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Abstract

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The paper shows the intimate relationship between the proposal and the *method of control variates*. It next relates the proposal to the estimation of coefficients in a reliability polynomial and indicates how this concept can be used to improve computing efficiency in certain cases. It also describes a procedure that determines the p vector, to be used in the sampling experiment, that minimizes a bound on the worst case variance. The paper also derives individual and simultaneous confidence intervals that hold for every fixed sample size K. An example illustrates how the proposal works in an s-t connectedness problem.

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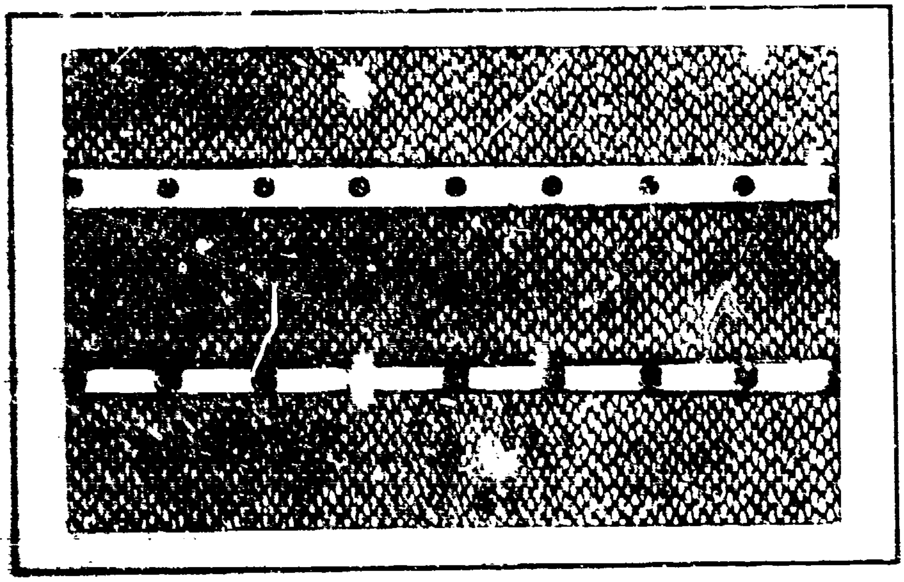

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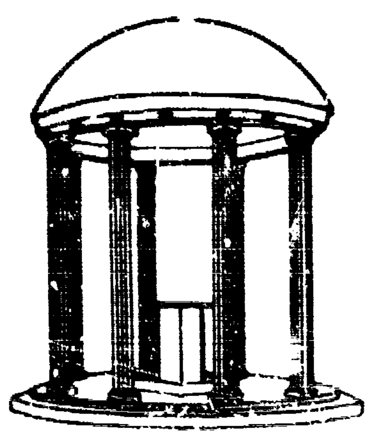

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Sensitivity Analysis for the System Reliability Function

George S. Fishman

Technical Report No. UNC/OR/TR-87/6

December 1987

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Abstract

Sensitivity analysis is an integral part of virtually every study of system reliability. This paper describes a Monte Carlo sampling plan for estimating this sensitivity in system reliability to changes in component reliabilities. The unique feature of the approach is that sample data collected on K independent replications using a specified component reliability vector **p** are transformed by an *importance function* into unbiased estimates of system reliability for each component reliability vector **q** in a set of vectors $\mathcal{Q}$. Moreover, this importance function together with available prior information about the given system enables one to produce estimates that require considerably less computing time to achieve a specified accuracy for all $|\mathcal{Q}|$ reliability estimates than a set of $|\mathcal{Q}|$ crude Monte Carlo sampling experiments would require to estimate each of the $|\mathcal{Q}|$ system reliabilities separately. As the number of components in the system grows, the relative efficiency continues to favor the proposed method.

The paper shows the intimate relationship between the proposal and the *method of control variates*. It next relates the proposal to the estimation of coefficients in a reliability polynomial and indicates how this concept can be used to improve computing efficiency in certain cases. It also describes a procedure that determines the **p** vector, to be used in the sampling experiment, that minimizes a bound on the worst case variance. The paper also derives individual and simultaneous confidence intervals that hold for every fixed sample size K. An example illustrates how the proposal works in an s-t connectedness problem.

Key Words: s-t reliability, system reliability, importance sampling, Monte Carlo method, control variates

Introduction

Sensitivity analysis, which represents an integral part of virtually every study of system reliability, measures variation in this quantity in response to changes in component reliabilities or in system design. Replacing old components with new ones with higher reliabilities affects system reliability. As time elapses, system reliability deteriorates when a non replacement policy for component failures is in force. Deleting, adding or rearranging components all affect system reliability. Sampling variation in component reliability estimates induce sampling variation in the corresponding system reliability estimate. Having access to a model that accurately predicts these changes in system behavior allows one to make considerably more well informed decisions for maintaining or enhancing performance.

This paper presents a method for estimating variation in system reliability in response to variation in component reliabilities. It describes a Monte Carlo sampling plan that on each replication provides sample data that contribute to the estimation of system reliability for each of w sets of distinct component reliabilities. The sets may represent alternative component replacement plans, deteriorating component reliabilities at a succession of time points or extremal points of simultaneous component reliability interval estimates (Fishman 1987). For purposes of exposition, we focus on s–t reliability but emphasize that the concepts discussed here also apply to other definitions of system reliability. (keywords:

To understand the significance of this approach, we first discuss the computation of s–t reliability at a point. Consider a system representable by an undirected network $G = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ denotes the set of nodes all of which function perfectly and $\mathcal{E}$ denotes the set of edges each of which fails randomly and independently. The concept of s–t reliability principally focuses on the probability that at least one path of functioning edges (components) connects nodes s and $t \in \mathcal{V}$, and it is this quantity that we wish to compute.

Since the exact computation of s–t reliability from a single set of component reliabilities belongs to the class of NP–hard problems (Valiant 1979), attempts to shed light on this computation have had to exploit special structure, rely on bounds or use the

coefficients, polynomials, confidence intervals), Monte Carlo method. The polynomial time algorithm of Agrawal and Satyanarayana (1984) for the exact reliability computation for series–parallel systems exemplifies a highly beneficial use of special structure. Bounds such as those of Esary and Proschan (1966), Van Slyke and Frank (1972) and Ball and Provan (1983) offer interval approximations to the reliability. The Monte Carlo method also exploits special structure and bounds. In particular, the sampling plans in Van Slyke and Frank (1972), Kumamoto, Tanaka and Inoue (1977), Easton and Wong (1980), Kumamoto, Tanaka, Inoue and Henley (1980), Karp and Luby (1983) and Fishman (1986) describe how to derive statistical estimates of reliability that are generally more accurate than a crude Monte Carlo sampling experiment would provide based on the same amount of work.

With regard to the exact computation of s–t reliability for $w>1$ sets of alternative component reliabilities, the corresponding time complexity has the same form as that for a single reliability computation increased by the multiplicative factor w. Also, whereas any of the aforementioned Monte Carlo proposals allow one to estimate s–t reliability for each of the w points, regrettably an observation on a trial for one point in no way contributes to estimating system reliability at the remaining $w-1$ points. The present paper overcomes this last inadequacy. It describes a Monte Carlo sampling plan that on each trial generates data that contribute to estimating all w system reliabilities simultaneously. Most importantly, these estimates, at all w points, are considerably more accurate than corresponding estimates that crude sampling can produce for the same amount of work.

The proposed method exploits *importance sampling*, a technique that Kahn (1950) and Kahn and Harris (1951) first described for reducing the variance of a Monte Carlo estimator of a point. The present account extends this technique to reliability function estimation and, in particular, shows how one can use knowledge available to the analyst before experimentation to enhance the accuracy of the estimated function at all w points for a given sample size K. Section 1 introduces relevant network nomenclature. Section 2 then describes system reliability estimation at a point using crude Monte Carlo sampling, as a baseline, and a highly efficient alternative method described in Fishman (1986). Section 3 extends the alternative method to simultaneous estimation at all w points, Section 4 shows how to assess the statistical efficiency of the proposed method and Section 5 shows the intimate relationship between the proposed importance sampling technique and the *method of control variates.*

Section 6 offers an alternative interpretation of the estimation procedure that reveals its relationship to estimating coefficients in a reliability polynomial and shows how this representation may save computation time when a small or moderate number of components vary their reliabilities. Section 7 discusses how to perform the importance sampling *optimally* given the set of component reliability vectors of interest. Sections 8 and 9 derive individual and simultaneous confidence intervals. Section 10 describes essential steps for implementation and Section 11 provides a comprehensive example that illustrates many of the features of the proposed technique.

1. Problem Setting

Consider a network $G = (\mathcal{V},\mathcal{E})$ with node set $\mathcal{V}$ and edge set $\mathcal{E}$. Assume that nodes function perfectly and that edges fail randomly and independently. Let

$$r = \text{number of distinct types of edges}$$

$$q_i = \text{probability that a node of type i functions} \quad i=1,\dots,r$$

$$q = (q_1,\dots,q_r)$$

$$\mathcal{E}_i = \text{set of edges of type i}$$

$$k_i = |\mathcal{E}_i| = \text{number of edges of type i}$$

$$k = (k_1,\dots,k_r)$$

$$e_{ij} = \text{jth edge of type i} \qquad j=1,\dots,k_i;\ i=1,\dots,r$$

$$x_{ij} = 1 \quad \text{if edge } e_{ij} \text{ functions}$$

$= 0$ otherwise

$$x_i = \sum_{j=1}^{k_i} x_{ij} = \text{number of functioning edges of type } i$$

$$x = (x_{11},\dots,x_{1k_1};\dots;x_{r1},\dots,x_{rk_r})$$

$\mathcal{X}$ = set of all edge states x

$$P(x,k,q) = \prod_{i=1}^{r} \prod_{j=1}^{k_i} [x_{ij}q_i + (1-x_{ij})(1-q_i)] = \prod_{i=1}^{r} q_i^{x_i}(1-q_i)^{k_i - x_i} \tag{1}$$

= probability mass function (p.m.f.) of state $x \in \mathcal{X}$

$\phi(x) = 1$ if the system functions when in state x

$= 0$ otherwise

$$g(q) = \sum_{x \in \mathcal{X}} \phi(x)\, P(x,k,q) \tag{2}$$

= probability that the system functions

We also assume that G describes a *coherent* system. A system of components is coherent if its structure function $\{\phi(x)\}$ is nondecreasing and each component is relevant (Barlow and Proschan 1981, p. 6).

For present purposes, the system functions ($\phi(x)=1$) when at least one minimal s–t path ($s,t \in \mathcal{V}$) exists and fails ($\phi(x)=0$) when no such path exists. Let $\mathcal{Q}$ denote a set of w component reliability vectors of interest. Then the purpose of analysis is to estimate the s–t reliability function $\{g(q),\ q \in \mathcal{Q}\}$.

2. Estimation at a Point

Crude Monte Carlo sampling offers a baseline against which potentially more efficient sampling plans can be compared. Let $X^{(1)},\dots,X^{(K)}$ denote K independent samples drawn from $\{P(x,k,q),\ x \in \mathcal{X}\}$. Then

$$\bar{g}_K(q) = \frac{1}{K} \sum_{i=1}^{K} \phi(X^{(i)}) \tag{3}$$

is an unbiased estimator of $g(q)$ with

$$\operatorname{var} \bar{g}_K(q) = g(q)[1-g(q)]/K. \tag{4}$$

To compute $\bar{g}_K(q)$, one performs K trials on each of which sampling X from $\{P(x,k,q)\}$ takes $O(|\mathcal{E}|)$ time and determination of $\phi(X)$ takes $O(\max(|\mathcal{V}|,|\mathcal{E}|))$ time, using a depth-first search as described in Aho, Hopcroft and Ullman (1974). These are worst case times. One can also show that the mean total computation time has the form

$$T(\bar{g}_K(q)) = \alpha_0 + K[\alpha_1 + \alpha_2 |\mathcal{E}| + \alpha_3(\mathcal{X},k,q)]$$

where

$$\alpha_3(\mathcal{X},k,q) = \sum_{x \in \mathcal{X}} P(x,k,q)\, C(x)$$

and

$$C(x) = \text{expected search time given the component state vector } x$$

The quantities α_0, α_1, α_2 and $\alpha_3(\mathcal{X},k,q)$ are machine dependent.

All Monte Carlo methods described in the previously cited references improve on the variance (4). In particular, the method described in Kumamoto, et al. (1977) and Fishman (1986) achieves this reduction by exploiting bounds on the structure function $\{\phi(x)\}$ and it is this approach that we now describe and later extend in Section 3. We follow the development in Fishman (1986).

