prime}\right)$. Again by Lemma 2.1 .2 b$) \Rightarrow \mathrm{a}$ ) and the inductive assumption, $L$ is trivial on $X$.
2.1.4. Remark. If $\operatorname{dim} X \geqq 3$ Weil proves Lemma 2.1.3iii) for all $m \geqq 1$ [16, Lemmas 3 and 4]. Assuming this fact the above proof then gives the result in the sharper form as in Remark 2.1.1 (cf. [16, No. 12, Theorem 2]).

## 3. A General Form of the Lemma of Enriques-Severi

A coherent sheaf $F$ on $X$ is called reflexive if the natural map $F \rightarrow F^{* *}$ of $F$ into its double dual is an isomorphism.
3.1. Lemma. Let $F$ be a coherent sheaf on $X$. The following are equivalent.
a) $F$ is reflexive.
b) Locally, i.e. on each of the open sets $U$ of some covering of $X$, there is an exact sequence

$$
0 \rightarrow F \mid U \rightarrow F_{1} \rightarrow T \rightarrow 0
$$

with $F_{1}$ free and $T$ torsion free.
c) Locally there is an exact sequence

$$
0 \rightarrow F \mid U \rightarrow F_{1} \rightarrow F_{0} \rightarrow Q \rightarrow 0
$$

where $F_{1}, F_{0}$ are free, i.e. $F$ is a $2 n d$ syzygy.

### 3.1.1. Corollary. Let $F$ be reflexive. Then

i) it satisfies the condition $S_{2}$ (cf. [1, Definition 2.1]),
ii) its restriction to $q_{\mathrm{m}}^{-1}(s)$ is reflexive for a general $s \in S_{\mathrm{m}}$.

Proof. b$) \Rightarrow \mathrm{c}$ ): Any torsion free module over a domain is a submodule of a free module.
c) $\Rightarrow$ b): Trivial.
$b) \Rightarrow$ a): Follows from [2, Sect. 4, Theorem 7ii)].
$\mathrm{a}) \Rightarrow \mathrm{b}$ ): Write $F^{*}$ as a quotient of a free module:

$$
0 \rightarrow K \rightarrow F_{1} \rightarrow F^{*} \rightarrow 0 .
$$

Taking duals and using $F=F^{* *}$ we get sequences as in b).
Part i) of the corollary follows from c).
Part ii) follows by noting that the restriction of the sequence in b) to any hyperplane section which does not pass through any of the associated primes of $T$ remains exact.
3.2. Proposition (General Enriques-Severi Lemma, cf. [6, Corollary 7.8]). Let $X \subset \mathbb{P}^{N}$ be a nonsingular projective variety of dimension $n \geqq 2$. Let $F$ be a coherent reflexive sheaf on $X$. Then there is an $m_{0}$ such that if $\mathbf{m}=\left(m_{1}, \ldots, m_{t}\right), 1 \leqq t \leqq n-1$, with each $m_{i} \geqq m_{0}$, then there is a nonempty open subset $U$ of $S_{\mathbf{m}}$ such that for $s=\left(s_{1}, \ldots, s_{t}\right) \in U$ the restriction map $H^{0}(X, F) \rightarrow H^{0}\left(X_{s}, F / X_{s}\right)$ is surjective where $X_{s}=q_{\mathbf{m}}^{-1}(s)$ is the subscheme of $X$ defined by the ideal generated by $s_{1}, \ldots, s_{t}$.
Proof. It is enough to find an $m_{0}$, depending only on $F$, such that $H^{1}\left(X_{s}, F(-l)\right)=0$ for $l \geqq m_{0}$ for a general $s \in S_{\mathbf{m}}$ for all $m=\left(m_{1}, \ldots, m_{t}\right)$ with $m_{i} \geqq 1$ and all $t \leqq n-1$. For, then from the exact sequence

$$
0 \rightarrow I_{r} \otimes F \rightarrow \mathcal{O}_{X_{r-1}} \otimes F \rightarrow \mathcal{O}_{X_{r}} \otimes F \rightarrow 0
$$

