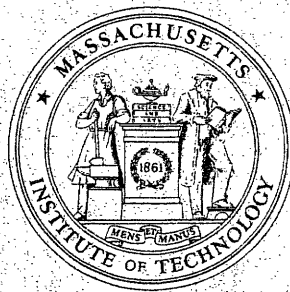


OPERATIONS RESEARCH CENTER

working paper



**MASSACHUSETTS INSTITUTE
OF TECHNOLOGY**

Sensitivity Analysis for Variational
Inequalities

by

Yuping Qiu and Thomas L. Magnanti

OR 163-87

June 1987

Sensitivity Analysis for Variational Inequalities

Yuping Qiu

Thomas L. Magnanti

Operations Research Center

Massachusetts Institute of Technology

Cambridge, MA 02139

December 1986

Revised June 1987

Abstract

In this paper we study the behavior of the local solutions of perturbed variational inequalities, governed by perturbations to both the variational inequality function and the feasible region. Assuming appropriate second-order and regularity conditions, we show that the perturbed local solution set is nonempty, Lipschitz continuous, and directionally differentiable. Even when the directional differentiability is not guaranteed, we are still able to describe and characterize first-order information concerning the perturbed local solution set. We also discuss relations to nonlinear programming sensitivity analysis.

Key Words. Sensitivity analysis, Variational inequalities, Nonlinear programming.

1. Introduction

In this paper we consider sensitivity analysis for the variational inequality problem stated in the following standard form:

VI: find $x \in \Omega$ such that $F(x)^T(x' - x) \geq 0$ for any $x' \in \Omega$.

In this formulation, the ground set $\Omega = \{x \in \mathbb{R}^n \mid g(x) \geq 0, h(x) = 0\}$, and F , g , and h are functions from \mathbb{R}^n to \mathbb{R}^n , from \mathbb{R}^n to \mathbb{R}^m , and from \mathbb{R}^n to \mathbb{R}^l respectively. As has long been recognized, a variational inequality is an alternative formulation for a number of well-known problems, such as convex programming problems, complementarity problems and fixed point problems. In practice, variational inequalities are useful tools for modeling various equilibria in economics and transportation science. Examples include spatial market equilibrium problems and traffic equilibrium problems. This paper deals mainly with the behavior of the local solutions of such variational inequalities with respect to smooth perturbations of the underlying problem data. Due to the nature of the variational inequality problem, the sensitivity analysis we perform is closely related to that for optimization problems. The main results of this paper are also applicable to perturbed optimization problems.

The first basic results of nonlinear programming sensitivity analysis assumed strict complementary slackness, linear independence of the gradients of the binding constraints, and a second order sufficient condition (see Fiacco [1976], or [1983]). Assuming these conditions, Fiacco [1976] showed that the perturbed local solution is a once continuously differentiable function of the perturbation parameter. When the strict complementary slackness condition is relaxed, in general the perturbed local solution is no longer differentiable with respect to the perturbation parameter. However, if we strengthen the second order condition by imposing the so-called strong second order sufficient condition, then the perturbed local solution is again a Lipschitz continuous function, and is directionally differentiable at the point being considered (see Jittorntrum [1984] and Robinson [1980]). Moreover, if we replace the linear independence condition with the Mangasarian and Fromovitz constraint qualification, and assume the general strong second order sufficient condition, then the perturbed local solution is merely a continuous function (see Kojima [1980]). If we further weaken the strong second-order condition to a general second order sufficient condition, then the

perturbed local solution still exists but may not be unique, and in this case, the perturbed local solution set is Lipschitz continuous at the point being considered (see Robinson [1982]). For a more detailed survey, see Fiacco and Kyparisis [1984].

A number of authors have considered sensitivity and stability issues of variational inequalities with special linear structures. These variational inequalities arise as natural mathematical formulations of certain equilibrium problems (for example, traffic equilibrium and spatially separated economic markets). Recently, a couple of authors have also considered the sensitivity analysis for the general form of the variational inequality problem VI. Assuming strict complementary slackness, linear independence of the gradients of the binding constraints, and the second order sufficient condition, Tobin [1986] applied nonlinear programming sensitivity analysis results of Fiacco [1976] to variational inequalities. While assuming the linear independence condition and the strong second order sufficient condition, Kyparisis [1985] extended Robinson [1980]'s work on generalized equations — he showed that the perturbed stationary point in this case is a Lipschitz continuous function and is directionally differentiable at the point being considered. Although stated only for nonlinear optimization problems, most of the results obtained by Robinson [1982] are also valid for variational inequalities. In fact, Robinson showed that the perturbed local solutions to the variational inequality problem is Lipschitz continuous at the point being considered, assuming a regularity condition and a general second order sufficient condition.

In this paper, we study differentiability properties of perturbed local solutions for situations in which the perturbed local solution is a multifunction of the perturbation parameter. Thus we need to introduce the notion of differentiability for a point-to-set mapping at a given point. Assuming appropriate second-order and regularity conditions, we prove that the perturbed local solution set is nonempty and is directionally differentiable at the point being considered.

In the next section, we study various second-order conditions and constraint qualifications associated with variational inequalities, and explore some properties of local solutions. Then in Section 3, we conduct sensitivity analysis for general variational inequalities with nonlinear constraints. In this section, we also define directional differentiability for a point-to-set mapping at a

given point. Finally, Section 4 discusses the application of this work to sensitivity analysis for nonlinear programs.

2. First and second order conditions for variational inequalities

In this section, we first investigate some properties of the local solution to the variational inequality problem VI. We then consider a perturbed version of VI and present some continuity properties of the perturbed local solutions. For the moment we assume that F is once continuously differentiable and that g and h are twice continuously differentiable.

Definition 2.1. We say x^* is a *local solution* to variational inequality problem VI if for some neighborhood M of x^* , $F(x^*)^T(x' - x^*) \geq 0$ for any $x' \in \Omega \cap M$. Furthermore, if x^* is the only local solution in some neighborhood of x^* , we say x^* is an *isolated local solution* of VI.

Suppose x^* is a local solution to problem VI. Then clearly x^* is also a local minimum of the following nonlinear program, and vice versa,

$$\begin{aligned} & \text{minimize} && F(x^*)^T z \\ & \text{subject to} && g(z) \geq 0 \\ & && h(z) = 0. \end{aligned}$$

Let (u^*, v^*) denote the corresponding Lagrange multipliers. Also let

$$\begin{aligned} I_1(x^*) &= \{i \mid g_i(x^*) = 0, u_i^* > 0\}, \\ I_2(x^*) &= \{i \mid g_i(x^*) = 0, u_i^* = 0\}, \\ I_3(x^*) &= \{i \mid g_i(x^*) > 0, u_i^* = 0\}, \\ I(x^*) &= I_1(x^*) \cup I_2(x^*). \end{aligned}$$

Now suppose the *Mangasarian-Fromovitz constraint qualification* (MFCQ) holds at x^* , i.e., $\nabla h_j(x^*)$ for $j = 1, \dots, l$ are linearly independent and some vector $z \in \mathbb{R}^n$ satisfies

$$\begin{aligned} & \nabla g_i(x^*) z > 0 \text{ for } i \in I(x^*), \text{ and} \\ & \nabla h_j(x^*) z = 0 \text{ for } j = 1, \dots, l. \end{aligned}$$

Then the *generalized Karush-Kuhn-Tucker condition* (GKKT) follows immediately, i.e., some vectors u^* and v^* satisfy

$$F(x^*) - \nabla g(x^*)^T u^* - \nabla h(x^*)^T v^* = 0$$

$$u^{*T} g(x^*) = 0$$

$$g(x^*) \geq 0$$

$$h(x^*) = 0$$

$$u^* \geq 0.$$

For convenience, we let $L_d(x, u, v) \equiv F(x) - \nabla g(x)^T u - \nabla h(x)^T v$.

Definition 2.2. We say a point x is a *stationary point* of the variational inequality problem VI if for some vectors u and v , (x, u, v) satisfies the GKKT condition.

In general, such Lagrange multipliers u and v may not be unique. However, it is easy to see that the *Linear Independence* (LI) of the gradients of the binding constraints implies the uniqueness of the Lagrange multipliers. A standard theorem of the alternative also shows that LI implies MFCQ. Therefore, LI implies GKKT as well at any local solution point. The LI condition at a local solution point x^* also implies the following *second order necessary condition* (SONC):

$$-\nabla^2 g(x^*) u^* - \nabla^2 h(x^*) v^* \text{ is positive semidefinite on } T_1(x^*),$$

where $T_1(x^*) = \{z \mid \nabla g_i(x^*) z = 0 \text{ for } i \in I_1(x^*), \nabla g_i(x^*) z \geq 0 \text{ for } i \in I_2(x^*), \nabla h(x^*) z = 0\}$, which, because of the GKKT condition, can be restated as $T_1(x^*) = \{z \mid F(x^*)^T z = 0, \nabla g_i(x^*) z \geq 0 \text{ for } i \in I(x^*), \nabla h(x^*) z = 0\}$. Note $T_1(x^*)$ is well defined and does not depend upon the Lagrange multipliers u^* . Roughly speaking, the SONC states that the feasible region must satisfy a certain convexity property at a local solution point.

