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SEPARABLE INTERACTIONS AND LIQUID ^{3}He

V. PHASE DIAGRAM IN THE PRESENCE OF A HUBBARD INTERACTION

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A comparison is made between the various extrema of the Landau expansion of liquid ^{3}He derived in a previous paper. As an application the phase diagram is investigated in the presence of an external magnetic field assuming that the Hubbard interaction is small as compared to the pairing interaction of the BCS-type, and also in zero magnetic field for arbitrary strength of the Hubbard interaction.

1. Introduction

1.1. *Landau expansion*

In the preceding paper[1]) we have investigated the Landau expansion for liquid ^{3}He in terms of 3 complex vectors $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ describing the ordering of spin pairs with $\uparrow\uparrow$, $\downarrow\downarrow$ and $\uparrow\downarrow$, $\downarrow\uparrow$, respectively, for the various orientations of the wave vector $\boldsymbol{k}$. Assuming that the fourth order terms in the Landau expansion can be taken to be independent of the magnetic field b and therefore can be identified with those at $b = 0$, the fourth order part can be obtained from the generalized Landau expansion that follows from symmetry considerations and that has been expressed in terms of a 3×3 complex ordering matrix

A in ref. 2, see also refs. 3, 4 for a review on the theoretical aspects of liquid ^{3}He. The vectors $\boldsymbol{m}_1$, $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$ are given by their components

$$m_1^j = -A_{1j} + \mathrm{i}A_{2j}\,, \quad m_2^j = A_{1j} + \mathrm{i}A_{2j}\,, \quad m_3^j = A_{3j} \quad (j = 1, 2, 3)\,, \tag{1.1}$$

cf. also eq. (5.2) of ref. 5.

The free energy per unit volume is given by

$$f = \min_{\{\boldsymbol{m}_i\}} \Phi(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)\,, \tag{1.2}$$

in which $\Phi(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ is expanded up to 4th order terms in the ordering parameters $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, $\boldsymbol{m}_3$. Decomposing Φ into a part $\Phi_0(m_1, m_2, m_3)$ depending only on the lengths

$$m_i \equiv (\boldsymbol{m}_i \cdot \boldsymbol{m}_i^*)^{1/2} \tag{1.3}$$

of the vectors and a part $\Phi_1(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ depending also on the directions, i.e.

$$\Phi(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) \equiv \Phi_0(m_1, m_2, m_3) + \Phi_1(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)\,, \tag{1.4}$$

we have

$$\begin{aligned}\Phi_0(m_1, m_2, m_3) = {} & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + 2v(m_1^4 + m_2^4 + 2m_3^4) \\ & + 4v m_3^2(m_1^2 + m_2^2) + w_1(m_1^2 + m_2^2 + 2m_3^2)^2 \\ & + w_2(m_1^2 + m_3^2)(m_2^2 + m_3^2) + w_3(m_1^2 m_2^2 + 2m_3^4)\,,\end{aligned} \tag{1.5}$$

$$\begin{aligned}\Phi_1(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) = {} & v\{|\boldsymbol{m}_1 \cdot \boldsymbol{m}_1|^2 + |\boldsymbol{m}_2 \cdot \boldsymbol{m}_2|^2\} + (2v + w_3')|\boldsymbol{m}_3 \cdot \boldsymbol{m}_3|^2 \\ & + 4v(|\boldsymbol{m}_1 \cdot \boldsymbol{m}_3|^2 + |\boldsymbol{m}_2 \cdot \boldsymbol{m}_3|^2) \\ & + (4v - w_2)(|\boldsymbol{m}_1 \cdot \boldsymbol{m}_3^*|^2 + |\boldsymbol{m}_2 \cdot \boldsymbol{m}_3^*|^2) \\ & + 2(2v - w_3')\,\mathrm{Re}(\boldsymbol{m}_1 \cdot \boldsymbol{m}_2\, \boldsymbol{m}_3^* \cdot \boldsymbol{m}_3^*) \\ & + 2(4v - w_2 - 2w_3)\,\mathrm{Re}(\boldsymbol{m}_1 \cdot \boldsymbol{m}_3^*\, \boldsymbol{m}_2 \cdot \boldsymbol{m}_3^*) \\ & + w_3'|\boldsymbol{m}_1 \cdot \boldsymbol{m}_2|^2 + w_3|\boldsymbol{m}_1 \cdot \boldsymbol{m}_2^*|^2\,,\end{aligned} \tag{1.6}$$

cf. eqs. (1.6)–(1.9) of ref. 1. The coefficients u_1, u_2, u_3 of the second-degree part in (1.5) depend on the magnetic field b, and up to quadratic terms $\sim b^2$ they are given by

$$u_1 = \tfrac{1}{3}(t + A_c \eta b)\,, \quad u_2 = \tfrac{1}{3}(t - A_c \eta b)\,, \quad u_3 = \tfrac{1}{3}(t + 2b^2 B_c)\,, \tag{1.7}$$

where

$$t = \tfrac{1}{2}N(0)\,\frac{T - T_c}{T_c}\,, \quad A_c = \tfrac{1}{2}N(0)\ln(1.14\beta_c\hbar\omega)\,, \quad B_c = \tfrac{1}{2}N(0)\,\frac{7}{8}\,\frac{\beta_c^2}{\pi^2}\,, \tag{1.8}$$

cf. eqs. (1.10) and (1.11) of ref. 1. In eq. (1.8) T_c is the critical temperature in zero magnetic field, $\beta_c = 1/(kT_c)$, $N(0)$ is the density of states at the Fermi energy, and η is an asymmetry parameter in the density of states $N(\epsilon)$, i.e.

$$N(\epsilon) = N(0)(1 + \eta\epsilon)\,, \quad \text{for } |\epsilon| < \hbar\omega \quad (\eta b < 0)\,,$$

which has been introduced in ref. 6 to explain the splitting of the A phase in an external magnetic field. The coefficient $v = \frac{1}{15}B_c$ in (1.5) and (1.6) arises from a pairing-interaction of the BCS-type and may also contain a shift due to strong-coupling effects, the contributions with the coefficients w_1, w_2, w_3, w_3' are extra terms arising from a Hubbard interaction and will be specified later on in (1.19) and (1.20).

In contrast to previous papers[1,6–9]), we do not take into account a possible b-dependence in the coefficients of the 4th order terms in the Landau expansion, such as e.g. in eqs. (1.9) and (1.12) of ref. 1. In ref. 8, in which we treated the phase diagram in the absence of spin fluctuations, it was argued that this b-dependence may be important for determining the occurrence of certain phases. In fact, in the absence of spin fluctuations there is a big symmetry in the problem leading to a large degeneracy of phases. The b-dependence in the 4th order terms provides the symmetry breaking that is necessary to distinguish between the various phases. In the present paper, however, due to the spin fluctuation terms with w_1, w_2, w_3, w_3' a large part of the degeneracy at $b = 0$ is lifted and neglecting the b-dependence in the 4th order terms does not give rise to problems.

1.2. *One- and two-dimensional solutions of the gap equations*

In ref. 1 we analysed the possible extrema S with order parameters $\boldsymbol{m}_{1S}$, $\boldsymbol{m}_{2S}$, $\boldsymbol{m}_{3S}$ that are determined by the gap equations

$$\frac{\partial\Phi}{\partial\boldsymbol{m}_i}(\boldsymbol{m}_{1S}, \boldsymbol{m}_{2S}, \boldsymbol{m}_{3S}) = \boldsymbol{0}\,, \tag{1.9}$$

together with the value

$$f_S \equiv \Phi(\boldsymbol{m}_{1S}, \boldsymbol{m}_{2S}, \boldsymbol{m}_{3S}) = \tfrac{1}{2}u_1 m_{1S}^2 + \tfrac{1}{2}u_2 m_{2S}^2 + u_3 m_{3S}^2 \tag{1.10}$$

TABLE I
One-dimensional solutions S.

S	Vector $\neq 0$	Inner product	f_S	Conditions of existence
A1	$\boldsymbol{m}_1$	$\boldsymbol{m}_1 \cdot \boldsymbol{m}_1 = 0$	$-\frac{1}{4}u_1^2/(2v + w_1)$	$u_1 < 0$, $2v + w_1 > 0$
A3	$\boldsymbol{m}_3$	$\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = 0$	$-u_3^2/(4v + 4w_1 + w_2 + 2w_3)$	$u_3 < 0$, $4v + 4w_1 + w_2 + 2w_3 > 0$
R3	$\boldsymbol{m}_3$	$\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = m_3^2$	$-u_3^2/(6v + 4w_1 + w_2 + w)$	$u_3 < 0$, $6v + 4w_1 + w_2 + w > 0$

of the function Φ at the extrema, as a function of the external parameters of the system such as u_1, u_2, u_3. The value of the free energy f in (1.2) is determined by minimizing f_S over the different solutions S.

We have considered one-dimensional solutions with only one of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ different from zero, two-dimensional solutions with two of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ different from zero and three-dimensional solutions with $m_1 m_2 m_3 \neq 0$. The one-dimensional solutions are presented in table I, in which we have used the abbreviation $w \equiv 2w_3 + w_3'$. Apart from A1 there is another solution A1$'$ with $\boldsymbol{m}_2 \neq \boldsymbol{0}$, $\boldsymbol{m}_2 \cdot \boldsymbol{m}_2 = 0$ and $f_{A1'} = -\frac{1}{4}u_2^2/(2v + w_1)$, which does not occur as $f_{A1'} > f_{A1}$ for $\eta b < 0$. Furthermore, there are two "real" solutions R1$'$ and R1$'$ with

$$|\boldsymbol{m}_1 \cdot \boldsymbol{m}_1| = m_1^2 = -u_1^2/(6v + 2w_1) \tag{1.11}$$

for R1, and a similar relation with m_1 and u_1 replaced by m_2 and u_2 for R1$'$. These solutions do not occur in the phase diagram, since $f_{Rp} > f_{Ap}$ for $p = 1, 1'$.

In ref. 1 we have also treated the two-dimensional solutions with $\boldsymbol{m}_3 = \boldsymbol{0}$. For these solutions $f_S \equiv f_S(u_1, u_2)$ has the general value

$$f_S(u_1, u_2) = \frac{-(2v + w_1 + a_{11})u_2^2 + (2w_1 + w_2 + w_3 + a_{12})u_1 u_2 - (2v + w_1 + a_{22})u_1^2}{4(2v + w_1 + a_{11})(2v + w_1 + a_{22}) - (2w_1 + w_2 + w_3 + a_{12})^2} \tag{1.12}$$

and we have the general conditions of stability and existence

$$\begin{gathered} 2v + w_1 + a_{11} > 0\,, \quad 2v + w_1 + a_{22} > 0\,, \\ 4(2v + w_1 + a_{11})(2v + w_1 + a_{22}) - (2w_1 + w_2 + w_3 + a_{12})^2 > 0\,, \\ (2w_1 + w_2 + w_3 + a_{12})u_2 - 2(2v + w_1 + a_{22})u_1 > 0\,, \\ (2w_1 + w_2 + w_3 + a_{12})u_1 - 2(2v + w_1 + a_{11})u_2 > 0\,, \end{gathered} \tag{1.13}$$

TABLE II
Five two-dimensional solutions S with $\boldsymbol{m}_3 = \boldsymbol{0}$.

S	$\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_1\rvert$	$\lvert\boldsymbol{m}_2\cdot\boldsymbol{m}_2\rvert$	$\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_2\rvert$	$\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_2^*\rvert$	a_{11}	a_{22}	a_{12}
I	0	m_2^2	0	0	0	v	0
II	0	0	m_1m_2	0	0	0	w_3'
III	0	0	0	m_1m_2	0	0	w_3
IV	m_1^2	m_2^2	0	0	v	v	0
V	m_1^2	m_2^2	m_1m_2	m_1m_2	v	v	$w_3 + w_3'$

TABLE III
The two-dimensional solution VI.

Inner products			
$\boldsymbol{m}_1\cdot\boldsymbol{m}_1$	$\boldsymbol{m}_2\cdot\boldsymbol{m}_2$	$w_3'\boldsymbol{m}_1\cdot\boldsymbol{m}_2 = -w_3\boldsymbol{m}_1\cdot\boldsymbol{m}_2^*$	
$\gamma\,\dfrac{2vm_2^2-\gamma m_1^2}{4v^2-\gamma^2}$	$\gamma\,\dfrac{2vm_1^2-\gamma m_2^2}{4v^2-\gamma^2}$	$\pm 2v\{\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_1\rvert\,\lvert\boldsymbol{m}_2\cdot\boldsymbol{m}_2\rvert\}^{1/2}$	
Parameters			
a_{11}	a_{22}	a_{12}	γ
$-\dfrac{\gamma^2 v}{4v^2-\gamma^2}$	$-\dfrac{\gamma^2 v}{4v^2-\gamma^2}$	$\dfrac{4v^2\gamma}{4v^2-\gamma^2}$	$\dfrac{w_3w_3'}{w_3+w_3'}$

in which the coefficients a_{11}, a_{12} and a_{22} depend on the solution S. The values of a_{11}, a_{12} and a_{22}, as well as the inner products involving $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, are presented in tables II and III.

Apart from these solutions there is a solution I′ with $\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_1\rvert = m_1^2$, $\lvert\boldsymbol{m}_2\cdot\boldsymbol{m}_2\rvert = 0$, $\boldsymbol{m}_1\cdot\boldsymbol{m}_2 = \boldsymbol{m}_1\cdot\boldsymbol{m}_2^* = 0$, $a_{11} = v$, $a_{22} = a_{12} = 0$. For $\eta b < 0$ we have $f_{\mathrm{I}'} > f_{\mathrm{I}}$, so that I′ does not occur in the phase diagram. Other solutions which occur only under the condition $w_3 + w_3' = 0$ and which have been used in ref. 1 for the construction of three-dimensional solutions are not taken into consideration here. Solution I is a solution with a spontaneous magnetization, as f contains a term $\sim\eta b$, II is the so-called two-dimensional solution for the first time given in ref. 10 in the special case $b = 0$ ($m_1 = m_2$, $u_1 = u_2$), and III is the ABM solution[11]). In the special case $b = 0$, solution IV has been referred to as the bipolar phase[10]), solution V as the polar phase and solution VI may be regarded as belonging to the so-called axiplanar phase first considered by Mermin and Stare as cited in ref. 12.

For the two-dimensional solutions with $\boldsymbol{m}_2 = \boldsymbol{0}$, $f_{\mathrm{S}}(u_1, u_3)$ has the general value

$$f_{\mathrm{S}}(u_1,u_3) = \frac{-(4v+4w_1+w_2+2w_3)u_1^2 + 2(4v+4w_1+w_2+a_{13})u_1u_3 - 4(2v+w_1+a_{11})u_3^2}{4(4v+4w_1+w_2+2w_3)(2v+w_1+a_{11}) - (4v+4w_1+w_2+a_{13})^2} \tag{1.14}$$

TABLE IV
Two-dimensional solutions S with $\boldsymbol{m}_2 = \boldsymbol{0}$.