Suppose that there exist 0–1 binary functions $\{\phi_L(x), x \in \mathcal{X}\}$ and $\{\phi_U(x), x \in \mathcal{X}\}$ such that

$$\phi_L(x) \le \phi(x) \le \phi_U(x) \qquad \forall\, x \in \mathcal{X}$$

Then the system reliability $g(q)$ has lower and upper bounds $g_L(q)$ and $g_U(q)$, respectively, where

$$g_i(q) = \sum_{x \in \mathscr{S}} \phi_i(x)\, P(x,k,q) \qquad i \in \{L,U\}.$$

Suppose that one now samples $X^{(1)},\ldots,X^{(K)}$ independently from the alternative probability mass function

$$Q(x,k,q) = \left[\frac{\phi_U(x)-\phi_L(x)}{\Delta(q)}\right] P(x,k,q) \qquad x \in \mathscr{S} \tag{5}$$

where

$$\Delta(q) = g_U(q) - g_L(q).$$

Then

$$\hat{g}_K(q) = g_L(q) + \Delta(q)\frac{1}{K}\sum_{i=1}^{K} \phi(X^{(i)}) \tag{6}$$

is also an unbiased estimator of $g(q)$, but with variance

$$\operatorname{var} \hat{g}_K(q) = [g_U(q)-g(q)][g(q)-g_L(q)]/K \le \Delta^2(q)/4K. \tag{7}$$

Compared to crude Monte Carlo sampling, one has

$$\frac{\operatorname{var}\ \bar{g}_K(q)}{\operatorname{var}\ \hat{g}_K(q)} \ge 1/\left[\{g_L(q)[1-g_U(q)]\}^{\frac{1}{2}} - \{g_U(q)[1-g_L(q)]\}^{\frac{1}{2}}\right]^2 \tag{8}$$

$$\ge 1,$$

indicating that $\hat{g}_K(q)$ always has a variance no larger that var $\bar{g}_K(q)$.

Choosing Bounds

The choice of bounds $g_L(q)$ and $g_U(q)$ depends on the reliability computation under consideration. As an example for the s–t connectedness problem, let $\mathscr{P}_1,\ldots,\mathscr{P}_I$ denote edge-disjoint minimal s-t paths of G and let $\mathscr{C}_1,\ldots,\mathscr{C}_J$ denote edge-disjoint minimal s-t cutsets of G. Let

$$\phi_L(\mathbf{x}) = 1 - \prod_{m=1}^{I}\left[1 - \prod_{i=1}^{r} \prod_{\substack{j=1 \\ e_{ij}\in \mathscr{E}_i \cap \mathscr{P}_m}}^{k_i} x_{ij}\right]$$

$$\phi_U(\mathbf{x}) = \prod_{m=1}^{J}\left[1 - \prod_{i=1}^{r} \prod_{\substack{j=1 \\ e_{ij}\in \mathscr{E}_i \cap \mathscr{C}_m}}^{k_i} (1-x_{ij})\right]$$

which are lower and upper bounds, respectively, for $\phi(\mathbf{x})$. Then

$$g_L(\mathbf{q}) = 1 - \prod_{m=1}^{I}\left[1 - \prod_{i=1}^{r} q_i^{|\mathscr{E}_i \cap \mathscr{P}_m|}\right]$$

and

$$g_U(\mathbf{q}) = \prod_{m=1}^{J}\left[1 - \prod_{i=1}^{r} (1-q_i)^{|\mathscr{E}_i \cap \mathscr{C}_m|}\right]$$

are lower and upper bounds, respectively, on $g(\mathbf{q})$. One can determine $\mathscr{P}_1,\ldots,\mathscr{P}_I$ in $0(I|\mathscr{E}|)$ time using a network flow algorithm with unit capacities, as in Wagner (1975, p. 954), and $\mathscr{C}_1,\ldots,\mathscr{C}_J$ in $0(|\mathscr{E}|)$ time by beginning at node s and appropriately labeling arcs. Note that

$$I \le \text{size of the smallest minimal s–t cutset in G}$$

and

$$J \le \text{size of the smallest minimal s–t path in G.}$$

Moreover, the resulting form of $\{Q(\mathbf{x},\mathbf{k},\mathbf{q})\}$ in (5) enables one to use Procedure Q in Fishman (1986) to sample $\mathbf{x}$ in $0(|\mathscr{E}|)$ time.

To compute $\hat{g}_K(\mathbf{q})$ using precomputed bounds based on edge-disjoint minimal s-t paths and cutsets, one performs K trials on each of which sampling X from $\{Q,\mathbf{x},\mathbf{k},\mathbf{q})\}$ occurs in $O(|\mathscr{E}|)$ time using Procedure Q in Fishman (1986), and determination of $\phi(X)$ again takes $O(\max(|\mathscr{V}|,|\mathscr{E}|))$ time. Also, mean total time assumes the form

$$T(\hat{g}_K(q)) = \beta_0 + K[\beta_1+\beta_2|\mathscr{E}|+\alpha_3(\mathscr{X}_{01},k,p)/\Delta(q)]$$

where

$$\mathscr{X}_{01} = \{x\in\mathscr{X}: \phi_L(x)=0 \text{ and } \phi_U(x)=1\}$$

and $\beta_0,\dots,\beta_2$ denote machine dependent constants.

Observe that

$$K(q) = K \text{ var } \bar{g}_K(q)/\text{var } \hat{g}_K(q)$$

denotes the number of trials one would have to take with crude Monte Carlo to achieve the same variance that arises in K trials using $\{Q(x,k,p)\}$. Then $\Lambda_1(q) = T(\bar{g}_{K(q)}(q))/T(\hat{g}_K(q))$ measures the efficiency of $\hat{g}_K(q)$ relative to $\bar{g}_K(q)$ and for large K and $|\mathscr{E}|$ has the approximate form

$$\Lambda_1(q) \approx \left[\frac{\alpha_2+\alpha_3(\mathscr{X}, k, q)/|\mathscr{E}|}{\beta_2+\alpha_3(\mathscr{X},k,q)/\Delta(q)|\mathscr{E}|}\right]\frac{g(q)[1-g(q)]}{[g_U(q)-g(q)][g(q)-g_L(q)]} \tag{9}$$

where $\alpha_3(\mathscr{X},k,q)/|\mathscr{E}|)$ and $\alpha_3(\mathscr{X}_{01},k,q)/|\mathscr{E}|$ are bounded from above. A ratio greater than unity favors the alternative sampling plan. Experience (Fishman 1986a) has shown this usually to be the case by a large margin.

3. Estimation at a Set of Points

This section extends the technique based on bounds for a single point to estimation at a set of $w = |\mathscr{Q}|$ points. In particular, it shows that at least two estimators deserve attention and later Section 5 shows how a linear combination of these estimators is, in fact, a control variate estimator. Let $p = (p_1,\dots,p_r)$ $0<p_i<1$ for $i=1,\dots,r$ and let

$$R(x,k,q,p) = P(x,k,q)/P(x,k,p)$$

$$= \prod_{i=1}^{r} (q_i/p_i)^{x_i} \left[(1-q_i)/(1-p_i)\right]^{k_i-x_i}. \tag{10}$$

Lemma 1. Let X be sampled from the p.m.f. $\{Q(x,k,p)\}$ in (5). Then

$$E[\phi(X)\, R(X,k,q,p)] = [g(q)-g_L(q)]/\Delta(p) \tag{11}$$

and

$$E[\phi(X)\, R(X,k,q,p)]^2 = c(q,p)[g(q^*)-g_L(q^*)]/\Delta(p) \tag{12}$$

where

$$c(q,p) = \prod_{i=1}^{r} c_i^{k_i}$$

$$c_i = c(q_i,p_i) = q_i^2/p_i + (1-q_i)^2/(1-p_i) \qquad i=1,\ldots,r$$

and

$$q^* = (q_1^2/c_1p_1,\ldots,q_r^2/c_rp_r).$$

The Appendix contains the proof.

Theorem 1 shows how these properties relate to reliability estimation.

Theorem 1. Let X be sampled from the p.m.f. $\{Q(x,k,p)\}$ in (5) and let

$$\psi_a(x,q,p) = g_L(q) + \Delta(p)\, \phi(x)R(x,k,q,p) \tag{13}$$

and

$$\psi_b(x,q,p) = g_U(q) - \Delta(p)[1-\phi(x)]R(x,k,q,p)] \quad q\in\mathcal{Q}. \tag{14}$$

Then for each $q \in \mathscr{Q}$

$$E\psi_a(X,q,p) = E\psi_b(X,q,p) = g(q) \tag{15}$$

$$\text{var } \psi_a(X,q,p) = v_a(q,p) = c(q,p)\Delta(p)[g(q^*)-g_L(q^*)]-[g(q)-g_L(q)]^2 \tag{16}$$

$$\text{var } \psi_b(X,q,p) = v_b(q,p) = c(q,p)\Delta(p)[g_U(q^*)-g(q^*)]-[g_U(q)-g(q)]^2 \tag{17}$$

and

$$\text{cov}[\psi_a(X,q,p), \psi_b(X,q,p)] = v_{ab}(q,p) = [g_U(q)-g(q)][g(q)-g_L(q)]. \tag{18}$$

The proof follows from Lemma 1.

Observe that the ***importance function*** R corrects the expectations (15) to the desired value, thereby inducing the variances in (16) and (17). The implication is immediate. Let $X^{(i)}$ now denote the ith sample drawn from $\{Q(x,k,p)\}$ and observe that one now has two potential estimators of $g(q)$, namely

$$\hat{g}_{jK}(q,p) = \frac{1}{K}\sum_{i=1}^{K} \psi_j(X^{(i)},q,p) \tag{19}$$

with

$$\text{var } \hat{g}_{jK}(q,p) = \text{var } \psi_j(X,q,p)/K \qquad j \in \{a,b\}. \tag{20}$$

Most importantly, note that merely sampling with edge reliabilities **p** enables one to generate two unbiased estimators of the entire reliability function $\{g(q), q \in \mathscr{Q}\}$. By contrast, all Monte Carlo sampling plans cited in the introduction require $|\mathscr{Q}|$ separate experiments to estimate $\{g(q), q \in \mathscr{Q}\}$.

4. Efficiency

Measuring the statistical efficiency of $\{\hat{g}_{aK}(q,p), q \in \mathscr{Q}\}$ and $\{\hat{g}_{bK}(q,p), q \in \mathscr{Q}\}$ as estimators of $\{g(q),\ q\in\mathscr{Q}\}$ calls for a more elaborate analysis than that for estimation at a single point. In particular, the sobering observation that $c(q,p)$ in (16) and (17) increases exponentially with $|\mathscr{E}|$ makes one circumspect about the benefit of the proposed method as the size of G grows. We now show that this benefit is assured for finite $|\mathscr{Q}|$ and number of edge types r, provided that $p\in\mathscr{Q}$.

Let $\mathscr{Q} = \{q_1,\ldots,q_w\}$ where $q_j = (q_{1j},\ldots,q_{rj})$ and q_{ij} is the reliability assigned to components of type i in the jth component reliability vector for $j=1,\ldots,w$. Let $\mathscr{N} = \{1,\ldots,r\}$ and

$$\mathscr{N}^* = \{i\in\mathscr{N}:\ p_i\neq q_{ij} \text{ for at least one } j;\quad j=1,\ldots,|\mathscr{Q}|\},$$

so that $|\mathscr{N}^*|$ component reliability types vary in $\mathscr{Q}$. Algorithm A describes the steps for computing the estimates and provides the basis for measuring efficiency. In addition to computing $\{\hat{g}_{aK}(q,p),\ \hat{g}_{bK}(q,p);\ q\in\mathscr{Q}\}$, it computes $\{V[\hat{g}_{aK}(q,p)],\ V[\hat{g}_{bK}(q,p)];\ q\in\mathscr{Q}\}$ as unbiased estimators of $\{\text{var}\ \hat{g}_{aK}(q,p),\ \text{var}\ \hat{g}_{bK}(q,p);\ q\in\mathscr{Q}\}$. Observe that preprocessing in step 1 takes $O(|\mathscr{N}^*||\mathscr{Q}|)$ time, postprocessing in step 3 takes $O(|\mathscr{Q}|)$ time and, on each replication, sampling in step 2a takes $O(|\mathscr{E}|)$ time, summation in step 2c takes $O(\sum_{i\in\mathscr{N}^*} k_i) \le O(|\mathscr{E}|)$ time, determination of $\phi(X)$ in step 2b takes $O(\max|\mathscr{V}|,|\mathscr{E}|))$ and step 2d takes $O(|\mathscr{N}^*||\mathscr{Q}|)$ time. One can also show that the mean total time for K replications using Algorithm A has the form

$$T(\{\hat{g}_{aK}(q,p),\hat{g}_{bK}(q,p)\}) = \omega_0+\omega_1|\mathscr{N}^*||\mathscr{Q}|+\omega_2|\mathscr{Q}|+K[\omega_3+\beta_2|\mathscr{E}|+\alpha_3(\mathscr{S}_{01},k,p)/\Delta(p)$$
$$+\omega_4|\mathscr{N}^*||\mathscr{Q}|+\omega_5\sum_{i\in\mathscr{N}^*} k_i]$$

time where $\omega_0,\ldots,\omega_5$ denote machine dependent constants and β_2 is identical with β_2 in $T(\hat{g}_K(q))$. To reduce numerical error, all computation in step 3 should be performed in double precision arithmetic.

