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SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

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Abstract.

It is shown that the property of sensitive dependence on initial conditions in the sense of Guckenheimer, follows from the other two more technical parts of one of the most common recent definitions of chaotic systems. It follows that this definition applies to a broad range of dynamical systems, many of which should not be considered chaotic. We investigate the implications of sensitive dependence on initial conditions and its relation to dynamical properties such as rigidity, ergodicity, minimality and positive topological entropy. In light of these investigations and several examples which we exhibit, we propose a natural family of dynamical systems—χ-systems—as a better abstract framework for a general theory of chaotic dynamics.

§0. Introduction

The vague notion of Chaos has attracted a great deal of attention in recent years and several authors have tried to formalize it in various ways. One popular such attempt uses the definition of "sensitive dependence on initial conditions". A chaotic system is defined, according to this school, to be a compact metric space X, together with a continuous self map $T : X \longrightarrow X$ satisfying the following three properties:

(1) Topological transitivity: There exists a point x_0 in X whose orbit $\mathcal{O}(x_0) = \{T^n x_0 : n \in \mathbb{N}\}$ is dense in X. (It then follows that $\mathcal{O}(x)$ is dense for all x in a dense G_δ subset of X).
(2) The T-periodic points are dense in X.
(3) Sensitive dependence on initial conditions: there exists a positive ϵ such that for all $x \in X$ and all $\delta > 0$ there is some y which is within a distance δ of x and for some n, $d(T^n x, T^n y) > \epsilon$.

As far as we know the first to formulate (3) was J. Guckenheimer, [7], in his study on maps of the interval (he required the condition to hold for a set of positive Lebesgue measure). The phrase—sensitive dependence on initial conditions—was used by D. Ruelle [9] (see also [1]), to indicate some exponential rate of divergence of orbits of nearby points. This is not too far from our suggestion of χ-systems below. The above definition of chaos (which, we believe, was introduced in [2]) became standard in several monographs dealing with the subject of chaos. In some cases the discussion of various examples of chaotic systems is concluded by proving

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sensitivity (after the usually easy observation that the system in question satisfies the conditions (1) and (2) of the definition of chaos).

Now it turns out that in fact—unless the system is a cyclic permutation of a finite set of points—conditions (1) and (2) actually imply condition (3). (Corollary 1.4).

It is thus clear that some other condition should be added to (1) and (2) if one is to capture the notion of chaos. The most natural candidate for this (in the topological category, at least) is the requirement that T has positive topological entropy. This quantifies a rate for the sensitive dependence and says that many nearby points have orbits that diverge exponentially fast.

Returning to the second condition, that periodic points be dense, we find that it is unnecessarily restrictive. In lieu of this we suggest, for reasons to become clear to the reader later in this note, an analogous condition requiring only the existence of a T-invariant measure whose support is all of X. We call a topologically transitive system satisfying this latter condition an E-system and if in addition the system has positive topological entropy we call it a χ-system. When the condition "the periodic points are dense" in X is replaced instead by a condition requiring the almost periodic points to be dense, we get the intermediate notion of an M-system. Recall that a point x_0 in a dynamical system (X, T) is called "almost periodic" if its orbit closure is a minimal subset of (X, T). Equivalently for any $\epsilon > 0$, the set of $n \in \mathbb{N}$ such that $d(T^n x_0, x_0) < \epsilon$ has bounded gaps.

In section 1 we examine the condition (3) of sensitive dependence on initial conditions and show that any non-equicontinuous E-system necessarily satisfies this condition. In the next two sections the notions of E and M systems and their relation to positive entropy are considered. Alongside these considerations various examples are exhibited which support the view that the notion of χ-system is a natural one. To be sure in many physical systems such as the ones modeled by the Henon map, the dissipative part of the system precludes the possibility that there be an invariant measure with global support. For these kinds of systems the chaotic nature lies in the presence of an attractor that is chaotic.

Remark. After the submission of this paper, the note of J. Banks, J. Brooks,G. Cairns, G. Davis and P. Stacey, "On Devaney's Definition of Chaos", appeared in Amer. Math. Monthlly, **99** (1992),332-334. Its main result is equivalent to our corollary 1.4.