S	$\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_1\rvert$	$\lvert\boldsymbol{m}_3\cdot\boldsymbol{m}_3\rvert$	$\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_3\rvert$	$\lvert\boldsymbol{m}_1\cdot\boldsymbol{m}_3^*\rvert$	a_{11}	a_{13}
VII	m_1^2	0	0	0	v	0
VIII	0	0	m_1m_3	0	0	$4v$

and here we have the conditions of stability and existence

$$\begin{gathered}2v + w_1 + a_{11} > 0\,, \quad 4v + 4w_1 + w_2 + 2w_3 > 0\,, \\ 4(4v + 4w_1 + w_2 + 2w_3)(2v + w_1 + a_{11}) - (4v + 4w_1 + w_2 + a_{13})^2 > 0\,, \\ (4v + 4w_1 + w_2 + a_{13})u_3 - (4v + 4w_1 + w_2 + 2w_3)u_1 > 0\,, \\ (4v + 4w_1 + w_2 + a_{13})u_1 - 4(2v + w_1 + a_{11})u_3 > 0\,,\end{gathered} \tag{1.15}$$

in which a_{11} and a_{13} depend on the solution S. The values of a_{11} and a_{13}, as well as the inner products involving $\boldsymbol{m}_1$ and $\boldsymbol{m}_3$, are presented in table IV.

In the special case $b = 0$ phase VII reduces to the so-called ϵ-solution introduced in ref. 12, and both solutions VII and VIII have a spontaneous magnetization. Apart from VII and VIII there are two solutions with $\boldsymbol{m}_1 = \boldsymbol{0}$, namely VII′ with $\lvert\boldsymbol{m}_2\cdot\boldsymbol{m}_2\rvert = m_2^2$, $\boldsymbol{m}_3\cdot\boldsymbol{m}_3 = \boldsymbol{m}_2\cdot\boldsymbol{m}_3 = \boldsymbol{m}_2\cdot\boldsymbol{m}_3^* = 0$, and VIII′ with $\lvert\boldsymbol{m}_2\cdot\boldsymbol{m}_3\rvert = m_2m_3$, $\boldsymbol{m}_2\cdot\boldsymbol{m}_2 = \boldsymbol{m}_3\cdot\boldsymbol{m}_3 = \boldsymbol{m}_2\cdot\boldsymbol{m}_3^* = 0$.

1.3. *Three-dimensional solutions*

In ref. 1 we have also investigated the three-dimensional solutions with $m_1m_2m_3 \neq 0$ under the two following assumptions which we believe to cover the physically interesting cases:

i) The inner products $\boldsymbol{m}_p\cdot\boldsymbol{m}_q$ and $\boldsymbol{m}_p\cdot\boldsymbol{m}_q^*$ $(p, q = 1, 2, 3)$ of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ can be chosen to be real.

ii) The orientation of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ satisfy an inertia condition expressing a certain rigidity with respect to variations of the external parameters. More specifically we have assumed that the Landau expansion can be minimized under the conditions

$$\left.\begin{aligned}\boldsymbol{m}_r\cdot\boldsymbol{m}_3 &= m_rm_3\lambda_{r3}(m_1^2, m_2^2, m_3^2)\,, \\ \boldsymbol{m}_r\cdot\boldsymbol{m}_3^* &= m_rm_3\mu_{r3}(m_1^2, m_2^2, m_3^2)\,,\end{aligned}\right\} \quad (r = 1, 2)\,. \tag{1.16}$$

Using (1.16) and (1.10) we have determined the various possibilities S for the geometrical configuration, i.e. the directions of the vectors $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, $\boldsymbol{m}_3$ that minimize $\Phi(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ at fixed values of the lengths m_1, m_2, m_3 of the vectors. For each geometrical configuration S one can then insert the values of the linear products $\boldsymbol{m}_p \cdot \boldsymbol{m}_q$, $\boldsymbol{m}_p \cdot \boldsymbol{m}_q^*$ in $\Phi(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3)$ to obtain a 3-parameter function $\Phi_S(m_1, m_2, m_3)$ depending on the lengths. The values $f_S = f_S(u_1, u_2, u_3)$ for the various solutions can in principle be obtained by minimizing $\Phi_S(m_1, m_2, m_3)$ with respect to m_1, m_2, m_3. Due to the presence of terms like $\mathrm{Re}(\boldsymbol{m}_1 \cdot \boldsymbol{m}_2\, \boldsymbol{m}_3^* \cdot \boldsymbol{m}_3^*)$ and $\mathrm{Re}(\boldsymbol{m}_1 \cdot \boldsymbol{m}_3^*\, \boldsymbol{m}_2 \cdot \boldsymbol{m}_3^*)$ in eq. (1.6), it is not always easy to do this analytically.

Using the procedure sketched above we have obtained 8 three-dimensional solutions IX–XVI and the absolute values of the inner products are presented in table V, which yields in particular the following conditions of existence:

$$
\begin{aligned}
m_3^2 &\leqslant \left|\frac{w_3 + w_3'}{2v - w_3'}\right| m_1 m_2\,, \quad \text{for solution XI}\,, \\
m_3^2 &\geqslant \left|\frac{2v - w_3'}{2v + w_3'}\right| m_1 m_2\,, \quad \text{for solution XVI}\,.
\end{aligned}
\tag{1.17}
$$

In table V we presented only the absolute values of the inner products, in table VI we present some further details on the signs as they may be relevant for the evaluation of the free energy.

As it is not so easy to evaluate the corresponding values of $f_S = f_S(u_1, u_2, u_3)$, we shall restrict ourselves to the three-parameter functions $\Phi_S(m_1, m_2, m_3)$ that are obtained inserting the values of the inner products as given in table V and table VI into eqs. (1.4)–(1.6). The three parameter functions are then given by

$$
\begin{aligned}
\Phi_S(m_1, m_2, m_3) = {} & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (2v + w_1 + b_1) m_1^4 \\
& + (2v + w_1 + b_2) m_2^4 + (4v + 4w_1 + w_2 + 2w_3 + b_3) m_3^4 \\
& + (4v + 4w_1 + w_2 + b_4) m_1^2 m_3^2 \\
& + (4v + 4w_1 + w_2 + b_5) m_2^2 m_3^2 \\
& + (2w_1 + w_2 + w_3 + b_6) m_1^2 m_2^2 - 2b_7 m_3^2 m_1 m_2\,,
\end{aligned}
\tag{1.18}
$$

in which the values of the coefficients $b_1, b_2, \ldots, b_7$ depend on the solution S. The values are presented in table VII.

From the solutions presented in tables V–VII, solution X for $2v > w_3'$ is the BW solution as presented in ref. 8. In the special case $b = 0$ one has the relation $m_1^2 = m_2^2 = 2m_3^2$ which indeed is characteristic for an isotropic superfluid[13]), but for $b \neq 0$ the values of m_1, m_2, m_3 can be quite different. For

TABLE V

Absolute values of inner products for three-dimensional solutions S.

S	$\lvert \boldsymbol{m}_1 \cdot \boldsymbol{m}_1 \rvert$	$\lvert \boldsymbol{m}_2 \cdot \boldsymbol{m}_2 \rvert$	$\lvert \boldsymbol{m}_3 \cdot \boldsymbol{m}_3 \rvert$	$\lvert \boldsymbol{m}_1 \cdot \boldsymbol{m}_3 \rvert$	$\lvert \boldsymbol{m}_1 \cdot \boldsymbol{m}_3^* \rvert$	$\lvert \boldsymbol{m}_2 \cdot \boldsymbol{m}_3 \rvert$	$\lvert \boldsymbol{m}_2 \cdot \boldsymbol{m}_3^* \rvert$	$\lvert \boldsymbol{m}_1 \cdot \boldsymbol{m}_2 \rvert$	$\lvert \boldsymbol{m}_1 \cdot \boldsymbol{m}_2^* \rvert$
IX	m_1^2	m_2^2	m_3^2	m_1m_3	m_1m_3	m_2m_3	m_2m_3	m_1m_2	m_1m_2
X	0	0	m_3^2	0	0	0	0	0	0
XI	m_1^2	m_2^2	m_3^2	0	0	0	0	$\left\lvert \frac{2v - w_3'}{w_3 + w_3'} \right\rvert m_3^2$	$\left\lvert \frac{2v - w_3'}{w_3 + w_3'} \right\rvert m_3^2$
XII	m_1^2	m_2^2	m_3^2	0	0	0	0	m_1m_2	m_1m_2
XIII	0	0	0	m_1m_3	0	m_2m_3	0	0	m_1m_2
XIV	0	0	0	0	m_1m_3	0	m_2m_3	0	m_1m_2
XV	0	m_2^2	0	m_1m_3	0	0	0	0	0
XV′	m_1^2	0	0	0	0	m_2m_3	0	0	0
XVI	m_1^2	m_2^2	$\left\lvert \frac{2v - w_3'}{2v + w_3'} \right\rvert m_1m_2$	0	0	0	0	m_1m_2	m_1m_2

TABLE VI
Signs of inner products for three-dimensional solutions S.

S		Inner products
IX	with $\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = m_3^2$	$\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 = -m_1 m_2 \operatorname{sgn}[6v - w_2 - 2w_3 - w_3']$
X	with $\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = m_3^2$	$\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 = -m_1 m_2 \operatorname{sgn}[2v - w_3']$
XI	with $\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = m_3^2$	$\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 = -m_3^2 \dfrac{2v - w_3'}{w_3 + w_3'}$
XII	with $\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = m_3^2$	$\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 = -m_1 m_2 \operatorname{sgn}[2v - w_3']$
XVI	with $\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 = m_1 m_2$	$\boldsymbol{m}_3 \cdot \boldsymbol{m}_3 = -m_1 m_2 \dfrac{2v - w_3'}{2v + w_3'}$

$2v < w_3'$ the special case $b = 0$ leads to $m_1^2 = m_2^2 < 2m_3^2$ which describes an anisotropic superfluid. The solutions XIII and XIV can be regarded as three-dimensional extensions with $m_3 \neq 0$ of the ABM solution, and solution XIV has already been considered in ref. 8. Solution XVI which has also been treated in ref. 8 can be considered as a three-dimensional extension of the ϵ solution VII[12]). For a more complete description of the bifurcations of three-dimensional solutions at lower-dimensional solutions, see fig. 1 at the end of ref. 1. Finally, it should be noted that none of the solutions IX–XVI has a spontaneous magnetization. This is clear from the apparent symmetry in table VI between $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ corresponding to the ordering of ↑↑ and ↓↓ spin pairs, except for the solutions XV and XV′. A more detailed calculation, cf. appendix C, shows that the corresponding f_{XV} and $f_{XV'}$ do not contain terms $\sim \eta b$.

1.4. *Spin fluctuation parameters*

Due to the large number of phases it is a very hard task to study the complete phase diagram as a function of external parameters for arbitrary values of the coefficients w_1, w_2, w_3, w_3'. In order to get some insight in the values of these coefficients we considered in ref. 5 a model hamiltonian consisting of a kinetic energy term, a Zeeman term, a pairing interaction of the BCS-type, and a contact term of the Hubbard type, cf. eqs. (1.1)–(1.4) of ref. 1. As a consequence of a theorem due to Bogolubov Jr.[14–16]), the free energy of the model can be expressed as the minimum of the free energy of a reference system taken with respect to the order parmeters of the system. The reference system is descibed by a hamiltonian containing the kinetic energy, the Zeeman term, and the complete Hubbard term, but a bilinear approximation to the pairing hamiltonian. Using a perturbation calculation for the Hubbard term in the hamiltonian of the reference system, one can evaluate the coefficients w_1, w_2, w_3, w_3' exactly up to a certain order in the coupling constant I of the Hubbard hamiltonian.

TABLE VII

Coefficients in three parameter function for three-dimensional solutions S.

S	b_1	b_2	b_3	b_4	b_5	b_6	b_7
IX	v	v	$2v + w_3'$	$8v - w_2$	$8v - w_2$	$w_3 + w_3'$	$\|6v - w_2 - 2w_3 - w_3'\|$
X	0	0	$2v + w_3'$	0	0	w_3'	$\|2v - w_3'\|$
XI	v	v	$2v + w_3' - \dfrac{(2v - w_3')^2}{w_3 + w_3'}$	0	0	0	0
XII	v	v	$2v + w_3'$	0	0	$w_3 + w_3'$	$\|2v - w_3'\|$
XIII	0	0	0	$4v$	$4v$	w_3	0
XIV	0	0	0	$4v - w_2$	$4v - w_2$	w_3	$\|4v - w_2 - 2w_3\|$
XV	0	v	0	$4v$	0	0	0
XV'	v	0	0	0	$4v$	0	0
XVI	v	v	0	0	0	$w_3 + w_3' - \dfrac{(2v - w_3')^2}{2v + w_3'}$	0

In ref. 5 we evaluated these coefficients up to the order I^2. The results are given by, cf. eq. (1.13) of ref. 1,

$$\begin{aligned} w_1 &= (3.4)\tfrac{4}{15}\,\beta(2\pi I)^2 k_F^{-3}(\tfrac{1}{2}N(0))^4\,,\\ w_2 &= (-13.6)\tfrac{4}{15}\,\beta(2\pi I)^2 k_F^{-3}(\tfrac{1}{2}N(0))^4\,,\\ w_3 = w_3' &= (25.4)\tfrac{4}{15}\,\beta(2\pi I)^2 k_F^{-3}(\tfrac{1}{2}N(0))^4\,. \end{aligned} \tag{1.19}$$

The relation $w_3 = w_3'$ in eq. (1.19) is a consequence of the second order perturbation calculation and cannot be expected to hold when (higher) odd orders of perturbation are taken into account. In the investigations on the phase diagram which will be reported in this paper we shall not insist on the precise values of the coefficients w_1, w_2, w_3, w_3', but we shall assume only some global characteristics, such as that the signs of w_1, w_2, w_3, w_3' are correctly given by (1.19), i.e.

$$w_1 > 0\,,\quad w_2 < 0\,,\quad w_3 > 0\,,\quad w_3' > 0\,. \tag{1.20}$$

Eq. (1.20) may be plausible in view of the usual considerations about enhancement which suggest the results obtained in non-vanishing lowest order should be multiplied by appropriate (positive) enhancement factors. In connection with this it can be noted that the spin fluctuation model, in which the contributions from the Hubbard term are expressed in terms of dynamical susceptibilities with the use of certain statistical approximations, leads to the values $w_1 = (-0.25)\tfrac{2}{15}B_c\delta$, $w_2 = (0.5)\tfrac{2}{15}B_c\delta$, $2w_3 = (-1.25)\tfrac{2}{15}B_c\delta$, $2w_3' = (-0.25)\tfrac{2}{15}B_c\delta$, where δ is a (positive) parameter depending on the coupling constant I. The signs of these values, which have been derived in ref. 17 for unitary states, do not agree with (1.20), and the signs are not affected by considering only terms up to the order I^2.

In the present paper we shall work out two specific applications with regard to the phase diagrams of liquid ^{3}He in the presence of a magnetic field. In section 2 we shall address ourselves to the problem of the stability of the phase diagram in the absence of spin fluctuations as considered in ref. 8, i.e. we consider the problem of determining to which extent this phase diagram will undergo qualitative changes, in the presence of a small Hubbard interaction. In this limit it is safe to assume that the signs of w_1, w_2, w_3, w_3' are correctly given by (1.20), and w_1, w_2, w_3, w_3' can be taken to be small as compared to v. Under these conditions it is shown that the ABM phase does not occur in the phase diagram, in contrast to the situation described in refs. 6 and 8. In fact, in this case the ABM phase is less favourable than phase VI, as given in table III, which in the absence of spin fluctuations, ($w_3 = w_3' = 0$), is degenerate with the ABM phase. A second problem which is treated in detail is the case that $b = 0$, also for larger values of the coupling constant I of the Hubbard interaction. This will be done in section 3, where we also assume that the signs of w_1, w_2, w_3, w_3' are correctly given by (1.20). Finally, in section 4 we give some concluding remarks and we pay also some attention to the so-called profound effect[18]). We also discuss the case that the coefficients w_3 and w_3' are negative, as suggested by the spin fluctuation results of ref. 17, and show that the ABM phase can indeed occur under such conditions.