Algorithm A

Purpose: To estimate the reliability function $\{g(q),\ q\in\mathcal{Q}\}$.

Input: Network $G = (\mathcal{V},\mathcal{E})$; number of type of components r; k_i = number of components of type i for $i=1,\ldots,r$; sampling distribution $\{Q(x,k,p),\ x\in\mathcal{X}\}$; set of $\mathcal{N}^*$ = set of component types that vary in $\mathcal{Q}$; lower and upper bounds $\{g_L(q),\ g_U(q);\ \ q\in\mathcal{Q}\cup\{p\}\}$; and number of independent replications K.

Output: $\{\hat{g}_{aK}(q,p),\ \hat{g}_{bK}(q,p),\ V[\hat{g}_{aK}(q,p)],\ V[\hat{g}_{bK}(q,p)];\ \ q\in\mathcal{Q}\}$ as unbiased estimates of $\{g(q),\ g(q),\ \mathrm{var}\ \hat{g}_{aK}(q,p),\ \mathrm{var}\ \hat{g}_{bK}(q,p);\ \ q\in\mathcal{Q}\}$.

Method:

1. **Initialization**

   a. $\Delta(p) \leftarrow g_U(p) - g_L(p)$.

   b. For each $q\in\mathcal{Q}$:

      $S(q) = S_1(q) = V(q) = V_1(q) \leftarrow 0$.

      For each $i\in\mathcal{N}^*$:

      $\alpha_i(q) \leftarrow \ln[q_i(1-p_i)/p_i(1-q_i)]$ and $\beta_i(q) \leftarrow \ln[(1-q_i)/(1-p_i)]$.

2. **On each of K independent trials:**

   a. Sample X_{ij} $j=1,\ldots,k_i$ $i=1,\ldots,r$ from $\{Q(x,k,p)\}$ as in Fishman (1986).

   b. Determine $\phi(X)$.

   c. For each $i\in\mathcal{N}^*$: $X_i \leftarrow \sum_{j=1}^{k_i} X_{ij}$.

   d. For $q\in\mathcal{Q}$:

      $T(q) \leftarrow 0$.

      For each $i\in\mathcal{N}^*$: $T(q) \leftarrow T(q) + k_i\beta_i(q) + X_i\alpha_i(q)$.

      $T(q) \leftarrow \exp[T(q)]$; $S(q) \leftarrow S(q) + T(q)$; $V(q) \leftarrow V(q) + \phi(X)\ T(q)$;

      $S_1(q) \leftarrow S_1(q) + T(q)\ T(q)$; $V_1(q) \leftarrow V_1(q) + \phi(X)\ T(q)\ T(q)$.

3. **Computation of summary statistics**

   For each $q\in\mathcal{Q}$:

   $\hat{g}_{aK}(q,p) \leftarrow g_L(q) + \Delta(p)\ V(q)/K$.

   $\hat{g}_{bK}(q,p) \leftarrow g_U(q) - \Delta(p)\ [S(q)-V(q)]/K$.

   $V[\hat{g}_{aK}(q,p)] \leftarrow \Delta^2(p)\ \{[V_1(q)-K[V(q)/K]^2\}/K(K-1)$.

   $V[\hat{g}_{bK}(q,p)] \leftarrow \Delta^2(p)\{[S_1(q)-V_1(q)]-K[(S(q)-V(q))/K]^2\}/K(K-1)$.

End of procedure

Let us now compare this approach to estimating $\{g(q),\ q\in\mathscr{Q}\}$ with the alternative approach based on the $|\mathscr{Q}|$ point estimates $\{\bar{g}_{K(q,p)}(q),\ q\in\mathscr{Q}\}$ using (3), where one chooses the sample sizes $\{K(q,p),\ q\in\mathscr{Q}\}$ to achieve equal variances. That is,

$$\mathrm{var}\ \bar{g}_{K(q,p)}(q) = g(q)[1-g(q)]/K(q,p) \tag{21}$$

where

$$K(q,p) = K\ \lambda(q,p)$$

and

$$\lambda(q,p) = \frac{g(q)[1-g(q)]}{\min\limits_{j\in\{a,b\}} \mathrm{var}\ \psi_j(q,p)}.$$

Observe that

$$\lambda(p,p) = g(p)[1-g(p)]/[g_U(p)-g(p)][g(p)-g_L(p)]$$

and, except in special cases, for any edge type $i\in\mathscr{N}^*$

$$\lim_{k_i\to\infty} \lambda(q,p) = 0 \qquad \text{for } q\neq p.$$

This last limit follows from the growth of $c(q,p)$ with k_i.

Let

$$\lambda(p) = \sum_{q\in\mathscr{Q}} \lambda(q,p). \tag{22}$$

and observe that

$$\lim_{k_i\to\infty} \lambda(p) = \lambda(p,p). \tag{23}$$

Observe that the time ratio

$$\Lambda_1(\mathcal{Q},p) = \frac{T(\{\bar{g}_{\mathbb{K}(q,p)}(q)\})}{T(\{\hat{g}_{a\mathbb{K}}(q,p),\hat{g}_{b\mathbb{K}}(q,p)\})} \tag{24}$$

where

$$T(\{\bar{g}_{\mathbb{K}(q,p)}(q)\}) = \sum_{q\in\mathcal{Q}} T(\bar{g}_{\mathbb{K}(q,p)}(q))$$

measures the efficiency of the proposed method relative to using crude Monte Carlo sampling with (3) $|\mathcal{Q}|$ times to obtain estimates with equal variances var $\bar{g}_{\mathbb{K}(q,p)}(q) = \min_{j\in\{a,b\}}$ var $\hat{g}_{j\mathbb{K}}(q,p)$ for $q\in\mathcal{Q}$. As k_i increases, (24) assumes the form

$$\Lambda_1(\mathcal{Q},p) \approx \frac{\sum_{q\in\mathcal{Q}} [\alpha_2+\alpha_3(\mathcal{S},k,q)/k_i]\lambda(q,p)}{\beta_2+\alpha_3(\mathcal{S}_{01},k,p)/\Delta(p)k_i+\omega_5}. \tag{25}$$

Practice indicates that $\omega_5 << \beta_2$. Then provided $p\in\mathcal{Q}$, (25) generally satisfies $\Lambda_1(\mathcal{Q},p) \geq \Lambda_1(p)$ for large k_i ($i\in\mathcal{N}^*$), implying that Algorithm A is at least as efficient as (6) at a single point. As the example in Section 11 shows, the realized efficiency can be considerably greater.

5. The Optimal Estimator

The representations of $\hat{g}_{a\mathbb{K}}(q,p)$ and $\hat{g}_{b\mathbb{K}}(q,p)$ in (19) suggest that these quantities conceptually are alternative forms of a more comprehensive estimator. Theorem 2 confirms this observation.

Theorem 2. Let X denote a sample from $\{Q(x,k,p)\}$ and define

$$\psi(x,q,p,\Theta) = \Theta\psi_a(x,q,p) + (1-\Theta)\,\psi_b(x,q,p) \quad -\infty<\Theta<\infty. \tag{26}$$

Then for fixed $p \neq q$

i. $E\psi(X,q,p,\Theta) = g(q)$

ii. $\Theta^*(q,p) = [v_b(q,p)-v_{ab}(q,p)]/[v_a(q,p)+v_b(q,p)-2v_{ab}(q,p)]$

$$= 1/\{1+[v_a(q,p)-v_{ab}(q,p)]/[v_b(q,p)-v_{ab}(q,p)]\}$$

minimizes $\operatorname{var} \psi(X,q,p,\Theta)$

iii. $\operatorname{var} \psi(X,q,p,\Theta^*(q,p)) = [v_a(q,p)v_b(q,p)-v_{ab}^2(q,p)]/[c(q,p)\Delta(p)\Delta(q^*)-\Delta^2(q)]$.

The proof follows directly from the minimization of $\operatorname{var} \psi(X,q,p,\Theta)$ with respect to Θ. Therefore, among estimators of the form

$$\hat{g}_X(q,p,\Theta) = \Theta\hat{g}_{aX}(q,p) + (1-\Theta)\,\hat{g}_{bX}(q,p), \tag{27}$$

$\hat{g}_X(q,p,\Theta^*(q,p))$ has minimal variance.

Observe that

$$\Theta^*(q,p) = 1 \qquad \text{if } v_a(q,p) = v_{ab}(q,p) \neq v_b(q,p)$$

$$= 0 \qquad \text{if } v_b(q,p) = v_{ab}(q,p) \neq v_a(q,p),$$

revealing that $\hat{g}_{aX}(q,p)$ is optimal if

$$\operatorname{var} \hat{g}_{aX}(q,p) = \operatorname{var} \hat{g}_X(q)$$

and $\hat{g}_{bX}(q,p)$ is optimal if

$$\operatorname{var} \hat{g}_{bX}(q,p) = \operatorname{var} \hat{g}_X(q)$$

where $\hat{g}_K(q)$ and var $\hat{g}_K(q)$ are defined in (6) and (7) respectively.

Writing (26) in the alternative form

$$\psi(x,q,p,\Theta) = \psi_b(x,q,p) + \Theta[R(x,k,q,p)) - \Delta(q)] \tag{28}$$

reveals that $R(X^{(1)},k,q,p),\ldots,R(X^{(K)},k,q,p)$ act as *control variates* with known mean $\Delta(q)$ and, by appropriate substitution, that

$$\text{var } \psi(X,q,p,\Theta^*(q,p)) = \text{var } \psi_b(X,q,p)\{1-\text{corr}^2[\psi_b(X,q,p),R(X,k,q,p)]\},$$

where corr(A,B) denotes the coefficient of correlation between A and B. Note that this variance diminishes as the correlation between $\psi_b(X,q,p)$ and $R(X,k,q,p)$ increases in magnitude.

In practice, $\{\Theta^*(q,p), q \in \mathcal{Q}\}$ is unknown but can be estimated unbiasedly for $q \neq p$ by

$$\hat{\Theta}^*(q,p) = K\{V[\hat{g}_{bK}(q,p)] - Z_K(q,p)\}/[c(q,p)\Delta(p)\Delta(q^*) - \Delta^2(q)] \tag{29}$$

where

$$Z_K(q,p) = [g_U(q) - \hat{g}_{bK}(q,p)][\hat{g}_{aK}(q,p) - g_L(q)]/(K-1)$$

and $V[\hat{g}_{bK}(q,p)]$ is computed in step 3 of Algorithm A. Although one may incline to use the estimator

$$\hat{g}_K(,q,p,\hat{\Theta}^*(q,p)) = \hat{\Theta}^*(q,p)\,\hat{g}_{aK}(q,p) + [1-\hat{\Theta}^*(q,p)]\hat{g}_{bK}(q,p) \tag{30}$$

for $g(q)$, one quickly sees that

$$E\hat{g}_K(q,p,\hat{\Theta}^*(q,p)) = g(q) + \mathrm{cov}\,[\hat{\Theta}^*(q,p), \hat{g}_{aK}(q,p)]$$

$$- \mathrm{cov}\,[\hat{\Theta}^*(q,p), \hat{g}_{bK}(q,p)].$$

Therefore, $\hat{g}_K(q,p,\hat{\Theta}^*(q,p))$ generally is biased. While this bias diminishes as K increases, the rate of dissipation depends on the particular system under study. As a result, no general statement is possible regarding how large K must be in order to treat bias as incidental. Because of this limitation, the remainder of this paper focuses on choosing between $\hat{g}_{aK}(q,p)$ and $\hat{g}_{bK}(q,p)$. We return to the issue of choosing a point estimator for $g(q)$ in the example of Section 11.