§1. Sensitivity

In the sequel we call a pair (X, T), where (X, d) is a compact metric space and T a continuous map from X to itself a **system**. we say that a system (X, T) has **sensitive dependence on initial conditions** or more briefly, has **property** S, or is **sensitive**, if there exists an $\epsilon > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ with $d(T^n x, T^n y) > \epsilon$. When (X, T) does not have property S we say that it is a $\sim S$**-system**, or that it is **not sensitive**. Spelling this property out we have: for every $\epsilon > 0$ there exist an $x \in X$ and a neighborhood U of x such that for every $y \in U$ and every $n \in \mathbb{N}$, $d(T^n x, T^n y) \le \epsilon$. We observe that trivially (X, T) is $\sim S$ whenever X has an isolated point.

Let (X, T) be a transitive (=topologically transitive) system, we say that the system (X, T) is

(1) a P-**system** if the periodic points are dense in X.
(2) an M-**system** if the almost periodic points are dense in X.
(3) an E-**system** if there exists a T-invariant probability measure on X, which is positive on every non-empty open set.

Following [4], let us call a point x of X, **regular** if it is a generic point for some invariant measure ν and $\nu(U) > 0$ for every open neighborhood U of x. Considering condition (3). we see, using the ergodic decomposition of the invariant measure, that when this condition is satisfied, the set of regular points is dense in X. Conversely, when the regular points are dense in X, it is easy to construct an invariant probability measure which is positive on non-empty open sets. Let us call a system (X, T) **ergodic** if there exists an ergodic T-invariant measure μ on X whose support is all of X. Thus (X, T) is an E-system iff the regular points are dense in X, iff the union of the ergodic sub-systems of X is dense in X. An example in [10] shows that an E-system need not be ergodic. (The example constructed there is actually a P-system). Clearly every minimal system is ergodic.

It is now clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Notice that for a transitive system these conditions imply that (X, T) has no isolated points unless it is finite. Since for a transitive system with no isolated points T is necessarily onto we conclude that under each of these conditions T is onto.

Lemma 1.1. *For a topologically transitive system (X, T) with no isolated points, being $\sim S$ is equivalent to the following property: For every $\epsilon > 0$ there exists a transitive point (i.e. a point with dense orbit) $x_0 \in X$ and a neighborhood U of x_0 such that for every $y \in U$ and every $n \in \mathbb{N}$, $d(T^n x_0, T^n y) \le \epsilon$.*

Proof. Let ϵ be given and let x and U be as in the definition of the property $\sim S$. By transitivity there is a point x_0 in X whose orbit is dense; let $n_0 \in \mathbb{N}$ with $T^{n_0} x_0 \in U$. There exists a $\delta > 0$ such that $B_\delta(T^n x_0) \subset U$. Denote $x_1 = T^n x_0$, and $V = B_\delta(x_1)$, then it is clear that for every $y \in V$ and $n \in \mathbb{N}$, $d(T^n x_1, T^n y) \le 2\epsilon$. Since X has no isolated points the point x_1 is also transitive and the proof is complete. □

Recall that a system (X, T) is called **uniformly rigid** if there exists a sequence $n_k \nearrow \infty$ such that the sequence $\{T^{n_k}\}$ tends uniformly to the identity map on X.

lemma 1.2. *A topologically transitive system without isolated points which is not sensitive is uniformly rigid.*

Proof. Given an $\epsilon > 0$ there is, by the previous lemma, a transitive point x_0 and a neighborhood U of x_0 such that $d(T^n x_0, T^n y) \le \epsilon$ for every n, and every $y \in U$. Let now k satisfy $T^k x_0 \in U$, then $d(T^{n+k} x_0, T^n x_0) \le \epsilon$ for every n, and since x_0 is transitive it follows that $d(T^k z, z) \le \epsilon$ for every z in X. Applying this observation to a sequence of ϵ_i's that tend to zero gives a sequence of k_i's such that T^{k_i}tends uniformly to the identity. □

We actually proved more than was stated in the lemma. This additional information will be used in the proof of the following theorem. Since an E-system is either finite or has no isolated points, it follows that an E-system which is not sensitive is uniformly rigid.

