SEPARABILITY OF TORSION FREE GROUPS AND A PROBLEM OF J. H. C. WHITEHEAD

BY

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1. Introduction

Our investigation of locally free groups is motivated by a question posed by J. H. C. Whitehead which asks for a characterization of those groups G for which Ext $(G, Z) = 0$. Such a group G is called a Whitehead group or more simply a W-group. Stein [6], Rotman [5], Chase [1], [2] and Nunke [4] have investigated these groups and have established a number of conditions that are necessary in order that a group be a W-group. The most notable necessary conditions are that a W-group must be locally free, totally separable, slender and satisfy Rotman's density condition [5]. It is the purpose of this paper to consider separability conditions on a group G and to study their effect on the groups Ext (G, Z) and Ext (G, S) where $S = \sum_{\aleph_0} Z$. Specifically, we wish to find rather natural sufficient conditions on the group structure of a group G in order that G be a W-group. These conditions appear on the surface to be weaker than the obvious condition that G be free. In Section 3 we establish our most striking result which states that Ext $(G, S) = 0$ if and only if G is locally free and $\aleph_1$-coseparable (see definition below). We also show that if G is a locally free, totally $\aleph_1$-separable group, then Ext $(G, S) = 0$. Hence either of the above conditions is sufficient for G to be a W-group. Section 2 is devoted to characterizing locally free, coseparable groups as just those groups G such that Ext (G, Z) is torsion free. This result is essentially just a recasting of Chase's Theorem 4.2 [1] in terms of coseparability.

Throughout this paper all groups are abelian. For the most part, the terminology and notation is that of [3]. Let G be an $\aleph_1$-free group (i.e. all countable subgroups of G are free). G is called separable ($\aleph_1$-separable) if every finitely (countably) generated subgroup of G is contained in a finitely (countably) generated direct summand of G.[2] We call G coseparable ($\aleph_1$-coseparable) if every subgroup H of G with the property that G/H is finitely (countably) generated contains a direct summand K of G such that G/K is finitely (countably) generated. If every subgroup of G is separable ($\aleph_1$-separable), we call G totally separable ($\aleph_1$-separable). Following R. J. Nunke, G will be called locally free if G is both separable and $\aleph_1$-free. It

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[2] Observe that our definition of separability agrees with the definition of Fuchs [3] for $\aleph_1$-free groups.

should be noted that a free group is locally free, totally $\aleph_1$-separable and $\aleph_1$-coseparable. Unfortunately, we do not know whether or not either of the conditions that a locally free group G be totally $\aleph_1$-separable or $\aleph_1$-coseparable is equivalent to the condition that G be free. In the sequal, the symbols Z, S and P denote the additive group of integers, the direct sum of $\aleph_0$ copies of Z and the direct product of $\aleph_0$ copies of Z, respectively.

2. Locally free, coseparable groups

We begin our study of locally free coseparable groups with the following lemma.

LEMMA 2.1. *If a group G is locally free and totally separable then G is coseparable.*

Proof. Suppose that H is a subgroup of G such that G/H is finitely generated. Then $G = \{C, H\}$ where C is finitely generated. Hence C is free of finite rank. Since H is also separable, $H = K + B$ where B has finite rank and where $C \cap H \subseteq B$. Let $A = \{B, C\}$. If $k \in K \cap A$, then $k = b + c$ where $b \in B$ and $c \in C$. Hence $c = k - b \in H \cap C \subseteq B$. But this implies that $k \in B \cap K = 0$. Observing that $G = \{K, A\}$, we have that $G = K + A$, $K \subseteq H$ and that G/K is finitely generated. Thus G is coseparable.

Examination of this proof shows that, with obvious changes, one obtains

COROLLARY 2.2. *If a group G is locally free and totally $\aleph_1$-separable, then G is $\aleph_1$-coseparable.*

Further examination shows that it is not necessary to assume every subgroup of G is separable (or $\aleph_1$-separable), but only those subgroups whose cokernel is finitely generated (or countable). This last remark and Theorem 4.2 [1] prove the sufficiency of our next result.