6. An Alternative Representation

This section describes an alternative representation for $g(q)$ that offers considerable conceptual value for function estimation. Let

$$\mathscr{S}(z_1,\ldots,z_r) = \{x \in \mathscr{S} \mid \sum_{j=1}^{k_i} x_{ij} = z_i,\ i=1,\ldots,r\} \tag{31}$$

and

$$M(z_1,\ldots,z_r) = \sum_{x \in \mathscr{S}(z_1,\ldots,z_r)} \phi(x) \tag{32}$$

= the number of possible ways that the system can function ($\phi(x)=1$) when $z_1,\ldots,z_r$ components of types $1,\ldots,r$, respectively, function and $k_1-z_1,\ldots,k_r-z_r$ components of types $1,\ldots,r$, respectively, fail.

Then one can write

$$g(q) = \sum_{z_1=0}^{k_1} f_{z_1}(k_1,q_1)\ldots \sum_{z_r=0}^{k_r} f_{z_r}(k_r,q_r)u(z_1,\ldots,z_r) \tag{33}$$

where

$$f_z(k,q) = \binom{k}{z} q^z(1-q)^{k-z} \quad z=0,1,\ldots,k; \quad 0\le q\le 1, \quad k=1,2,\ldots$$

and

$$u(z_1,\ldots,z_r) = M(z_1,\ldots,z_r)/\prod_{i=1}^{r}\binom{k_i}{z_i}.$$

Observe that since we are working with a coherent system

$$u(0,\ldots,0) = 0$$

$$u(z_1,\ldots,z_j,\ldots,z_r) \le u(z_1,\ldots,z_j+1,\ldots,z_r) \qquad j=1,\ldots,r$$

$$u(k_1,\ldots,k_r) = 1,$$

so that $\{u(z_1,\ldots,z_r);\ 0\le z_i\le k_i,\ \ i=1,\ldots,r\}$ is a multivariate distribution function.

If one were to perform crude Monte Carlo sampling, then the estimation of $g(q)$ would be equivalent to estimating the coefficients $\{u(z_1,\ldots,z_r)\}$. In the case of $r=1$, $M(z_1)$ denotes the number of connecting cutsets of G when z_1 arcs function and k_1-z_1 arcs fail. Fishman (1987) describes a method of estimating $M(z_1)$ in this special case. As we now show, the present proposal corresponds to the implicit estimation of analogous quantities.

Let

$$u_j(z_1,\ldots,z_r) = \left[\prod_{i=1}^{r}\binom{k_i}{z_i}\right]^{-1} \sum_{x\in\mathscr{S}(z_1,\ldots,z_r)} \phi_j(x) \quad j\in\{L,U\} \tag{34}$$

$K_a(z_1,\ldots,z_r)$ = number of replications on which the system functions when $z_1,\ldots,z_r$ components of types $1,\ldots,r$, respectively, function and $k_1-z_1,\ldots,k_r-z_r$ components of types $1,\ldots,r$, respectively, fail (35)

and

$$K_b(z_1,\ldots,z_r) = \text{number of replications on which the system fails when } z_1,\ldots,z_r \text{ components of types } 1,\ldots,r, \text{ respectively, function and } k_1-z_1,\ldots,k_r-z_r \text{ components of types } 1,\ldots,r, \text{ respectively, fail.} \tag{36}$$

Then $\hat{g}_{aK}(q,p)$ and $\hat{g}_{bK}(q,p)$ in (6) have the equivalent forms

$$\hat{g}_{aK}(q,p) = g_L(q) + \Delta(p) \sum_{z_1=0}^{k_1} \ldots \sum_{z_r=0}^{k_r} R^*(z,k,q,p)\, K_a(z_1,\ldots,z_r)/K \tag{37}$$

and

$$\hat{g}_{bK}(q,p) = g_U(q) - \Delta(p) \sum_{z_1=0}^{k_1} \ldots \sum_{z_r=0}^{k_r} R^*(z,k,q,p)\, K_b(z_1,\ldots,z_r)/K \tag{38}$$

where

$$R^*(z,k,q,p) = \prod_{i=1}^{r} (q_i/p_i)^{z_i} [(1-q_i)/(1-p_i)]^{k_i-z_i}$$

$$EK_a(z_1,\ldots,z_r)/K = u_a(z_1,\ldots,z_r) = u(z_1,\ldots,z_r) - u_L(z_1,\ldots,z_r)$$

and

$$EK_b(z_1,\ldots,z_r)/K = u_b(z_1,\ldots,z_r) = u_U(z_1,\ldots,z_r) - u(z_1,\ldots,z_r).$$

Therefore, using $\hat{g}_{aK}(q,p)$ is equivalent to estimating the coefficients $\{u_a(z_1,\ldots,z_r)\}$ implicitly and using $\hat{g}_{bK}(q,p)$ is equivalent to estimating the coefficients $\{u_b(z_1,\ldots,z_r)\}$ implicitly.

Expressions (19), on the one hand, and (37) and (38), on the other, have beneficial and limiting features. If one uses Algorithm A, then the sample reliability functions $\{\hat{g}_{aK}(q,p),\ \hat{g}_{bK}(q,p);\ q \in \mathcal{Q}\}$ available for study have ordinates only at the points in $\mathcal{Q}$ specified in the sampling experiment. However, if one alternatively records $\{K_a(z_1,\ldots,z_r),$

$K_b(z_1,\ldots,z_r)\}$, then (37) and (38) enable one to construct the sample functions for any q in $[0,1]^r$ which may be of interest at any time after the sampling experiment terminates. Also, as Section 9 shows shortly, for a given network G, the widths of confidence intervals for $\{g(q),\ q\in\mathscr{Q}\}$ based on either $\{\hat{g}_{aK}(q),\ q\in\mathscr{Q}\}$ or $\{\hat{g}_{bK}(q),\ q\in\mathscr{Q}\}$ as in (19) increase with $|\mathscr{Q}|$ whereas confidence intervals based on $\{K_a(z_1,\ldots,z_r),\ K_b(z_1,\ldots,z_r)\}$ have widths independent of $|\mathscr{Q}|$.

This alternative approach requires a space of $\prod_{i\in\mathscr{N}^*} k_i$ counters to accumulate $\{K_a(\cdot)\}$ and a like space to accumulate $\{K_b(\cdot)\}$. Moreover, a modified Algorithm A based on this approach would replace the time components $O(K|\mathscr{N}^*||\mathscr{Q}|)$ and $O(K\sum_{i\in\mathscr{N}^*} k_i)$ in $T(\{\hat{g}_{aK}(q,p),\ \hat{g}_{bK}(q,p)\})$ by $O(|\mathscr{Q}|\prod_{i\in\mathscr{N}^*} k_i) \le O(|\mathscr{E}|/|\mathscr{N}^*|)^{|\mathscr{N}^*|})$ thus eliminating their dependence on the sample size K. If $\prod_{i\in\mathscr{N}^*} k_i$ is large, this may limit the extent to which one can store the sample sums $\{K_a(\cdot),\ K_b(\cdot)\}$. However, when they are storable, their availability offers considerable post-experimental discretion for computing quantities of interest.

7. Choosing the Sampling Vector p

The forms of the variances (16) and (17) clearly indicate that the choice of p affects the statistical accuracies of $\hat{g}_{aK}(q,p)$ and $\hat{g}_{bK}(q,p)$. While no unequivocal rule exists for choosing p, minimizing $\max_{q\in\mathscr{Q}}\,[\min_{j\in\{a,b\}} \operatorname{var} \hat{g}_{jK}(q,p)]$ is one reasonable objective. Unfortunately, the unknown variances render this minimization impossible. An immediate alternative uses the upper bound

$$c(q,p)\,\Delta(p)\,\Delta(q^*) \ge \max_{j\in\{a,b\}} \operatorname{var} \hat{g}_{jK}(q,p)$$

and finds, by grid search, the p that minimizes $\Delta(p)\,\max_{q\in\mathscr{Q}} c(q,p)\,\Delta(q^*)$. However, a considerably better bound $h(q,p)$ exists when either $p \leq q$ or $p \geq q$ so that sampling with the p that minimizes $\max_{q \in \mathcal{Q}} h(q,p)$ can produce considerably better worst case results.

This section derives $h(q,p)$ explicitly for var $\psi_b(X,q,p)$. It makes use of the observation that for a coherent system $p \leq q$ for all $q \in \mathcal{Q}$ implies $g(p) \leq g(q)$ and $g_j(p) \leq g_j(q)$ for $j \in \{L,U\}$ and $p \geq q$ for all $q \in \mathcal{Q}$ implies $g(p) \geq g(q)$ and $g_j(p) \geq g_j(q)$ for $j\{L,U\}$. A completely analogous approach holds for var $\hat{g}_{aK}(q,p)$. Also, the Appendix extends the analysis (Theorem 4 and 5) to cases in which the coefficients of variation

$$\gamma_{1j}(q,p) = [\text{var } \psi_j(X,q,p)]^{\frac{1}{2}}/g(q)$$

and

$$\gamma_{2j}(q,p) = [\text{var } \psi_j(X,q,p)]^{\frac{1}{2}}/[1-g(q)] \qquad j \in \{a,b\}$$

are the criteria of accuracy.

Lemma 2. Define

$$h_1(z) = h_1(z,q,p) = c(q,p)\Delta(p)\Delta(q^*) - [g_U(q)-z]^2$$

and (39)

$$h_2(z) = h_2(z,q,p) = c(q,p)\Delta(p)[g_U(q^*)-z] - [g_U(q)-z]^2$$

$$-\infty < z < \infty.$$

Then for either $p \leq q$ or $p \geq q$

$$\text{var } \psi_b(X,q,p) \le h(q,p) = \max\left[\max_{g_L(q)\le z\le g_L(q^*)} h_1(z), \max_{g_L(q^*)\le z\le g_U(q)} h_2(z)\right] \quad (40a)$$

if $g_L(q) \le g_L(q^*) \le g_U(q)$

$$= \max_{g_L(q)\le z\le g_U(q)} h_1(z) \qquad \text{otherwise.} \quad (40b)$$

The Appendix contains the proof.

Theorem 3. For h_1 and h_2 as defined in (39), $p \ge q$ or $p \ge q$, and

$$z^* = g_U(q) - c(q,p)\Delta(p)/2,$$

$$\text{var } \psi_b(X,q,p) \le h(q,p) = \max[h_1(g_L(q^*)), h_2(\max(z^*, g_L(q^*)))]$$

if $g_L(q) \le g_L(q^*) \le g_U(q)$ (41)

$$= h_1(g_U(q)) \qquad \text{otherwise.}$$

See the Appendix for the proof. To derive the analogous upper bound for var $\psi_a(X,q,p)$, one replaces z by $1-z$, $g_L(q)$ by $1-g_U(q)$, $g_U(q)$ by $1-g_L(q)$, $g_L(q^*)$ by $1-g_U(q^*)$ and $g_U(q^*)$ by $1-g_L(q^*)$ everywhere in (38), (39) and (40).

Recall from Section 4 that choosing p from $\mathcal{Q}$ is beneficial from the viewpoint of efficiency as any of the k_i grows. Then one can compute $\max_{q\in\mathcal{Q}} h(q,p)$ by enumeration for every p in $\mathcal{Q}$ and select the p that minimizes the maxima. In total $|\mathcal{Q}|^2$ points are evaluated. As the example in Section 11 shows, this method of choosing p can lead to significant improvements in statistical efficiency.

8. Individual Confidence Intervals

Although the distribution of $[\hat{g}_{jk}(q,p)-g(q)]/[\text{var } \hat{g}_{jk}(q,p)]^{\frac{1}{2}}$ converges to the standard normal distribution as $K \to \infty$, this result, at best, can only lead to a rough confidence interval for $g(q)$. To avoid the errors of approximation inherent in the normal approach to confidence intervals, we use an alternative technique.

Theorem 6. Let

$$\mathscr{S}_a = \{x \in \mathscr{S}: \ \phi_L(x)=0, \ \phi(x)=1\},$$

$$R_a(k,q,p) = \max_{x \in \mathscr{S}_a} R(x,k,q,p),$$

$$Y_K(q,p) = [\hat{g}_{aK}(q,p) - g_L(q)]/\Delta(p)R_a(k,q,p),$$

$$m(z,\omega) = z \log(\omega/z) + (1-z) \log [(1-\omega)/(1-z)] \qquad 0<z,\omega<1,$$

let $\omega(z,\delta/2,K)$ denote the solution to $m(z,\omega) = \frac{1}{K} \ln(\delta/2)$ for fixed $z \in (0,1]$ and $\delta \in (0,1)$, and let

$$\begin{aligned} \omega^*(z,\delta/2,K) &= \omega(z,\delta/2,K) && \text{if } 0<z\leq 1 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{42}$$

Then, the interval

$$(g_L(q)+\Delta(p)R_a(k,q,p)\omega^*(Y_K(q,p),\delta/2,K),\ g_L(q)+\Delta(p)R_a(k,q,p)\omega^*(1-Y_K(q,p),\delta/2,K)) \tag{43}$$

covers $g(q)$ with probability $> 1 - \delta$.