2. Phase diagram under small perturbations of the Hubbard type

2.1. *Non-occurring phases*

In the introduction we have presented a large collection of solutions of the gap equations (1.9), namely A1–A3, I–XVI, which, apart from all one- and two-dimensional solutions, with one of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ equal to zero, include all three-dimensional solutions with real inner products satisfying the inertia condition (1.16). In this section we investigate the possibility of changes in the phase diagram as obtained in ref. 8 in the absence of spin fluctuation effects, under a small perturbation of the Hubbard type. For sufficiently small Hubbard interaction many of these solutions can be ruled out as possible candidates for the phase diagram. More specifically, the following remarks can be made:

i) Considering the case that $\eta b < 0$, we have $f_{\mathrm{A1'}} > f_{\mathrm{A1}}$, and $f_{\mathrm{I'}} > f_{\mathrm{I}}$, as stated in section 1, implying that the phases

A1′, I′

do not occur for $\eta b < 0$.

ii) The phases

A3, R3, IV, XIII, XIV

can never lead to an absolute minimum of the Landau expansion. This will be proved in appendix A.

iii) Under the conditions $w_3 > -2v$, $w_3' > -2v$, which include the inequalities (1.20) as a special case, the phases

V, IX, XI, XII

do not occur, as will be shown in appendix B.

iv) Finally, for sufficiently small values of w_1, w_2, w_3, w_3' satisfying (1.20) one can rule out the phases

$$\mathrm{VII}, \mathrm{VII}', \mathrm{VIII}, \mathrm{VIII}', \mathrm{XV}, \mathrm{XV}', \mathrm{XVI}\,.$$

This is shown in appendix C.

2.2. *Phase diagram for one- and two-dimensional phases*

Taking into account the remarks (i)–(iv) we are left with the phases

$$\mathrm{A1},\ \mathrm{I},\ \mathrm{II},\ \mathrm{III},\ \mathrm{VI},\ \mathrm{X} \tag{2.1}$$

and in the remainder of this section we shall compare the f_S for these solutions S. From table I, we have

$$f_{A1} = \frac{-u_1^2}{4(2v + w_1)}\,. \tag{2.2}$$

Furthermore, from (1.11) and tables II and III we have

$$\begin{aligned} f_{\mathrm{I}} = &-\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{5v + 4w_1 + w_2 + w_3 - v^2/(5v - w_2 - w_3)} \\ &-\tfrac{1}{2}v\,\frac{(u_1+u_2)(u_1-u_2)}{(5v + 4w_1 + w_2 + w_3)(5v - w_2 - w_3) - v^2} \\ &-\tfrac{1}{4}\,\frac{(u_1-u_2)^2}{5v - w_2 - w_3 - v^2/(5v + 4w_1 + w_2 + w_3)}\,, \end{aligned} \tag{2.3}$$

$$f_{\mathrm{II}} = -\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{4v + 4w_1 + w_2 + w_3 + w_3'} - \tfrac{1}{4}\,\frac{(u_1-u_2)^2}{4v - w_2 - w_3 - w_3'}\,, \tag{2.4}$$

$$f_{\mathrm{III}} = -\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{4v + 4w_1 + w_2 + 2w_3} - \tfrac{1}{4}\,\frac{(u_1-u_2)^2}{4v - w_2 - 2w_3}\,, \tag{2.5}$$

$$\begin{aligned} f_{\mathrm{VI}} = &-\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{4v + 4w_1 + w_2 + w_3 + 2\gamma v/(2v+\gamma)} \\ &-\tfrac{1}{4}\,\frac{(u_1-u_2)^2}{4v - w_2 - w_3 - 2\gamma v/(2v-\gamma)}\,, \quad (\gamma = w_3 w_3'/(w_3 + w_3'))\,. \end{aligned} \tag{2.6}$$

The corresponding values of m_1^2 and m_2^2 at the minimum can be inferred from the relations $m_1^2 = \partial f_S/\partial u_1$, $m_2^2 = \partial f_S/\partial u_2$ for the solutions S = A1, I, II, III and VI.

From the conditions (1.13) the first three inequalities are trivially satisfied for sufficiently small w_1, w_2, w_3, w_3', the fourth inequality yields $u_1 \leq 0$ and from the last one we obtain

$$\frac{u_1 - u_2}{u_1 + u_2} \leq \frac{4v - w_2 - w_3 - a_{12} + 2a_{11}}{4v + 4w_1 + w_2 + w_3 + a_{12} + 2a_{11}} \tag{2.7}$$

in which the equality sign corresponds to a possible bifurcation of solution S with the A1 phase. Comparing the values of a_{11}, a_{12}, as given in tables II, III, it is easily seen that the right-hand side of (2.7) is largest for the bifurcation of phase I and A1, so that one may anticipate a second-order transition A1→I at the value

$$\frac{u_1 - u_2}{u_1 + u_2} = \frac{4v - w_2 - w_3}{4v + 4w_1 + w_2 + w_3} \,. \tag{2.8}$$

Furthermore, we may have a second order transition from phase VI to phase I, when $\boldsymbol{m}_2 \cdot \boldsymbol{m}_2 = -m_2^2$ for phase VI as mentioned in eq. (2.20) of ref. 1. Using the values of $\boldsymbol{m}_2 \cdot \boldsymbol{m}_2$ given in table III we obtain the condition $2vm_2^2 = \gamma m_1^2$, and inserting the values $m_1^2 = \partial f_{\mathrm{VI}}/\partial u_1$, $m_2^2 = \partial f_{\mathrm{VI}}/\partial u_2$ that are obtained from eq. (2.6), or equivalently from (2.3), we find for the second order transition VI→I

$$\begin{aligned} &2v[-2(2v + w_1)u_2 + (2w_1 + w_2 + w_3)u_1] \\ &\quad = \gamma[-2(3v + w_1)u_1 + (2w_1 + w_2 + w_3)u_2] \,. \end{aligned} \tag{2.9}$$

Comparing (2.9) and (2.8) it is easy to show that the transition I→VI takes place at a larger value of u_2/u_1, i.e. a smaller value of b, than the transition I→A1. Hence, we get the following picture, as given in table VIII, when only the phases A1, I and VI and the normal liquid phase N with $m_1 = m_2 = m_3 = 0$ are taken into account.

Finally, it can be shown that for sufficiently small values of w_1, w_2, w_3, w_3' satisfying (1.20) the phases II and III do not occur. In fact, from eq. (1.18) with $m_3 = 0$ and $b_1 = a_{11}$, $b_2 = a_{22}$, $b_6 = a_{12}$, as given by tables II and III, we have

$$\Phi_{\mathrm{III}} - \Phi_{\mathrm{I}} = m_1^2 m_2^2 [w_3 - v m_2^2 / m_1^2] \,, \tag{2.10}$$

$$\Phi_{\mathrm{III}} - \Phi_{\mathrm{VI}} = \frac{v\gamma^2}{4v^2 - \gamma^2} (m_1^2 - m_2^2)^2 + \left(w_3 - \frac{2\gamma v}{2v + \gamma}\right) m_1^2 m_2^2 \,. \tag{2.11}$$

Evaluating the right-hand side of (2.10) at the minimum of solution III, we have

TABLE VIII
Phase diagram for one- and two-dimensional solutions in the presence of a small Hubbard interaction.

Phase	Conditions of existence
VI	$u_1 < 0, \quad 0 < \frac{u_1 - u_2}{u_1 + u_2} < B$
I	$u_1 < 0, \quad B < \frac{u_1 - u_2}{u_1 + u_2} < \frac{4v - w_2 - w_3}{4v + 4w_1 + w_2 + w_3}$
A1	$u_1 < 0, \quad \frac{4v - w_2 - w_3}{4v + 4w_1 + w_2 + w_3} < \frac{u_1 - u_2}{u_1 + u_2}$
N	$u_1 > 0, \quad B \equiv \left(\frac{2v - \gamma}{2v + \gamma}\right) \frac{4v - w_2 - w_3 - 2\gamma v/(2v - \gamma)}{4v + 4w_1 + w_2 + w_3 + 2\gamma v/(2v + \gamma)}$

$$\Phi_{\mathrm{III}} - \Phi_{\mathrm{I}} = m_1^2 m_2^2 \left[-\frac{2v - 2w_3}{2v + 2w_3} + \frac{u_1 - u_2}{u_1 + u_2} \frac{4v + 4w_1 + w_2 + 2w_3}{4v - w_2 - 2w_3} \right] > 0 , \tag{2.12}$$

as follows from the inequalities $2v\gamma/(2v - \gamma) \leqslant w_3$, $B \leqslant (u_1 - u_2)/(u_1 + u_2)$, of which the first one is satisfied for w_3, $w_3' > 0$, $w_3' < 2v$, and the second one is the existence of condition of phase I, as can be seen from table VIII.

Furthermore, using the inequalities $2v\gamma/(2v + \gamma) \leqslant w_3$, $2v \geqslant \gamma$ of which the first one is identically satisfied for w_3, $w_3' > 0$, and the second one follows from table VIII for phase VI, in combination with the second line of (1.13) and the fact that $(u_1 - u_2)/(u_1 + u_2)$ must be positive, we also obtain $f_{\mathrm{III}} > f_{\mathrm{VI}}$, implying that phase III does not occur. By a completely analogous line of reasoning, i.e. replacing some of the w_3 at appropriate places by w_3', one can also rule out phase II. This means that for sufficiently small w_1, w_2, w_3, w_3' satisfying (1.20), the phase diagram of the one- and two-dimensional phases is completely determined by table VIII.

2.3. *The BW phase*

To complete the phase diagram under the influence of a small Hubbard term we must also take into account the BW solution X given by (1.18) and table VII. We have

$$\Phi_{\mathrm{X}} = (u_1 + \lambda^2 u_2) m_1^2 + 2u_3 m_3^2 + C_{11} m_1^4 + C_{33} m_3^4 + C_{13} m_1^2 m_3^2 , \tag{2.13}$$

in which we have introduced

$$C_{11} = (2v + w_1)(1 + \lambda^4) + (2w_1 + w_2 + w_3 + w_3')\lambda^2 \,,$$

$$C_{13} = (4v + 4w_1 + w_2)(1 + \lambda^2) - 2|2v - w_3'|\lambda \,, \tag{2.14}$$

$$C_{33} = 6v + 4w_1 + w_2 + 2w_3 + w_3' \,, \quad \lambda \equiv m_2/m_1 \,.$$

The free energy value f_{X} can be expressed as

$$f_{\mathrm{X}} = \min_{\lambda} f(\lambda) \,, \quad f(\lambda) \equiv \min_{m_1, m_3} \Phi_{\mathrm{X}}(m_1, m_3) \,, \tag{2.15}$$

and the gap equations for solution X are equivalent to $\partial \Phi_{\mathrm{X}}/\partial m_1 = 0$, $\partial \Phi_{\mathrm{X}}/\partial m_3 = 0$, $\mathrm{d}f/\mathrm{d}\lambda = 0$. The condition that these equations have a solution $m_1 m_3 \neq 0$ is given by

$$\begin{vmatrix} \frac{1}{2}(1+\lambda^2) + \frac{1}{2}(1-\lambda^2)\,\dfrac{u_1 - u_2}{u_1 + u_2} & \dfrac{2u_3}{u_1 + u_2} & \lambda\left\{1 - \dfrac{u_1 - u_2}{u_1 + u_2}\right\} \\ 2C_{11} & C_{13} & \dfrac{\mathrm{d}C_{11}}{\mathrm{d}\lambda} \\ C_{13} & 2C_{33} & \dfrac{\mathrm{d}C_{13}}{\mathrm{d}\lambda} \end{vmatrix} = 0 \,, \tag{2.16}$$

from which λ may be solved at given values of u_1, u_2, u_3, v, w_1, w_2, w_3, w_3'.

To study the first order phase transitions between phase X and the other phases VI, I, A1, we have to use the relation $f_{\mathrm{X}} = f_{\mathrm{S}}$ with S = A1, I, VI. Here f_{X} is given by (1.10), in which m_1^2, m_3^2 and λ can be solved from the gap equations for m_1 and m_3 and (2.16). Furthermore, from (2.2), (2.3) and (2.6) we have

$$f_{\mathrm{S}} = -\tfrac{1}{2} f_{11} u_1^2 - \tfrac{1}{2} f_{12} u_1 u_2 - \tfrac{1}{2} f_{22} u_2^2 \,, \tag{2.17}$$

in which the constants f_{11}, f_{12}, f_{22} for the phases S = A1, I, and VI are given by

phase A1: $\quad f_{11} = \dfrac{1}{4v + 2w_1} \,, \quad f_{12} = f_{22} = 0 \,;$

phase I: $\quad \tfrac{1}{4}(f_{11} + f_{12} + f_{22}) = \tfrac{1}{2}[5v + 4w_1 + w_2 + w_3 - v^2/(5v - w_2 - w_3)]^{-1} \,,$

$$\tfrac{1}{4}(f_{11} + f_{22} - f_{22}) = \tfrac{1}{2}[5v - w_2 - w_3 - v^2/(5v + 4w_1 + w_2 + w_3)]^{-1} \,,$$

$$\tfrac{1}{2}(f_{11} - f_{22}) = \tfrac{1}{2} v[(5v + 4w_1 + w_2 + w_3)(5v - w_2 - w_3) - v^2]^{-1} \,;$$

$$\text{phase VI:}\quad \tfrac{1}{4}(f_{11}+f_{12}+f_{22}) = \tfrac{1}{2}[4v+4w_1+w_2+w_3+2\gamma v/(2v+\gamma)]^{-1}\,,$$

$$\tfrac{1}{4}(f_{11}+f_{22}-f_{12}) = \tfrac{1}{2}[4v-w_2-w_3-2\gamma v/(2v-\gamma)]^{-1}\,,$$

$$f_{11}=f_{22}\,. \tag{2.18}$$

Using (2.17) and (2.18) we have another linear relation in m_1^2 and m_3^2 and combining it with the gap equations for m_1 and m_3, we obtain the condition

$$\begin{vmatrix} \begin{matrix}(f_{11}+f_{22}+f_{12}) \\ +\tfrac{1}{4}(f_{11}+f_{22}-f_{12})(u_1-u_2)^2/(u_1+u_2)^2 \\ +\tfrac{1}{2}(f_{11}-f_{22})(u_1-u_2)/(u_1+u_2)\end{matrix} & \tfrac{1}{2}(1+\lambda^2)+\tfrac{1}{2}(1-\lambda^2)\dfrac{u_1-u_2}{u_1+u_2} & \dfrac{2u_3}{u_1+u_2} \\ \tfrac{1}{2}(1+\lambda^2)+\tfrac{1}{2}(1-\lambda^2)\dfrac{u_1-u_2}{u_1+u_2} & 2C_{11} & C_{13} \\ \dfrac{2u_3}{u_1+u_2} & C_{13} & 2C_{33}\end{vmatrix} = 0 \tag{2.19}$$

Using (2.16) and (2.19) the first-order transition between BW and S = A1, I, VI at fixed v, w_1, w_2, w_3, w_3' can be determined by solving for each λ-value with $0<\lambda<1$, $(u_1-u_2)/(u_1+u_2)$ and $2u_3/(u_1+u_2)$ from (2.16) and (2.19). In doing so, the phase S should be chosen such that the value of $(u_1-u_2)/(u_1+u_2)$ is in agreement with the condition for phase S as specified by table VIII, i.e. $(u_1-u_2)/(u_1+u_2)$ should lie in the region of the phase diagram where S occurs. In the limiting case $\lambda\uparrow 1$ we have from (2.16) $(u_1-u_2)/(u_1+u_2)\to 0$ and $2u_3/(u_1+u_2)$ can be solved from (2.19).