THEOREM 2.3. *Let G be a reduced group. Then G is a locally free, coseparable group if and only if* Ext (G, Z) *is torsion free.*

Proof. It remains to show that if G is locally free and coseparable then Ext (G, Z) is torsion free. Let H be a subgroup of G such that $G/H \cong Z/pZ$ for some prime p. Then $G = K + F$ where F is free of finite rank and where $K \subseteq H$. It follows that $H = K + (H \cap F)$ is locally free since both K and $H \cap F$ are necessarily locally free. Applying 4.2 [1], we have that Ext (G, Z) is torsion free.

COROLLARY 2.4. *There are locally free groups which are not coseparable. In particular, P is not coseparable.*

COROLLARY 2.5. *If the Continuum Hypothesis holds, there is a locally free, coseparable group which is not free.*

This corollary is just a restatement of Chase's Theorem 4.4 [2] in terms of coseparability.

3. Locally free $\aleph_1$-coseparable groups

We now prove our main theorem.

THEOREM 3.1. *G is a locally free, $\aleph_1$-coseparable group if and only if* $\operatorname{Ext}(G, S) = 0$.

Proof. Suppose that G is a locally free, $\aleph_1$-coseparable group and suppose that

$$0 \to S \xrightarrow{i} M \xrightarrow{j} G \to 0$$

represents an element of $\operatorname{Ext}(G, S)$ where i is assumed to be the inclusion homomorphism. Let H be a subgroup of M maximal with respect to the property that $S \cap H = 0$. Hence, $H \cong j(H)$ and $| G/j(H) | \leq \aleph_0$ because M/H is an essential extension of $S = (S + H)/H$, and so both have the same injective envelope. Therefore, $G = K + B$ where $K \subseteq j(H)$ and B is a countable free group. Let $C = j^{-1}(B)$, $E = j^{-1}(K) \cap H$ and let $x \in M$. Then $j(x) = k + b$ where $k \in K$ and $b \in B$. Since $K \subseteq j(H)$, we have that $j^{-1}(K) \subseteq H + S$. Therefore, there are elements $e \in E$ and $c \in C$ such that $j(e) = k$ and $j(c) = b$. Since $j(x - (e + c)) = 0$, then

$$x - (e + c) \in S \subseteq C,$$

which implies that $x \in \{E, C\}$. Thus $M = \{E, C\}$. If $x \in E \cap C$, then

$$j(x) \in B \cap K = 0,$$

which implies $x \in S \cap E \subseteq S \cap H = 0$. Hence, $M = E + C$ where $S \subseteq C$ and $| C | \leq \aleph_0$. But C/S is isomorphic to a countable subgroup of G and hence C/S is free. It follows that S is a direct summand of M, i.e.,

$$0 \to S \xrightarrow{i} M \xrightarrow{j} G \to 0$$

splits. Thus $\operatorname{Ext}(G, S) = 0$.

Now suppose that $\operatorname{Ext}(G, S) = 0$ and suppose that H is a subgroup of G such that $| G/H | \leq \aleph_0$. Then there is a countable free subgroup A of G such that $G = \{A, H\}$. We form the short exact sequence

$$0 \to C \xrightarrow{i} A + H \xrightarrow{\nu} G \to 0$$

where $A + H$ is the outer direct sum of A and H and $\nu : (a, h) \to a + h$. Clearly,

$$C = \ker \nu = \{(x, -x) \mid x \in A \cap H\}$$

is a countable free group. Since $\operatorname{Ext}(G, C) = 0$, it follows that there is a homomorphism $\theta : G \to A + H$ such that $\nu\theta = 1_G$. Let π be the natural projection of $A + H$ onto A and define $\psi : G \to A$ by $\psi = \pi\theta$. Since A is a countable free group, $G = \ker \psi + F$ where F is countable. It remains only to show that $\ker \psi \subseteq H$. Now $g \in \ker \psi$ if and only if $\theta(g) = (h, 0) \in H + C$. But this implies that $g = \nu\theta(g) = \nu(h, 0) = h \in H$.