on $X_{r-1}$ corresponding to a general $s$ (cf. Proposition 1.5), noting that $I_{r}=\mathcal{O}_{\mathbf{P N}}\left(-m_{r}\right) \otimes \mathcal{O}_{X_{r-1}}$, it follows that

$$
H^{0}\left(X_{r-1}, F \mid X_{r-1}\right) \rightarrow H^{0}\left(X_{r}, F \mid X_{r}\right)
$$

is surjective for $m_{r} \geqq m_{0}$.
By duality $H^{1}\left(X_{s}, F(-l)\right)^{*} \approx \operatorname{Ext}^{n^{\prime}-1}(F(-l), \omega)$ where $\omega$ is the canonical line bundle of $X_{s}$ and $n^{\prime}=\operatorname{dim} X_{s}[1$, Sect. I, 1.3, p. $5 ;$ Corollary IV, 5.6, p. 81]. We have a spectral sequence $\quad H^{p}\left(\mathscr{E} x t^{q}(F(-l), \omega)\right) \Rightarrow \operatorname{Ext}^{p+q}(F(-l), \omega) \quad$ (cf. $\quad[1$, Proposition 2.4]). Since $\mathscr{E} x t^{q}(F(-l), \omega)=\mathscr{E} x t^{q}(F, \omega) \otimes \mathcal{O}(l)$, the spectral sequence degenerates for large $l$ (depending on $s$ ). Then

$$
\operatorname{Exx}^{n^{\prime}-1}(F(-l), \omega)=H^{0}\left(\mathscr{E} x t^{n^{\prime}-1}(F, \omega) \otimes \mathcal{O}(l)\right)
$$

But since $F$ is $S_{2}$ on $X_{s}\left(\right.$ Corollary 3.1.1) $\mathscr{E} x t^{n^{\prime}-1}(F, \omega)=0$ (cf. [1, Theorem 5.19 and Proposition 5.20]). Therefore we can find an $m_{0}$ (depending only on $F$ ) such that $H^{1}\left(X_{s}, F(-l)=0\right.$ for $l \geqq m_{0}$ for a general $s=\left(s_{1}, \ldots, s_{t}\right)$ in $S_{(1, \ldots, 1)}$ (i.e. $\operatorname{deg} s_{i}=1$, for every $i$ ) for all $t \leqq n-1$.

Now suppose $H^{1}\left(X_{s}, F(-l)\right)=0$ for $l \geqq m_{0}$ for a general $s \in S_{\mathrm{m}}$ with $\mathbf{m}=\left(m_{1}, \ldots, m_{r}, 1, \ldots, 1\right)$. We will prove that $H^{1}\left(X_{s^{\prime}}, F(-l)\right)=0$ for $l \geqq m_{0}$ for a general $s^{\prime} \in S_{\mathbf{m}^{\prime}}$ where $\mathbf{m}^{\prime}=\left(m_{1}, \ldots, m_{r}, d, 1, \ldots, 1\right)$. [When $r=0 \mathbf{m}=(1, \ldots, 1)$ and the assumption is true by the choice of $m_{0}$ so we can start the induction.] It is easy to see that a permutation of the sequence $\boldsymbol{m}^{\prime}$ does not affect anything relevant. So we rewrite $\mathbf{m}^{\prime}$ as $\left(m_{1}, \ldots, m_{r}, 1, \ldots, 1, d\right)$. For $s=\left(s_{1} \ldots, s_{t}\right) \in S_{\mathbf{m}}$ denote by $s^{\prime}$ the point $\left(s_{1}, \ldots, s_{t}^{d}\right)$ of $S_{\mathbf{m}^{\prime}}$. Let $Y$ be the subscheme of $X$ defined by $s_{1}, \ldots, s_{t-1}$. Let $X_{s}$ (resp. $X_{s^{\prime}}$ ) be the subscheme of $Y$ defined by $s_{1}, \ldots, s_{t}$ (resp. $s_{1}, \ldots, s_{t}^{d}$ ). On $Y$ we have the exact sequence (cf. Proposition 1.5)