The phase transition between BW and S = VI, I is a first-order transition, but the transition between BW and A1 may be of second order. At such a second order transition λ and m_3 tend simultaneously to zero, and from the gap equations for m_1 and m_3 one has the bifurcation condition

$$2\,\frac{u_3}{u_1} = \frac{4v+4w_1+w_2}{4v+2w_1}\,. \tag{2.20}$$

Apart from (2.20) one has the condition $\mathrm{d}^2f/\mathrm{d}\lambda^2>0$ at $\lambda=0$, which ensures that the BW solution with $\lambda\to 0$, $m_3\to 0$ corresponds to a minimum. Considering the gap equations at very small values of λ and using the condition $\mathrm{d}^2f/\mathrm{d}\lambda^2>0$ at $\lambda=0$, it can be shown that the BW solution occurs at lower b values, for which $2u_3/u_1$ is larger than the r.h.s. of (2.20).

Using the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial}{\partial\lambda} + \frac{\partial}{\partial m_1}\frac{\mathrm{d}m_1}{\mathrm{d}\lambda} + \frac{\partial}{\partial m_3}\frac{\mathrm{d}m_3}{\mathrm{d}\lambda} \tag{2.21}$$

and solving $\mathrm{d}m_1/\mathrm{d}\lambda$ and $\mathrm{d}m_3/\mathrm{d}\lambda$ from the equations $\mathrm{d}/\mathrm{d}\lambda(\partial\phi/\partial m_1) = 0$ and $\mathrm{d}/\mathrm{d}\lambda(\partial\phi/\partial m_3) = 0$ one obtains the relations

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} = \frac{\partial^2\phi}{\partial\lambda^2} + \frac{2\dfrac{\partial^2\phi}{\partial m_1\,\partial\lambda}\dfrac{\partial^2\phi}{\partial m_3\,\partial\lambda}\dfrac{\partial^2\phi}{\partial m_1\,\partial m_3} - \left(\dfrac{\partial^2\phi}{\partial m_1\,\partial\lambda}\right)^2\dfrac{\partial^2\phi}{\partial m_3^2} - \left(\dfrac{\partial^2\phi}{\partial m_3\,\partial\lambda}\right)^2\dfrac{\partial^2\phi}{\partial m_1^2}}{\dfrac{\partial^2\phi}{\partial m_1^2}\dfrac{\partial^2\phi}{\partial m_3^2} - \left(\dfrac{\partial^2\phi}{\partial m_1\,\partial m_3}\right)^2} \tag{2.22}$$

in which $\phi = \Phi_X$. Evaluating the second derivatives in (2.22) with (2.14) and (2.19) one obtains the condition

$$\frac{1}{4v + 2w_1}\, m_1^{-4}\,\frac{\mathrm{d}^2 f}{\mathrm{d}\lambda^2} = -2\frac{u_2}{u_1} + \frac{2(2w_1 + w_2 + w_3 + w_3')}{4v + 2w_1} - C \geqslant 0\,, \tag{2.23}$$

with

$$C \equiv \frac{4(2v - w_3')^2}{4(2v + w_1)(6v + 4w_1 + w_2 + 2w_3 + w_3') - (4v + 4w_1 + w_2)^2}\,. \tag{2.24}$$

On the basis of (2.23) one can investigate the order of the transition in the limit $b \downarrow 0$, and also at the second-order line between I and A1. In the limit $b \downarrow 0$ we have $2u_3 \downarrow u_1 + u_2$ and with the bifurcation condition (2.20) we obtain $2(w_3 + w_3') \geqslant C$ as the condition to have a second-order transition in the limit $b \downarrow 0$. At the second-order line between I and A1, we have $u_2/u_1 = (2w_1 + w_2 + w_3)/(4v + 2w_1)$ leading to the condition $2w_3' \geqslant C$ in order that the BW→A1 transition is of second order. Combining the results stated above we obtain table IX.

TABLE IX
Order of BW→A1 transition.

Condition for w_3, w_3'	Order of transition
$2(w_3 + w_3') \leqslant C$	first order
$2w_3' \leqslant C \leqslant 2(w_3 + w_3')$	second order for $b \leqslant b_t$, first order for $b \geqslant b_t$
$2w_3' \geqslant C$	second order

2.4. *Phase diagrams*

We now present some phase diagrams in the case of a small Hubbard interaction. The phase diagrams are plotted in the xy-plane, in which x and y are defined by

$$x \equiv \frac{-tB_c}{A_c^2\eta^2}\,, \quad y \equiv \frac{-bB_c}{A_c\eta} \rightarrow \frac{u_1-u_2}{u_1+u_2} = \frac{y}{x}\,, \quad \frac{2u_3}{u_1+u_2} = 1 - \frac{2y^2}{x}\,. \tag{2.25}$$

In figs. 1–4 we present the phase diagrams for the values $w_1 = 0$, $w_1 = 0.03v$, $w_1 = 0.05v$, $w_1 = 0.10v$, taking $w_2/w_1 = -4$ and $w_3/w_1 = w_3'/w_1 = (25.4)/(3.4)$, in accordance with the second-order results of (1.19). The second-order phase transitions have been indicated by solid curves and the first-order transitions by dashed curves. The second-order lines between VI and I and between I and A1 have been obtained from table VIII, the second-order line between BW and A1 as far as it occurs has been found from (2.20), and the first-order lines between BW and S = VI, I, A1 have been evaluated from (2.16) and (2.19). In terms of normal temperature and magnetic field variables, i.e. t and b, the phase transition lines have much larger slopes, as the y coordinates have to be multiplied by $-(A_c\eta)/B_c$ and the x coordinates by the much smaller factor $(A_c\eta)^2/B_c$.

Fig. 1 is the phase diagram in the absence of spin fluctuations which has been presented in ref. 8. In this case there is no phase I, and phase VI is degenerate with other two-dimensional phases such as II or III. When one considers phase II, the transition between BW and II is of second-order at sufficiently large x and there is a tricritical point[6,8]), with $y/x = 0.922$, $y^2/x = 0.092$, $\lambda = 0.202$, at which the transition becomes first-order. When one chooses another (degenerate) two-dimensional phase the transition between BW and this phase is always first-order, and the point $y/x = 0.922$, $y^2/x = 0.092$ separates a regime of first-order transitions without latent heat from a regime of first-order transitions with latent heat. Furthermore for y values smaller than the one at the critical point, the value of λ remains constant along the first-order line. Apart from the critical point, there is a critical endpoint of $y \sim 0.05$, $x \sim 0.05$, where the second order line between A1 and the two-dimensional phase meets the first order lines between BW and the two-dimensional phase and between BW and A1. On the first-order line between BW and A1, the λ value decreases from 0.202 at the critical end point to $\lambda \sim 0.14$ in the limit $y \downarrow 0$.

In fig. 2 we have plotted the phase diagram for a relatively small value $w_1 = 0.03v$. In this case phase VI is no longer degenerate with the two-dimensional phase II and the ABM phase III, but the difference between these phases may be rather small and not so easy to detect. Apart from this, phase I has appeared in a rather limited area of the phase diagram. There is a critical

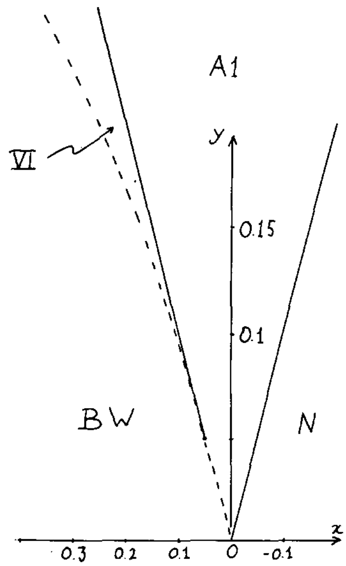

Fig. 1. Phase diagram in the xy-plane in the absence of a Hubbard interaction. The solid curves are second-order transitions, the dashed curves first-order transitions.

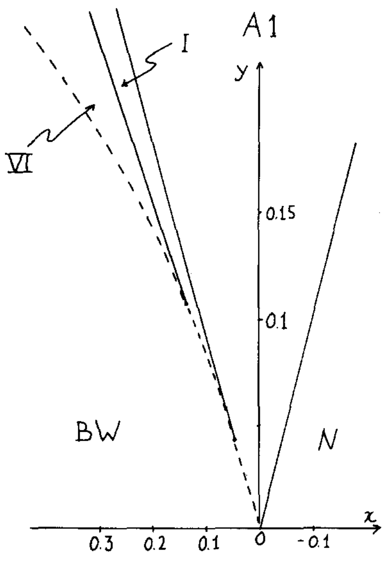

Fig. 2. Phase diagram in the case that $w_1 = 0.03v$, $w_2/w_1 = -4$, $w_3/w_1 = w_3'/w_1 = (25.4)/(3.4)$. The solid curves are second-order transitions, the dashed curves first-order transitions.

end point at $y \sim 0.11$, $x \sim 0.14$ at which the second order line between VI and I meets the first order lines between BW and VI, and between BW and I. There is another critical end point at $y \sim 0.04$, $x \sim 0.05$ at which the second order line I→A1 meets the first order lines BW→I and BW→A1.

In fig. 3 we have given the phase diagram for a slightly larger value $w_1 = 0.05v$. The phase diagram is analogous to the one for $w_1 = 0.03v$, apart from the fact that at small values of y the transition between BW and A1 is of second order. There is a tricritical point at $y \sim 0.03$, $x \sim 0.03$, at which the second-order line BW→A1 changes into a first-order line. Apart from that there are two critical end points, one at $y \sim 0.05$, $x = 0.06$ with regard to the phases BW, I and A1, and another one at $y \sim 0.13$, $x \sim 0.20$ with regard to the phases BW, VI and I.

In fig. 4 we have presented the phase diagram at $w_1 = 0.10v$. Here the transition between BW and A1 is always of second order and at $y \sim 0.10$, $x \sim 0.13$ there is a bicritical point at which the two second-order lines between I and A1, and between BW and A1, meet the first-order line between BW and I. Finally at $y \sim 0.17$, $x \sim 0.40$, there is a critical end point at which the second-order line VI→I meets the first-order lines BW→VI, and BW→I.

As a conclusion one may state that the phase diagram obtained in ref. 8 with phase VI as the two-dimensional phase remains qualitatively the same under small perturbations of the Hubbard type, apart from the occurrence of phase I in a very small region of the phase diagram. For slightly larger values of the

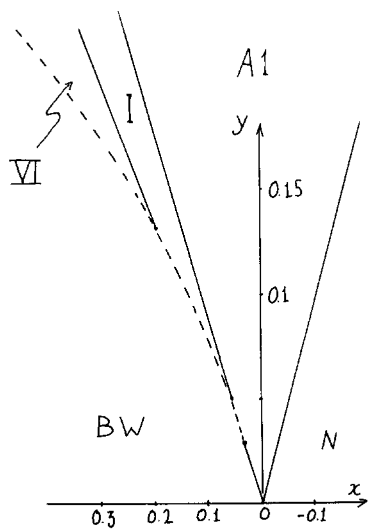

Fig. 3. Phase diagram in the case that $w_1 = 0.05v$, $w_2/w_1 = -4$, $w_3/w_1 = w_3'/w_1 = (25.4)/(3.4)$. The solid curves are second-order transitions, the dashed curves first-order transitions.

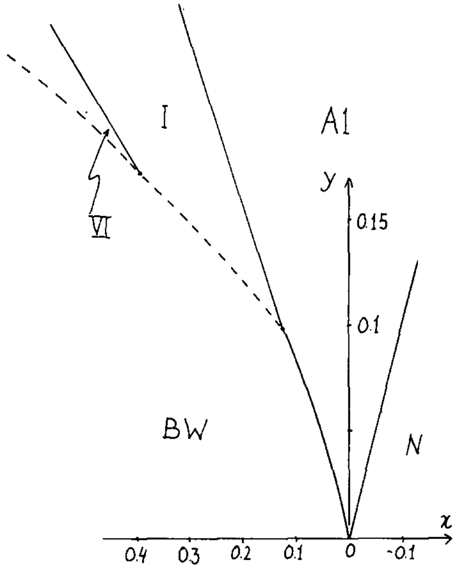

Fig. 4. Phase diagram in the case that $w_1 = 0.10v$, $w_2/w_1 = -4$, $w_3/w_1 = w_3'/w_1 = (25.4)/(3.4)$. The solid curves are second-order transitions, the dashed curves first-order transitions.

Hubbard interaction the first-order BW→ A1 transition changes into a second-order transition. On the other hand, considering small negative values of w_3, w_3', the phase diagram with phase III for $w_3' > w_3$ and with phase II for $w_3' < w_3$ will remain qualitatively the same as the one presented in ref. 8, see also the discussion at the end of section 4, but negative values of w_3, w_3' are not in agreement with the signs in (1.20) obtained by second-order perturbation treatment of the Hubbard interaction.

3. Phase diagram for $b = 0$

In the previous section we have shown that in the presence of a sufficiently small Hubbard interaction many of the solutions A1–A3, I–XVI do not occur. In fact, we showed that one only needs to take into account the phases A1, I, VI and X for a qualitative description of the phase diagram. For $b = 0$, however, there are many simplifying features. In this section we study the phase diagram for $b = 0$ also for larger values of w_1, w_2, w_3, w_3' under the restrictions

$$w_2 < 0\,, \quad w_3 > 0\,, \quad w_3' > 0\,, \quad q \equiv -w_2/w_3 < 1\,. \tag{3.1}$$

The first three inequalities in (3.1) express the assumption that the signs of w_2, w_3, w_3' are correctly given by (1.20) and $q<1$ is well obeyed in a large range of parameter values including the ones in (1.19).