Theorem 7. Let

$$\mathscr{S}_b = \{x \in \mathscr{S}: \phi(x)=0, \ \phi_U(x)=1\},$$

$$R_b(k,q,p) = \max_{x \in \mathscr{S}_b} R(x,k,q,p),$$

and

$$Z_K(q,p) = [g_U(q)-\hat{g}_{bK}(q,p)]/\Delta(p)R_b(k,q,p).$$

Then the interval

$$(g_U(q)-\Delta(p)R_b(k,q,p)\omega^*(1-Z_K(q,p),\delta/2,K),g_U(q)-\Delta(p)R_b(k,q,p)\omega^*(Z_K(q,p),\delta/2,K)) \qquad (44)$$

covers $g(q)$ with probability $> 1-\delta$.

Proof of Theorems 6 and 7. Inspection of (13) and (14) makes clear that

$$pr[g_L(q) \leq \psi_a(X,q,p) \leq g_L(q)+ \Delta(p)R_a(k,q,p)] = 1$$

and (45)

$$pr[g_U(q)-\Delta(p)R_b(k,q,p) \leq \psi_b(X,q,p) \leq g_U(q)] = 1.$$

The resulting confidence intervals follow from Theorem 1 in Fishman (1988)∙

Although these intervals generally are wider than the corresponding normal confidence intervals would be for given K and δ, they are free of the error of approximation inherent in normal intervals.

To use these intervals in practice, one needs to know $\{R_a(k,q,p),R_b(k,q,p);\ q\in\mathcal{Q}\}$. Theorems 8 and 9 formulate mathematical programs aimed at computing these quantities. Since experience with several networks for the s-t connectedness problem with $\mathcal{Q} = \{q_1\leq \dots \leq q_r\}$ has shown that $p = q_1$ usually minimizes the worst case bound (42), we focus on the case $p\leq q$,

Theorem 8. Let $\mathcal{D}$ denote the set of all minimal s-t cutsets of smallest cardinality, let

$$\mathscr{E}(\mathscr{K}^*) = \bigcup_{i\in\mathscr{K}^*} \mathscr{E}_i$$

$$\mathscr{M}_a = \{\mathscr{P}_i,\ i=1,\ldots,I:\ \mathscr{P}_i \subseteq \mathscr{E}(\mathscr{K}^*)\}$$

$$\mathscr{N}_a = \{\mathscr{C} \in \mathscr{D}:\ \mathscr{C} \subseteq \mathscr{E}(\mathscr{K}^*) \text{ and } |\mathscr{C}| = |\mathscr{M}_a|\}$$

$$a_i = \log\,[q_i(1-p_i)/p_i(1-q_i)] \qquad\qquad i\in\mathscr{K}^*$$

and assume $q_i \geq p_i$ for $\forall\, i\in\mathscr{K}^*$. Then

$$R_a(\mathbf{k},\mathbf{q},\mathbf{p}) = \prod_{i\in\mathscr{K}^*} (q_i/p_i)^{k_i - \sum_{j\in\mathscr{E}_i} z_i^*}\,[(1-q_i)/(1-p_i)]^{\sum_{j\in\mathscr{E}_i} z_j^*} \tag{46}$$

where $\mathbf{z}^*$ solves the integer program

$$\min_{\mathbf{z}} \sum_{i\in\mathscr{K}^*} a_i \sum_{j\in\mathscr{E}_i} z_j \tag{47a}$$

subject to

$$\sum_{j\in\mathscr{P}} z_j \geq 1 \qquad\qquad \forall\, \mathscr{P}\in\mathscr{M}_a \tag{47b}$$

$$\sum_{j\in\mathscr{C}} z_j \leq |\mathscr{C}| - 1 \qquad\qquad \forall\, \mathscr{C}\in\mathscr{N}_a \tag{47c}$$

and

$$z_j \in \{0,1\} \qquad\qquad \forall\, j\in \mathscr{E}(\mathscr{K}^*). \tag{47d}$$

The Appendix contains the proof.

Theorem 9. Let $\mathscr{T}$ denote the set of all minimal s–t paths of smallest cardinality, let

$$\mathscr{M}_b = \{\mathscr{C}_i,\ i=1,\ldots,J:\ \mathscr{C}_i \subseteq \mathscr{E}(\mathscr{K}^*)\}$$

$$\mathscr{N}_b = \{\mathscr{P}\in\mathscr{T}:\ \mathscr{P} \subseteq \mathscr{E}(\mathscr{K}^*),\ |\mathscr{P}| = |\mathscr{M}_b|\}$$

and assume $q_i \geq p_i$ for $\forall\, i \in \mathscr{N}^*$. Then

$$R_b(\mathbf{k},\mathbf{q},\mathbf{p}) = \prod_{i \in \mathscr{N}^*} (q_i/p_i)^{k_i - \sum_{j \in \mathscr{E}_i} z_j^*} [(1-q_i)/(1-p_i)]^{\sum_{j \in \mathscr{E}_i} z_j^*} \tag{48}$$

where $\mathbf{z}^*$ solves the integer program

$$\min_{\mathbf{z}} \sum_{i \in \mathscr{N}^*} a_i \sum_{j \in \mathscr{E}_i} z_i \tag{49a}$$

subject to

$$\sum_{j \in \mathscr{C}} z_j \leq |\mathscr{C}| - 1 \qquad \forall\, \mathscr{C} \in \mathscr{M}_b \tag{49b}$$

$$\sum_{j \in \mathscr{P}} z_j \geq 1 \qquad \forall\, \mathscr{P} \in \mathscr{N}_b \tag{49c}$$

and

$$z_j \in \{0,1\} \qquad \forall\, j \in \mathscr{E}(\mathscr{N}^*). \tag{49d}$$

The proof follows analogously to that for Theorem 8.

Recall that since $\mathscr{P}_1,\ldots,\mathscr{P}_I$ are edge disjoint, $I \leq I^* \equiv$ the size of the minimal s–t cutset of smallest cardinality. Therefore, if $|\mathscr{M}_a| < I^*$, then $|\mathscr{N}_a| = 0$ so that the constraints in (47c) vanish and

$$R_a(\mathbf{k},\mathbf{q},\mathbf{p}) = [\prod_{i \in \mathscr{N}^*} (q_i/p_i)] \exp(-\sum_{\mathscr{P} \in \mathscr{M}_a} \min_{i \in \mathscr{P}} a_i). \tag{50}$$

The case of $|\mathscr{M}_a| = I^*$ requires more detail. If the minimal s–t cutsets in $\mathscr{N}_a$ are edge–disjoint, then (47) has the form of a transportation problem with $|\mathscr{N}_a| \leq J^* \equiv$ the size of the minimal s–t path of smallest cardinality and can be solved using a special purpose algorithm as in Dantzig (1963, p. 308). If the cutsets are not edge–disjoint, $\mathscr{N}_a$ potentially can have an exponential number of members, limiting one's capacity to enumerate them all. This possibility suggests an iterative approach.

Suppose one begins by relaxing (47c). This gives the candidate solution (50). If the set of arcs chosen there do not form a minimal s–t cutset in $\mathscr{D}$, then the problem is solved. If they do form a cutset $\mathscr{C}^*$, then one activates the corresponding constraint. Let i^* denote the edge in $\mathscr{C}^*$ with the largest a_i. Then $[\prod_{i\in\mathscr{K}^*}(q_i/p_i)]\exp(-\sum_{\mathscr{P}\in\mathscr{M}_a}\sum_{i\in\mathscr{P}\setminus\{i^*\}}\min a_i)$ solves the problem provided that the selected edges do not form a minimal s–t cutset in $\mathscr{D}$. If they do form a cutset, then continued iteration becomes more complicated and one may elect to drop one of the edge–disjoint paths $\mathscr{P}_0$ in $\mathscr{M}_a$ from the lower bound $g_L(q)$ thereby reducing the size of $\mathscr{M}_a$ and making

$$R_a(k,q,p) = [\prod_{i\in\mathscr{K}^*}(q_i/p_i)]\exp(-\sum_{\mathscr{P}\in\mathscr{M}_a\setminus\mathscr{P}_0}\sum_{i\in\mathscr{P}}\min a_i)$$

the solution.

The solution to (49) proceeds in an analogous manner. If $\mathscr{M}_b < J^*$, then

$$R_b(k,q,p) = [\prod_{i\in\mathscr{K}^*}(q_i/p_i)]\exp(-\sum_{\mathscr{C}\in\mathscr{M}_b}\sum_{i\in\mathscr{C}}\min a_i). \tag{51}$$

If $|\mathscr{M}_b| = J^*$, then one can either drop a cutset $\mathscr{C}_0$ in $\mathscr{M}_b$ from the upper bound $g_U(q)$ and use

$$R_b(k,q,p) = [\prod_{i\in\mathscr{P}^*}(q_i/p_i)]\exp(-\sum_{\mathscr{C}\in\mathscr{M}_b\setminus\mathscr{C}_0}\sum_{i\in\mathscr{C}}\min a_i)$$

as the solution, or again proceed iteratively. With (49c) relaxed, (51) is the candidate solution. If the set of selected edges do not form a path in $\mathscr{N}_b$, then (51) is the minimum. If they do form a path $\mathscr{P}^*$ with edge i^* giving the largest a_i, then the solution $[\prod_{i\in\mathscr{P}^*}(q_i/p_i)]\exp(-\sum_{\mathscr{C}\in\mathscr{M}_b}\sum_{i\in\mathscr{C}\setminus\{i^*\}}\min a_i)$ needs to be checked, etc. One anticipates that choosing edge-disjoint paths $\mathscr{P}_1,\ldots,\mathscr{P}_I$ and cutsets $\mathscr{C}_1,\ldots,\mathscr{C}_J$ such that $\mathscr{N}_a$ and $\mathscr{N}_b$ are empty generally will have small effect on the bounds $g_L(q)$ and $g_U(q)$ for large networks.

9. Simultaneous Confidence Intervals

Although each confidence interval in Section 8 holds with probability $> 1-\delta$, the joint confidence intervals for $\{g(q), q\in\mathscr{Q}\}$ hold simultaneously only with probability $> 1-|\mathscr{Q}|\delta$. This result follows from a Bonferroni inequality. See Miller (1981, p. 8). To restore the joint confidence level to $1-\delta$, one replaces $\log(\delta/2)$ by $\log(\delta/2|\mathscr{Q}|)$ in (43) and (44) and determines the corresponding solutions. The effect of this substitution is to increase the constant of proportionality in the approximate interval widths from $[2\log(2/\delta)]^{\frac{1}{2}}$ to $[2\log(2|\mathscr{Q}|/\delta)]^{\frac{1}{2}}$ (see Fishman 1986). For $\delta=.01$ and $|\mathscr{Q}|=20$ one has $[\log(2|\mathscr{Q}|/\delta)/\log(2/\delta)]^{\frac{1}{2}} = 1.25$. For $\delta=.01$ and $|\mathscr{Q}|=100$, it is 1.37 and for $\delta=.01$ and $|\mathscr{Q}|=1000$ it is 1.52. Moreover, if $\mathscr{Q}$ denotes a continuous region in the $|\mathscr{E}|$-dimensional hypercube $(0,1)^{|\mathscr{E}|}$, then the resulting confidence intervals have infinite widths and are therefore useless.

An alternative approach derives simultaneous confidence intervals for $\{g(q), q\in\mathscr{Q}\}$ using the representation of $g(q)$ in (44). In particular, it implicitly finds simultaneous confidence intervals for the coefficients $\{u_j(z_1,\ldots,z_r)\}$ of which there are $N \leq \prod_{i\in \mathscr{N}^*} k_i$ in (34). For convenience of notation we take $|\mathscr{N}^*| = r$ but note the relatively straightforward adjustment for $|\mathscr{N}^*| < r$. Let $z = (z_1,\ldots,z_r)$ and recall the definitions of $K_a(z)$ and $K_b(z)$ in (35) and (36). Then $\{(\omega^*(K_j(z)/K,\delta/2N,K),\ \omega^*(1-K_j(z)/K,\delta/2N,K);\ \forall z\}$, where $\omega^*(\cdot,\cdot,\cdot)$ is defined in (42), provide confidence intervals for $\{u_j(z)\}$ that hold simultaneously with probability $> 1-\delta$.