$$
0 \rightarrow I \rightarrow \mathcal{O}_{Y} \rightarrow \mathcal{O}_{X_{s}} \rightarrow 0,
$$

where $I \approx \mathscr{C}_{\mathbb{P}^{N}}(-1) \mid Y$ is the ideal generated by $s_{t}$ in $\mathcal{O}_{Y}$. For a general $s$ the above sequence tensored with $F$ remains exact (Proposition 1.5) and hence $\operatorname{Tor}_{\mathscr{O}_{Y}}^{1}\left(F, \mathcal{O}_{\mathbf{Y}} / I\right)=0$. Therefore $\operatorname{Tor}_{\mathscr{O}_{\mathbf{Y}}}^{1}\left(F, \mathcal{O}_{Y} / I^{d}\right)=0$ and hence $0 \rightarrow I^{d} \rightarrow \mathcal{O}_{\boldsymbol{Y}} \rightarrow \mathcal{O}_{X_{s^{\prime}}} \rightarrow 0$ tensored with $F$ remains exact. Therefore by Proposition $1.5 p_{\mathbf{m}^{\prime}}^{*}(F)$ is flat over $s^{\prime}$ (for a general $s$ ). On $X_{s^{\prime}}$ we have the exact sequence $0 \rightarrow I / I^{d} \rightarrow \mathcal{O}_{X_{s^{\prime}}} \rightarrow \mathcal{O}_{X_{s}} \rightarrow 0$. Since $\operatorname{Tor}_{\mathscr{O}_{Y}}^{1}\left(F, \mathcal{O}_{X_{s}}\right)=0$ tensoring this with $F(-l)$ gives an exact sequence:

$$
0 \rightarrow F(-l) \otimes\left(I / I^{d}\right) \rightarrow F(-l) / X_{s^{\prime}} \rightarrow F(-l) / X_{s} \rightarrow 0 .
$$

For $l \geqq m_{0} H^{1}\left(X_{s}, F(-l)\right)=0$ by induction assumption. From the exact sequence $0 \rightarrow I^{d-1} / I^{d} \rightarrow I / I^{d} \rightarrow I / I^{d-1} \rightarrow 0$ using induction on $d$ one can see that $H^{1}\left(X_{s}, F(-l) \otimes\left(I / I^{d}\right)\right)=0$ for $l \geqq m_{0}$. Therefore $H^{1}\left(X_{s^{\prime}}, F(-l)\right)=0$ for $l \geqq m_{0}$. But $p_{\mathbf{m}^{\prime}}^{*}(F)$ is flat at $s^{\prime}$ and hence by semicontinuity $H^{1}\left(X_{u}, F(-l)\right)=0$ for $l \geqq m_{0}$ for a general $u \in S_{\mathbf{m}^{\prime}}$. Thus the proof of the proposition is complete.

Remark. If $F$ is locally free the proof is much simpler. For, in that case $H^{i}(X, F(-l)) \approx H^{n-i}\left(X, F^{*} \otimes \omega(l)\right)^{*}=0,0 \leqq i \leqq n-1$ for $l \geqq m_{0}$. Then for any hyperplane section $Y$ of $X$ tensoring the exact sequence $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}_{X} \rightarrow \mathcal{O}_{Y} \rightarrow 0$ with $F(-l)$ gives $H^{1}(Y, F(-l))=0$ for $l \geqq m_{0}$.

## 4. Vector Bundles on Families of Curves

4.1. Let $V$ be a vector bundle on a nonsingular projective curve $C$ over a field $K$. If $V$ is not semistable there is a unique proper subbundle $V_{1}$ of $V$ such that

1) $\mu\left(V_{1}\right) \geqq \mu(W)$ for all subbundles $W$ of $V$.
2) If $\mu(W)=\mu\left(V_{1}\right)$ then $\mathrm{rk} W \leqq \mathrm{rk} V_{1}$.