3.1. *Free energy of phases at $b=0$*

We now present a list of the free energies of the possible phases at $b=0$ together with their regions of existence in the space of spin fluctuation parameters satisfying (3.1). In section 2 we mentioned that the phases

A1′, I′

cannot occur for $\eta b<0$. For $\eta b=0$ these phases are degenerate with the phases A1 and I respectively. Here we only consider the phases that pertain for infinitesimal $\eta b<0$. Furthermore it was shown in appendices A and B that the phases

A3, R3, IV, IV′ V, IX, XI–XIV

do not lead to an absolute minimum of the Landau expansion under the condition (1.20) or (3.1). This is so for $b\neq 0$, but for $b=0$ phase XI is degenerate with phase X at $2v<w_3'$, and phase XIV is degenerate with A1 for $4v<w_2+2w_3$. From the remaining phases, phase VII is degenerate with VII′ and phase VIII is degenerate with VIII′ at $b=0$; cf. also eqs. (C.8) and (C.10). When both phases XV and XV′ exist, phase XV is favorable with respect to XV′ for $\eta b<0$, cf. eq. (C.13), but for $b=0$ both phases are degenerate, and phase XVI is degenerate with VI at $b=0$. Finally, comparing eqs. (2.4) and (2.5) with (2.6), it is clear that for $\eta b=0$ $(u_1=u_2)$

$$f_{\mathrm{II}}>f_{\mathrm{VI}}\,,\quad f_{\mathrm{III}}>f_{\mathrm{VI}}\,. \tag{3.2}$$

Hence, we are left with the phases

$$\text{A1, I, VI, VII, VIII, X, XV}\,. \tag{3.3}$$

All these phases S exist under the condition $u<0$ $(u_1=u_2=u_3\equiv u,\, b=0)$, together with (3.1) and some additional conditions which we denote by $C_{\mathrm{S}}>0$ depending on the type of solution. The free energy of the different phases can be expressed as

$$f_{\mathrm{S}}=\frac{-u^2}{N_{\mathrm{S}}}\quad (N_{\mathrm{S}}>0)\,, \tag{3.4}$$

and in table X we present N_S and C_S for the phases (3.3), and also for the phases which are degenerate at $b = 0$, but will not occur for $\eta b < 0$. N_{A1} has been given in eq. (2.2), and N_I and N_{VI} follow from (2.3) and (2.6). $C_I = 0$ describes the bifurcation at $b = 0$ between the phases I and A1, the first condition $C_{VI} = 0$ decribes the bifurcation at $b = 0$ between the phases VI and I, cf. table VIII, and the second condition $C_{VI} = 0$ the bifurcation at $b = 0$ between the phases XVI and VII, cf. eq. (C.17). N_{VII} and N_{VIII} can be inferred from (1.14) and table IV. $C_{VII} = 0$ is the bifurcation at $b = 0$ between the phases XVI and VII and $C_{VIII} = 0$ is the bifurcation condition at $b = 0$ between VIII and A3, which, however, does not occur in practice. N_X and N_{XI} can be found taking the minimum of (1.18) with $u_1 = u_2 = u_3 = u$, and $b_1, \ldots, b_7$ as given by table VII for the solutions X and XI. The condition $C_X = 0$ for $2v < w_3'$ follows from the requirement that $m_3^2 > 0$, whereas solution XI exists under the condition $2w_3 + 3w_3' - 2v > 0$ which is automatically satisfied when $C_X > 0$ and $w_3' > 2v$. Finally, N_{XV} follows directly from eq. (C.13) of appendix C, and $C_{XV} = 0$ turns out to be important for determining the phase diagram at $b = 0$.

Remark. For the sake of completeness we also list the denominators N_S, together with the degenerate solutions at $b = 0$ for the solutions which do not occur under the condition (3.1), cf. table XI. For these (degenerate) solutions

TABLE X

Possible phases S under the condition (3.1) at $b = 0$, together with the degenerate phases, the denominators N_S and additional conditions of existence $C_S > 0$.

S	Degenerate phase	N_S	C_S
A1	A1′, XIV [1])	$8v + 4w_1$	–
I	I′	$5v + 4w_1 + w_2 + w_3 - v^2/(5v - w_2 - w_3)$	$4v - w_2 - w_3$
VI	XVI	$4v + 4w_1 + w_2 + w_3 + \dfrac{2\gamma v}{2v + \gamma}$	$\begin{cases} 4v - w_2 - w_3 - \dfrac{2\gamma v}{2v - \gamma} \\ 8v - w_2 - w_3 - \dfrac{8w_3' v}{2v + w_3'} \end{cases}$
VII	VII′	$4v + 4w_1 + w_2 + w_3 + \dfrac{8v - w_2 - 2w_3}{8v - w_2 - 2w_3} w_3$	$-\left\{8v - w_2 - w_3 - \dfrac{8w_3' v}{2v + w_3'}\right\}$
VIII	VIII′	$8v + 4w_1 + w_2^2/(4v + w_2 - 2w_3)$	$w_3 - 2v$
X	XI [2])	$4v + 4w_1 + w_2 + w_3 + w_3' - \dfrac{\{w_3 + w_3' + \lvert 2v - w_3'\rvert\}^2}{2v + 3w_3 + 2w_3' + \lvert 4v - 2w_3'\rvert}$	$2w_3 + 2w_3' - v$ [2])
XV	XV′	$\frac{16}{3}v + 4w_1 + w_2 + \frac{2}{3}w_3$	$(w_3 - v)(5v - w_2 - w_3) - 3v^2$

[1]) under the condition $4v < w_2 + 2w_3$.
[2]) under the condition $2v < w_3'$.

TABLE XI
The remaining solutions S at $b=0$, together with the degenerate solutions and the denominators N_S for the free energies

S	Degenerate solutions	N_S
R1	R1′, IX ($6v < w_2 + 2w_3 + w_3'$)	$12v + 4w_1$
II	XII ($w_3' < 2v$)	$4v + 4w_1 + w_2 + w_3 + w_3'$
III	A3, XIII ($w_3 < 2v$), XIV ($w_2 + w_3 < 4v$)	$4v + 4w_1 + w_2 + 2w_3$
IV	IV′, XII ($w_3' > 2v$)	$6v + 4w_1 + w_2 + w_3$
V	R3, IX ($6v > w_2 + 2w_3 + w_3'$)	$6v + 4w_1 + w_2 + 2w_3 + w_3'$

we have not investigated, however, which solution will be the most favorable one for $\eta b < 0$.

The tables X and XI contain the complete information on the denominators N_S for all solutions S at $b=0$, independent of the values of the spin fluctuation parameters. From the solutions presented in table XI, the solutions R1, R1′, A3, R3, IV, XIII and XIV can never occur, see also appendix A.

3.2. *Phase diagram at* $b=0$

In order to describe the phase diagram we introduce the quantities

$$z \equiv \tfrac{1}{2}w_3/v\,, \quad z' \equiv \tfrac{1}{2}w_3'/v\,, \quad -qz \equiv \tfrac{1}{2}w_2/v\,, \tag{3.5}$$

where we take $z>0$, $z'>0$, $0<q<1$, in agreement with (3.1). We first show that phase A1 will not occur under these conditions. Next we investigate the phases I, VII, VIII and XV, for which N_S does not depend on z', and finally the two phases VI and X with denominators N_S depending on z, z' and q will be studied.

Phase A1. Comparing A1 with VIII and XV, we have

$$f_{A1} < f_{VIII} \rightarrow 4v + w_2 - 2w_3 > 0 \rightarrow z < (1 + \tfrac{1}{2}q)^{-1} < 1\,, \tag{3.6}$$

$$f_{A1} < f_{XV} \rightarrow \tfrac{8}{3}v - w_2 - \tfrac{2}{3}w_3 < 0 \rightarrow 2 < (1 - \tfrac{3}{2}q)z\,. \tag{3.7}$$

Eq. (3.6) implies that for $z>1$, where VIII exists, we cannot have A1. Eq. (3.7) cannot be satisfied for $0<z<1$, and if XV does not exist for $z<1$, we have $f_{XV} > f_{VII}$, so that A1 does not occur at $b=0$.

Phases I, VII, VIII, XV. Using the results of table X it is straightforward to show that

$$\tfrac{1}{2}(N_{\mathrm{I}} - N_{\mathrm{XV}})/v = -\tfrac{1}{3}F(z)/(\tfrac{5}{2} + (q-1)z)\,, \tag{3.8}$$

$$\tfrac{1}{2}(N_{\mathrm{VII}} - N_{\mathrm{XV}})/v = -\tfrac{4}{3}F(z)/(4 + (q+2)z)\,, \tag{3.9}$$

$$\tfrac{1}{2}(N_{\mathrm{VIII}} - N_{\mathrm{XV}})/v = -\tfrac{4}{3}F(z)/((q+2)z - 2)\,, \tag{3.10}$$

where

$$F(z) = (1-q)z^2 - \tfrac{1}{2}(6-q)z + 2\,. \tag{3.11}$$

For every q with $0<q<1$, $F(z)$ has two zeros z_- and z_+ which are explicitly given by

$$z_\pm = \frac{1}{4}\,\frac{6-q}{1-q} \pm \left\{\frac{1}{16}\left(\frac{6-q}{1-q}\right)^2 - \frac{2}{1-q}\right\}^{1/2}, \tag{3.12}$$

so that in particular $-1+\sqrt{3}<z_-<1$, $z_+>2/(1-q)$. It is clear that the condition $C_{\mathrm{XV}}>0$ is equivalent to $F(z)<0$. $F(z)<0$, together with the conditions $C_{\mathrm{I}}>0$ and $C_{\mathrm{VIII}}>0$, implies that $N_{\mathrm{I}}>N_{\mathrm{XV}}$, $N_{\mathrm{VII}}>N_{\mathrm{XV}}$, $N_{\mathrm{VIII}}>N_{\mathrm{X}}$ in eqs. (3.8)–(3.10), so that phase XV will be favorable in comparison with the phases I, VII and VIII for $z_-<z<z_+$. For $z>z_+$ the conditions $C_{\mathrm{I}}>0$ and $C_{\mathrm{XV}}>0$ do not hold and we can only have the solutions VII and VIII. It is easy to see that $N_{\mathrm{VIII}}<N_{\mathrm{VII}}$ from eqs. (3.9) and (3.10), so that phase VIII will be more favorable for $z>z_+$. For $0<z<z_-$ only the solutions I and VII exist, and from (3.8) and (3.9) we have $N_{\mathrm{VII}}<N_{\mathrm{I}}$ for $z<1$, so that phase VII will be more favorable for $0<z<z_-$.

As far as the phases I, VII, VIIII, XV are concerned, we have the following picture. Phase I does not occur, phase VII occurs for $0<z<z_-$, phase XV for $z_-<z<z_+$ and phase VIII for $z>z_+$.

Phases VI and X. From table X it is straightforward to derive the following expressions:

$$\tfrac{1}{2}(N_{\mathrm{X}} - N_{\mathrm{XV}})/v = z' - 1\,, \tag{3.13}$$

$$\tfrac{1}{2}(N_{\mathrm{X}} - N_{\mathrm{VIII}})/v = z' - 1 + \frac{4}{3}\,\frac{F(z)}{(2+q)z-2} \equiv \delta(z,z')\,, \tag{3.14}$$

$$\tfrac{1}{2}(N_{\mathrm{VI}} - N_{\mathrm{XV}})/v = \frac{1}{3}\,\frac{z'((z+1)^2-3) + z(z-2)}{z+z'+zz'} \equiv \frac{1}{3}\,\frac{\alpha(z,z')}{z+z'+zz'}\,, \tag{3.15}$$

$$\tfrac{1}{2}(N_{\mathrm{VI}} - N_{\mathrm{X}})/v = \frac{z+1}{z+z'+zz'}\,[(z'+1)(z+1) - 3z'^2 - 1]$$
$$\equiv \frac{(z+1)\beta(z,z')}{z+z'+zz'}\,, \tag{3.16}$$

$$\tfrac{1}{2}(N_{\mathrm{VI}} - N_{\mathrm{VII}})/v = \frac{z[2-(1-\frac{1}{2}q)z(z'+1)]}{(2(1+\frac{1}{2}q)z)(z+z'+zz')} = \frac{2\gamma(z,z')}{z+z'+zz'}\,. \tag{3.17}$$

From (3.13) we see that X can only occur for $z'<1$, and XV only for $z'>1$. Using (3.15) and (3.16) it is clear that we may have phase VI under the condition $\beta(z,z')<0$ for $0<z'<1$, and $\alpha(z,z')<0$ for $z'>1$. Both conditions imply that $z\leq 1$, so that there cannot be a phase transition between VI and VIII. From eq. (3.17) and $z<z_-$ we find that phase VII may only occur for $z'>1$, so that there is no phase transition between X and VII.

We can now discuss the possible phase transitions. For $0<z<1$, $0<z'<1$, the only possible phases are X and VI and there is a first-order transition which is given by $\beta(z,z')=0$, $0<z'<1$. For $0<z<1$, $z'>1$, we can have the phases VI, VII and XV. Phase VI is most favorable for a region containing $z'>\frac{1}{3}$, $z=0$ and $z'=1$, $z<1$, and XV is most favorable for a region containing $1<z<z_+$, $z'>1$. Phase VII will actually occur, since $\alpha(z,z')>0$ for $z=z_-$ at sufficiently large z'. There is a second-order transition between VII and XV at $z=z_-$, a first-order transition between VI and VII at $\gamma(z,z')=0$, and a first-order transition between VI and XV for $\alpha(z,z')=0$. Note that $\gamma(z,z')=0$ corresponds to $C_{\mathrm{VII}}=0$, so that $\gamma(z,z')=0$ describes also the second-order transition between XVI and VII. For $z>1$, $z'>1$ the only possible phases are XV and VIII and there is a second-order transition at $z=z_+$. Finally, for $z>1$, $0<z'<1$, we have only X and VIII, and there is a first-order transition at $\delta(z,z')=0$.

On the basis of the considerations given above we have obtained the phase diagram at $b=0$ under the condition (3.1), as given in fig. 5. The phase transition lines $\gamma=0$, $\alpha=0$, $\beta=0$, $\delta=0$, $z=z_+$, $z=z_-$ have been indicated in this figure. The transition line $z'=1$ between X and XV is a first-order transition, since the vector $\boldsymbol{m}_3$ changes discontinuously from a real vector to a vector with $\boldsymbol{m}_3\cdot\boldsymbol{m}_3=0$, although the lengths m_1, m_2, m_3 of the order vectors are continuous at this transition. Fig. 5 contains also the special points P, Q, R. At the critical end point P, the second-order line between XV and VII meets the first-order lines between XV and VI and between VI and VII. When phase VI, however, is replaced by the degenerate phase XVI, P becomes a bicritical point, at which the first-order line between XVI and XV meets the second-order lines between XV and VII and between XVI and VII. Q is always a

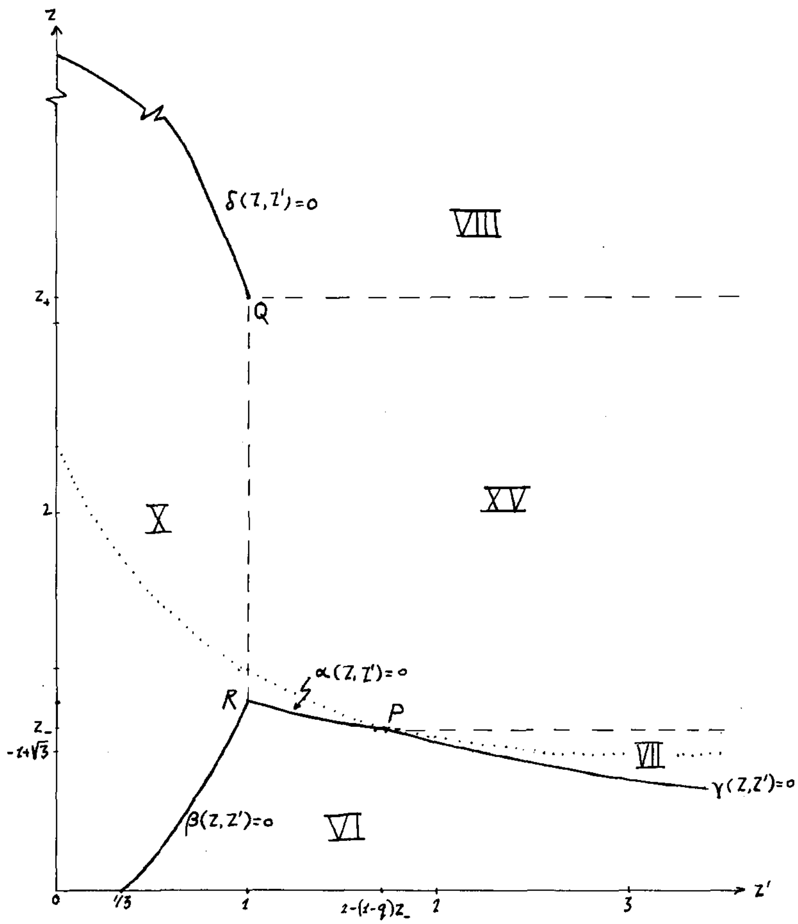

Fig. 5. Phase diagrams at $b=0$ for fixed $q=-\frac{1}{2}w_2/v$, $0<q<1$, and positive $z=\frac{1}{2}w_3/v$, $z'=\frac{1}{2}w_3'/v$. The dashed lines are second-order transitions and the solid lines are first-order transitions, P and Q are critical end points and R is a triple point. The dotted lines are the continuations of $\alpha=0$ and $\gamma=0$.

critical end point at which the second-order line between VIII and XV meets the first-order lines between X and XV and between X and VIII, and R is a triple point at which three first-order lines between X and VI, VI and XV, and XV and X come together.