Now suppose the GKKT condition holds at x^* for some u^* and v^* . If $g_i(\cdot)$ for $i = 1, \dots, m$ are quasi-concave at x^* , i.e., $g_i(x) \geq g_i(x^*)$ implies $\nabla g_i(x^*)(x - x^*) \geq 0$, and $h_j(x)$ for $j = 1, \dots, l$ are affine functions, then x^* is a local solution to VI. However, for more general functions g and h , we need the following *second order sufficient condition* (SOSC) to ensure that x^* is a local solution:

$-\nabla^2 g(x^*)u^* - \nabla^2 h(x^*)v^*$ is positive definite on $T_1(x^*)$.

Loosely speaking, the SOSC is imposed only upon the feasible region about x^* . So for any small perturbation of the function F , we do not expect that such a condition will ensure the existence of a local solution near x^* to the perturbed problem.

If $\nabla_x L_d(x^*, u, v)$ is positive definite on $T_1(x^*)$ for each (u, v) for which (x^*, u, v) satisfies (GKKT), then we say that the problem satisfies the *general second order condition* (GSOC) at x^* . (If in particular, the Lagrange multipliers are unique at x^* , then we say the *second order condition* (SOC) holds at x^* .) Robinson [10] has studied the behavior of perturbed solutions for the general optimization problems under similar assumptions. Some of his results are also valid for variational inequalities — the only necessary change is to replace ∇f , the gradient of the objective function f , with our general vector function F . In this section, we will paraphrase some of his results in terms of variational inequalities.

Theorem 2.1. Suppose x^* is a local solution of VI. Assume the problem satisfies MFCQ and GSOC at x^* . Then x^* is an isolated local solution.

Proof. By Robinson [10, Theorem 2.4], x^* is an isolated stationary point. By the continuity of $g, \nabla g, h$ and ∇h , MFCQ is valid in a neighborhood of x^* (see Robinson [12, Theorem 3]). Since MFCQ is a sufficient condition for GKKT, any local solution near x^* must also be a stationary point and, therefore, x^* must be an isolated local solution.

□

We now consider the following perturbed version of variational inequality problem VI:

VI(ε): find $x \in \Omega(\varepsilon)$ satisfying $F(x, \varepsilon)^T(x' - x) \geq 0$ for any $x' \in \Omega(\varepsilon)$.

In this formulation, $\Omega(\varepsilon) = \{x \mid g(x, \varepsilon) \geq 0, h(x, \varepsilon) = 0\}$ and $\varepsilon \in \mathbb{R}^k$ is a perturbation parameter. Throughout this paper, we assume x^* is a local solution (or a stationary point, as distinguished by context) to problem VI(ε^*). We now list some basic assumptions needed in this paper, which all

concern the local properties of the functions F , g and h near (x^*, ε^*) . The variables (x, ε) appearing in the following assumptions are restricted to a neighborhood of (x^*, ε^*) .

Assumption 2.1. $g(\cdot, \varepsilon)$ and $h(\cdot, \varepsilon)$ are differentiable, and $F(\cdot, \cdot)$, $\nabla_x g(\cdot, \cdot)$, $\nabla_x h(\cdot, \cdot)$ are continuous. Moreover, $F(\cdot, \varepsilon^*)$ is once continuously differentiable, $g(\cdot, \varepsilon^*)$ and $h(\cdot, \varepsilon^*)$ are twice continuously differentiable.

Assumption 2.2. (convergence condition) For some $L > 0$,

$$\begin{aligned} \|F(x, \varepsilon) - F(x, \varepsilon^*)\| &\leq L \|\varepsilon - \varepsilon^*\|, \\ \|g(x, \varepsilon) - g(x, \varepsilon^*)\| &\leq L \|\varepsilon - \varepsilon^*\|, \\ \|h(x, \varepsilon) - h(x, \varepsilon^*)\| &\leq L \|\varepsilon - \varepsilon^*\|, \\ \|\nabla_x g(x, \varepsilon) - \nabla_x g(x, \varepsilon^*)\| &\leq L \|\varepsilon - \varepsilon^*\|, \\ \|\nabla_x h(x, \varepsilon) - \nabla_x h(x, \varepsilon^*)\| &\leq L \|\varepsilon - \varepsilon^*\|. \end{aligned}$$

Assumption 2.3. $F(\cdot, \cdot)$, $\nabla_x g(\cdot, \cdot)$ and $\nabla_x h(\cdot, \cdot)$ are differentiable at (x^*, ε^*) .

Assumptions 2.1 and 2.2 are essential to ensure the Lipschitz continuity of the perturbed local solution set. Assumption 2.3 is needed only in studying the differentiability property of the perturbed local solution set. We also rely upon the following notation for carrying out the sensitivity analysis. Let

$$\begin{aligned} S(\varepsilon) &= \{x \mid x \text{ is a local solution to VI}(\varepsilon)\}, \\ SP(\varepsilon) &= \{x \mid x \text{ is a stationary point of VI}(\varepsilon)\}, \\ K(x, \varepsilon) &= \{(u, v) \mid (x, u, v) \text{ satisfies the GKKT corresponding to } \varepsilon\}. \end{aligned}$$

If the constraints $g(\cdot, \varepsilon)$ are quasi-concave and the constraints $h(\cdot, \varepsilon)$ are linear for each ε , then $SP(\varepsilon) \subseteq S(\varepsilon)$. On the other hand, if MFCQ is valid over the entire feasible region, then $S(\varepsilon) \subseteq SP(\varepsilon)$. The next theorem presents a continuity property regarding the point-to-set mappings $SP(\cdot)$ and $K(\cdot, \cdot)$ near the point being considered. Although Robinson [10] has stated this result only for optimization problems, it is also valid for variational inequalities.

Theorem 2.2. Suppose Assumption 2.1 is satisfied near (x^*, ε^*) . If GKKT and MFCQ hold at x^* for $\varepsilon = \varepsilon^*$, then for some neighborhoods M of x^* and N of ε^* , $K(\cdot, \cdot)$ and $SP(\cdot) \cap M$ are upper semicontinuous on $M \times N$ and N respectively. Furthermore, $K(\cdot, \cdot)$ is uniformly bounded on $M \times N$.

Proof. See Robinson [10, Theorem 2.3].

□

In general, with these assumptions the local perturbed stationary point set $SP \cap M$ may be empty for any small perturbation. Therefore, we need additional second order conditions to ensure the existence of perturbed stationary points or perturbed local solutions.

In the following theorem, we assume a local convexity condition and that the MFCQ condition holds at (x^*, ε^*) . Therefore in this case, the stationary points coincide with the local solutions in a neighborhood of x^* .

Theorem 2.3. Suppose Assumption 2.1 is satisfied at (x^*, ε^*) . Assume GKKT, MFCQ and GSOC hold at (x^*, ε^*) . Also assume that $g(\cdot, \varepsilon)$ is locally quasi-concave at x^* and $h(\cdot, \varepsilon)$ is affine for ε near ε^* . Then for some neighborhoods M of x^* and N of ε^* , $S(\varepsilon) \cap M = SP(\varepsilon) \cap M \neq \emptyset$ for each $\varepsilon \in N$.

Proof. Since $g(\cdot, \varepsilon)$ is locally quasi-concave and $h(\cdot, \varepsilon)$ is affine for ε near ε^* , and since MFCQ holds at (x^*, ε^*) , for some neighborhoods M_1 of x^* and N_1 of ε^* , $\Omega(\varepsilon) \cap M_1$ is a convex set, and MFCQ holds over $\Omega(\varepsilon) \cap M_1$ for each $\varepsilon \in N_1$ (see Robinson [12, Theorem 3]). Hence, $S(\varepsilon) \cap M_1 = SP(\varepsilon) \cap M_1$ for each $\varepsilon \in N_1$.

We now proceed to prove that the perturbed local solution set is nonempty by contraposition. Let δ be positive number and B be an n -dimensional unit ball, and consider the following locally restricted variational inequality problem

VI (ε, δ) : find $x \in \Omega(\varepsilon) \cap \{x^* + \delta B\}$ satisfying

$$F(x, \varepsilon)^T (x' - x) \geq 0 \quad \text{for any } x' \in \Omega(\varepsilon) \cap \{x^* + \delta B\}.$$

Note that $\Omega(\varepsilon) \cap \{x^* + \delta B\}$ is compact for all $\varepsilon \in N_1$ and is convex for δ sufficiently small. So in this case, $VI(\varepsilon, \delta)$ always has a solution. Furthermore, if this solution is in the interior of $\{x^* + \delta B\}$, then it is also a local solution to the original problem $VI(\varepsilon)$.