We call $V_{1}$ the $\beta$-subbundle of $V$ (cf. [7,5]). Because of the uniqueness the $\beta$ subbundle is defined over the base field $K$ even when it is not algebraically closed [7].
4.2. Let $f: D \rightarrow S$ be a flat family of nonsingular projective curves over an integral scheme $S$. Let $V$ be a vectorbundle over $D$. Then $\left\{s \in S \mid V_{s}\right.$ is semistable on $\left.D_{s}\right\}$ is an open subset of $S[12,8]$. When all the $V_{s}$ are not semistable, or equivalently when $V_{s_{0}}$ is not semistable for $s_{0}$ the generic point of $S$, we have the $\beta$-subbundle $W_{s_{0}}$ of $V_{s_{0}}$ defined over $K_{s_{0}}$ the function field of $S$. Then $W_{s_{0}}$ gives a section of the corresponding quot scheme over the generic point and hence gives a section over
an open subset $U^{\prime} \subset S$. Thus we have a subbundle $W$ of $V / f^{-1}\left(U^{\prime}\right)$ extending $W_{s 0}$. By semicontinuity $W_{s}$ remains the $\beta$-subbundle of $V_{s}$ for $s$ in an open subset $U$ of $U^{\prime}$.
4.3. Proposition. Let $A$ be a discrete valuation ring with quotient field $K$ and residue field $k$. Let $S=\operatorname{Spec} A$. Let $f: D \rightarrow S$ be a flat family of projective curves such that $D$ and $D_{K}$, the generic fibre, are nonsingular and the special fibre $D_{k}$ is reduced with nonsingular components $D_{k}^{1}, \ldots, D_{k}^{r}$ Let $V$ be a vector bundle on $D$. Let $\mu_{\kappa}$ $=\max \left\{\mu(W) \mid W\right.$ subbundle of $\left.V_{K} \rightarrow D_{K}\right\}$ and $\mu_{k}^{i}=\max \{\mu(W) \mid W$ subbundle of $\left.V / D_{k}^{i} \rightarrow D_{k}^{i}\right\}$. Then $\mu_{K} \leqq \sum_{i=1}^{r} \mu_{k}^{i}$.
Proof. Let $W_{K}$ be a subbundle of $V_{K}$ with $\mu\left(W_{K}\right)=\mu_{K}$. By the completeness of the Quot scheme the exact sequence $0 \rightarrow W_{K} \rightarrow V_{K} \rightarrow Q_{K} \rightarrow 0$ can be extended to an exact sequence on $D: 0 \rightarrow W_{A} \rightarrow V \rightarrow Q_{A} \rightarrow 0$ with $Q_{A}$ torsion free [3, Lemma 3.7]. By Lemma 3.1 $W_{A}$ is reflexive and is $S_{2}$ (Corollary 3.1.1). Therefore $D$ being nonsingular of dimension $2, W_{A}$ is locally free (as follows for e.g. from [1, Theorem 5.19]). Let $W_{k}^{i}=W_{A} \mid D_{k}^{i}$ and $V_{k}^{i}=V \mid D_{k}^{i}$. We then have $\operatorname{deg} W_{k}=\sum_{i=1}^{r} \operatorname{deg} W_{k}^{i}$ and $\mathrm{rk} W_{k}=\operatorname{rk} W_{k}^{i}$. (Where $\operatorname{deg} W_{k}$ is computed on the reducible curve $D_{k}$ and $\operatorname{deg} W_{k}^{i}$ on the irreducible curve $D_{k}^{i}$.) Similarly $\operatorname{deg} V_{k}=\sum_{i=1}^{r} \operatorname{deg} V_{k}^{i}$ and $\mathrm{rk} V_{k}=\mathrm{rk} V_{k^{\prime}}^{i}$.