3.3. *Phase diagram for $b\neq 0$ at the triple point $z=z'=1$*

When an external magnetic field is taken into account, the situation is more complicated and other phases such as I and A1 may appear in the phase diagram. As an example we shall treat the phase diagram as a function of temperature and magnetic field at the triple point R ($z = z' = 1$), where the phases X, XV and VII are in equilibrium at $b = 0$.

In the case that $z = z' = 1$, it can be shown that only the phases X, I, VI and A1 can occur in the phase diagram for $b \neq 0$, see appendix D for some details. We now first consider the BW solution X. From the explicit expression for Φ_X that follows from (1.18) and table VII, one obtains the following gap equations:

$$m_1\{u_1 + (4v + 2w_1)m_1^2 + (2w_1 + w_2 + w_3 + w_3')m_2^2 + (4v + 4w_1 + w_2)m_3^2\} = 0\,,$$

$$m_2\{u_2 + (4v + 2w_1)m_2^2 + (2w_1 + w_2 + w_3 + w_3')m_1^2 + (4v + 4w_1 + w_2)m_3^2\} = 0\,, \tag{3.18}$$

$$m_3\{2u_3 + (4v + 4w_1 + w_2)(m_1^2 + m_2^2) + 2(6v + 4w_1 + w_2 + 2w_3 + w_3')m_3^2\} = 0\,.$$

Due to the fact that Φ_X does not contain a term $\sim m_3^2 m_1 m_2$ in the special case $z' = 1$, we must consider a BW phase with $m_1 m_2 m_3 \neq 0$, and a different phase (BW$'$) with $m_2 = 0$, $m_1 m_3 \neq 0$.

In the case that $m_1 m_2 m_3 \neq 0$ we obtain

$$\frac{-(m_1^2 - m_2^2)v}{u_1 + u_2} = \frac{y}{x}\,, \quad \frac{-(m_1^2 + m_2^2)v}{u_1 + u_2} = \frac{1}{10}\left(1 + \frac{y^2}{x}\right),$$
$$\frac{-m_3^2 v}{u_1 + u_2} = \frac{1}{40}\left(1 - 4\,\frac{y^2}{x}\right) \tag{3.19}$$

and, since $m_2^2 > 0$, one has the condition

$$\frac{y}{x} \leq \frac{1}{10}\left(1 + \frac{y^2}{x}\right), \tag{3.20}$$

in which the equality sign describes the bifurcation to the phase BW$'$ with $m_2 = 0$. Inserting (3.19) in (1.10), we obtain

$$\frac{f_{BW}v}{(u_1 + u_2)^2} = -\frac{3}{80} + \frac{1}{20}\,\frac{y^2}{x} - \frac{1}{10}\left(\frac{y^2}{x}\right) - \frac{1}{4}\left(\frac{y}{x}\right)^2. \tag{3.21}$$

On the other hand, for $m_2 = 0$ one finds

$$\frac{-m_1^2 v}{u_1+u_2} = \frac{1}{23}\left(2+3\,\frac{y}{x}+2\,\frac{y^2}{x}\right), \qquad \frac{-m_3^2 v}{u_1+u_2} = \frac{\frac{5}{2}-2\,\frac{y}{x}-9\,\frac{y^2}{x}}{92}, \tag{3.22}$$

leading to the result

$$\frac{f_{\mathrm{BW}'}v}{(u_1+u_2)^2} = -\frac{13}{368}-\frac{1}{23}\,\frac{y}{x}-\frac{3}{92}\left(\frac{y}{x}\right)^2+\frac{5}{92}\,\frac{y^2}{x}-\frac{1}{23}\,\frac{y^3}{x^2}-\frac{9}{92}\left(\frac{y^2}{x}\right)^2. \tag{3.23}$$

For the other phases A1, I and VI, one has from (2.1), (2.3) and (2.6)

$$\frac{f_{\mathrm{A1}}v}{(u_1+u_2)^2} = -\frac{1}{36}\left(1+\frac{y}{x}\right)^2, \tag{3.24}$$

$$\frac{f_{\mathrm{I}}v}{(u_1+u_2)^2} = -\frac{1}{27}-\frac{1}{54}\,\frac{y}{x}-\frac{7}{108}\left(\frac{y}{x}\right)^2, \tag{3.25}$$

$$\frac{f_{\mathrm{VI}}v}{(u_1+u_2)^2} = -\frac{3}{80}-\frac{1}{4}\left(\frac{y}{x}\right)^2. \tag{3.26}$$

From the results (3.21), (3.23)–(3.26) one can work out table XII of phase-transition lines.

The phase diagram in the special case $z = z' = 1$ has been plotted in fig. 6. The phase diagram presented in fig. 6 contains a bicritical point at $x = \frac{2}{3}$, $y = \frac{1}{3}$, where the two second-order lines between I and A1, and between (BW)′ and

TABLE XII
Phase-transition lines in the special case that $z = z' = 1$

Phase transition	Order of transition	Transition line
VI→I	2	$y = \frac{1}{20}x$
I→A1	2	$y = \frac{1}{2}x,\ x > \frac{2}{3}$
BW′→A1	2	$5x = 18y^2 + 4y,\ x < \frac{2}{3}$
BW→BW′	2	$\frac{1}{10}x = y - \frac{1}{10}y^2$
BW→I	1	$\left(\frac{y^2}{x}\right)^2 - \frac{1}{2}\,\frac{y^2}{x} + \frac{1}{216}\left(1-20\,\frac{y}{x}\right)^2 = 0$
BW′→I	1	$-\frac{17}{972}+\frac{62}{243}\,\frac{y}{x}-\frac{80}{243}\left(\frac{y}{x}\right)^2-\frac{5}{9}\,\frac{y^2}{x}+\frac{4}{9}\,\frac{y^3}{x^2}+\left(\frac{y^2}{x}\right)^2 = 0$

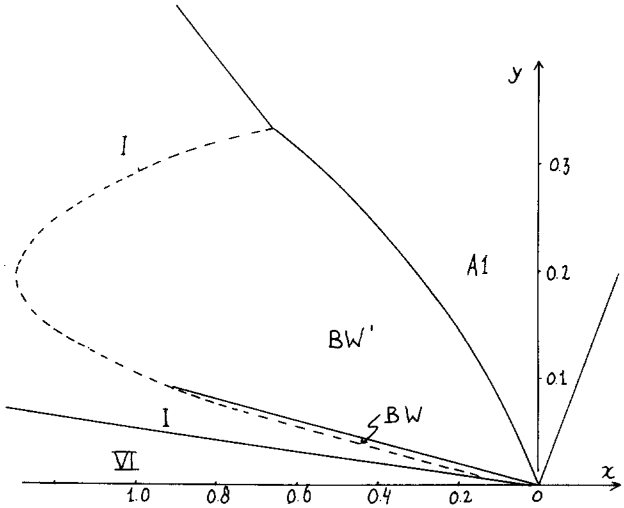

Fig. 6. Phase diagram in the case that $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3 = w_3' = 2v$. The solid curves are second-order transitions, the dashed curves first-order transitions.

A1 meet the first-order line between BW′ and I, and a critical end point at $x \sim 0.95$, $y \sim 0.95$, where the second-order line BW→BW′ meets the first-order lines BW→I, BW′→I. This critical end point can only exist for $w_3' = 2v$ and for all other values $w_3' \neq 2v$ there is only one BW phase with $m_1 m_2 m_3 \neq 0$.

4. Concluding remarks

In a sequence of papers, cf. also refs. 7, 8, 5, 1, we have presented a systematic study of the phases that can occur in liquid ^{3}He in the presence of a magnetic field and taking also into account the contributions from a contact term of the Hubbard type. This has been done on the basis of the Landau expansion (1.4)–(1.6) in terms of the 18 real order parameters, i.e. the three complex vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ describing the ordering of spin pairs ↑↑, ↓↓ and ↑↓, ↓↑, respectively, with explicit values of the coefficients. The coefficients u_1, u_2, u_3 of the second-degree part have been given in eq. (1.7), the coefficient v in (1.5), (1.6) arises from a pairing interaction with $l = 1$ of the BCS-type, and the coefficients w_1, w_2, w_3, w_3' are the contributions from spin fluctuations. In eq. (1.19) we have presented the explicit results for w_1, w_2, w_3, w_3' at $b = 0$ up to order I^2 which have been obtained in ref. 5 on the basis of a second-order perturbation calculation of the Hubbard interaction.

As the analysis of the 18 order parameter problem is a very complicated task, we have assumed that the inner products of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ can be chosen to be real and furthermore we have applied a so-called inertia condition (1.16) which expresses a certain rigidity of the geometrical configuration of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ with respect to changes in external parameters, such as temperature, magnetic field and pressure. In this way we have obtained one-dimensional solutions R1, R1′, A1, A1′, A3, in which only one of the vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ is non-vanishing, two-dimensional solutions I–VIII with two non-vanishing vectors, as well as three-dimensional solutions IX–XVI for which all three vectors are different from zero.

As it turned out to be too complicated to investigate the phase diagram on the basis of the complete Landau expansion for arbitrary values of the coefficients w_1, w_2, w_3, w_3', we have used the second-order results for the coefficients, as given in (1.19), as a guideline, thereby assuming that higher order contributions will not affect the signs of these coefficients, but merely change the second-order values by certain (positive) enhancement factors.

In the present paper we have dealt with two separate problems. The first problem is concerned with the stability of the phase diagram in the absence of spin fluctuations, as given in ref. 8. For this purpose we have investigated the phase diagram in the presence of a magnetic field for small values of the coupling constant I of the Hubbard interaction. In this limit it is safe to assume that the signs of the coefficients w_1, w_2, w_3, w_3' are correctly given by (1.20). Taking $\eta b < 0$ we have shown that for sufficiently small values of these coefficients only the phases A1, I, VI and X can occur. The corresponding phase digrams have been presented in figs. 1–4. Here X is the extension of the BW phase to finite values of the magnetic field, as already considered in ref. 8, and there are two new phases I and VI, which have not been considered before as candidates for the phase diagram. In the limit $w_3, w_3' \to 0$, phase VI becomes degenerate with the ABM phase (III) or the planar phase (II), and the region in which I occurs shrinks to zero. For small values of the spin fluctuation parameters it may not be easy to distinguish the phase diagram from a diagram in which the phases VI and I are replaced by the ABM or the planar phase.

The second problem we have dealt with is the phase diagram at zero magnetic field, also for larger values of the spin fluctuation parameters under the restriction that w_2, w_3 and w_3' satisfy eq. (3.1), in which the signs of w_2, w_3, w_3' correspond to the ones in (1.20), and the inequality $q < 1$ is obeyed in a large range of coefficient values, including the values (1.19). In that case only the phases VI, VII, VIII, X and XV play a role in the phase diagram. Phase X is again the BW phase, VII is the so-called ϵ-solution introduced in ref. 12, and VI, VIII and XV are new phases. The resulting phase diagram, as a function of $z = \frac{1}{2}w_3/v$, $z' = \frac{1}{2}w_3'/v$, at a fixed value of $q \equiv -w_2/w_3$ with $0 < q < 1$, has been presented in fig. 5. The phase diagram of fig. 5 contains some special critical points, and in fig. 6 we have presented the phase diagram as a function of temperature and magnetic field at the special value $z = z' = 1$, corresponding to a triple point at $b = 0$, at which the phases VI, X and XV are in equilibrium.

On the basis of these considerations one may atempt to give some discussion of the so-called profound effect, i.e. the effect of a small magnetic field on the phase diagram as a function of temperature T and pressure p. When p increases, the parameters w_3 and w_3' can be expected to increase as well. Starting from the BW phase for $b = 0$ at low pressure, one could expect, at a certain pressure p_0, to have a phase transition to one of the ordered phases VI, VIII or XV, as displayed in fig. 5. If we assume that w_3'/w_3 would be independent of $I \equiv I(p)$, and therefore be equal to 1 as in (1.19), this transition would take place at the point $(z, z') = (1, 1)$ in fig. 5. However, for w_3 slightly larger than w_3', one would cross the line $z' = 1$ and have a transition form X to XV, and, for w_3 slightly smaller than w_3', one would cross the curve $\beta(z, z') = 0$ in fig. 5 and have a transition from X (BW) to VI. At this stage one may speculate that the second possibility could occur in practice, as VI is a two-dimensional phase with only $\upuparrows$ and $\downdownarrows$ ordering, but in the presence of a magnetic field one may anticipate a more complicated behaviour.

Let us discuss, on the basis of figs. 4 and 6, what may happen at a small value of the magnetic field, when w_3 and w_3' increase from the values of fig. 4 to values of the order of $2v$. Comparing fig. 4 and fig. 6 one may anticipate no qualitative changes in the second-order phase transitions between BW and A1 and between I and A1 and also at the bicritical point where these transition lines meet. Furthermore there is a second-order transition line between VI and I, the slope of which decreases upon increasing w_3 and w_3', to a finite value $\frac{1}{20}$ at $w_3 = w_3' = 2v$. On the other hand at values of w_3, w_3' very close to the transition line $\beta(z, z') = 0$ in fig. 5 one may expect the transition between BW and VI, and also the critical end point on the second-order line between VI and I to occur at very small values of y. These features are displayed in fig. 7, in which we have plotted the phase diagram as a function of temperature and magnetic field at the values $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3 = 1.90v$, $w_3' = 1.94v$. (In this case again we can only have the ordered phases X, I, VI and A1, as discussed at the end of appendix D.)

Note that in this case the transition line between the phases BW and I has a rather complicated behaviour with a maximum value of x at $x \sim 0.98$. Furthermore, the value of $\lambda \equiv m_2/m_1$, in which m_1 and m_2 denote the order parameters of the BW phase, is very small on a large part of this transition line. This feature is reminiscent of what happens at the special value $w_3' = 2v$, for which the BW phase is split into a phase with $m_1 m_2 m_3 \neq 0$ and another phase BW' with $m_2 = 0$, as shown in fig. 6.

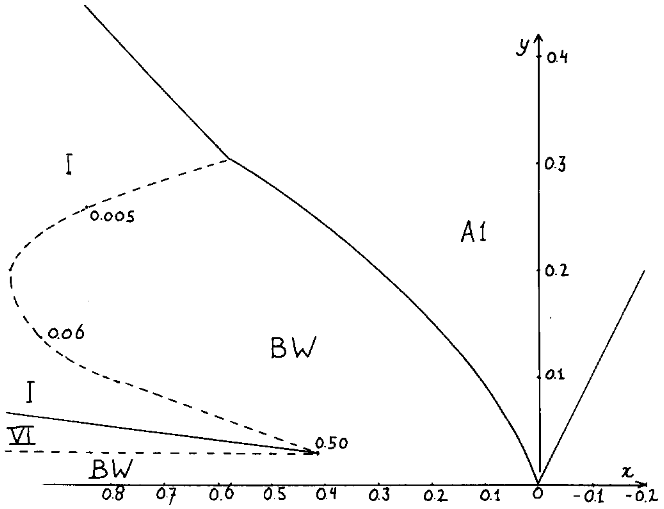

Fig. 7. Phase diagram in the case that $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3 = 1.90v$, $w_3' = 1.94v$. The solid curves are second-order transitions, the dashed curves first-order transitions. The numbers indicate the values of $\lambda \equiv m_2/m_1$ in the BW phase at the transition lines.