Theorem 1.3. *An E-system which is not sensitive is necessarily a minimal equicontinuous system.*

Proof. [1] Let (X,T) be a non-sensitive E-system. If X has an isolated point it is finite and minimal and we are done. Otherwise, given $\epsilon > 0$, as in the previous proof, there exists a transitive point x_0 and a neighborhood U of x_0 such that $d(T^n x_0, T^n y) \leq \epsilon$ for every n and every $y \in U$. By assumption there exists a point $z \in U$, generic for some ergodic measure μ on X with $\mu(U) > 0$.

Let $A = \{n \in \mathbb{N} : T^n z \in U\}$. Since z is generic for μ, it follows that A has positive upper density. A well known fact (see for example [6],page 75), implies now that $A - A = \{a - a' : a, a' \in A\}$, is syndetic (i.e. has bounded gaps). Let $k, l \in A$ and suppose $k > l$. Then $T^l z \in U$ implies $\forall n, d(T^{n+l} z, T^n x_0) \leq \epsilon$ and $z \in U$ implies $\forall n, d(T^{n+l} z, T^{n+l} x_0) \leq \epsilon$. Hence $\forall n, d(T^{n+l} x_0, T^n x_0) \leq 2\epsilon$, hence $\forall w \in X, d(T^l w, w) \leq 2\epsilon$. Similarly we get $\forall w \in X, d(T^k w, w) \leq 2\epsilon$. Put together these yield

$$\forall w, d(T^k w, T^l w) = d(T^{k-l} T^l w, T^l w) \leq 4\epsilon$$

Since T is onto we get $d(T^{k-l} w, w) \leq 4\epsilon$ for all $w \in X$. We have now proved that for every $\epsilon > 0$ there exists a syndetic subset B of $\mathbb{N}$ with

$$n \in B \Longrightarrow Sup_{w \in X} d(T^n w, w) \leq \epsilon$$

This is Bohr's almost periodicity condition which is well known to be equivalent to equicontinuity. Finally, transitivity implies that (X,T) is minimal. □

Corollary 1.4. *A P-system which is not sensitive is a cyclic permutation on a finite set.*

Proof. This is an immediate corollary of theorem 1.3. Since however, a direct proof will only take few lines, let us write it down. Let x_0 be a transitive point and U a neighborhood of x_0 with $d(T^n x_0, T^n y) \leq \epsilon, \forall y \in U, \forall n$. There exists a periodic point $z \in U$, say $T^k z = z$. Then

$$\forall n, m \ d(T^{n+mk} x_0, T^n x_0) < d(T^{n+mk} x_0, T^{n+mk} z) + d(T^n z, T^n x_0) < 2\epsilon$$

Hence $d(T^{mk} w, w) < 2\epsilon$ for every $w \in X$, and every m. This implies equicontinuity; transitivity implies minimality and since the periodic points are dense, X must be finite. □

By lemma 1.2 a transitive $\sim S$ system with no isolated points is uniformly rigid and by theorem 1.3 such systems are already minimal equicontinuous if they are E-systems. A natural question to ask is what kind of transitive $\sim S$ uniformly rigid systems can arise. Here is a partial answer showing that a wide variety do arise.

Proposition 1.5. *Any transitive uniformly rigid system (X,T) has an extension (Y,S) that is transitive, uniformly rigid, with no isolated points and is not sensitive.*

Proof. We assume that T^{n_i} tends uniformly to the identity map and that x_0 has a dense orbit. Define for $x, x' \in X, \rho(x,x') = Sup_{n \in \mathbb{N}} d(T^n x, T^n x')$ and notice that by rigidity, the sequence $\rho(T^{n_i} x, x)$ tends to 0 with i. Let now $\Omega = (X \times \mathbb{R})^{\mathbb{N}}$. For

[1] We thank H. Furstenberg for his help in the following proof

$\bar\omega \in \Omega$ we denote by $\bar\omega = (\xi, \omega)$ the decomposition into $\xi \in X^{\mathbb{N}}$ and $\omega \in \mathbb{R}^{\mathbb{N}}$. For $\bar\omega, \bar\omega' \in \Omega$ let

$$\hat{d}(\bar\omega, \bar\omega') = \sum_{k=0}^{\infty} 2^{-k}\{d(\xi(k), \xi'(k)) + |\omega(k) - \omega'(k)|\}.$$