Since $Q_{A}$ is not locally free only at finitely many points of $D, W_{k}^{i}$ is a subsheaf of $V_{k}^{i}$. It is easy to see that this implies $\mu\left(W_{k}^{i}\right) \leqq \mu_{k}^{i}\left[13\right.$, Sect. 4]. By flatness $\operatorname{deg} W_{K}$ $=\operatorname{deg} W_{k}$ and $\mathrm{rk} W_{\mathrm{K}}=\mathrm{rk} W_{k}$. Therefore

$$
\mu_{\mathrm{K}}=\sum_{i=1}^{r} \mu\left(W_{k}^{i}\right) \leqq \sum_{i=1}^{r} \mu_{k}^{i} .
$$

4.3.1. Corollary. If $V \mid D_{k}^{i}$ is semistable for all i, then $V_{K}$ is semistable.

Proof. Let $W_{K}$ be a subbundle of $V_{K}$. Since $V_{k}^{i}$ are semistable $\mu_{k}^{i} \leqq \mu\left(V_{k}^{i}\right)$. By the proposition $\quad \mu\left(W_{K}\right) \leqq \sum_{i=1}^{r} \mu_{k}^{i} \quad$ But $\quad \sum_{i=1}^{r} \mu\left(V_{k}^{i}\right)=\left(\sum_{i=1}^{r} \operatorname{deg} V_{k}^{i}\right) / \mathrm{rk} V=\mu\left(V_{k}\right)=\mu\left(V_{K}\right)$. Therefore $\mu_{K} \leqq \mu\left(V_{K}\right)$ proving the semistability of $V_{K}$.

## 5. A Degenerating Family of Curves

5.1. Notation. We fix a sequence $\left(\alpha_{1}, \ldots, \alpha_{n-1}\right)$ of integers with each $\alpha_{i} \geqq 2$. We let $\alpha=\alpha_{1} \cdot \ldots \cdot \alpha_{n-1}$. For a positive integer $m$ we denote by ( $m$ ) the sequence $\left(\alpha_{1}^{m}, \ldots, \alpha_{n-1}^{m}\right)$.
5.2. Proposition. Let $l=m+r, r>0$. Let $U_{m} \subset S_{(m)}$ and $U_{l} \subset S_{(l)}$ be nonempty open subsets. Then there is a point $s \in S_{(l)}$ and a nonsingular curve $C$ in $S_{(l)}$ passing through $s$ such that
i) $C-\{s\} \subset U_{1}$.
ii) $q_{(1)}^{-1}(C)$ is nonsingular and $q_{(l)}^{-1}(C) \rightarrow C$ is flat.
iii) The fibre $q_{(1)}^{-1}(s)$ is a reduced curve with $\alpha^{r}$ nonsingular components $C_{1}, C_{2}, \ldots$ which intersect transversally and at most two of which pass through any point of $X$, with each $C_{i}$ a fibre of $q_{(m)}$ over a point of $U_{m}$.

Proof. Let $S^{i}=\left(S_{\alpha_{i}^{m}}\right)^{x^{r}}$ and $S=S^{1} \times \ldots \times S^{n-1}$. Let $\pi_{i}: S^{i} \rightarrow S_{\alpha_{j}}$ be the multiplication map sending $\left(s_{1}, s_{2}, \ldots\right)$ to $s_{1} \cdot s_{2} \cdot \ldots$ Let $\pi: S \rightarrow S_{(l)}$ be the product $\pi_{1} \times \ldots \times \pi_{n-1}$. Let $u_{1}, u_{2}, \ldots$ be the $\alpha^{r}$ projections $S \rightarrow S_{(m)}$ corresponding to different choices of one factor (from among $\alpha_{i}^{r}$ ) from each $S^{i}(i=1, \ldots, n-1$ ).


For a general $s \in S$ the curve $q_{(l)}^{-1}(\pi(s))$ has the $\alpha^{p}$ nonsingular irreducible components $q_{(m)}^{-1}\left(u_{j}(s)\right)$. We claim that there is a nonempty open subset $T$ of $S$ such that if $s \in T$ then $q_{(l)}^{-1}(\pi(s))$ is a reduced curve in $X$ satisfying the conditions of iii). For, the condition that $q_{(m)}^{-1}\left(u_{j}(s)\right)$ is nonsingular is a nonempty open condition on $s$ as are the conditions that for the families $u_{j}^{*}\left(Z_{(m)}^{\prime}\right)$ [which are flat over $u_{j}^{-1}\left(S_{(m)}^{\prime}\right)$, cf. Proposition 1.5], $q_{(m)}^{-1}\left(u_{j}(s)\right)$ intersect transversally [EGA IV/4, Remark 17.13.4(ii)] and at most two pass through a point [EGAIV/3, Theorem 12.2.4(vi)] as $j$ varies. When these conditions are satisfied $q_{(l)}^{-1}(\pi(s))$ is Cohen-Macaulay and generically reduced and hence reduced [1, Lemma 2.3]. To satisfy the last condition of iii) we have only to intersect with the open sets $u_{j}^{-1}\left(U_{m}\right)$.