Consequently, if the theorem is not valid, then for some sequence $\delta_n \downarrow 0$, there are sequences $\{\varepsilon_k^n\}_k$ and $\{x_k^n\}_k$ with x_k^n solving $VI(\varepsilon_k^n, \delta_n)$ and that satisfy $\varepsilon_k^n \rightarrow \varepsilon^*$ and $\|x_k^n - x^*\| = \delta_n$. Without loss of generality we assume $x_k^n \rightarrow x^n$. Letting $k \rightarrow +\infty$, we thus obtain a sequence $\{x^n\}$ satisfying $\|x^n - x^*\| = \delta_n$ and x^n solves $VI(\varepsilon^*, \delta_n)$. Notice that the restricted feasible region $\Omega(\varepsilon^*) \cap \{x^* + \delta_n B\} = \{x \mid g(x, \varepsilon^*) \geq 0, -\delta_n \|x - x^*\|^2 + \delta_n^3 \geq 0, h(x, \varepsilon^*) = 0\}$. It is easy to see that MFCQ holds for the enlarged system over $\Omega(\varepsilon^*) \cap \{x^* + \delta_n B\}$ as long as δ_n is small enough. Thus, the GKKT condition applies at any solution point of $VI(\varepsilon^*, \delta_n)$. In particular, for each x^n , some vectors u^n , v^n , and w^n satisfy

$$\begin{aligned} F(x^n, \varepsilon^*) - \nabla_x g(x^n, \varepsilon^*)^T u^n - \nabla_x h(x^n, \varepsilon^*)^T v^n + 2\delta_n (x^n - x^*)^T w^n &= 0, \\ (u^n)^T g(x^n, \varepsilon^*) &= 0, \\ (w^n)^T (-\delta_n \|x - x^*\|^2 + \delta_n^3) &= 0, \\ g(x^n, \varepsilon^*) &\geq 0, \\ -\delta_n \|x - x^*\|^2 + \delta_n^3 &\geq 0, \\ h(x^n, \varepsilon^*) &= 0, \\ u^n \geq 0, w^n \geq 0. \end{aligned}$$

We now view δ_n as a perturbation parameter in the new system. When $\delta_n = 0$, the new system reduces to the original GKKT condition corresponding to ε^* . By Theorem 2.2, $K(x^n, \delta_n)$ is upper semicontinuous and uniformly bounded at $(x^*, 0)$. Therefore, without loss of generality we assume that $u^n \rightarrow u^*$, $v^n \rightarrow v^*$, and $w^n \rightarrow w^*$, and that (x^*, u^*, v^*) satisfies the original GKKT condition corresponding to ε^* .

Now let $(x_t, u_t, v_t, w_t) = (1 - t)(x^*, u^*, v^*, w^*) + t(x^n, u^n, v^n, w^n)$, and consider the following function

$$\begin{aligned} s(t) &= (x^n - x^*)^T [L_d(x_t, u_t, v_t, \varepsilon^*) + 2\delta_n (x_t - x^*) w_t] + (u^n - u^*)^T g(x_t, \varepsilon^*) + \\ &\quad + (v^n - v^*)^T h(x_t, \varepsilon^*) + (w^n - w^*) [-\delta_n \|x_t - x^*\|^2 + \delta_n^3] \end{aligned}$$

It is easy to verify that $s(0) \geq (w^n - w^*) \delta_n^3$ and $s(1) \leq 0$. So by the mean-value theorem, for some $0 \leq t_n \leq 1$, $s'(t_n) = s(1) - s(0) \leq (w^* - w^n) \delta_n^3$. Some simple algebraic manipulation shows that $s'(t_n) = (x^n - x^*)^T [\nabla_x L_d(x_{t_n}, u_{t_n}, v_{t_n}, \varepsilon^*) + 2\delta_n w_{t_n}] (x^n - x^*)$. Now notice that $\|x^n - x^*\| = \delta_n$. If we let z be any limiting vector of the sequence $\{(x^n - x^*)/\delta_n\}_n$, then $\|z\| = 1$ and $z \in T_1(x^*)$. Also, the inequality $s'(t_n) \leq (w^* - w^n) \delta_n^3$ implies that $\lim s'(t_n)/\delta_n^2 \leq 0$ and consequently that

$$z^T \nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) z \leq 0,$$

which contradicts the GSOC assumption. □

The next theorem indicates that the local perturbed stationary points (and hence the perturbed local solutions under the MFCQ assumption) are actually Lipschitz continuous if in addition, Assumption 2.2 is satisfied. We let $d(x, A)$ denote the distance from point x to set A , i.e., $d(x, A) = \inf \{\|x - y\| : y \in A\}$.

Theorem 2.4. Suppose Assumptions 2.1 and 2.2 are satisfied at (x^*, ε^*) . Assume GKKT, MFCQ and GSOC hold at (x^*, ε^*) . Then for some neighborhoods M of x^* and N of ε^* , and for some constant $\mu > 0$,

$$d[(x, u, v), \{x^*\} \times K(x^*, \varepsilon^*)] \leq \mu \|\varepsilon - \varepsilon^*\|$$

for each $\varepsilon \in N$, $x \in SP(\varepsilon) \cap M$, and $(u, v) \in K(x, \varepsilon)$.

Proof. By Robinson [10, Theorem 4.2 and Corollary 4.3]. □

3. Sensitivity analysis for variational inequalities

We begin this section with a few definitions. As we mentioned previously, the local perturbed solution to $VI(\varepsilon)$ may not be unique under the assumptions invoked in this paper. Therefore, for any ε in any neighborhood M of x^* , $\varepsilon \rightarrow S(\varepsilon) \cap M$ is generally a point-to-set mapping. For the purpose of our analysis, we define Lipschitz continuity and directional differentiability for a point-to-set mapping $T(\varepsilon)$ at a given point (x^*, ε^*) in the following sense.

Definition 3.1. A point-to-set mapping $T(\varepsilon)$ from \mathbb{R}^k to \mathbb{R}^n is said to be *Lipschitz continuous* at (x^*, ε^*) if for some neighborhood N of ε^* and some number $L > 0$,

$$\|x(\varepsilon) - x^*\| \leq L \|\varepsilon - \varepsilon^*\| \quad \text{for any } x(\varepsilon) \in T(\varepsilon) \text{ and } \varepsilon \in N.$$

Definition 3.2. A point-to-set mapping $T(\varepsilon)$ from \mathbb{R}^k to \mathbb{R}^n is said to be *differentiable in the direction* ε_0 at (x^*, ε^*) if there is a vector $d(\varepsilon_0) \in \mathbb{R}^n$ satisfying the property that for any $x(\varepsilon^* + t\varepsilon_0) \in T(\varepsilon^* + t\varepsilon_0)$,

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \left[x(\varepsilon^* + t\varepsilon_0) - x^* \right] = d(\varepsilon_0).$$

The mapping $T(\varepsilon)$ is said to be *directionally differentiable* at (x^*, ε^*) if it is differentiable in every direction $\varepsilon_0 \in \mathbb{R}^k$.

These definitions are natural extensions of the same notions for point-to-point mappings and have clear geometrical meanings — when the mapping is single valued, these definitions are exactly the usual ones for functions. By our definition, differentiability is a strong property that requires all points in $T(\varepsilon)$ converge to a common point along the same direction and with the same rate. For example, $T(\varepsilon) = \{\varepsilon\}$ is differentiable at $(0, 0)$ while $T(\varepsilon) = [0, \varepsilon]$ is not. In general, when a point-to-set mapping $T(\cdot)$ is not differentiable along direction ε_0 at (x^*, ε^*) , we let

$$D(\varepsilon_0) = \{d \mid d \text{ is the limit of some convergent sequence of the form } [x(\varepsilon^* + t_k \varepsilon_0) - x^*] / t_k\}.$$

To be more precise, for $t > 0$ we first let

$$T_d(t, \varepsilon_0) = \{ [x(\varepsilon^* + t\varepsilon_0) - x^*] / t \mid x(\varepsilon^* + t\varepsilon_0) \in T(\varepsilon^* + t\varepsilon_0) \} \text{ for } t > 0.$$

Then we define

$$D(\varepsilon_0) = \overline{\lim}_{t \rightarrow 0} T_d(t, \varepsilon_0) \equiv \{d \mid \exists d(t_k) \in T_d(t_k, \varepsilon_0) \text{ such that } d(t_k) \rightarrow d \text{ as } t_k \downarrow 0\}.$$

Clearly, $D(\varepsilon_0)$ also contains the first order information about the limiting behavior of $T(\cdot)$ at (x^*, ε^*) . For example, $T(\varepsilon) = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = \varepsilon^2\}$ is not differentiable at $(0, 0)$ but $D(1) = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ (we let $\varepsilon_0 = 1$ since ε is a scale parameter), which means that the set $T(\varepsilon)$

converges to $x^* = 0$ along all directions with the same rate. Notice that $D(\varepsilon_0)$ is a closed set and has the following simple properties.

Lemma 3.1. If the mapping $T(\cdot)$ is Lipschitz continuous at (x^*, ε^*) with Lipschitz constant L , then $D(\varepsilon_0) \subseteq \{x \in \mathbb{R}^n \mid \|x\| \leq L\}$ is uniformly bounded in any direction ε_0 . Furthermore, $T(\cdot)$ is differentiable in direction ε_0 if and only if $D(\varepsilon_0)$ is a singleton.

A Polyhedral Version of the Problem

We first briefly summarize the results obtained by Qiu and Magnanti [8] for the case in which the feasible region mapping is a constant polyhedron, i.e., $\Omega(\varepsilon) \equiv P = \{x \in \mathbb{R}^n \mid Ax \geq b, Cx = d\}$. We then try to generalize these results to the general case for which the feasible region mapping is defined by a set of parameterized nonlinear constraints.