Let $\bar\omega_0$ be the point of Ω whose n-th coordinate is $(T^n x_0, \rho(T^n x_0, x_0))$ and let Y be the orbit closure of $\bar\omega_0$ under the shift map S of Ω

The points $\bar\omega \in Y$ have the form $\bar\omega(k) = (T^k x, \omega(k))$ for some $x \in X$, and

$$(S\bar\omega)(n) = (T^{n+1}x, \omega(n+1)).$$

It turns out, as is always the case for a transitive system, that in checking the non-sensitivity we will be dealing with only one point $\bar\omega_0$. Given $\epsilon > 0$, let U be the neighborhood of $\bar\omega_0$ defined by

$$U = \{\bar\omega \in Y : \omega(0) < \epsilon/2\}$$

Since $\bar\omega_0$ has a dense orbit, in order to verify that for all $\bar\omega \in U$ and all n

$$\hat{d}(S^n\bar\omega_0, S^n\bar\omega) \le \epsilon$$

it suffices to do so for points $\bar\omega$ of the form $S^j\bar\omega_0$. Suppose then that $S^{j_0}\bar\omega_0 \in U$. Since $\rho(T^{j_0}x_0, x_0) < \epsilon/2$ we have $d(T^{i+j_0}x_0, T^i x_0) < \epsilon/2$ for all $i \ge 0$, hence also $\rho(T^{i+j_0}x_0, T^i x_0) < \epsilon/2$ for all $i \ge 0$. By the triangle inequality we find that $|\rho(T^i x_0, x_0) - \rho(T^{i+j_0}x_0, x_0)| \le \epsilon/2$ for all $i \ge 0$. For any n we therefore have

$$\begin{aligned}
&\hat{d}(S^n\bar\omega_0, S^n(S^{j_0}\bar\omega_0)) = \hat{d}(S^n\bar\omega_0, S^{n+j_0}\bar\omega_0)\\
&= \sum_{k=0}^{\infty} 2^{-k}\{d(T^{k+n}x_0, T^{k+n+j_0}x_0) + |\rho(T^{k+n}x_0, x_0) - \rho(T^{k+n+j_0}x_0, x_0)|\}\\
&\le \sum_{k=0}^{\infty} 2^{-k}\{\epsilon/2 + \epsilon/2\} = 2\epsilon.
\end{aligned}$$

We observe that the only reason for requiring (X, T) to be rigid, is to make sure that the point $\bar\omega_0$, is not an isolated point. □

Since there are plenty of uniformly rigid systems which are sensitive—e.g. every uniformly rigid weakly mixing minimal system, (see [8] e.g. for the existence of these), is sensitive according to theorem 1.3—it follows from proposition 1.5 that an extension of a sensitive system with no isolated points, need not itself be sensitive (see however corollary 1.7 below).

It is not hard to see that whenever we deal with a transitive system with no isolated points, the condition $\sim S$ is equivalent to the condition:

$$\exists x\ \forall \epsilon > 0\ \exists \delta > 0 \forall\ y \in B_\delta(x)\ \forall n\ d(T^n x, T^n y) < \epsilon$$

A map $\pi : X \longrightarrow Y$ is called **semi-open** if the image under π of every non-empty open set has a non-empty interior.

Lemma 1.6. *Let (X,T) be a transitive $\sim S$-system with no isolated points and let $\pi : X \longrightarrow Y$ be a semi-open homomorphism of systems, then Y is a $\sim S$-system.*

Proof. Given $\epsilon > 0$ there exist $x \in X$ and $\delta > 0$ such that for every $x' \in B_\delta(x)$ and for every $n, d(T^n x, T^n x') \leq \epsilon$. The map $\pi^{-1} : Y \longrightarrow 2^X$ is an upper-semicontinuous map; therefore there exists a dense G_δ subset Y_0 of Y where π^{-1} is continuous. Since π is semi-open it follows that $X_0 = \pi^{-1}(Y_0)$ is a dense G_δ subset of X. Let $x_0 \in X_1 \cap X_0 \cap B_\delta(x)$, where X_1 is the dense G_δ subset of transitive points in X. There exists an $\eta > 0$ such that $B_\eta(x_0) \subset B_\delta(x)$ and then for every $x' \in B_\eta(x_0)$ and every $n, d(T^n x_0, T^n x') \leq 2\epsilon$. Since $y_0 = \pi(x_0)$ is a continuity point for π^{-1}, y_0 is in the interior of the set $\pi(B_\eta(x_0))$. Thus there exists a $\theta > 0$ with $B_\theta(y_0) \subset \text{int } \pi(B_\eta(x_0))$. If $y' \in B_\theta(y_0)$ then there exists $x' \in B_\eta(x_0)$ with $\pi(x') = y'$ whence, for every n, $d(T^n x_0, T^n x') \leq 2\epsilon$ and finally also $d(T^n y', T^n y_0) \leq 2\epsilon$. □