Observe that all coefficients $u_j(z)$ are nonnegative and that $\{\omega^*(K_j(z)/K,\delta/2N,K),$ $\omega^*(1-K_j(z)/K,\delta/2N,K\}$ are independent of q. Therefore, for all $q\in \mathscr{Q}$

$$\begin{aligned}&(g_L(q)-\Delta(p)\sum_{z_1=0}^{k_1}\cdots\sum_{z_r=0}^{k_r}R^*(z,k,q,p)\omega^*(K_a(z)/K,\delta/2N,K),\\ &g_L(q)+\Delta(p)\sum_{z_1=0}^{k_1}\cdots\sum_{z_r=0}^{k_r}R^*(z,k,q,p)\omega^*(1-K_a(z)/K,\delta/2N,K))\end{aligned} \tag{52}$$

simultaneously covers $\{g(q),\ q\in \mathscr{Q}\}$ with probability $> 1-\delta$ and likewise

$$\begin{aligned}&(g_U(q)+\Delta(p)\sum_{z_1=0}^{k_1}\cdots\sum_{z_r=0}^{k_r}R^*(z,k,q,p)\omega^*(1-K_b(z)/K,\delta/2N,K),\\ &g_U(q)-\Delta(p)\sum_{z_1=0}^{k_1}\cdots\sum_{z_r=0}^{k_r}R^*(z,k,q,p)\omega^*(K_b(z)/K,\delta/2N,K))\end{aligned} \tag{53}$$

simultaneously covers $\{g(q),\ q\in \mathscr{Q}\}$ with probability $> 1-\delta$.

The most desirable feature of this alternative approach is that the resulting intervals are unaffected in width or confidence level by the size of $\mathscr{Q}$. However, since the number of quantities $K_j(z)$ to be collected is $O(\prod_{i\in \mathscr{N}^*} k_i)$, this alternative approach becomes less feasible to implement as the k_i and r increase.

10. Steps for Implementation

To implement the proposed sampling plan to estimate reliability for s–t connectedness, one proceeds as follows:

1. Determine a set of edge–disjoint minimal s–t paths $\mathcal{P}_1,\ldots,\mathcal{P}_I$.
2. Determine a set of edge–disjoint minimal s–t cutsets $\mathcal{C}_1,\ldots,\mathcal{C}_J$.
3. Compute $\{g_L(q), g_U(q);\ q\in\mathcal{Q}\}$.
4. Determine a sampling vector p from $\mathcal{Q}$ as in Section 7.
5. Using Algorithm A, perform K independent replications.
6. For each $q\in\mathcal{Q}$: compute $R_b(k,q,p)$ if $V[\hat{g}_{aK}(q,p)] > V[\hat{g}_{bK}(q,p)]$; otherwise compute $R_a(k,q,p)$ (Section 8).
7. Using the bounds $\{R_a(k,q,p);\ q\in\mathcal{Q}\}$ or $\{R_b(k,q,p);\ q\in\mathcal{Q}\}$ in step 6, compute individual or simultaneous confidence intervals for $\{g(q),\ q\in\mathcal{Q}\}$ (Sections 8 and 9).

Although these steps require more work than crude Monte Carlo function estimation does, one can develop computer programs with sufficient generality to compute all quantities in steps 1 through 7 for many different network designs. Reusing the programs enables one to distribute the fixed cost of their development over all such network, making the cost per network incidental.

11. Example

An analysis of the network in Fig. 1 illustrates the proposed method. The network has 30 edges and 20 nodes. Also, the example assumes r=1 so that all edges have identical reliabilities, allowing us to write q=q. Note that any other specification with r>1 can also

Insert Fig. 1 about here.

be accommodated easily. The objective is to estimate $\{g(q),\ q=.80+.01(i-1)\quad i=1,\ldots,20\}$

where $g(q)$ = probability that nodes $s=1$ and $t=20$ are connected when edge reliabilities are q. For sampling, we use $p=p$, again merely as a convenience. The selected edge–disjoint paths and cutsets are

$$\mathcal{P}_1 = \{3,9,18,27,28\} \qquad \mathcal{P}_2 = \{1,5,12,21,29\}$$

$$\mathcal{P}_3 = \{2,7,15,24,30\} \qquad \mathcal{C}_1 = \{1,2,3\}$$

$$\mathcal{C}_2 = \{28,29,30\} \qquad \mathcal{C}_3 = \{11,12,14,15,17,18\}$$

$$\mathcal{C}_4 = \{4,5,6,7,8,9\} \qquad \mathcal{C}_5 = \{19,21,22,24,25,27\}.$$

As a preliminary step, Table 1 shows the worst case upper bound on var $\psi_b(X,q,p)$ as given in (41). Observe that the choice $p=.80$ minimizes this worst case bound and it is

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Insert Tables 1,2, and 3 about here.

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this component reliability that we use for sampling. A parallel analysis for var $\psi_a(X,q,p)$ also chose $p=.80$.

Table 2 compares the estimates of var $\hat{g}_{aK}(q,p)$ and var $\hat{g}_{bK}(q,p)$ for a sample size $K = 1048576$ and shows the estimated control variate coefficient $\hat{\Theta}^*(q,p)$, as in (29). These results strongly favor relying on $\hat{g}_{bK}(q,p)$, if the choice is between this quantity and $\hat{g}_{aK}(q,p)$. Table 3 shows the resulting estimates in col. 1 along with variance estimates in col. 5 and individual 99% confidence intervals in cols. 6–8. In contrast to the exact results in col. 3 which took slightly more than one hour each to compute, all results in cols. 1,2,4 and 5 took 72.7 minutes in total, or 4.16 milliseconds per replication. Computation of the confidence intervals took incidental time. Whereas the calculated exact results in col. 3 were accurate to sixteen significant digits (reduced to four digits here for comparative purposes), the confidence intervals suggest an accuracy to two significant digits at the .99 level. If two significant digits is acceptable for purposes of analysis, then the Monte Carlo approach clearly prevails.

Table 4 shows the effect of sampling at an arbitrary point p=.90 rather than at p=.80. Although sampling at p=.90 does produce better results for $q \geq p$, the deficiency of sampling at p=.80 in this interval is considerably less than the corresponding deficiency for sampling with p=.90 for $.80 \leq q \leq .89$.

Insert Fig. 2 about here.

Figure 2 displays several variance ratios that reveal how $\{\hat{g}_{bK}(q,.80)\}$ performs compared to the crude estimator $\{\bar{g}_K(q)\}$ in (3), the estimator $\{\hat{g}_K(q)\}$ in (6), and the approximately optimal estimator $\{\hat{g}_K(q,p,\hat{\Theta}^*(q,.80))\}$ in (30). First, note that $\{\hat{g}_{bK}(q,p)\}$ performs almost as well as $\{\hat{g}_K(q,p,\hat{\Theta}^*(q,.80))\}$. Second, observe that uniformly superior ratio for $\{\hat{g}_{bK}(q,p)\}$ when compared to $\{\bar{g}_K(q)\}$. In particular, note that these ratios exceed 100 for $q \geq .95$.

We now turn to the efficiency measure (22). Since $V[\hat{g}_{aK}(q,.80)] > V[\hat{g}_{bK}(q,.80)]$ for all $q \in \mathcal{Q}$, Fig. 2 makes clear that $\Lambda_1(p) > 10^5$, indicating the clear superiority of $\{\hat{g}_{bK}(q,.80)\}$ over the crude estimator $\{\bar{g}_K(q)\}$ in (3).

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Table 1†

$\max_{q \in \mathcal{Q}} h(q,p)$ for $\mathcal{Q} = \{.8+.01(i-1)\ i=1,\dots,20\}$ and $p \in \{.5+.02(i-1)\ i=1,\dots,25\}$

p	q.	$\max_{q \in \mathcal{Q}} h(q,p)$	p	q.	$\max_{q \in \mathcal{Q}} h(q,p)$
.50	.89	.3801D+03	.76	.83	.7496D–01
.52	.89	.1561D+03	.78	.84	.4985D–01
.54	.89	.6667D+02	.80	.84	.3234D–01
.56	.89	.2953D+02	.82	.80	.8818D–01
.58	.88	.1363D+02	.84	.80	.1090D+00
.60	.88	.6519D+01	.86	.80	.1642D+00
.62	.88	.3222D+01	.88	.80	.3255D+00
.64	.88	.1545D+01	.90	.80	.6961D+00
.66	.87	.8754D+00	.92	.80	.5410D+01
.68	.87	.4824D+00	.92	.80	.9111D+02
.70	.86	.2790D+00	.96	.80	.1400D+05
.72	.84	.1709D+00	.98	.80	.7866D+09
.74	.83	.1116D+00			

† q. = q in $\mathcal{Q}$ at which h(q,p) achieves its maximum for specified p.

Table 2

Comparison of $V[\hat{g}_{aK}(q,p)]$ and $V[\hat{g}_{bK}(q,p)]$

($p=.80$, $K=1048576$)

q	$\frac{V[\hat{g}_{aK}(q,p)]}{V[\hat{g}_{bK}(q,p)]}$	$\hat{\Theta}^*(q,p)$	q	$\frac{V[\hat{g}_{aK}(q,p)]}{V[\hat{g}_{bK}(q,p)]}$	$\hat{\Theta}^*(q,p)$
.80	1.00	–	.90	214.0	.0001
.81	1.58	–.4941	.91	319.0	.0005
.82	3.04	–.1799	.92	475.5	.0006
.83	6.07	–.0852	.93	713.8	.0006
.84	11.61	–.0440	.94	1097.0	.0005
.85	20.97	–.0233	.95	1755.0	.0003
.86	35.92	–.0122	.96	2995.0	.0002
.87	58.78	–.0060	.97	5727.0	.0001
.88	92.67	–.0026	.98	13665.0	.0001
.89	142.10	–.0008	.99	57108.0	.0000

Table 3

Reliability Estimation

($p=.80$, $K=1048576$)

q	$1-g_U(q)$	$1-g_L(q)$	$1-g(q)$ †	$1-\hat{g}_{bK}(q)]$ ††	$V[\hat{g}_{bK}(q)]$ ††	Individual 99% Confidence Intervals for $1-g(q)$: Lower	Upper	Width
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
.80	.1612D-01	.3039D+00	.3624D-01	.3624D-01	.5135D-08	.3601D-01	.3648D-01	.4665D-03
.81	.1381D-01	.2763D+00	.2976D-01	.2976D-01	.3280D-08	.2954D-01	.2998D-01	.4455D-03
.82	.1173D-01	.2492D+00	.2421D-01	.2420D-01	.2107D-01	.2400D-01	.2441D-01	.4179D-03
.83	.9874D-02	.2227D+00	.1949D-01	.1948D-01	.1360D-08	.1929D-01	.1968D-01	.3850D-03
.84	.8225D-02	.1969D+00	.1552D-01	.1551D-01	.8818D-09	.1534D-01	.1569D-01	.3483D-03
.85	.6773D-02	.1722D+00	.1221D-01	.1220D-01	.5726D-09	.1205D-01	.1235D-01	.3089D-03
.86	.5503D-02	.1485D+00	.9473D-02	.9464D-02	.3711D-09	.9332D-02	.9600D-02	.2681D-03
.87	.4404D-02	.1262D+00	.7241D-02	.7232D-02	.2387D-09	.7120D-02	.7348D-02	.2274D-03
.88	.3462D-02	.1053D+00	.5440D-02	.5432D-02	.1514D-09	.5339D-02	.5527D-02	.1877D-03
.89	.2666D-02	.8611D-01	.4006D-02	.3998D-02	.9372D-10	.3924D-02	.4075D-02	.1504D-03
.90	.2002D-02	6867D-01	.2880D-02	.2873D-02	.5599D-10	.2816D-02	.2932D-02	.1162D-03
.91	.1459D-02	.5314D-01	.2011D-02	.2006D-02	.3179D-10	.1964D-02	.2050D-02	.8614D-04
.92	.1025D-02	.3962D-01	.1355D-02	.1351D-02	.1680D-10	.1321D-02	.1382D-02	.6062D-04
.93	.6862D-03	.2818D-01	.8720D-03	.8689D-03	.8042D-11	.8497D-03	.8897D-03	.3999D-04
.94	.4321D-03	.1884D-01	.5284D-03	.5263D-03	.3350D-11	.5147D-03	.5389D-03	.2425D-04
.95	.2500D-03	.1158D-01	.2947D-03	.2934D-03	.1142D-11	.2871D-03	.3003D-03	.1311D-04
.96	.1280D-03	.6293D-01	.1456D-03	.1449D-03	.2883D-12	.1421D-03	.1481D-03	.6006D-05
.97	.5400D-04	.2819D-01	.5937D-04	.5908D-04	.4454D-13	.5809D-04	.6021D-04	.2120D-05
.98	.1600D-04	.8869D-03	.1702D-04	.1695D-04	.2750D-14	.1673D-04	.1720D-04	.4652D-06
.99	.2000D-05	.1177D-03	.2062D-05	.2055D-05	.1743D-16	.2041D-05	.2073D-05	.3211D-07

† Provided by J.S. Provan using an algorithm based on cutset enumeration.