Fig. 7 can also give a qualitative understanding of the profound effect, as the transition line between BW and VI has a rather low minimum $y \sim 0.03$ at a rather large value of $x \sim 0.6$. Let us consider a small fixed value b' of the magnetic field and a pressure p_1 which is slightly smaller than the pressure p_0 at which transition at $b = 0$ between BW and VI takes place. When b' is small enough, the minimum of the transition curve $b(p_1)$ between VI and BW will occur at a value larger than b', and on increasing the temperature from a lower value to the critical temperature T_c (i.e. $x = 0$), one only observes a transition BW→A1 in the immediate neighbourhood of T_c. However, when p increases, the transition line between BW and VI will go down, and at a certain value p_2, the minimum of the transition curve $b(p_2)$ will coincide with the value b'. Hence, for all p values satisfying $p_2 < p < p_0$, the minimum $b(p_2)$ will lie below the chosen value b', and there will be a phase transition BW→VI at a relatively large value of x, corresponding to a temperature well below T_c.

The arguments given above can also be justified analytically for values of w_3, w_3' which are very close to $2v$. In fact, taking $w_3 = 2v(1-\alpha)$, $w_3' = 2v(1-\alpha')$, one may evaluate f_{BW} up to linear terms in α, α'. Using the relation

$$f_{BW} - f^0_{BW} = -(4\alpha + 2\alpha')vm_3^{0^4} - 2(\alpha + \alpha')vm_1^{0^2}m_2^{0^2} - 4|\alpha'|vm_3^{0^2}m_1^0 m_2^0 , \tag{4.1}$$

in which m_1^0, m_2^0, m_3^0 and f^0_{BW} refer to the BW solution at $\alpha = \alpha' = 0$, as given by (3.19) and (3.21), one finds

$$\frac{(f_{\mathrm{BW}} - f^0_{\mathrm{BW}})v}{(u_1 + u_2)^2} = -\frac{6\alpha - 5\alpha' + 4|\alpha'|}{800} + \frac{\alpha}{100}\frac{y^2}{x} - \frac{(9\alpha + 5\alpha')}{200}\left(\frac{y^2}{x}\right) + \tfrac{1}{2}(\alpha + \alpha')\left(\frac{y}{x}\right)^2 - \frac{|\alpha'|}{20}\left[\left(1 - 4\frac{y^2}{x}\right)\left\{\frac{1}{100}\left(1 + \frac{y^2}{x}\right)^2 - \left(\frac{y}{x}\right)^2\right\}^{1/2} - \frac{1}{10}\right]. \tag{4.2}$$

Comparing (4.2) to the relation

$$\frac{(f_{\mathrm{VI}} - f^0_{\mathrm{VI}})v}{(u_1 + u_2)^2} = -\frac{(\alpha' + 10\alpha)}{800} + (\alpha' + \tfrac{1}{2}\alpha)\left(\frac{y}{x}\right)^2, \tag{4.3}$$

in which again f^0_{VI} refers to the solution VI at $\alpha = \alpha' = 0$, one can derive the relation

$$(\alpha' - \tfrac{1}{2}\alpha)x = \tfrac{5}{2}y^2 - \{\tfrac{25}{4}y^4 - (10y^4 + 50\alpha'' y^2)(\alpha' - \tfrac{1}{2}\alpha)\}^{1/2} \qquad (\alpha'' \equiv \alpha - \tfrac{1}{2}\alpha') \tag{4.4}$$

for the transition line between BW and VI in linear approximation for α and α'.

Eq. (4.4) implies that the transition between BW and VI at $b = 0$ will take place at the values $\alpha' = \frac{1}{2}\alpha$, and for α' values slightly larger than $\frac{1}{2}\alpha$, the minimum value of y is equal to $\{8\alpha''(\alpha' - \frac{1}{2}\alpha)\}^{1/2}$ and occurs at a relatively large value of x, i.e. $x \sim 20\alpha''$. The coordinates of the critical end point at the second-order line between VI and I are given by $x \sim 10\alpha''$, $y \sim \frac{1}{2}\alpha''$ in this approximation. (For extremely small values of $\alpha' - \frac{1}{2}\alpha$, i.e. $\alpha' - \frac{1}{2}\alpha \ll \alpha''$, it is possible that x has also a minimum value on the transition line between BW and VI, before it reaches the critical end point.)

Although the features mentioned above may give a qualitative account, it certainly does not give a detailed understanding. First of all, the precise behaviour of the spin fluctuation coefficients as a function of pressure is not known, and in the absence of more detailed data on the coefficients of the Landau expansion, in the case of a contact interaction of the Hubbard type, one does not know the point on the curve $\beta(z, z') = 0$ in fig. 5, where the phase transition BW→VI at $b = 0$ will take place. The precise location of this point can be important for actual estimates of the coordinates of the minimum of the transition curve BW$\rightarrow$VI at finite small values of b. Furthermore, one should have rather precise estimates on the relation between the coordinates x and y, and the temperature differences and magnetic field values, respectively, which occur in practice. In connection with this, it is hard to say to which extent other phases such as phase I (and possibly other phases at larger values of x and y), may be of importance in a discussion of the profound effect. It should be noted that the transition lines in a realistic temperature-magnetic field diagram would have a much steeper behaviour than the one displayed in figs. 6 and 7, as the y coordinate has to be multiplied by $-(A_c\eta)/B_c$ and the x coordinate by a much smaller factor $(A_c\eta)^2/B_c$.

Finally, the phase diagrams presented in figs. 1–7 are based on the assumption that the signs of the parameters w_3, w_3' are correctly given by the second-order values of (1.19). In particular, this assumption is sufficient to rule out the ABM phase as a candidate to occur in the phase diagram. From a theoretical point of view, starting from the Hubbard hamiltonian, and considering the usual ideas about enhancement, it is not easy to imagine that this assumption is not true. But, it may be noted that the ABM phase can occur indeed when the parameters w_3, w_3' are negative, as suggested by the spin fluctuation results of ref. 17. Therefore, we shall discuss this case as well.

First of all it is straightforward to show that at small negative values of w_3 and w_3' only the phases A1, II, III and X can occur in the phase diagram in the presence of an external field. From these phases III has always a smaller free energy than phase II when $w_3 < w_3'$. Considering the case $w_3 < w_3'$, one has only the phases A1, III and X, implying that the phase diagram in the absence of spin fluctuations as presented in fig. 1 is stable under small perturbations w_1, w_2, w_3, w_3' with $w_3 < 0$, $w_3' < 0$ in the Landau expansion. Of course, in such a case the two-dimensional phase has to be identified with the ABM phase, i.e. phase VI in fig. 1 must be replaced by III. We shall not present explicit results on phase diagrams for negative w_3, w_3' in the presence of a magnetic field. Such phase diagrams have been given in ref. 19 under the assumption that only the phases BW, ABM and A1 are important and this assumption has been corroborated in the appendices A, B and E of the present paper, at least for small (negative) values of w_3 and w_3'.

Secondly, it is not hard to extend the phase diagram of fig. 5 at zero magnetic field to negative values of w_3 and w_3'. In fact, one will have the ABM phase in the region bounded by the line $z' = -1$, $z < -1$, where we have a transition to phase VI and the line $1 + 4z - 3z' = 0$ for $-1 < z' < 0$, where we have a transition to phase X. Apart from that there is a phase transition between phase VI and phase V at the curve $(1+z)(1+z') = 1$ for $z < -1$, $z' < -1$, and between phase VI and phase X at the line $z = -1$, $z' < -1$. (The solutions VII, VIII, XV, I and II do not occur at $b = 0$. For the solutions VIII and XV this follows from table X, for the solutions I and VII this has been shown in appendix E, and the phases X, VI and V can be shown to be more favourable than phase II for $z' < z$.)

The phase diagram at $b = 0$ as a function of z and z' for negative z and z' is presented in fig. 8.

The spin fluctuation results of ref. 17 with $z' = \frac{1}{5}z$ would indicate a first-order transition between BW and ABM at the value $z = -\frac{5}{17}$, $z' = -\frac{1}{17}$. Finally, the profound effect in the case of negative w_3 and w_3' may be explained quite easily by the fact that the transition curve between BW on the one hand, and ABM and A1 on the other hand in fig. 1 moves down to the x-axis for z and z' values such that $\frac{4}{3}z - z' + \frac{1}{3}$ is a small positive number, so that at zero field BW is only slightly more favourable than ABM. Assuming no other phases to occur under such conditions, this will provide the explanation that the phase transition at a small fixed magnetic field may occur at temperatures which are substantially lower than the transition temperature at zero field.

With regard to the actual situation in ^{3}He one might note that the experimental results for the specific heat jumps[20]) do not give direct support to the signs of the Landau parameters as suggested in eq. (1.20), see also ref. 21. One even could raise the question whether the Hubbard hamiltonian with a contact interaction would give rise to an appropriate description[22]). One should not even exclude the possibility that the spin fluctuation model which is usually justified by means of an approximate diagrammatic analysis[23-26]), starting from

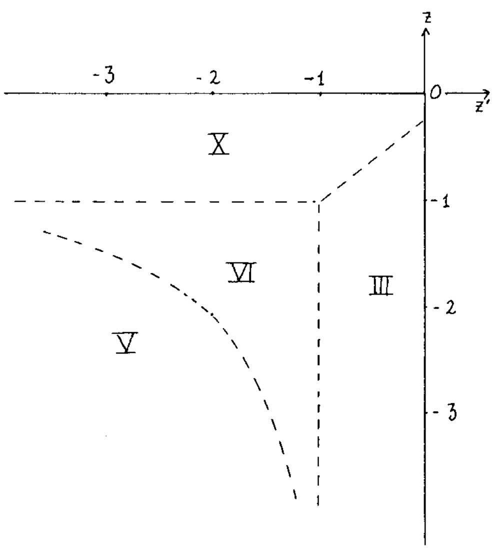

Fig. 8. Phase diagram for negative $z = \frac{1}{2}w_3/v$ and $z' = \frac{1}{2}w_3'/v$. The dashed lines are first-order transitions.

a contact interaction of the Hubbard type, could produce more realistic results than a contact interaction itself. Future experiments could give some detailed information on the values of the spin fluctuation coefficients so that one can make more concrete theoretical predictions on the regions in the phase diagram which apply to the situation in liquid ^{3}He. From fig. 8 one can note that phase VI may also occur for negative values of z and z', provided that z and z' are large enough.

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Appendix A

In this appendix we show that the solutions A3, R3, IV, XIII, XIV do not lead to an absolute minimum of the Landau expansion.

i) *Solutions A3 and R3.* Considering f_{A3} as given by table I, and taking into account that $2u_3 \geq u_1 + u_2$ as follows from (1.7), we immediately have $f_{A3} \geq f_{III}$ $(b = 0)$, implying that A3 does not lead to an absolute minimum. The same conclusion applies to R3, since $f_{R3} \geq f_V$ $(b = 0)$.

ii) *Solution IV.* Considering (1.18) with $m_3 = 0$, $b_1 = a_{11}$, $b_2 = a_{22}$ and $b_6 = a_{12}$, it follows from table II that $\Phi_{IV} = \Phi_I + vm_1^4$ at all values of m_1 and m_2, so that solution IV does not occur.

iii) *Solution XIII.* From (1.18) and table VII one has the expression

$$\begin{aligned}\Phi_{XIII} = u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (4v + 4w_1 + w_2 + 2w_3)\{\tfrac{1}{4}(m_1^2 + m_2^2)^2 + m_3^4\} \\ +(8v + 4w_1 + w_2)m_3^2(m_1^2 + m_2^2) + \tfrac{1}{4}(4v - w_2 - 2w_3)(m_1^2 - m_2^2)^2 .\end{aligned} \tag{A.1}$$

In order that Φ_{XIII} at fixed $m_1^2 - m_2^2$ has a minimum corresponding to a mixed phase with $m_1^2 + m_2^2 > 0$, $m_3^2 > 0$, the coefficient of $m_3^2(m_1^2 + m_2^2)$ should not be larger than twice the square root of the product of the coefficients of

$(m_1^2+m_2^2)^2$ and of m_3^4, leading to the condition $w_3 \geqslant 2v$. In order that Φ_{XIII} has a minimum with $m_2^2>0$ the coefficient of $(m_1^2-m_2^2)^2$ should be positive, i.e. $4v-w_2-2w_3>0$. From the gap equations corresponding to Φ_{XIII} we have $m_1^2-m_2^2=(u_2-u_1)/(4v-w_2-2w_3)$, implying that $m_1^2 \geqslant m_2^2$. Comparing solution XIII with solution XV we have

$$\Phi_{\text{XIII}}-\Phi_{\text{XV}}=-vm_2^4+4vm_2^2m_3^2+w_3m_1^2m_2^2 \tag{A.2}$$

which is non-negative at $w_3 \geqslant v$. Hence, solution XIII does not occur.

iv) *Solution XIV*. From (1.18) and table VII one can write down an explicit expression for Φ_{XIV}. From the gap equations corresponding to Φ_{XIV} one can show that

$$\begin{aligned}&u_1+u_2-2u_3-(4v-w_2-2w_3)(m_1-m_2\,\mathrm{sgn}(4v-w_2-w_3))^2\\&\quad\times\left\{1+\frac{m_3^2}{m_1m_2}\,\mathrm{sgn}(4v-w_2-2w_3)\right\}=0\end{aligned} \tag{A.3}$$

implying with (1.7) that $4v-w_2-2w_3<0$. Considering the extrema of Φ_{XIV} with $m_1m_2m_3=0$ in the case that $4v-w_2-2w_3<0$, i.e.

$$\begin{aligned}&(u_1+u_2-2u_3)+(4v-w_2-2w_3)\left(\frac{m_3^2}{m_1m_2}-1\right)(m_1+m_2)^2=0\,,\\&(u_1-u_2)+(4v-w_2-2w_3)\left(1-\frac{m_3^2}{m_1m_2}\right)(m_1^2-m_2^2)=0\,,\\&(u_1+u_2+2u_3)+(16v+8w_1)(m_1^2+m_2^2+2m_3^2)\\&\quad+(4v-w_2-2w_3)\left(\frac{m_3^2}{m_1m_2}-1\right)(m_1-m_2)^2=0\,.\end{aligned} \tag{A.4}$$

From the first eq. (A.4) one finds that $m_3^2/(m_1m_2) \leqslant 1$, and the second and third eqs. (A.4) yield the inequalities

$$m_1^2-m_2^2 \leqslant 0\,,\quad m_1^2+m_2^2+2m_3^2 \leqslant -\frac{(u_1+u_2+2u_3)}{16v+8w_1}\,. \tag{A.5}$$

From (1.10), (1.7) and (A.5) it is easy to show that

$$f_{\text{XIV}} \geqslant -\tfrac{1}{4}\,\frac{(u_1+u_2)(u_1+u_2+2u_3)}{16v+8w_1}\,, \tag{A.6}$$

so that phase XIV at $b \neq 0$ is less favorable than the A1 solution. On the other hand, for $b = 0$ the equality sign in (A.6) holds so that in that case XIV and A1 are degenerate.