Let $s \in T$. By Proposition $1.5 q_{(l)}$ is flat at $\pi(s)$. Therefore on the fibre over $s, q_{(l)}$ fails to be smooth only at the singular points of the fibre [1, Theorem 1.8]. At these singular points because of the conditions in iii) the differential of $q_{(l)}$ has a two dimensional kernel. Therefore for a general curve $C$ through $s$ defined by regular parameters at $s, q_{(l)}^{-1}(C)$ is nonsingular. Therefore we can find a $C$ satisfying the conditions of the proposition.

## 6. Restriction to Curves

In this section we will prove the following theorem.
6.1. Theorem. Let $V$ be a semistable torsion free sheaf on $X$ (with respect to the polarisation $H$ ). Let $Y_{(m)}$ be the generic curve of type ( $m$ ) (Sects. 1.4 and 5.1). Then there is an $m_{0}$ such that for $m \geqq m_{0}$ the restriction of $V$ to $Y_{(m)}\left(i . e . \varphi_{(m)}^{*} p_{(m)}^{*} V\right.$, Sects. 1.1 and 1.3) is semistable, or equivalently for $m \geqq m_{0}$ and for $s$ in a nonempty open subset of $S_{(m)}, V \mid q_{(m)}^{-1}(s)$ is semistable.
6.2. Remark. Conversely if $V \mid q_{(m)}^{-1}(s)$ is semistable then $V$ is semistable as follows from the fact that the degree of a sheaf on $X$ is determined by its restriction to $q_{(m)}^{-1}(s)$. Therefore from the abovè theorem it follows that $V$ restricted to a general complete intersection subvariety of type $\left(\alpha_{1}^{m}, \ldots, \alpha_{t}^{m}\right), 1 \leqq t \leqq n-1$ with $m \geqq m_{0}$ is also semistable (with respect to the induced polarisation).
6.3. Since $V$ is a torsion free sheaf there is an open subset $U$ of $X$ with $\operatorname{codim}(X-U) \geqq 2$ such that $V / U$ is a vector bundle. Therefore $V / Y_{(m)}\left(\right.$ i.e. $\left.\varphi_{(m)}^{*} p_{(m)}^{*} V\right)$ is a vector bundle.
6.4. Proposition. If $V / Y_{(m)}$ is semistable then $V / Y_{(l)}$ is semistable for any $l \geqq m$.

Proof. As $V / Y_{(m)}$ is semistable there is an open set $U_{m}$ of $S_{(m)}$ such that $V / q_{(m)}^{-1}(s)$ is a semistable vector bundle for $s \in U_{m}$ (cf. Sect.4.2). Apply Proposition 5.2 with this $U_{m}$ and $U_{l}=S_{(l)}^{\prime \prime}$ to get a degenerating family $q_{(l)}^{-1}(C) \rightarrow C$. The lemma now follows from Corollary 4.3 .1 applied to the vector bundle $p_{(l)}^{*}(V)$ on $q_{(l)}^{-1}(C)$.
6.5. Proposition. If $V / Y_{(m)}$ is not semistable for every $m$ then $V$ is not semistable.

Proof. If $V \mid Y_{(m)}$ is not semistable we can find a nonempty open subset $U_{m}$ of $S_{(m)}^{\prime \prime}$ such that (i) $p_{(m)} q_{(m)}^{-1}\left(U_{m}\right) \subset U$ so that $p_{m}^{*}(V) \mid q_{(m)}^{-1}\left(U_{m}\right)$ is a vector bundle, (ii) for $s \in U_{m}, V \mid q_{(m)}^{-1}(s)$ is not semistable, and (iii) there is a subbundle $W_{m}$ of $p_{m}^{*}(V) \mid q_{(m)}^{-1}\left(U_{m}\right)$ such that for $s \in U_{m}, W_{m} \mid q_{m}^{-1}(s)$ is the $\beta$-subbundle of $V \mid q_{(m)}^{-1}(s)$ (Sects. 4.1 and 4.2).