The following more general statement is implied in the proof of the above theorem.

COROLLARY 3.2. *Let F be a free group of infinite rank α. Then* Ext $(G, F) = 0$ *if and only if the following conditions are satisfied:*

(i) *Every subgroup of G of cardinality less than or equal to α is free.*

(ii) *Every subgroup of G of index α contains a direct summand of G of index α.*

COROLLARY 3.3. *A locally free, $\aleph_1$-coseparable group is a W-group.*

Our next corollary is a consequence of Corollary 2.2.

COROLLARY 3.4. *If a group G is locally free and totally $\aleph_1$-separable, then* Ext $(G, S) = 0$.

We now consider the necessity of the hypothesis of Corollary 3.4. In this direction, we prove the following result.

THEOREM 3.5. *If* Ext $(G, S) = 0$ *and if G is $\aleph_1$-separable, then G is totally $\aleph_1$-separable.*

Proof. Let H be a subgroup of G and let A be a countable subgroup of H. By hypothesis $G = B + K$ where B is a countable subgroup of G containing A. Let π be the natural projection of G onto K restricted to H. Therefore,

$$0 \to B \cap H \xrightarrow{i} H \xrightarrow{\pi} \pi(H) \to 0$$

is exact and $B \cap H$ is isomorphic to a direct summand of S. For any subgroup C of G, Ext $(C, S) = 0$ since there is an exact sequence

$$0 = \operatorname{Ext}(G, S) \to \operatorname{Ext}(C, S) \to 0.$$

Hence, Ext $(\pi(H), S) = 0$ and therefore Ext $(\pi(H), B \cap H) = 0$. Hence,

$$0 \to B \cap H \xrightarrow{i} H \xrightarrow{\pi} \pi(H) \to 0$$

splits and thus $B \cap H$ is a countable direct summand of H containing A.

THEOREM 3.6. *Let G be a locally free, $\aleph_1$-separable group. If a subgroup H of G is $\aleph_1$-separable whenever G/H is a countable group, then G is totally $\aleph_1$-separable.*

Proof. Observe that our proof of Theorem 3.4 demands only the hypothesis of this present theorem to insure that Ext $(G, S) = 0$. Hence, by Theorem 3.5, G is totally $\aleph_1$-separable.

We conclude our discussion of locally free totally $\aleph_1$-separable groups with the next theorem.

THEOREM 3.7. *If G and H are locally free, totally $\aleph_1$-separable groups, then for any extension*

$$0 \to H \xrightarrow{i} M \xrightarrow{j} G \to 0$$

of H by G, M is also locally free and totally $\aleph_1$-separable.

Proof. We may assume that i is the inclusion homomorphism. Since there is an exact sequence

$$0 = \text{Ext}(G, S) \xrightarrow{i^*} \text{Ext}(M, S) \xrightarrow{j^*} \text{Ext}(H, S) = 0,$$

it follows that $\text{Ext}(M, S) = 0$. By Theorem 3.5, it is enough to show that M is $\aleph_1$-separable. Therefore, let A be a countable subgroup of M. Since $A/(A \cap H)$ is isomorphic to a subgroup of G and since $|A/A(\cap H)| \leq \aleph_0$, we have that $A = A \cap H + A_0$. Since H is totally $\aleph_1$-separable, $H = H_0 + H_1$ where H_0 is countable and contains $A \cap H$. Let K be a subgroup of M which is maximal with respect to the property that $K \cap \{H, A_0\} = 0$. It follows that $|M/(H_1 + K)| \leq \aleph_0$. Hence, there is a countable free subgroup C of M containing $\{H_0, A_0\}$ such that

$$M = \{C, H_1 + K\}.$$

Since both H_1 and K are $\aleph_1$-separable, it is easily seen that $H_1 + K$ is $\aleph_1$-separable. Hence, $H_1 + K = Y + X$ where Y is countable and contains $C \cap (H_1 + K)$. Set $F = \{C, Y\}$. It is straightforward to check that $M = F + X$ and that F is countable. Moreover, $F = \{C, Y\} \supseteq \{H_0, A_0\} \supseteq A$.