Corollary 1.7. *If $\pi : X \longrightarrow Y$ is a semi-open homomorphism where X is transitive with no isolated points and Y is sensitive, then so is X.*

§2. Entropy and χ-systems

As was shown in the previous section, a transitive $\sim S$-system with no isolated points is uniformly rigid. In particular this implies that it has zero (topological) entropy. Thus if (X,T) is transitive, has no isolated points (e,g, when it is an E-system), and has positive entropy then it has sensitive dependence on initial conditions. In the other direction it is easy to produce an example of an infinite P-system which is sensitive and of zero entropy. One way to build such an example is as follows.

Example 2.1.

The system X will be the orbit closure of a point ω_0 in the shift system $(\{0,1\}^{\mathbb{N}}, \sigma)$. To ensure that X has zero topological entropy we insist that for all $k \geq 1$, and $n \geq 0$

$$\omega_0(n \cdot 3^{k+1} + i) = \omega_0(n \cdot 3^{k+1} + i + 3^k) \quad 0 \leq i < 3^k.$$

By stage k, $\omega_0(i)$ will be defined for some indices i, and we will have the property that if A_k denotes the block $\omega_0(i), 0 \leq i < 3^k$, for some sequence of n_j tending to infinity, ω_0 will also be defined for $n_j + m$, $0 \leq m < j \cdot 3^k$, and will equal $A_k \cdot A_k \cdots A_k$ (j-times) there. These repetitions don't conflict with the previous requirement and it is easy to see that this guarantees property P. There is enough freedom left to construct a <u>non</u> periodic such ω_0 whose orbit closure has then all of the desired properties. In particular the sensitivity follows from the expansiveness of all subshifts of symbolic dynamical systems. □

Let us call an E-system of positive entropy an **E^+-system** or a **χ-system**. Recall that a system (X,T) is ergodic if there exists an ergodic measure on X whose topological support is all of X.

Proposition 2.2.

(1) *The product of a weakly mixing system (X,T) and an E-system (Y,T) is topologically transitive.*
(2) *A product of two E-systems which is transitive is an E-system.*

(3) *The product of an E-system and a χ-system, when transitive, is a χ-system. In particular this is the case when at least one of the factors is topologically weakly mixing.*
(4) *For a χ-system (X, T) we have:*

$$h(X, T) = Sup\{h(Y, T) : (Y, T) \text{ an ergodic subsystem of } (X, T)\}$$

Proof. (1) Let $A, B \subset X, U, V \subset Y$ be nonempty open sets. We have to show that for some $l \in \mathbb{Z}$, $T^l A \cap B \neq \emptyset$ and also $T^l U \cap V \neq \emptyset$. Let $W = \cup_{n\in\mathbb{Z}} T^n U$, then W is a nonempty T-invariant open subset of Y. By assumption there exists a T-invariant probability measure μ on Y which assigns positive measure to every nonempty open set, and in particular $\mu(W) = a > 0$. Since Y is transitive the set $O = W \cap V$ is a nonempty open subset and we have $\mu(O) = b > 0$. We now choose a positive integer N such that

$$\mu(\bigcup_{|n|\leq N} T^n U) > a - b/2.$$

Now the system (X, T) is topologically weakly mixing, hence by [5] the set $N(A, B) = \{k \in \mathbb{Z} : T^k A \cap B \neq \emptyset\}$ contains arbitrarily long intervals. We can therefore find some $j \in \mathbb{Z}$ with

$$T^{j+k} A \cap B \neq \emptyset, \qquad \forall |k| \leq N.$$

By T-invariance of μ we have

$$\mu(T^j(\bigcup_{|n|\leq N} T^n U)) = \mu(\bigcup_{|n|\leq N} T^n U) > a - b/2.$$

This implies $T^j(\bigcup_{|n|\leq N} T^n U) \cap V \neq \emptyset$, and there exists n_0 with $|n_0| \leq N$ and $T^{j+n_0} U \cap V \neq \emptyset$ as well as $T^{j+n_0} A \cap B \neq \emptyset$. This completes the proof of (1).