†† Computedas in Algorithm A

Table 4

$$\Lambda(q) = \text{var } \hat{g}_{bK}(q,.90)/\text{var } \hat{g}_{bK}(q,.80)^{\dagger}$$

(K=1048576)

q	$\Lambda(q)$	$[\Lambda(q)]^{\frac{1}{2}}$	q	$\Lambda(q)$	$[\Lambda(q)]^{\frac{1}{2}}$
.80	39.18	6.26	.90	.9821	.9910
.81	46.80	6.84	.91	.7072	.8410
.82	29.78	5.46	.92	.2796	.5288
.83	18.87	4.34	.93	.4117	.6416
.84	11.96	3.46	.94	.3334	.5774
.85	7.61	2.76	.95	.2808	.5299
.86	4.88	2.21	.96	.2452	.4951
.87	3.17	1.78	.97	.2217	.4709
.88	2.09	1.45	.98	.2061	.4540
.89	1.41	1.18	.99	.1989	.4460

† Estimated by $V[\hat{g}_{bK}(q,.90)]/V[\hat{g}_{bK}(q,.80)]$.

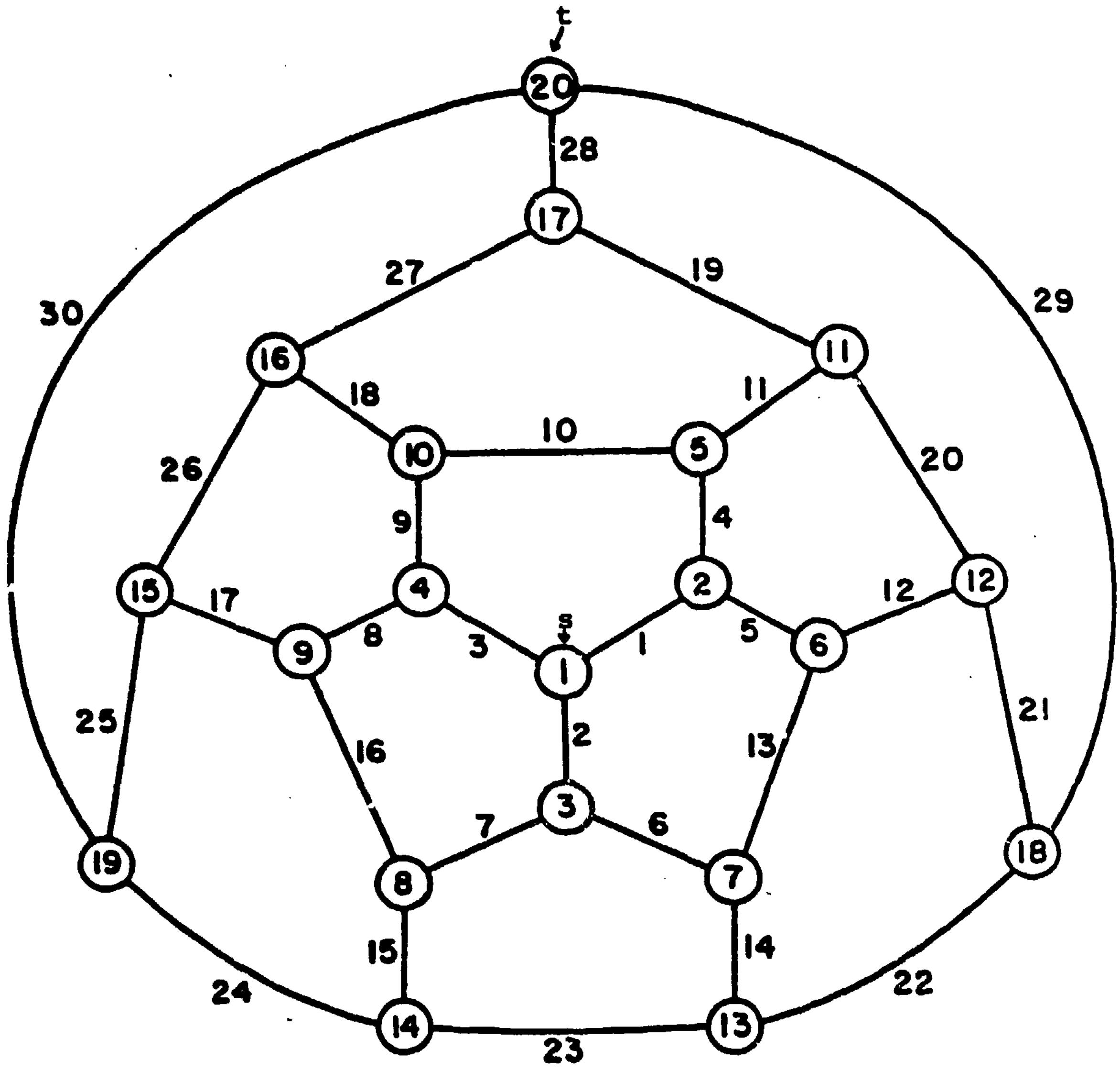
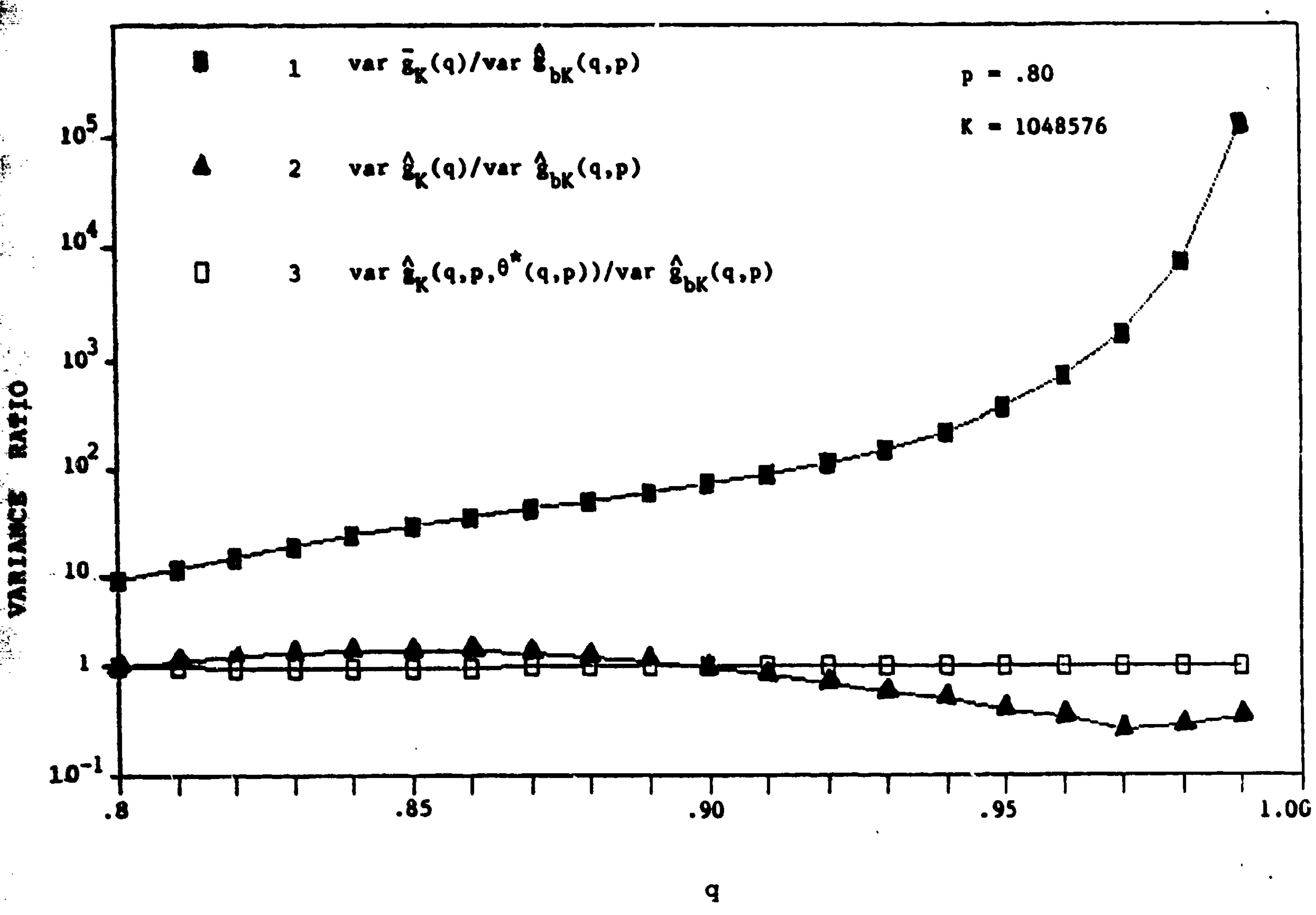


Fig. 1 Network

All component reliabilities are identical.
$p = .80$
$\ell = \{.80 + .01(i\text{-}1) \quad i=1,\ldots,20\}$
$K = 2^{20} = 1048576$
Lower bound based on 3 edge-disjoint paths
Upper bound based on 5 edge-disjoint cutsets

Fig. 2 Variance Ratios for Alternative Estimators



Appendix

Proof of Lemma 1. Observe that

$$E[\phi(X)R(X,k,q,p)] = \sum_{x\in\mathcal{S}} \phi(x)R(x,k,q,p)Q(x,k,p)$$

$$= \sum_{x\in\mathcal{S}} \phi(x)\,\frac{P(x,k,q)}{P(x,k,p)}\left[\frac{\phi_U(x)-\phi_L(x)}{\Delta(p)}\right]P(x,k,p).$$

Since $\phi(x)\phi_L(x) = \phi_L(x)$ and $\phi(x)\phi_U(x) = \phi(x)$, one has

$$E[\phi(X)R(X,k,q,p)] = [g(q)-g_L(q)]/\Delta(p).$$

Also,

$$E[\phi(X)R(X,k,q,p)]^2 = \sum_{x\in\mathcal{S}} \phi(x)R^2(x,k,q,p)Q(x,k,p)$$

$$= \sum_{x\in\mathcal{S}} \phi(x)\,\frac{P^2(x,k,q)}{P(x,k,p)}\left[\frac{\phi_U(x)-\phi_L(x)}{\Delta(p)}\right].$$

Observe that

$$P^2(x,k,q)/P(x,k,p) = \prod_{i=1}^{r}(q_i^2/p_i)^{x_i}[(1-q_i)^2/(1-p_i)]^{k_i-x_i},$$

$$= c(q,p)\{\prod_{i=1}^{r}(q_i^2/c_ip_i)^{x_i}[(1-q_i)^2/c_i(1-p_i)]^{k_i-x_i}\},$$

so that $\{P^2(x,k,q)/c(q,p)P(x,k,p),\ x\in\mathcal{S}\}$ is a p.m.f. Expression (12) follows.

Proof of Lemma 2. We restrict z to $[g_L(q), g_U(q)]$. Consider the case $p\geq q$ which implies that $q^*\leq q$ so that $g(q^*) \leq g(q)$ and $g_L(q^*) \leq g_L(q)$. In this case $h_1(z) \leq h_2(z)$ so that (40b) gives the tightest upper bound.

Since $p\leq q$ implies $q\leq q^*$ so that $g(q) \leq g(q^*)$ and $g_j(q) \leq g_j(q^*)$ for $j\in\{L,U\}$, one has $g(q^*) \geq \max[g_L(q^*), g(q)]$. Also, either $g_L(q) \leq g_L(q^*) \leq g(q)$ or $g_L(q^*) \geq g_U(q)$. Since $g_L(q^*)$

$\geq g_U(q)$ implies $g_L(q^*) \geq g(q)$, $h_1(z) \leq h_2(z)$ so that (40b) gives the tightest bound. If $g_L(q) \leq g_L(q^*) \leq g_U(q)$, it is not clear whether $g(q) \leq g_L(q^*)$ or $g(q) \geq g_L(q^*)$. Therefore, (40a) gives the best bound.