Let $r_{m}=\operatorname{rk} W_{m}$ and $\beta_{m}=\mu\left(W_{m} \mid q_{m}^{-1}(s)\right), s \in U_{m}$. By Proposition 2.1 there is a unique line bundle $L_{m}$ on $X$ such that $L_{m} \mid Y_{(m)}$ i.e. $\varphi_{(m)}^{*} p_{(m)}^{*} L_{m}$ is isomorphic to $\left(\operatorname{det} W_{m}\right) \mid Y_{(m)}$. Let $d_{m}=$ degree of $L_{m}$ on $X$. We then have the following lemma.
6.5.1. Lemma. As a function of $m, d_{m}$ is bounded.

Proof. By Proposition 5.2 we have a degenerating family of curves $q_{(m+1)}^{-1}(C) \rightarrow C$ with all components of the singular fibre in $U_{m}$. Applying Proposition 4.3 to this family we get $\beta_{m+1} \leqq \alpha \cdot \beta_{m}$. But $\beta_{m}=d_{m} \alpha^{m} / r_{m}$. Therefore $d_{m+1} / r_{m+1} \leqq d_{m} / r_{m}$. Since $1 \leqq r_{m} \leqq \mathrm{rk} V$ this shows that $d_{m}$ remains bounded above. Since $W_{m}$ contradicts the semistability of $V \mid Y_{(m)}$ we have $d_{m} / r_{m}>\operatorname{deg} V / \mathrm{rk} V$ which proves $d_{m}$ is bounded below.

Now $d_{m}$ being bounded we can find a subsequence $Q$ of the sequence of natural numbers such that $d_{q}=d$ and $r_{q}=r$ are constants for $q \in Q$. Then $\beta_{l}=\alpha^{l-m} \cdot \beta_{m}$ for all $l>m l, m \in Q$.
6.5.2. Lemma. The line bundles $L_{q}, q \in Q$, are all isomorphic on $X$.

Proof. Let $m, l \in Q$ with $l>m$. Using Proposition 5.2 we can construct a degenerating family of curves $D \rightarrow \operatorname{Spec} A, A$ a discrete valuation ring with quotient field $K$, with all the components of the special fibre $D_{k}$ in $U_{m}$ and the generic fibre $D_{K}$ in $U_{l}$. Extend the $\beta$-subbundle $W_{l} \mid D_{K}$ of $p_{(l)}^{*}(V) \mid D_{K}$ to a subsheaf (with torsion free quotient) $\tilde{W}_{l}$ of $p_{(l)}^{*}(V) \mid D$ (as in the proof of Proposition 4.3). Then since $\beta_{l}=\alpha^{l-m} \beta_{m}$ the restriction of $W_{l}$ to any component of $D_{k}$ is the $\beta$-subbundle there. Therefore $\operatorname{det} \tilde{W}_{l}$, which is isomorphic to $L_{l}$ on $D_{K}$, is isomorphic to $L_{m}$ on each component of $D_{k}$. Thus the two line bundles $\operatorname{det} \tilde{W}_{l}$ and $L_{l}$ on $D$ are isomorphic on $D_{K}$ and have the same degree (since $\beta_{l}=\alpha^{l-m} \beta_{m}$ ) on each component of $D_{k}$ and hence are isomorphic on $D$. Therefore $L_{l}$ and $L_{m}$ are isomorphic on the components of $D_{k}$ and thus on $q_{(m)}^{-1}(s)$ for a general $s \in S_{(m)}$. Therefore $L_{l}\left|Y_{(m)} \approx L_{m}\right| Y_{(m)}$ and hence $L_{l} \approx L_{m}$ by Proposition 2.1.
6.5.3. Lemma. When $m \in Q$ is sufficiently large, for a general $s \in U_{m}$ there is a subsheaf $\tilde{W}$ of $V$ (which depends on $s$ ) such that $\tilde{W}\left|q_{(m)}^{-1}(s)=W_{m}\right| q_{(m)}^{-1}(s)$.