(2) and (3) are now clear. For (4) we recall that the measure theoretical entropy considered as a function on the space of invariant probability measures on X satisfies the formula:

$$h(\mu) = \int h(\omega) dP(\omega)$$

where $\mu = \int \omega dP(\omega)$ is the ergodic decomposition of the invariant measure μ. Combining this with the variational principle we get our result. (See for example [3],p.78). □

Part (1) of proposition 2.2. generalizes the result of [5] asserting that the product of a weakly mixing system (X, T) and a minimal system (Y, S) is topologically transitive. Taking (Y, S) to be the one point compactification of the translation on $\mathbb{Z}$ it is easy to see that the assumption of topological transitivity of Y is not enough for this result to hold. We remark, without giving the details, that if one takes (X, T) to be a weakly mixing rigid minimal system (see [8]), and (Y, S) the system constructed in proposition 1.5. then, although the system (Y, S) is transitive and pointwise recurrent, the product system $(X \times Y, T \times S)$ is not transitive.

In the parallelism between topological dynamics and ergodic theory, minimal corresponds to ergodic and M-systems correspond to E-systems. In the next section we shall examine the possibility of substituting M-systems and minimal systems for E-systems and ergodic systems respectively in the above proposition.

§3. Entropy and M-systems

Lemma 3.1. *A transitive product of two M-systems is an M-system.*

Proof. It suffices to show that if (X, T) and (Y, T) are minimal and $(X \times Y, T \times T)$ is transitive then it is an M-system. Let $Z \subset X \times Y$ be an arbitrary minimal subset of $(X \times Y, T \times T)$. Clearly Z projects onto all of X and since $I \times T^n$ commutes with $T \times T$, $(I \times T^n)(Z)$ is also minimal. Since (Y, T) is minimal it follows that $\cup(I \times T^n)(Z)$ is dense in $X \times Y$ as required. □

Let us call an M-system of positive entropy an M^+**-system**.

Corollary 3.2.

(1) *A transitive product of an M-system and an M^+-system is an M^+-system. In particular this is the case when at least one of the systems is topologically weakly mixing.*

(2) *The product of two minimal systems of which at least one is of positive entropy and at least one weakly mixing, is an M^+-system.*

Proof. These statements follow from lemma 3.1., proposition 2.2.(1) and the fact that every M-system is an E-system. □

Are there examples of M^+-systems which are not product of minimal systems and whose subset of periodic points is empty? (The shift on $\{0,1\}^{\mathbb{Z}}$ is an example of a P-system of positive entropy which is not a product of minimal sets). The answer to this question is yes and we next describe an example of such an M^+-system which is moreover the set of non wandering points of a C^∞ map.

Example 3.3.

Let (Ω, σ) be the shift on two symbols, say 0 and 1. Let $\mathbb{T}$ be the circle group, realized as $\mathbb{R}$ *mod* 1. Let $X = \Omega \times \mathbb{T}$ and let $\varphi : \Omega \longrightarrow \mathbb{T}$ be defined by:

$$\varphi(\omega) = \begin{cases} \alpha & \text{when } \omega(0) = 0 \\ \beta & \text{when } \omega(0) = 1 \end{cases}$$

where α, β and 1 are rationally independent. Now define the skew product map $T : X \longrightarrow X$ by $T(\omega, t) = (\sigma\omega, t + \varphi(\omega))$. Let ω be a periodic point of period k. Suppose the block defining ω has m zeroes and n ones, and let $\gamma = m\alpha + n\beta$. Since $\sigma^k\omega = \omega$, we have $T^k(\omega, t) = (\omega, t + \gamma)$ and we conclude that $\{\omega, \sigma\omega, \cdots, \sigma^{k-1}\omega\} \times \mathbb{T}$ is a minimal subset of (X, T). This observation implies that the almost periodic points are dense in X. It now follows that for every transitive point ω_0 of Ω, the point (ω_0, t) for any t , is a transitive point of X. We conclude that (X, T) is an M^+-system (hence also a χ-system), with no periodic points.