Proof of Theorem 3. The function h_1 has its unrestricted maximum at $z = g_U(q)$. Therefore,

$$\max_{g_L(q) \leq z \leq g_U(q)} h_1(z) = h_1(g_U(q)) \qquad \text{if } g_L(q^*) \notin [g_L(q), g_U(q)]$$

and

$$\max_{g_L(q) \leq z \leq g_L(q^*)} h_1(z) = h_1(g_L(q^*)) \qquad \text{if } g_L(q^*) \in [g_L(q), g_U(q)].$$

The function h_2 is concave with its maximum at $z^* < g_U(q)$. Therefore,

$$\max_{g_L(q^*) \leq z \leq g_U(q)} h_2(z) = h_2(z^*) \qquad \text{if } g_L(q^*) \leq z^* < g_U(q)$$

$$= h_2(g_L(q^*)) \qquad \text{if } z^* \leq g_L(q^*).$$

Then Theorem 3 follows directly from Lemma 2.

Theorem 4. Define $w_1(z) = h_1(z)/z^2$ and $w_2(z) = h_2(z)/z^2$ for h_1 and h_2 as in (39) for $-\infty < z < \infty$. Let $z_1 = g_U(q) - c(q,p)\Delta(p)\Delta(q^*)/g_U(q)$, $z_2 = [g_U^2(q) - c(q,p)\Delta(p)\Delta(q^*)]/[g_U(q) - c(q,p)]\Delta(p)/2]$ and $b_2 = 2g_U(q) - c(q,p)\Delta(p)$. Then

$\Gamma_{1b}($[illegible]$) = w^*(q,p) = \max_{g_L(q) \leq z \leq g_U(q)} w_1(z) \qquad$ if $g_L(q^*) \notin [g_L(q), g_U(q)]$

$$= \max\left[\max_{g_L(q) \leq z \leq g_L(q^*)} w_1(z), \max_{g_L(q^*) \leq z \leq g_U(q)} w_2(z)\right]$$

if $g_L(q^*) \in [g_L(q), g_U(q)]$

where

i.

$$\max_{g_L(q) \le z \le g_U(q)} w_1(z) = w_1(g_L(q)) \qquad \text{if } z_1 \le g_L(q)$$

$$= w_1(z_1) \qquad \text{if } z_1 \ge g_L(q),$$

ii.

$$\max_{g_L(q) \le z \le g_U(q^*)} w_1(z) = w_1(g_L(q)) \qquad \text{if } z_1 \le g_L(q)$$

$$= w_1(z_1)) \qquad \text{if } g_L(q) \le z_1 \le g_L(q^*)$$

$$= w_1(g_L(q^*)) \qquad \text{if } z_1 \ge g_L(q^*)$$

and

iii.

$$\max_{g_L(q^*) \le z \le g_U(q)} w_2(z) = w_2(g_L(q^*)) \qquad \text{if } z_2 < 0 \text{ and } b_2 > 0,$$

$$= w_2(g_U(q)) \qquad \text{if } z_2 < 0 \text{ and } b_2 < 0$$

$$= w_2(g_L(q^*)) \qquad \text{if } 0 \le z_2 \le g_L(q^*) \text{ and } b_2 > 0$$

$$= w_2(z_2) \qquad \text{if } g_L(q^*) \le z_2 \le g_U(q) \text{ and } b_2 > 0$$

$$= w_2(g_U(q)) \qquad \text{if } z_2 \ge g_U(q) \text{ and } b_2 > 0$$

$$= w_2(g_U(q)) \qquad \text{if } 0 \le z_2 \le g_L(q^*) \text{ and } b_2 < 0$$

$$= \max\left[w_2(g_L(q^*)), w_2(g_U(q))\right]$$

$$\text{if } g_L(q^*) \le z_2 \le g_U(q) \text{ and } b_2 < 0$$

$$= w_2(g_L(q^*)) \qquad \text{if } z_2 \ge g_U(q) \text{ and } b_2 < 0.$$

Proof of parts i and ii. Since

$$w_1(z) = h_1(g)/z^2 = a_1/z^2 + b_1/z - 1$$

where

$$a_1 = c(q,p)\Delta(p)\Delta(q^*) - g_U^2(q) \text{ and } b_1 = 2g_U(q),$$

then

$$\frac{dw_1}{dz} = -(2a_1/z+b_1)/z^2 = -b_1(-z_1/z+1)/z^2$$

and

$$\frac{d^2w_1}{dz^2} = 2(3a_1/z+b_1)/z^3 = 2b_1(-3z_1/2z+1)/z^3.$$

If $z_1 \leq 0$, then w_1 is convex and decreasing on $[0,\infty)$ and w_1 has its maximum at $z = g_L(q)$. If $z_1 > 0$, then w_1 is concave on $[0, 3z_1/2]$ and $z_1 \leq g_U(q)$ so that w_1 has its maximum on $[g_L(q), g_U(q)]$ at $z = z_1$ if $z_1 \in [g_L(q), g_U(q)]$ and at $z = g_L(q)$ otherwise. The result for part ii follows immediately.

Proof of iii. Since

$$w_2(z) = h_2(z)/z^2 = a_2/z^2 + b_2/z - 1$$

where

$$a_2 = c(q,p)\Delta(p)\, g_U(q^*) - g_U^2(q) \text{ and } b_2 = 2g_U(q) - c(q,p)\Delta(p),$$

then

$$\frac{dw_2}{dz} = -b_2(-z_2/z+1)/z^2$$

and

$$\frac{d^2w_2}{dz^2} = 2b_2(-3z_2/z+1)/z^3.$$

Consider the interval $[g_L(q^*), g_U(q)]$. If $z_2<0$ and $b_2>0$ then w_2 is convex on $[0,\infty)$ and its maximum occurs at $z = g_L(q^*)$. If $z_2<0$ and $b_2<0$ then w_2 is concave on $[0,\infty)$ and has its maximum at $z = g_U(q)$. If $z_2>0$ and $b_2>0$, then w_2 is concave on $[0,3z_2^2]$ so that the maximum occurs at $z = g_L(q^*)$ if $z_2 \leq g_L(q^*)$, $z = z_2$ if $g_L(q^*) \leq z_2 \leq g_U(q)$ and at $z = g_U(q)$ if $z_2 \geq g_U(q)$. If $z_2 > 0$ and $b_2 < 0$, the maximum occurs at $z = g_U(q)$ if $z_2 \leq g_L(q^*)$, at $z = g_L(q^*)$ if $z_2 \geq g_U(q)$ and at $z = \max[w_2(g_L(q^*)), w_2(g_U(q))]$ if $g_L(q^*) \leq z_2 \leq g_U(q)$.

Theorem 5. Let $w_3(z) = h_1(z)/(1-z)^2$ and $w_4(z) = h_2/(1-z)^2$ for h_1 and h_2 as defined in (39). Let

$$z_3 = g_U(q) + c(q,p)\Delta(p)\,\Delta(q^*)/[1-g_U(q)],$$

$$z_4 = 1 - 2\{c(q,p)\Delta(p)[1-g_U(q^*)] + [1-g_U(q)]^2\}/\{c(q,p)\Delta(p) + 2[1-g_U(q)]\}.$$

Then

$$\gamma^2_{2b}(q,p) \leq w^{**}(q,p) = \max_{g_L(q) \leq z \leq g_U(q)} w_3(z) \qquad \text{if } g_L(q^*) \notin [g_L(q), g_U(q)]$$

$$= \max\left[\max_{g_L(q) \leq z \leq g_L(q^*)} w_3(z), \max_{g_L(q^*) \leq z \leq g_U(q)} w_4(z)\right]$$

$$\text{if } g_L(q^*) \in [g_L(q), g_U(q)]$$

where

$$\max_{g_L(q) \leq z \leq g_U(q)} w_3(z) = w_3(g_L(q)) \qquad \text{if } z_3 \geq 1$$

$$= w_3(g_U(q)) \qquad \text{if } z_3 < 1$$

$$\max_{g_L(q) \leq z \leq g_L(q^*)} w_3(z) = w_3(g_L(q^*)) \qquad \text{if } z_3 \geq 1$$

$$= w_3(g_U(q)) \qquad \text{if } z_3 < 1$$

$$\max_{g_L(q^*) \le z \le g_U(q)} w_4(z) = w_4(g_L(q^*)) \qquad \text{if } z_4 \le g_L(q^*)$$

$$= w_4(z_4) \qquad \text{if } g_L(q^*) \le z_4 \le g_U(q)$$

$$= w_4(g_U(q)) \qquad \text{if } z_4 \ge g_U(q).$$

Proof of Theorem 5. Since

$$w_3(z) = a_3/(1-z)^2 + b_3/(1-z) - 1$$

where

$$a_3 = c(q,p)\Delta(p)\ \Delta(q^*) - [1-g_U(q)]^2$$

and

$$b_3 = 2[1-g_U(q)],$$

then

$$\frac{dw_3}{dz} = \frac{b_3}{(1-z)^2}\left[2a_3/b_3(1-z)+1\right] = \frac{b_3}{(1-z)^2}\left[-(1-z_3)/(1-z)+1\right]$$

and

$$\frac{d^2w}{dz^2} = \frac{2b_3}{(1-z)^3}\left[3a_3/b_3(1-z)+1\right] = \frac{2b_1}{(1-z)^3}\left[-3(1-z_3)/2(1-z)+1\right].$$

Observe that $z_3 = 1 + 2a_3/b_3 > g_U(q)$. If $z_3 \ge 1$, then w_3 is convex on $(-\infty,1]$, having its maximum on $[g_L(q), g_U(q)]$ and on $[g_L(q), g_L(q^*)]$ at $z = g_L(q)$. If $z_3 \le 1$, then w_3 is convex on $[(3z_3-1)/2,1]$ and w_1 has its maximum on $[g_L(q), g_U(q)]$ at $z = g_U(q)$ and on $[g_L(q), g_L(q^*)]$ at $z = g_L(q^*)$, establishing i and ii.

Proof of iii. Since

$$w_4(z) = a_4/(1-z)^2 + b_4/(1-z) - 1$$

where

$$a_4 = c(q,p)\Delta(p)[1-g_U(q^*)] + [1-g_U(q)]^2$$

and

$$b_4 = 2[1-g_U(q)] + c(q,p)\Delta(p),$$

then

$$\frac{dw_2}{dz} = \frac{b_4}{(1-z)^2}\left[2a_4/b_4(1-z)+1\right] = \frac{b_4}{(1-z)^2}\left[-(1-z_4)/(1-z)+1\right]$$

and

$$\frac{d^2w_2}{dz^2} = \frac{2b_4}{(1-z)^2}\left[3a_4/b_4(1-z)+2\right] = \frac{2b_4}{(1-z)^3}\left[-3(1-z_4)/2(1-z)+1\right].$$

Consider the interval $[g_L(q^*), g_U(q)]$. Since $z_4<1$ w_2 is concave on $[-(1-z_4)/2,1]$. Then w_4 has its maximum at $z = g_L(q^*)$ if $z_4 \le g_L(q^*)$, at $z = z_4$ if $g_L(q^*) \le z_2 \le g_U(q)$ and at $z = g_U(q)$ if $z_4 \ge g_U(q)$.

Proof of Theorem 8. Observe that

$$R^*(\mathbf{x},\mathbf{k},\mathbf{q},\mathbf{p}) = \prod_{i\in\mathscr{H}^*} (q_i/p_i)^{x_i} [(1-q_i)/(1-p_i)]^{k_i-x_i}$$

has the alternative form

$$R^*(\mathbf{x},\mathbf{k},\mathbf{q},\mathbf{p}) = [\prod_{i\in\mathscr{H}^*} (q_i/p_i)^{k_i}] \exp[-\sum_{i\in\mathscr{H}^*} a_i \sum_{j\in\mathscr{E}_i} (1-y_j)]$$

with

$$\sum_{j\in\mathscr{E}_i} y_i = x_i \qquad i\in\mathscr{H}^*.$$

The condition $\phi_L(\mathbf{x}) = 0$ requires that

$$\sum_{j\in\mathscr{P}} (1-y_i) \ge 1 \qquad \forall\ \mathscr{P}\in\mathscr{M}_a$$

and the condition $\phi(\mathbf{x}) = 1$ requires that

$$\sum_{j\in\mathscr{C}} (1-y_j) \le |\mathscr{C}| - 1 \qquad \forall\ \mathscr{C}\in\mathscr{N}_a.$$

Since

$$\max_{y} \exp[-\sum_{i\in\mathscr{N}^*} a_i \sum_{j\in\mathscr{S}_i} (1-y_i)] = \exp[-\min_{y} \sum_{i\in\mathscr{N}^*} a_i \sum_{j\in\mathscr{S}_i} (1-y_j)],$$

one has the integer program (47) with $y_j = (1-z_j)$.