Suppose x^* is a solution to the problem $VI(\varepsilon^*)$. Let

$$T_1 = \{x \mid F(x^*, \varepsilon^*)^T x = 0, A_i x \geq 0 \text{ for } i \in I(x^*), Cx = 0\},$$

where A_i is the i th row vector of matrix A . We now invoke the following basic assumptions concerning the function $F(\cdot, \cdot)$ near (x^*, ε^*) . We assume for some neighborhoods U of x^* and V of ε^* ,

(a) $F(\cdot, \cdot)$ is continuous on $U \times V$,

(b) For some $L > 0$,

$$\|F(x, \varepsilon) - F(x, \varepsilon^*)\| \leq L \|\varepsilon - \varepsilon^*\| \text{ for any } x \in \{x^* + T_1\} \cap U, \varepsilon \in V,$$

(c) $F(\cdot, \cdot)$ is differentiable at (x^*, ε^*) .

With additional second order assumptions, we are now able to obtain some properties of the local perturbed solution set.

Proposition 3.1. Suppose $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on T_1 , i.e., GSOC is satisfied. Then for some neighborhood M of x^* , $S(\varepsilon) \cap M$ is nonempty and is Lipschitz continuous at (x^*, ε^*) . Furthermore, for any direction $\varepsilon_0 \in \mathbb{R}^k$, the set $D(\varepsilon_0)$ is contained in the solution set $S(\varepsilon_0)^\perp$ of the following linear variational inequality problem

$VI(\varepsilon_0)^\perp$: find $x \in T_1$ satisfying $[\nabla_x F(x^*, \varepsilon^*)x + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0]^T(x' - x) \geq 0$ for any $x' \in T_1$.

Here we try to characterize the set $D(\varepsilon_0)$ that contains first order information of the perturbed solutions. The theorem shows only that $D(\varepsilon_0)$ is contained in $S(\varepsilon_0)^\perp$. But we further conjecture that $D(\varepsilon_0) = S(\varepsilon_0)^\perp$ for any ε_0 . Notice that $S(\varepsilon_0)^\perp$ is also a compact set if we assume the hypotheses of Proposition 3.1. To see this, suppose some sequence $\{x^n\}$ satisfies $x^n \in S(\varepsilon_0)^\perp$ and $x^n \rightarrow \infty$. Since $x = 0$ is in T_1 , we obtain

$$-[\nabla_x F(x^*, \varepsilon^*)x^n + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0]^T x^n \geq 0 \quad \text{for all } n.$$

Let z be a limiting vector of the sequence $\{x^n / \|x^n\|\}$. Then $z \in T_1$ and $\|z\| = 1$. But from the inequality, we have $z^T \nabla_x F(x^*, \varepsilon^*)z \leq 0$, which is a contradiction to the GSOC assumption. So $S(\varepsilon_0)^\perp$ is bounded. It is easy to see that $S(\varepsilon_0)^\perp$ is a closed set.

Example 3.1. Consider the following three-dimensional example

$VI(\varepsilon)$: find $x \in P$ satisfying $F(x, \varepsilon)^T(x' - x) \geq 0$ for any $x' \in P$

where $P = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$, $0 \leq \varepsilon < \infty$, and $F(x, \varepsilon) = (x_1 + x_2 - \varepsilon, x_1 + x_2 - \varepsilon, 1)^T$.

Note that $x^* = (0, 0, 0)$ is the unique solution to $VI(0)$ and that the hypotheses of Proposition 3.1 are satisfied. In this case, the perturbed solution set is given by $S(\varepsilon) = \{x \mid x_1 + x_2 = \varepsilon, x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$ for $0 \leq \varepsilon < \infty$, which is not differentiable at (x^*, ε^*) . However, it is easy to verify that $D(\varepsilon_0) = S(\varepsilon_0)^\perp = \{x \mid x_1 + x_2 = \varepsilon_0, x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$ for $\varepsilon_0 \geq 0$.

The next theorem shows that if the general second order condition is strengthened slightly, then the local perturbed solution set is directionally differentiable. We let $\text{span}(T_1)$ denote the linear subspace spanned by the set T_1 .

Proposition 3.2. Suppose $\nabla_x F(x^*, \varepsilon^*)$ is positive definite on $\text{span}(T_1)$. Then for some neighborhood M of x^* , $S(\varepsilon) \cap M$ is nonempty and is directionally differentiable at (x^*, ε^*) for any direction ε_0 . Furthermore, the derivative $d(\varepsilon_0)$ uniquely solves $VI(\varepsilon_0)^\perp$.

Example 3.2. Consider a one-dimensional example with $P = \{x \in \mathbb{R}^1 \mid x \geq 0\}$, $0 \leq \varepsilon < 1$, and let F be given by

$$F(x, \varepsilon) = \begin{cases} 0 & 0 \leq x < \varepsilon^2 \\ (x - \varepsilon^2)/(1 - \varepsilon) & \varepsilon^2 \leq x < \varepsilon \\ x & \varepsilon \leq x < \infty. \end{cases}$$

It is easy to verify that $x^* = 0$ uniquely solves $VI(0)$ and that this problem satisfies the hypotheses of Proposition 3.2. The perturbed solution set in this case is $S(\varepsilon) = [0, \varepsilon^2]$, which is indeed differentiable at $(0, 0)$. However, we notice that the perturbed solutions are not unique in this case. The next result shows that the local perturbed solution would be unique if we further impose some first-order smoothness conditions on the function F .

Proposition 3.3. Assume the hypotheses of Proposition 3.2. Also assume that $F(\cdot, \varepsilon)$ is differentiable in a neighborhood of x^* for ε near ε^* , and that $\nabla_x F(\cdot, \cdot)$ is continuous at (x^*, ε^*) . Then for some neighborhoods M of x^* and N of ε^* , $S(\varepsilon) \cap M$ is single valued for each $\varepsilon \in N$ and is directionally differentiable at (x^*, ε^*) .

The Nonlinear Problem

We now consider the problem $VI(\varepsilon)$ with perturbed nonlinear constraints. From this point on, we always assume that the problem satisfies Assumptions 2.1-2.3 at (x^*, ε^*) . By Theorem 2.4, MFCQ and GSOC would imply that for some neighborhoods M of x^* and N of ε^* , and for some $\mu > 0$,

$$\|x(\varepsilon) - x^*\| \leq \mu \|\varepsilon - \varepsilon^*\| \quad \text{for any } x(\varepsilon) \in SP(\varepsilon) \cap M, \varepsilon \in N.$$

Furthermore, for each $(u(\varepsilon), v(\varepsilon)) \in K(x(\varepsilon), \varepsilon)$, some $(u^*, v^*) \in K(x^*, \varepsilon^*)$ satisfies

$$\|(u(\varepsilon), v(\varepsilon)) - (u^*, v^*)\| \leq \mu \|\varepsilon - \varepsilon^*\|.$$

Now suppose ε_0 is a vector in \mathbb{R}^k . Let x^\perp be a vector in the set $D(\varepsilon_0)$ associated with mapping $SP(\varepsilon) \cap M$ and direction ε_0 . By definition, x^\perp is the limit of some convergent sequence of the form

$$\{[x(\varepsilon^* + t_n \varepsilon_0) - x^*]/t_n\}_{n \in \mathbb{N}}, \text{ where } x(\varepsilon^* + t_n \varepsilon_0) \in SP(\varepsilon^* + t_n \varepsilon_0) \cap M.$$

We now want to derive some constraints that the vector x^L must satisfy.

Lemma 3.2. Suppose $[x^n - x^*]/t_n \rightarrow x^L, [(u^n, v^n) - (u^*, v^*)]/t_n \rightarrow (u^L, v^L)$, where $x^n \in SP(\varepsilon^* + t_n \varepsilon_0) \cap M$ and $(u^n, v^n) \in K(x^n, \varepsilon^* + t_n \varepsilon_0)$. Then (x^L, u^L, v^L) satisfies the following system

$$\begin{aligned}
& \nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) x^L + \nabla_\varepsilon L_d(x^*, u^*, v^*, \varepsilon^*) \varepsilon_0 - \nabla_x g(x^*, \varepsilon^*)^T u^L - \nabla_x h(x^*, \varepsilon^*)^T v^L = 0 \\
& \nabla_x g_i(x^*, \varepsilon^*) x^L + \nabla_\varepsilon g_i(x^*, \varepsilon^*) \varepsilon_0 = 0 \quad \text{if } i \in I_1, \text{ or } i \in I_2 \text{ and } u_i^L > 0 \\
& \nabla_x g_i(x^*, \varepsilon^*) x^L + \nabla_\varepsilon g_i(x^*, \varepsilon^*) \varepsilon_0 \geq 0 \quad \text{if } i \in I_2 \text{ and } u_i^L = 0 \\
& \nabla_x h(x^*, \varepsilon^*) x^L + \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0 = 0 \\
& u_i^L \text{ UIS for } i \in I_1, u_i^L \geq 0 \text{ for } i \in I_2, u_i^L = 0 \text{ for } i \in I_3.
\end{aligned} \tag{3.1}$$

Proof. For convenience, let $\varepsilon^n = \varepsilon^* + t_n \varepsilon_0$. Since GKKT holds for ε^* and each $\varepsilon^* + t_n \varepsilon_0$, we have the following relations for n sufficiently large,

$$\begin{aligned}
& [L_d(x^n, u^n, v^n, \varepsilon^n) - L_d(x^*, u^*, v^*, \varepsilon^*)]/t_n = 0 \\
& [g_i(x^n, \varepsilon^n) - g_i(x^*, \varepsilon^*)]/t_n = 0 \quad \text{if } i \in I_1, \text{ or } i \in I_2 \text{ and } u_i^L > 0 \\
& [g_i(x^n, \varepsilon^n) - g_i(x^*, \varepsilon^*)]/t_n \geq 0 \quad \text{if } i \in I_2 \text{ and } u_i^L = 0 \\
& [h(x^n, \varepsilon^n) - h(x^*, \varepsilon^*)]/t_n = 0 \\
& [u_i^n - u_i^*]/t_n \geq 0 \quad \text{if } i \in I_2 \\
& [u_i^n - u_i^*]/t_n = 0 \quad \text{if } i \in I_3.
\end{aligned}$$

Letting $n \rightarrow \infty, n \in \mathbb{N}$, we observe that (x^L, u^L, v^L) satisfies the desired system (3.1).