Although we cannot show that $\text{Ext}(G, S) = 0$ implies that G is $\aleph_1$-separable, we can show that $\text{Ext}(G, S) = 0$ implies a slightly weaker statement.

THEOREM 3.8. *If* $\text{Ext}(G, S) = 0$ *and if* A *is a countable subgroup of* G *then there is a decomposition* $G = B + H$ *where* B *is countable and* $A \cap H = 0$.

Proof. Let K be a subgroup of G maximal with respect to the property that $A \cap K = 0$. Hence, $|G/K| \leq \aleph_0$ which implies by Theorem 3.1 that $G = H + B$ where $H \subseteq K$ and $|B| \leq \aleph_0$. Since $A \cap H \subseteq A \cap K = 0$, the proof is complete.

COROLLARY 3.9. *If* $\text{Ext}(G, S) = 0$ *and if rank* $(G) \geq \aleph_0$, *then* G *contains a free direct summand of rank* $\aleph_0$.

Proof. Using the notation in Theorem 3.8, choose A such that rank $(A) = \aleph_0$. Since $A \cap H = 0$, we have that $\aleph_0 = \text{rank}(A) \leq \text{rank}(B)$. However, $|B| \leq \aleph_0$ implies that rank $(B) = \aleph_0$.

We now give several results on the structure of groups which satisfy $\text{Ext}(G, S) = 0$. These results exhibit rather strong properties for such groups.

THEOREM 3.10. *If* $\text{Ext}(G, S) = 0$ *and if* H *is a subgroup of* G *such that* G/H *is countable, then* $S + H \cong S + G$.

Proof. If rank $(G) \leq \aleph_0$, then G is free and the conclusion follows. Hence, suppose rank $(G) > \aleph_0$. Since $|G/H| \leq \aleph_0$, there is a countable free subgroup A of G such that $\text{rank}(A) = \text{rank}(A \cap H) = \aleph_0$ and $G = \{A, H\}$. We now use the fact that the short exact sequence

$$0 \to \text{Ker}\,\nu \to A + H \xrightarrow{\nu} G \to 0$$

(defined in the proof of Theorem 3.1) splits. The proof is complete when one notes that $S \cong A \cong \text{Ker } \nu$.

THEOREM 3.11. *If the group G has infinite rank and if* Ext $(G, S) = 0$, *then* $G \cong G + S$.

Proof. By Corollary 3.9 G contains a direct summand H such that $G \cong H + S$ and by Theorem 3.10, $H + S \cong G + S$.

THEOREM 3.12. *If G is an uncountable group satisfying* Ext $(G, S) = 0$, *then $G \cong H$ for each subgroup H of G with a countable cokernel.*

Proof. Observing that H is also an uncountable group satisfying Ext $(H, S) = 0$, we apply Theorem 3.10 and Theorem 3.11 to obtain

$$H \cong S + H \cong S + G \cong G.$$

COROLLARY 3.13. *If G and K are groups of infinite rank satisfying* Ext $(G, S) = 0$ *and* Ext $(K, S) = 0$ *and if $G/A \cong K/B$ where A and B are countable subgroups of G and K, respectively, then $G \cong K$.*

Proof. We may assume that both G and K are uncountable. Let φ be the isomorphism of G/A onto K/B and let H be a subgroup of G maximal with respect to the property that $H \cap A = 0$. It follows that $|G/H| \leq \aleph_0$ since G/H is an essential extension of $A = (A + H)/H$. Hence we also have that $|K/\varphi(H)| \leq \aleph_0$ where we are identifying H with $(H + A)/A$. Let L be a subgroup of K such that $B \subseteq L$ and $L/B = \varphi(H)$. Hence $L = B + J$ since Ext $(\varphi(H), B) = 0$. It follows that $|K/J| \leq \aleph_0$ and that $J \cong \varphi(H) \cong H$. By Theorem 3.12, $G \cong H \cong J \cong K$.

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