To see that (X, T) is not a product of minimal systems we observe that if $\underline{0}$ and $\underline{1}$ are the fixed points of σ, then the restrictions of T to the sets $\{\underline{0}\} \times \mathbb{T}$ and $\{\underline{1}\} \times \mathbb{T}$ form two disjoint minimal sets. This implies that no non-trivial minimal system can be a factor of (X, T).

Finally to exhibit (X, T) as the subset of non wandering points of a C^∞ map, define a skew product map T on $\mathbb{R} \times \mathbb{T}$ by $T(s, t) = (f(s), t + \varphi(s))$, where f and φ are the C^∞ functions given by the following:

$$f(t) = 2 - 3t^2$$

and

$$\varphi(t) = \begin{cases} \alpha & t \leq -1/2 \\ \text{a } C^\infty \text{ interpolation between these values on } [-1/2, 1/2]. & \\ \beta & t \geq 1/2 \end{cases}$$

□

The two-shift (Ω, σ) satisfies the following relation:

$$h(\Omega, \sigma) = Sup\{h(Y, \sigma) : (Y, \sigma) \text{ a minimal subset of} (X, \sigma)\}$$

This formula is true for torus automorphisms, for the horseshoe map and many other chaotic systems where finite type sub-shifts appear as subsystems. Can one hope for it to hold for every M-system? The answer is no as the following example shows.

Example 3.4.

We construct a transitive system (X, T) with (i) positive topological entropy, such that (ii) all minimal subsets of (X, T) have zero entropy but none the less (iii) the almost periodic points are dense. In fact we construct a single point of $\{0, 1\}^{\mathbb{Z}}$ whose orbit closure has the desired properties. First let us see how to achieve (i) and (ii).

Set $\omega(n) = 0$ for all n in the set

$$A = \bigcup_{k=1}^{\infty} \bigcup_{n \neq 0} \{n \cdot 10^k, n \cdot 10^k + 1, \ldots, n \cdot 10^k + k\}$$

For $n \notin A$ let $\omega(n)$ be independently equal to either zero or one, and let X denote the orbit closure of ω under the shift T. Now X is a random set (depending on the outcome of the $\omega(n)$ for $n \notin A$), and it is easy to see that with probability one (X, T) has positive topological entropy. On the other hand, since $\omega(n) = 0$ for $n \in A$, it is clear that the only minimal set in (X, T) is the fixed point $\zeta(n) \equiv 0$.

This construction will now be modified so that the periodic points will be dense. To this end $\omega(n)$ will be changed to $\hat{\omega}(n)$ only for some $n \in A$. Thus the positivity of the topological entropy will not be affected at all. For each n, we will ensure that we have a periodic point p_n that contains the block $\hat{\omega}(i)$ for $-n \leq i \leq n$, where $\hat{\omega}$ denotes the new point that we are constructing. In the inductive definition if a_n denotes the period of p_n, then the basic block of p_{n+1} will consist of the central $2(n+1)+1$ block of $\hat{\omega}$ followed by $10 \cdot a_n$ zeroes. Thus the period of p_{n+1} is

$$a_{n+1} = 2(n+1) + 1 + 10 \cdot a_n.$$

We insert longer and longer repetitions of this basic block well inside zero blocks of $\hat{\omega}$. In this way we can guarantee that the only "new" points that are obtained in the orbit closure of $\hat{\omega}$ are the periodic points that we are trying to insert. It is fairly easy to verify now that the only minimal sets are the finite orbits of the periodic points which clearly have zero entropy. In fact, the nature of our construction is such that the only blocks in $\hat{\omega}$ of length greater than $2 \cdot a_n$ that do not contain a sub-block of n-consecutive zeroes are those arising from periodic points p_i with $i < n$.

We remark in passing that $\omega(n)$ provides an example showing that sequences with positive density may have only trivial minimal sets in their orbit closure, thus proving that Szemeredi's theorem cannot be established using only the dynamical van der Warden theorem on arithmetic progressions. □

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