□

For now, in order to ensure that there is always a convergent sequence of the form

$$\{[(u^n, v^n) - (u^*, v^*)]/t_n\},$$

we assume that $K(x^*, \varepsilon^*)$ is a singleton. Notice that LI is a sufficient condition for such a requirement. Recently, Kyparisis [7] showed that this uniqueness condition is equivalent to the following *strict Mangasarian-Fromovitz constraint qualification (SMFCQ)*:

$\nabla_x g_i(x^*, \varepsilon^*), i \in I_1, \nabla_x h_j(x^*, \varepsilon^*), j = 1, \dots, l$ are linearly independent and for some $z \in \mathbb{R}^c$,

$$\nabla_x g_i(x^*, \varepsilon^*) z > 0, \quad i \in I_2$$

$$\nabla_x g_i(x^*, \varepsilon^*) z = 0, \quad i \in I_1$$

$$\nabla_x h_j(x^*, \varepsilon^*) z = 0, \quad j = 1, \dots, l.$$

Note that we do not need this condition for problems defined on polyhedral sets (see Proposition 3.1).

Theorem 3.1. Suppose GKKT, SMFCQ and SOC hold at (x^*, ε^*) . Then for any direction $\varepsilon_0 \in \mathbb{R}^k$, $D(\varepsilon_0)$ is a compact set and is contained in $S(\varepsilon_0)^\perp$, the solution set of the following linear variational inequality problem

VI $(\varepsilon_0)^\perp$: find $x \in \Omega^\perp$ satisfying

$$[\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) x + \nabla_\varepsilon L_d(x^*, u^*, v^*, \varepsilon^*) \varepsilon_0]^\top (x' - x) \geq 0 \quad \text{for any } x' \in \Omega^\perp$$

where $\Omega^\perp = \{x \mid F(x^*, \varepsilon^*) x + (u^*)^\top \nabla_\varepsilon g(x^*, \varepsilon^*) \varepsilon_0 + (v^*)^\top \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0 = 0, \nabla_x g_i(x^*, \varepsilon^*) x + \nabla_\varepsilon g_i(x^*, \varepsilon^*) \varepsilon_0 \geq 0 \text{ for } i \in I(x^*), \nabla_x h(x^*, \varepsilon^*) x + \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0 = 0\}$.

Proof. The compactness of $D(\varepsilon_0)$ follows immediately from Theorem 2.4 and Lemma 3.1. Now consider any convergent sequence $[x^n - x^*] / t_n \rightarrow x^\perp$ with $x^n \in \text{SP}(\varepsilon^* + t_n \varepsilon_0) \cap M$ and $n \in \mathbb{N}$. Since $K(x^*, \varepsilon^*) = \{(u^*, v^*)\}$ is a singleton, by Theorem 2.4,

$$\|(u^n, v^n) - (u^*, v^*)\| \leq \mu \|\varepsilon_0\| t_n \quad \text{for some } (u^n, v^n) \in K(x^n, \varepsilon^* + t_n \varepsilon_0).$$

Thus, for some subsequence $N' \subseteq \mathbb{N}$, $[(u^n, v^n) - (u^*, v^*)] / t_n \rightarrow (u^\perp, v^\perp)$, $n \in N'$. By Lemma 3.2, any such $(x^\perp, u^\perp, v^\perp)$ satisfies the system (3.1), which can be restated as follows if we view (u^\perp, v^\perp) as a dual solution,

$$[\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) x^\perp + \nabla_\varepsilon L_d(x^*, u^*, v^*, \varepsilon^*) \varepsilon_0]^\top (x' - x^\perp) \geq 0 \quad \text{for any } x' \in \Omega^0$$

where $\Omega^0 = \{x \mid \nabla_x g_i(x^*, \varepsilon^*) x + \nabla_\varepsilon g_i(x^*, \varepsilon^*) \varepsilon_0 = 0 \text{ for } i \in I_1, \nabla_x g_i(x^*, \varepsilon^*) x + \nabla_\varepsilon g_i(x^*, \varepsilon^*) \varepsilon_0 \geq 0 \text{ for } i \in I_2, \nabla_x h(x^*, \varepsilon^*) x + \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0 = 0\}$.

Now we need show only that $\Omega^\perp = \Omega^0$.

$\Omega^0 \subseteq \Omega^\perp$: Suppose $x \in \Omega^0$. Then

$$\begin{aligned} & F(x^*, \varepsilon^*)^\top x + (u^*)^\top \nabla_\varepsilon g(x^*, \varepsilon^*) \varepsilon_0 + (v^*)^\top \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0 \\ &= [\nabla_x g(x^*, \varepsilon^*)^\top u^* + \nabla_x h(x^*, \varepsilon^*)^\top v^*]^\top x + (u^*)^\top \nabla_\varepsilon g(x^*, \varepsilon^*) \varepsilon_0 + (v^*)^\top \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0 \\ &= [\nabla_x g(x^*, \varepsilon^*) x + \nabla_\varepsilon g(x^*, \varepsilon^*) \varepsilon_0]^\top u^* + [\nabla_x h(x^*, \varepsilon^*) x + \nabla_\varepsilon h(x^*, \varepsilon^*) \varepsilon_0]^\top v^* \\ &= 0. \end{aligned}$$

Thus, $x \in \Omega^\perp$.

$\Omega^\perp \subseteq \Omega^0$: Suppose $x \in \Omega^\perp$. Then

$$\begin{aligned} 0 &= F(x^*, \varepsilon^*)^T x + (u^*)^T \nabla_{\varepsilon} g(x^*, \varepsilon^*) \varepsilon_0 + (v^*)^T \nabla_{\varepsilon} h(x^*, \varepsilon^*) \varepsilon_0 \\ &= [\nabla_x g(x^*, \varepsilon^*) x + \nabla_{\varepsilon} g(x^*, \varepsilon^*) \varepsilon_0]^T u^* + [\nabla_x h(x^*, \varepsilon^*) x + \nabla_{\varepsilon} h(x^*, \varepsilon^*) \varepsilon_0]^T v^* \\ &= [\nabla_x g(x^*, \varepsilon^*) x + \nabla_{\varepsilon} g(x^*, \varepsilon^*) \varepsilon_0]^T u^*. \end{aligned}$$

But this equality and the fact that $\nabla_x g(x^*, \varepsilon^*) x + \nabla_{\varepsilon} g(x^*, \varepsilon^*) \varepsilon_0 \geq 0$ for any $i \in I$ and $u_i^* > 0$ imply $\nabla_x g_i(x^*, \varepsilon^*) x + \nabla_{\varepsilon} g_i(x^*, \varepsilon^*) \varepsilon_0 = 0$ for $i \in I_1$. Thus, $x \in \Omega^0$.

□

The theorem shows that $D(\varepsilon_0) \subseteq S(\varepsilon_0)^\perp$, which is a partial characterization of the set $D(\varepsilon_0)$. We suspect that $D(\varepsilon_0) = S(\varepsilon_0)^\perp$ is always the case.

Conjecture 3.1. Assume the hypotheses of Theorem 3.1. Then $D(\varepsilon_0) = S(\varepsilon_0)^\perp$ for any direction $\varepsilon_0 \in \mathbb{R}^k$.

In the next theorem, we show that the conjecture is true if the LI condition is satisfied at (x^*, ε^*) and $VI(\varepsilon_0)^\perp$ (or system (3.1)) satisfies the *strict complementary slackness condition* (SCS), that is, for any $x^L \in S(\varepsilon_0)^\perp$, there is some dual solution (u^L, v^L) satisfying $u_i^L > 0$ if $i \in I_2$ and $\nabla_x g_i(x^*, \varepsilon^*) x^L + \nabla_{\varepsilon} g_i(x^*, \varepsilon^*) \varepsilon_0 = 0$.

Theorem 3.2. Assume the hypotheses of Theorem 3.1. Suppose that LI holds at (x^*, ε^*) and that SCS holds at each solution point of $VI(\varepsilon_0)^\perp$. Then $D(\varepsilon_0) = S(\varepsilon_0)^\perp$.

Proof. Suppose $x^L \in S(\varepsilon_0)^\perp$. Then for some dual solution (u^L, v^L) , (x^L, u^L, v^L) satisfies system (3.1) and the SCS condition. We want to show that some functions $x(t) \in SP(x^* + t\varepsilon_0) \cap M$ and $(u(t), v(t)) \in K(x(t), \varepsilon^* + t\varepsilon_0)$ satisfy $(x'(0), u'(0), v'(0)) = (x^L, u^L, v^L)$. And then, $x^L \in D(\varepsilon_0)$ follows immediately.