Proof. Let $L$ be the common line fundle $L_{q}, q \in Q$. Let $U$, with $\operatorname{codim}(X-U) \geqq 2$, be the open subset on which $V$ is a vector bundle (Sect. 6.3). Extend $\bigwedge^{r}(V \mid U)$ on $U$ to a reflexive sheaf $F$ on $X$. Consider the reflexive sheaf $\operatorname{Hom}(L, F)=L^{*} \otimes F$. The Grassmann bundle of $r$ dimensional subspaces of the fibres of $V \mid U$ is embedded in $\mathbb{P}\left({ }_{\wedge}^{r}(V \mid U)\right)=\mathbb{P}\left(L^{*} \otimes \stackrel{r}{\wedge}(V \mid U)\right)$. Let $\Sigma \subset L^{*} \otimes \bigwedge_{\wedge}^{r}(V \mid U)$ be the cone over it. For $\phi \in H^{0}\left(X, L^{*} \otimes F\right)$ let $\Sigma(\phi)=\{x \in U \mid \phi(x) \in \Sigma\}$. As $\phi$ varies over the finite dimensional space $H^{0}\left(X, L^{*} \otimes F\right)$ the corresponding $\Sigma(\phi)$ form a bounded family of subvarieties of $U$. Then it is easy to see that there is an $N_{0}$ such that if $\Sigma(\phi) \neq U$, then $\Sigma(\phi)$ does not contain any $q_{(m)}^{-1}(s)$ for $m \geqq N_{0}$. By Proposition 3.2 there is an $N_{1}$ such that for $m \geqq N_{1}, H^{0}\left(X, L^{*} \otimes F\right) \rightarrow H^{0}\left(q_{(m)}^{-1}(s), L^{*} \otimes F\right)$ is surjective for a general $s$. Let $m \in Q$ with $m \geqq \max \left(N_{0}, N_{1}\right)=m_{0}$. Let $\bar{\phi} \in H^{0}\left(q_{(m)}^{-1}(s), L^{*} \otimes F\right)$ correspond to the $\beta$-subbundle $W_{m} \mid q_{(m)}^{-1}(s)$ and $\phi$ its lift in $H^{0}\left(X, L^{*} \otimes F\right)$. Then $\Sigma(\phi)=U$ and on the open set $U^{\prime}$ where $\phi$ is nonzero $\phi$ gives a subbundle $W$ of $V$ extending $W_{m} \mid q_{(m)}^{-1}(s)$. Extend $W$ on $U^{\prime}$ to $\tilde{W} \subset V$ on $X$.

To complete the proof of Proposition 6.5 we have only to note that $U^{\prime} \supset q_{(m)}^{-1}(s)$ and since $W \mid q_{(m)}^{-1}(s)$ contradicts the semistability of $V \mid q_{(m)}^{-1}(s), \tilde{W}$ contradicts the semistability of $V$.

Now Propositions 6.4 and 6.5 together immediately imply Theorem 6.1.
6.6. Remarks. i) In our proof $m_{0}$ depends on $V$. If $m_{0}$ can be chosen to depend only on the Chern classes of $V$ boundedness of the family of semistable bundles with fixed Chern classes would follow.
ii) If $m_{0}$ works for a $V$, clearly it would do for any small deformation of $V$. Therefore it follows that for any bounded family of sheaves there is a single curve $C$ on which all of them are semistable.
iii) If char $k=0$ it follows from the result of [13] relating unitary representations of Fuchsian groups and semistable vector bundles on the corresponding curves that on a curve $X$ if $V$ is semistable then any associated bundle (for e.g. the exterior powers, symmetric powers etc.) is also semistable. When char $k=0$, from this and Theorem 6.1 (and Remark 6.2) it follows immediately that the same result holds for higher dimensional $X$ as well. This result has also been proved in [9] and [15] by other methods.

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