Consider the following system of equations

$$L_d(x, u, v, \varepsilon) = 0$$

$$g_i(x, \varepsilon) = 0 \quad \text{if } i \in I_1, \text{ or } i \in I_2 \text{ and } u_i^L > 0$$

$$u_i = 0 \quad \text{if } i \in I_3, \text{ or } i \in I_2 \text{ and } u_i^L = 0$$

$$h(x, \varepsilon) = 0.$$

For simplicity of notation, we let $y = (x, u, v)$, and let $H(y, \varepsilon) = 0$ denote this system of equations.

We also assume $\{1, \dots, i_1\} = I_1 \cup \{i \mid i \in I_2 \text{ and } u_i^L > 0\}$. Now with this notation, the Jacobian matrix of H with respect to $y = (x, u, v)$ is given by

$$J = \begin{bmatrix} \nabla_x L_d & -\nabla_x g_1 & \cdots & -\nabla_x g_m & -\nabla_x h_1 & \cdots & -\nabla_x h_l \\ \hline \nabla_x g_1 & 0 & & 0 & & & \\ \vdots & & & & & & \\ \nabla_x g_{i_1} & & & & & & 0 \\ 0 & & & 1 & & & \\ \vdots & & & & \ddots & & \\ 0 & 0 & & & & & 1 \\ \hline \nabla_x h_1 & & & & & & \\ \vdots & & & & & & \\ \nabla_x h_l & & 0 & & & & 0 \end{bmatrix}$$

It is not hard to show that SOC and LI imply J is nonsingular at (x^*, u^*, v^*) .

We now consider equations of the form $H(y^* + t y^L + J^T z, \varepsilon^* + t \varepsilon_0) = 0$ with t and z as variables. Notice $(t, z) = (0, 0)$ satisfies the equations. The Jacobian matrix of this system at $(0, 0)$ with respect to z is JJ^T , which is nonsingular. By the Implicit Function Theorem, for some differentiable function $z(t)$ in a neighborhood of $t = 0$, $(t, z(t))$ satisfies the equations and $z(0) = 0$. Furthermore, differentiating both sides of $H(y^* + t y^L + J^T z(t), \varepsilon^* + t \varepsilon_0) = 0$ and using the chain rule shows that $z'(0)$ is determined by the equations $J y^L + JJ^T z'(0) + K \varepsilon_0 = 0$, where

$$K = [\nabla_\varepsilon L_d(x^*, u^*, v^*, \varepsilon^*)^T, \nabla_\varepsilon g_1(x^*, \varepsilon^*)^T, \dots, \nabla_\varepsilon g_{i_1}(x^*, \varepsilon^*)^T, 0, \dots, 0, \nabla_\varepsilon h(x^*, \varepsilon^*)^T]^T.$$

Note that each row of K is the gradient of an equation in the system $H(y, \varepsilon) = 0$ with respect to ε . Since $y^L = (x^L, u^L, v^L)$ satisfies system (3.1), we have $Jy^L + K\varepsilon_0 = 0$, which in turn implies that $z'(0) = 0$. Therefore, if we let

$$y(t) \equiv (x(t), u(t), v(t)) = y^* + ty^L + J^T z(t),$$

then $y'(0) = y^L = (x^L, u^L, v^L)$.

Finally, in order to show that $x(t) \in SP(x^* + t\varepsilon_0) \cap M$ and $(u(t), v(t)) \in K(x(t), \varepsilon^* + t\varepsilon_0)$, we need to prove only that $g_i(t) \equiv g_i(x(t), \varepsilon^* + t\varepsilon_0) \geq 0$ ($t > 0$) for $i \in I_2$ and $u_i^L = 0$, and that $u_i(t) \geq 0$ ($t > 0$) for $i \in I_2$ and $u_i^L > 0$. Since we assume SCS holds for $VI(\varepsilon_0)^\perp$ at (x^L, u^L, v^L) , by the chain rule we have

$$g_i'(0) = \nabla_x g_i(x^*, \varepsilon^*) x^L + \nabla_\varepsilon g_i(x^*, \varepsilon^*) \varepsilon_0 > 0 \quad \text{for } i \in I_2 \text{ and } u_i^L = 0, \text{ and}$$

$$u_i'(0) = u_i^L > 0 \quad \text{for } i \in I_2 \text{ and } u_i^L > 0.$$

Therefore, for t positive and small enough, the desired property is guaranteed.

Thus we have shown that $S(\varepsilon_0)^\perp \subseteq D(\varepsilon_0)$. On the other hand, by Theorem 3.1, $D(\varepsilon_0) \subseteq S(\varepsilon_0)^\perp$. Hence, $D(\varepsilon_0) = S(\varepsilon_0)^\perp$.

□

It is worth noting that the SCS condition for the linear problem $VI^-(\varepsilon_0)$ does not imply the SCS condition for the original problem $VI(\varepsilon^*)$ at x^* . The following example illustrates this fact.

Example 3.3. Consider a problem $VI(\varepsilon)$ with feasible region $\Omega = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$ for $\varepsilon \geq 0$ and function $F(x, \varepsilon) = (x_1 + 2x_2 - \varepsilon, 2x_1 + x_2 - \varepsilon, 1)^\top$. Note that $x^* = (0, 0, 0)$ is the unique solution to problem $VI(0)$ and that the SCS condition is not satisfied at x^* . However, it is not hard to verify that in this case the linear problem $VI(\varepsilon_0)^\perp$ is specified by $\Omega^\perp = \Omega$ and $F^\perp = F(x, \varepsilon_0)$ for any $\varepsilon_0 > 0$, which has a unique solution $x^L = (\varepsilon_0/3, \varepsilon_0/3, 0)$. It is obvious that the SCS condition holds at x^L .

Lemma 3.1 shows that the local perturbed stationary point set is directionally differentiable if and only if the set $D(\varepsilon_0)$ is single valued for any $\varepsilon_0 \in \mathbb{R}^k$. Therefore, we will now strengthen the conditions of Theorem 3.2 to make $S(\varepsilon_0)^+$ a singleton.

Theorem 3.3. Assume the hypotheses of Theorem 3.1. If $\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)$ is positive definite on $\text{span}(T_1)$, then for some neighborhood M of x^* , $SP(\varepsilon) \cap M$ is directionally differentiable at (x^*, ε^*) for any direction ε_0 . Furthermore, the derivative $d(\varepsilon_0)$ uniquely solves the linear variational inequality problem $VI(\varepsilon_0)^+$.

Proof. We first show $VI(\varepsilon_0)^+$ has a unique solution. Suppose x^1 and $x^2 \in S(\varepsilon_0)^+$. Then $x^1 - x^2 \in \text{span}(T_1)$ and

$$[\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) x^1 + \nabla_\varepsilon L_d(x^*, u^*, v^*, \varepsilon^*) \varepsilon_0]^T (x^2 - x^1) \geq 0,$$

$$[\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) x^2 + \nabla_\varepsilon L_d(x^*, u^*, v^*, \varepsilon^*) \varepsilon_0]^T (x^1 - x^2) \geq 0.$$

Adding these two inequalities, we obtain $(x^1 - x^2)^T \nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) (x^1 - x^2) \leq 0$. Since we assume that $\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)$ is positive definite on $\text{span}(T_1)$, $x^1 = x^2$. By Theorem 3.1, $D(\varepsilon_0) \subseteq S(\varepsilon_0)^+$. Therefore, $D(\varepsilon_0)$ is also a singleton.

□

The next corollary follows immediately if we notice that

$$\begin{aligned} \text{span}(T_1) &\subseteq T_2 = \{x \in \mathbb{R}^n \mid \nabla_x g_i(x^*, \varepsilon^*) x = 0 \text{ for } i \in I_1, \nabla_x h(x^*, \varepsilon^*) x = 0\} \\ &= \{x \in \mathbb{R}^n \mid F(x^*, \varepsilon^*)^T x = 0, \nabla_x h(x^*, \varepsilon^*) x = 0\}. \end{aligned}$$

Corollary 3.1. Assume the hypotheses of Theorem 3.1. Also assume that the *strong second order condition* (SSOC) holds at (x^*, ε^*) , i.e., $\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)$ is positive definite on T_2 . Then for some neighborhood M of x^* , $SP(\varepsilon) \cap M$ is directionally differentiable at (x^*, ε^*) .

Notice that the directional derivative $d(\varepsilon_0)$ uniquely solves the variational inequality problem $VI(\varepsilon_0)^+$. By applying Theorems 2.4 and 3.3 to $VI(\varepsilon_0)^+$, we know that $d(\varepsilon_0)$ is locally Lipschitz

continuous and directionally differentiable at each point $\varepsilon_0 \in \mathbb{R}^k$. For the case in which the feasible region is unperturbed, we can further show that the directional derivative $d(\varepsilon_0)$ is globally Lipschitz continuous with respect to ε_0 . Suppose (e_1, \dots, e_r) is an orthonormal basis of the subspace $\text{span}(T_1)$. Let $E = [e_1, \dots, e_r]$ be an $n \times r$ matrix with e_1, \dots, e_r as its columns, and let

$$G = E^T \nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) E,$$

the projected (on to $\text{span}(T_1)$) matrix of $\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)$. Then, $\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)$ is positive definite on $\text{span}(T_1)$ if and only if G is positive definite. We let λ_{\min} denote the minimum eigenvalue of the symmetric matrix $[G + G^T]/2$. For any arbitrary matrix A , the norm of A is usually defined as

$$\|A\| = \sup_{\|x\|=1} \{\|Ax\|\}.$$

The next corollary shows that the derivative $d(\varepsilon_0)$ of the local perturbed stationary point set is Lipschitz continuous with respect to ε_0 if the feasible region is not perturbed.

Corollary 3.2. Assume the hypotheses of Theorem 3.3. Further assume that $\Omega(\varepsilon) \equiv \Omega = \{x \mid g(x) \geq 0, h(x) = 0\}$. Then for any $\varepsilon_0', \varepsilon_0'' \in \mathbb{R}^k$,

$$\|d(\varepsilon_0') - d(\varepsilon_0'')\| \leq [\|\nabla_\varepsilon F(x^*, \varepsilon^*)\| / \lambda_{\min}] \|\varepsilon_0' - \varepsilon_0''\|.$$

Proof. Since g and h are independent of ε , $VI(\varepsilon_0)^\perp$ becomes

$$\text{find } x \in \Omega^\perp \text{ satisfying } [\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)x + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0]^\top (x' - x) \geq 0 \text{ for any } x' \in \Omega^\perp$$

where $\Omega^\perp = \{x \mid F(x^*, \varepsilon^*)x = 0, \nabla_x g_i(x^*)x \geq 0 \text{ for } i \in I, \nabla_x h(x^*)x = 0\}$.

Therefore, we have

$$\begin{aligned} & [\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)d(\varepsilon_0') + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0']^\top [d(\varepsilon_0') - d(\varepsilon_0'')] \leq 0 \leq \\ & \leq [\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)d(\varepsilon_0'') + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0'']^\top [d(\varepsilon_0') - d(\varepsilon_0'')]. \end{aligned}$$

Since $\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)$ is positive definite on $\text{span}(T_1)$, this inequality implies

$$\begin{aligned} & \lambda_{\min} \|d(\varepsilon_0') - d(\varepsilon_0'')\|^2 \leq [d(\varepsilon_0') - d(\varepsilon_0'')]^\top \nabla_x L_d(x^*, u^*, v^*, \varepsilon^*) [d(\varepsilon_0') - d(\varepsilon_0'')] \\ & \leq [(\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)d(\varepsilon_0') + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0') - (\nabla_x L_d(x^*, u^*, v^*, \varepsilon^*)d(\varepsilon_0'') + \\ & \quad + \nabla_\varepsilon F(x^*, \varepsilon^*)\varepsilon_0'')]^\top [d(\varepsilon_0') - d(\varepsilon_0'')] \\ & \leq [\nabla_\varepsilon F(x^*, \varepsilon^*)(\varepsilon_0' - \varepsilon_0'')]^\top [d(\varepsilon_0') - d(\varepsilon_0'')] \\ & \leq \|\nabla_\varepsilon F(x^*, \varepsilon^*)\| \|d(\varepsilon_0') - d(\varepsilon_0'')\| \|\varepsilon_0' - \varepsilon_0''\|. \end{aligned}$$

Dividing both sides by $\|d(\varepsilon_0') - d(\varepsilon_0'')\|$, we obtain the desired inequality.

□

In the previous theorems, the perturbed stationary point need not be unique (see Example 3.1 or 3.2). The next theorem describes a condition that ensures the uniqueness of the perturbed stationary point.

Theorem 3.4. Assume the hypotheses of Corollary 3.1. Suppose that $F(\cdot, \varepsilon)$, $\nabla_x g(\cdot, \varepsilon)$, $\nabla_x h(\cdot, \varepsilon)$ are differentiable in a neighborhood of x^* for each ε near ε^* and that $\nabla_x F(\cdot, \cdot)$, $\nabla_x^2 g(\cdot, \cdot)$, $\nabla_x^2 h(\cdot, \cdot)$ are continuous at (x^*, ε^*) . Then for some neighborhood M of x^* and N of ε^* , $SP(\varepsilon) \cap M$ is single valued for each $\varepsilon \in N$.

Proof. We prove the theorem by showing the contraposition. Suppose some sequences $\{x^n\}$, $\{y^n\}$, and $\{\varepsilon^n\}$ satisfy $x^n \neq y^n$, $x^n, y^n \in SP(\varepsilon^n)$, and $x^n, y^n \rightarrow x^*$, $\varepsilon^n \rightarrow \varepsilon^*$. Let (u_x^n, v_x^n) be a vector in $K(x^n, \varepsilon^n)$ and (u_y^n, v_y^n) a vector in $K(y^n, \varepsilon^n)$. By Theorem 2.4, both (u_x^n, v_x^n) and (u_y^n, v_y^n) approach (u^*, v^*) as n approaches ∞ . Without loss of generality we assume that $(x^n - y^n) / \|x^n - y^n\| \rightarrow z$. Notice that z satisfies $\|z\| = 1$, $\nabla_x g_i(x^*, \varepsilon^*) z = 0$ for $i \in I_1(x^*)$, and $\nabla_x h(x^*, \varepsilon^*) z = 0$. Therefore, we have $z \neq 0$ and $z \in T_2$. Now let $(x(t), u(t), v(t)) = t(x^n, u_x^n, v_x^n) + (1-t)(y^n, u_y^n, v_y^n)$ and consider the following function

$$s(t) = L_d(x(t), u(t), v(t), \varepsilon^n)^T (x^n - y^n) + g(x(t), \varepsilon^n)^T (u_x^n - u_y^n) + h(x(t), \varepsilon^n)^T (v_x^n - v_y^n).$$

Notice that $s'(t) = (x^n - y^n)^T \nabla_x L_d(x(t), u(t), v(t), \varepsilon^n) (x^n - y^n)$. Since the GKKT condition holds for both (x^n, u_x^n, v_x^n) and (y^n, u_y^n, v_y^n) , it is easy to verify that $s(0) \geq 0$ and $s(1) \leq 0$. By the mean-value theorem, for some $0 \leq t_n \leq 1$, we have

$$s'(t_n) = (x^n - y^n)^T \nabla_x L_d(x(t_n), u(t_n), v(t_n), \varepsilon^n) (x^n - y^n) \leq 0.$$

Dividing both sides by $\|x^n - y^n\|^2$ and taking the limit as $n \rightarrow \infty$, we find a contradiction to the GSSOC assumption and thereby complete the proof.

□

Theorems 3.3 and 3.4 extend a recent result by Kyparisis [6, Theorem 4.3]. Specifically, we weakened his LI condition to the SMFCQ condition. We now discuss the possibility of further relaxing the SMFCQ condition. In the prior analysis, we use the fact that for each sequence $\{ (x^n, \varepsilon^n) \}_{n \in \mathbb{N}}$ satisfying $x^n \in SP(\varepsilon^n)$ and $x^n \rightarrow x^*$, $\varepsilon^n \rightarrow \varepsilon^*$, there is a vector $(u^*, v^*) \in K(x^*, \varepsilon^*)$ and a sequence $(u^n, v^n) \in K(x^n, \varepsilon^n)$, $n \in \mathbb{N}$ for which $\| (u^n, v^n) - (u^*, v^*) \| / \|\varepsilon^n - \varepsilon^*\|$ is uniformly bounded. And, this property is guaranteed by our assumption that $K(x^*, \varepsilon^*)$ is a singleton. We hope this assumption can somehow be relaxed. Now let us consider some of the properties of the mapping $K(\cdot, \cdot)$. We notice that for each $x(\varepsilon) \in SP(\varepsilon)$, $x(\varepsilon)$ solves the following linearized problem:

$$\begin{aligned} & \text{minimize} && F(x(\varepsilon), \varepsilon)z && (3.2) \\ & \text{subject to} && \nabla_x g(x(\varepsilon), \varepsilon)z \geq \nabla_x g(x(\varepsilon), \varepsilon)x(\varepsilon) - g(x(\varepsilon), \varepsilon) \\ & && \nabla_x h(x(\varepsilon), \varepsilon)z = \nabla_x h(x(\varepsilon), \varepsilon)x(\varepsilon) - h(x(\varepsilon), \varepsilon). \end{aligned}$$

For purpose of convenience, let

$$\begin{aligned} a(x(\varepsilon), \varepsilon) &= \nabla_x g(x(\varepsilon), \varepsilon)x(\varepsilon) - g(x(\varepsilon), \varepsilon), \text{ and} \\ b(x(\varepsilon), \varepsilon) &= \nabla_x h(x(\varepsilon), \varepsilon)x(\varepsilon) - h(x(\varepsilon), \varepsilon). \end{aligned}$$

Then the set of Lagrange multipliers $K(x(\varepsilon), \varepsilon)$ is specified by the solution set of the dual problem of (3.2):

$$\begin{aligned} & \text{maximize} && a(x(\varepsilon), \varepsilon)^T u + b(x(\varepsilon), \varepsilon)^T v && (3.3) \\ & \text{subject to} && \nabla_x g(x(\varepsilon), \varepsilon)^T u + \nabla_x h(x(\varepsilon), \varepsilon)^T v = F(x(\varepsilon), \varepsilon) \\ & && u \geq 0. \end{aligned}$$

Therefore, if MFCQ and GSOC are satisfied at (x^*, ε^*) , then by Theorem 2.4, $K(x(\varepsilon), \varepsilon)$ is a bounded polyhedron for $(x(\varepsilon), \varepsilon)$ in a neighborhood of (x^*, ε^*) and for each $(u(\varepsilon), v(\varepsilon)) \in K(x(\varepsilon), \varepsilon)$, there is some $(u_{\varepsilon^*}, v_{\varepsilon^*}) \in K(x^*, \varepsilon^*)$ for which

$$\| (u(\varepsilon), v(\varepsilon)) - (u_{\varepsilon^*}, v_{\varepsilon^*}) \| \leq \mu \|\varepsilon - \varepsilon^*\|.$$

We observe from the proof of Theorem 3.1 that if for each sequence $\{ (x^n, \varepsilon^n) \}_{n \in \mathbb{N}}$ satisfying $x^n \in SP(\varepsilon^n)$ and $x^n \rightarrow x^*$ and $\varepsilon^n \rightarrow \varepsilon^*$, we can somehow select a subsequence $N' \subseteq \mathbb{N}$, and a sequence $\{ (u^n, v^n) \}_{n \in N'}$ satisfying $(u^n, v^n) \in K(x^n, \varepsilon^n)$ for $n \in N'$ and a vector $(u^*, v^*) \in K(x^*, \varepsilon^*)$ so that

$$\| (u^n, v^n) - (u^*, v^*) \| \leq \rho \| \varepsilon^n - \varepsilon^* \| \text{ for some } \rho > 0,$$

then all the results we obtained in this section are still valid. We also notice that this property is satisfied in the following two cases:

(i) some basic optimal solution (u^*, v^*) to the dual problem (3.3) corresponding to ε^* has nondegenerate u variables,

or, more generally,

(ii) for some subsequence $N' \subseteq N$ and a basis $B(\cdot, \cdot)$ of (3.3), $B(x^n, \varepsilon^n)$ for $n \in N'$ is a sequence of optimal bases and $B(x^*, \varepsilon^*)$ is invertible.

4. Relations to Sensitivity Analysis for Optimization Problems

As has long been recognized, a function F from \mathbb{R}^n to \mathbb{R}^n can be written as the gradient of some function f from \mathbb{R}^n to \mathbb{R}^1 (i.e., $F(x) = \nabla f(x)$) if and only if the Jacobian matrix $\nabla F(x)$ is symmetric for all x . And in this special case, the GKKT condition for problem VI becomes the ordinary *Karush-Kuhn-Tucker condition* (KKT) for the following optimization problem:

$$\begin{aligned} \text{MIN:} \quad & \text{minimize } f(x) \\ & \text{subject to } g(x) \geq 0 \\ & \quad \quad \quad h(x) = 0 \end{aligned}$$

Let L be the usual Lagrange function associated with problem MIN, i.e.,

$$L(x, u, v) = f(x) - g(x)^T u - h(x)^T v.$$

Then by definition, $L_d(x, u, v) = \nabla_x L(x, u, v)$ and $\nabla_x L_d(x, u, v) = \nabla_x^2 L(x, u, v)$. The optimization problem MIN and the variational inequality problem VI (with $F = \nabla f$) are intimately related, though results for one problem class need not translate directly into useful results for the other. For example, as we have noticed, the stationary points for these two problems are identical. However, the conditions for when a stationary point x^* is a local solution for MIN or is a local solution to VI are different. Typically, optimization theory imposes a second order condition ($\nabla_x^2 L(x^*, u^*, v^*)$ is positive definite on T_1) to ensure that x^* is a local solution to MIN. On the other hand, as we observed

previously, a convexity condition (local quasi-concavity of the constraints) will ensure that x^* is a local solution to VI. In general, these second-order and convexity conditions do not imply each other.

Note, however, that whenever the MFCQ is valid, every local solution to MIN is a stationary point. Consequently, when conducting sensitivity analysis for local minimum (i.e., studying properties such as Lipschitz continuity, directional differentiability, and uniqueness, but not existence), we can rely upon results developed for the broader class of stationary points. Since all of our results apply to stationary points of MIN (or, equivalently, of VI with $F = \nabla f$), they provide results concerning perturbed local minima to MIN. In particular, our results specify conditions under which the set of perturbed local minima is directionally differentiable. In contrast, in the context of optimization problems, using weaker conditions (MFCQ instead of our SMFCQ), Robinson [10] has already established Lipschitz continuity, but not differentiability. Jittorntrum [3], evoking a stronger condition (LI instead of SMFCQ) has established the uniqueness and directional differentiability of the perturbed solutions to MIN.

We also note that when applied to optimization problems MIN with linear constraints, our companion paper (Qiu and Magnanti [8]) provides an alternative proof, with somewhat weaker conditions, for the Lipschitz continuity and directional differentiability of the perturbed local solution set. Moreover, the results in that paper will also apply to optimization problems with auxiliary variables that appear only in the constraints.

To conclude this section, we give one example to illustrate the property stated in Theorem 3.1.

Example 4.1. Consider the following perturbed minimization problem MIN (ε):

$$\begin{aligned} & \text{minimize} && (x_1 + x_2)^2 / 2 - \varepsilon (x_1 + x_2) + x_3 \\ & \text{subject to} && x_1 \geq 0, x_2 \geq 0, x_3 = 0 \end{aligned}$$

where $0 \leq \varepsilon < \infty$. Since ε is a scale parameter, we let the perturbation direction $\varepsilon_0 = 1$. It is easy to verify that $x^* = (0, 0, 0)$ solves MIN (0) and that $(u_1^*, u_2^*) = (0, 0)$ and $v^* = 1$ are the corresponding Lagrange multipliers. Notice that the problem satisfies the hypotheses of Theorem 3.1 at (x^*, ε^*) . The perturbed solution set in this case is given by

$$S(\varepsilon) = \{x \mid x_1 + x_2 = \varepsilon, x_1 \geq 0, x_2 \geq 0, x_3 = 0\},$$

which is not differentiable at $\varepsilon^* = 0$. However, just as proved by Theorem 3.1, it is possible to show that the set $D(1) = \{x \mid x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$ is the solution set of the following linear variational inequality problem

$VI(\varepsilon_0)^\perp$: find $x \in \Omega^\perp$ satisfying

$$[\nabla_{xx}^2 L(x^*, u^*, v^*, \varepsilon^*)x + \nabla_{x\varepsilon}^2 L(x^*, u^*, v^*, \varepsilon^*)\varepsilon_0]^T (x' - x) \geq 0 \quad \text{for any } x' \in \Omega^\perp,$$

where $\Omega^\perp = \Omega$ in this special case.

References

- [1] Fiacco, A.V., Introduction to Sensitivity and Stability Analysis in Nonlinear Programming, Academic Press, New York, 1983.
- [2] Fiacco, A.V. and J. Kyparisis, "Sensitivity Analysis in Nonlinear Programming Under Second Order Assumptions," Lecture Notes in Control and Information Sciences, 66, pp. 74-97, 1985.
- [3] Jittorntrum, J., "Solution Point Differentiability Without Strict Complementarity in Nonlinear Programming," Mathematical Programming Study 21, pp. 127-138, 1984.
- [4] Kojima, M., "Strongly Stable Stationary Solutions in Nonlinear Programs," in: S.M. Robinson, ed., Analysis and Computation of Fixed Points, pp. 93-138, Academic Press, New York, 1980.
- [5] Kyparisis, J., "Perturbed Solutions of Variational Inequality Problems Over Polyhedral Sets," Preprint, Department of Decision Sciences, Florida International University, Miami, FL 33199, 1986.
- [6] Kyparisis, J., "Sensitivity Analysis Framework for Variational Inequalities," Preprint, Department of Decision Sciences, Florida International University, Miami, FL 33199, 1985.
- [7] Kyparisis, J., "On Uniqueness of Kuhn-Tucker Multipliers in Nonlinear Programming," Mathematical Programming 32, pp. 242-246, 1985.

- [8] Qiu, Y. and T.L. Magnanti, "Sensitivity Analysis for Variational Inequalities Defined on Polyhedral Sets," Working Paper OR 154-87, Operations Research Center, Massachusetts Institute of Technology, 1987.
- [9] Robinson, S.M., "Strongly Regular Generalized Equations," Mathematics of Operations Research 5 (1), pp. 43-62, 1980.
- [10] Robinson, S.M., "Generalized Equations and Their Solutions, Part II: Applications to Nonlinear Programming," Mathematical Programming Study 19, pp. 200-221, 1982.
- [11] Robinson, S.M., "Implicit B-Differentiability in Generalized Equations," Technical Summary Report No. 2854, Mathematics Research Center, University of Wisconsin-Madison, 1985.
- [12] Robinson, S.M., "Stability Theory for Systems of Inequalities, Part II: Differentiable Nonlinear Systems," SIAM Journal on Numerical Analysis 13 (1976) 497-513.
- [13] Tobin, R.L., " Sensitivity Analysis for Variational Inequalities," Journal of Optimization Theory and Applications 48 (1), pp. 191-204, 1986.