Appendix B

In this appendix we show that the solutions V, IX, XI and XII do not occur under the condition (1.20) for the signs of w_2, w_3, w_3'.

i) *Solution V.* Considering (1.18) with $m_3 = 0$, $b_1 = a_{11}$, $b_2 = a_{22}$, $b_6 = a_{12}$, and using table II, we have

$$\Phi_{\mathrm{V}} = \Phi_{\mathrm{II}} + v(m_1^4 + m_2^4) + w_3 m_1^2 m_2^2 = \Phi_{\mathrm{III}} + v(m_1^4 + m_2^4) + w_3' m_1^2 m_2^2 \,, \tag{B.1}$$

implying that V does not occur, if w_3 or w_3' is larger than $-2v$.

ii) *Solution IX.* For $6v - w_2 - 2w_3 - w_3' > 0$ one can compare IX with solution XII. From (1.18) and table VII we have

$$\Phi_{\mathrm{IX}} - \Phi_{\mathrm{XII}} > \begin{cases} (4v + 2w_3) m_3^2 (m_1^2 + m_2^2) & (w_3' < 2v) \,, \\ (8v - w_2) m_3^2 (m_1 - m_2)^2 + 4(w_3 + w_3') m_3^2 m_1 m_2 \,, \end{cases} \tag{B.2}$$

implying that IX does not occur under the conditions $6v - w_2 - 2w_3 - w_3' > 0$, $w_3 > -2v$, $8v - w_2 > 0$.

For $6v - w_2 - 2w_3 - w_3' < 0$, one can compare IX with solution XIV. This leads to

$$\Phi_{\mathrm{IX}} - \Phi_{\mathrm{XIV}} \geqslant v(m_1^2 - m_2^2)^2 + (2v + w_3')(m_3^2 - m_1 m_2)^2 + 4v m_3^2 (m_1 + m_2)^2 \,, \tag{B.3}$$

so that IX does not occur when $6v - w_2 - 2w_3 - w_3' < 0$, $w_3' > -2v$.

iii) *Solution XI.* For $w_3' < 2v$ we compare XI with solution X. Using the relation $(w_3 + w_3') \boldsymbol{m}_1 \cdot \boldsymbol{m}_2 + (2v - w_3') m_3^2 = 0$, given in table VI, we have

$$\Phi_{\mathrm{XI}} - \Phi_{\mathrm{X}} = v(m_1^4 + m_2^4) - w_3' m_1^2 m_2^2 + (2v - w_3') m_3^2 (2 m_1 m_2 + \boldsymbol{m}_1 \cdot \boldsymbol{m}_2) \geqslant 0 \,. \tag{B.4}$$

For $w_3' > 2v$ we can compare XI with solution XV leading to

$$\Phi_{\mathrm{XI}} - \Phi_{\mathrm{XV}} = v(m_1^2 - 2m_3^2)^2 + (w_3' - 2v)(w_3 + 2v)(w_3 + w_3')^{-1} m_3^4 \,. \tag{B.5}$$

From (B.4) and (B.5) it follows that XI does not occur when $w_3 > -2v$.

iv) *Solution XII.* Comparing XII with X we have

$$\Phi_{\mathrm{XII}} - \Phi_{\mathrm{X}} = w_3 m_1^2 m_2^2 + v(m_1^4 + m_2^4) \,, \tag{B.6}$$

so that XII does not occur when $w_3 > -2v$.

Appendix C

In this appendix we show that the solutions VII, VII′, VIII, VIII′, XV, XV′ and XVI do not occur at sufficiently small values of w_1, w_2, w_3, w_3' satisfying (1.20).

i) *Solutions VII, VII′.* Using (1.14) with table IV, and also table I, it is straightforward to compare VII with the A1 solution. We find that VII can only have a lower free energy than A1, i.e. $f_{\mathrm{VII}} < f_{\mathrm{A1}}$, under the condition

$$\begin{aligned} &\frac{u_3}{u_1} > A \,, \\ &4A \equiv \frac{4v + 4w_1 + w_2}{3v + w_1} + \left\{ \left(\frac{4v + 4w_1 + w_2}{3v + w_1} \right)^2 \right. \\ &\qquad \left. + \frac{4v(4v + 4w_1 + w_2 + 2w_3) - (4v + 4w_1 + w_2)^2}{(3v + w_1)(2v + w_1)} \right\}^{1/2} . \end{aligned} \tag{C.1}$$

From $2u_3 > u_1 + u_2$ and (C.1) we have $(u_1 - u_2)/(u_1 + u_2) < -1 + A^{-1}$ which for sufficiently small w_1, w_2, w_3, w_3' must be smaller than the value B given in table VIII. As a result phase VII can only have a lower free energy than A1 in the region where phase VI exists. We therefore compare the solutions VII and VI. For this purpose we rewrite f_{VII} as

$$f_{\mathrm{VII}} = \frac{-u_3^2(4v - \frac{1}{2}w_2 + w_3) - \frac{1}{2}(u_1 - u_3)^2(4v + 4w_1 + w_2) + (u_3^2 - u_1^2)w_3}{(4v + 4w_1 + w_2 + w_3)(4v - \frac{1}{2}w_2 + w_3) + w_3(4v - \frac{1}{2}w_2 - w_3)} \,. \tag{C.2}$$

Considering VII as the special case $m_2 = 0$ of solution XVI, we obtain for its stability, using also the gap equations for m_1 and m_3, respectively,

$$0 < \tfrac{1}{2}\,\frac{\partial^2 \phi_{\mathrm{XVI}}}{\partial m_2^2} = \begin{cases} u_2 - u_1 - \{8v - w_2 - 2w_3 - (a+1)w_3'\}m_3^2\,, \\ u_1 + u_2 - 2u_3 + \{2w_3 + (a+1)w_3'\}m_1^2 - 4w_3 m_3^2\,, \end{cases} \tag{C.3}$$

with $a \equiv -1 + 8v/(2v + w_3')$. Combining both inequlities (C.3) we find

$$\frac{u_3 - u_1}{u_2 - u_1} < \frac{1}{2}\left[1 + \frac{2w_3 + (a+1)w_3'}{8v - w_2 - 2w_3 - (a+1)w_3'}\right]. \tag{C.4}$$

On the other hand, from the second inequality of (C.3) we have $\{2w_3 + (a + 1)w_3'\}m_1^2 > 4w_3 m_3^2$, and inserting the solutions for m_1^2 and m_3^2 from the gap equations for VII we obtain

$$\frac{u_3 - u_1}{u_3 + u_1} > \frac{4(4v + 2w_1 + \frac{1}{4}w_2) - \{2 + (a+1)w_3'/w_3\}(4v + 4w_1 + w_2 + w_3)}{8v - w_2 - 2w_3 - (a+1)w_3'}\,. \tag{C.5}$$

Inserting (C.4) and (C.5) in (C.2) one can derive the inequality

$$f_{\mathrm{VII}} > \frac{-u_3^2 - \frac{1}{4}(u_1 - u_2)^2 A'}{4v + 4w_1 + w_2 + w_3 + w_3(8v - w_2 - 2w_3)/(8v - w_2 + 2w_3)} \tag{C.6}$$

with a rather complicated explicit expression for A'. We shall not give this expression, but merely mention that $A' \to \frac{1}{2}$ in the limit that w_1, w_2, w_3, w_3' tend to zero.

Using the inequalities

$$4v + 4w_1 + w_2 + w_3 + \frac{8v - w_2 - 2w_3}{8v + w_2 + 2w_3}\,w_3$$
$$> \begin{cases} 4v + 4w_1 + w_2 + w_3 + \dfrac{2\gamma v}{2v + \gamma}\,, \\ A'\left(4\bar{v} - w_2 - w_3 - \dfrac{2\gamma v}{2v - \gamma}\right), \end{cases} \tag{C.7}$$

both of which are satisfied at sufficiently small values of w_1, w_2, w_3, w_3', we immediately obtain from (C.5) and (2.6) $f_{\mathrm{VII}} > f_{\mathrm{VI}}$, so that phase VII does not occur.

From eq. (C.2) for f_{VII} and a similar expression with u_1 replaced by u_2 for $f_{\mathrm{VII'}}$ we have

$$f_{\mathrm{VII}'} - f_{\mathrm{VII}} = \frac{(u_2 - u_1)\{\frac{1}{2}(2u_3 - u_1 - u_2)(4v + 4w_1 + w_2) - w_3(u_1 + u_2)\}}{(4v + 4w_1 + w_2 + w_3)(4v - \frac{1}{2}w_2 + w_3) + w_3(4v - \frac{1}{2}w_2 - w_3)} \tag{C.8}$$

so that $f_{\mathrm{VII}'} > f_{\mathrm{VII}}$ for $4v + 4w_1 + w_2 > 0$, $w_3 > 0$, $4v - \frac{1}{2}w_2 - w_3 > 0$.

ii) *Solutions VIII, VIII′*. Eq. (1.18) with $m_2 = 0$ can only have an absolute minimum with $m_1 m_3 \neq 0$ under the condition

$$p \equiv \frac{(4v + 4w_1 + w_2 + b_4)}{\{(2v + w_1 + b_1)(4v + 4w_1 + w_2 + 2w_3 + b_3)\}^{1/2}} < 2\,. \tag{C.9}$$

For solution VIII we have $b_1 = a_{11} = b_3 = 0$, $b_4 = a_{13} = 4v$, implying that (C.9) cannot be satisfied for small values of w_1, w_2, w_3, w_3'.

A similar argument can be applied with regard to solution VIII′ which can be obtained from VIII interchanging u_1 and u_2, and also m_1 and m_2. From the explicit expression for f_{VIII} which follows from (1.14) and table IV and a similar one for $f_{\mathrm{VIII}'}$, which can be obtained interchanging u_1 and u_2 in the expression for f_{VIII}, we have

$$f_{\mathrm{VIII}'} - f_{\mathrm{VIII}} = \frac{(u_2 - u_1)\{(4v - 2w_3)2u_3 - (4v + 4w_1 + w_2 + 2w_3)(u_1 + u_2 - 2u_3)\}}{4(4v + 4w_1 + w_2 + 2w_3)(2v + w_1) - (8v + 4w_1 + w_2)^2}\,, \tag{C.10}$$

implying that VIII′ is less favourable than VIII for $w_3 > 2v$ at all values of b, and for $w_3 < 2v$ at sufficiently large values of b, see also (1.7) and the stability conditions (1.15).

iii) *Solutions XV, XV′*. Eq. (1.18) at $m_2 > 0$ can only have a minimum $m_1 m_3 \neq 0$ under the condition (C.9). For solution XV we have $b_1 = b_3 = 0$, $b_4 = 4v$, so that (C.9) cannot be satisfied at small values of w_1, w_2, w_3, w_3'. Therefore, solution XV (and also XV′) cannot lead to an absolute minimum of the Landau expansion.

In order to investigate the relative stability of the solutions XV and XV′ we introduce the function

$$\begin{aligned}\Phi = {} & \tfrac{1}{6}(u_1 + u_2 + u_3)(m_1^2 + m_2^2 + 2m_3^2) + \tfrac{1}{12}(u_1 + u_2 - 2u_3)(m_1^2 + m_2^2 - 4m_3^2) \\ & + \tfrac{1}{4}(u_1 - u_2)(m_1^2 - m_2^2) + (\tfrac{4}{3}v + w_1 + \tfrac{1}{4}w_2 + \tfrac{1}{6}w_3)(m_1^2 + m_2^2 + 2m_3^2)^2 \\ & + \tfrac{1}{12}(w_3 - v)(m_1^2 + m_2^2 - 4m_3^2)^2 + \tfrac{1}{4}(5v - w_2 - w_3)(m_1^2 - m_2^2)^2 \\ & + \tfrac{1}{2}\epsilon v(m_1^2 - m_2^2)(m_1^2 + m_2^2 - 4m_3^2)\,,\end{aligned} \tag{C.11}$$

such that $\Phi = \Phi_{XV}$ for $\epsilon = -1$, and $\Phi = \Phi_{XV'}$, for $\epsilon = 1$. From the condition that Φ at fixed $m_1^2 + m_2^2 + 2m_3^2$ has an absolute minimum for $m_1^2 - m_2^2 \neq 0$, $m_1^2 + m_2^2 - 4m_3^2 \neq 0$, we obtain the inequality

$$(w_3 - v)(5v - w_2 - w_3) - 3v^2 \geqslant 0 . \tag{C.12}$$

On the other hand, from the gap equations for $m_1^2 - m_2^2$, $m_1^2 + m_2^2 + 2m_3^2$ and $m_1^2 + m_2^2 - 4m_3^2$ it is easy to show that the absolute minimum f of Φ for $m_1m_2m_3 \neq 0$ is given by

$$f = -\frac{\frac{1}{3}(u_1 + u_2 + u_3)^2}{16v + 12w_1 + 3w_2 + 2w_3} - \frac{\frac{1}{4}(w_3 - v)(u_1 - u_2)^2 + \frac{1}{12}(5v - w_2 - w_3)(u_1 + u_2 - 2u_3)^2 - \frac{1}{2}\epsilon v(u_1 - u_2)(u_1 + u_2 - 2u_3)}{(w_3 - v)(5v - w_2 - w_3) - 3v^2} \tag{C.13}$$

From (C.13) it is clear that, when XV with $\epsilon = -1$ and XV′ with $\epsilon = 1$ both exist, solution XV must have a lower free energy.

iv) *Solution XVI.* From (1.18) and table VII we have

$$\begin{aligned}\Phi_{XVI} = {} & u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + (3v + w_1)(m_1^4 + m_2^4) \\ & + (4v + 4w_1 + w_2 + 2w_3)m_3^4 + (4v + 4w_1 + w_2)m_3^2(m_1^2 + m_2^2) \\ & + \{2w_1 + w_2 + 2w_3 + (a+1)w_3' - 2v\}m_1^2m_2^2 ,\end{aligned} \tag{C.14}$$

in which a has been defined by (C.3). From (1.10) and the solutions with $m_1m_2m_3 \neq 0$ of the gap equations corresponding to Φ_{XVI} it is straightforward to show that

$$\begin{aligned} f_{XVI} = {} & -\frac{1}{4}\,\frac{(u_1 + u_2)^2}{4v + 4w_1 + w_2 + w_3 + 2\gamma v/(2v + \gamma)} - \frac{1}{4}\,\frac{(u_1 - u_2)^2}{8v - w_2 - 2w_3 - (a+1)w_3'} \\ & + \frac{1}{4}\,\frac{\{2w_3 + (a+1)w_3'\}(u_1 + u_2 - 2u_3)(u_1 + u_2 + 2u_3) - (4v + 4w_1 + w_2)(u_1 + u_2 - 2u_3)^2}{\{4w_3 + (a+1)w_3'\}\{4v + 4w_1 + w_2 + w_3 + 2\gamma v/(2v + \gamma)\}} \end{aligned} \tag{C.15}$$

and the last term on the right-hand side of (C.15) is non-negative, as can be shown using the solution of the gap equations for Φ_{XVI} in the condition $m_3^2 > 0$. Denoting the first two terms on the right-hand side by f', we have $f_{XVI} > f'$ and we shall now show that f' can never be smaller than the absolute minimum of the Landau expansion.

Comparing f' and f_{A1} we find that $f' < f_{A1}$, only under the condition that $(u_1 - u_2)/(u_1 + u_2) < A''$, in which A'' is a rather complicated expression. We shall not give this expression here, but only use the fact that $A'' \to \frac{1}{2}$ in the limit that w_1, w_2, w_3, w_3' tend to zero. For sufficiently small w_1, w_2, w_3, w_3' we therefore have $A'' < B$, in which B has been given in table VIII, so that we can have $f' < f_{A1}$, only in the region where phase VI exists. Using the inequlity

$$8v - w_2 - 2w_3 - (a+1)w_3' > 4v - w_2 - w_3 - 2\gamma v/(2v - \gamma)\,, \tag{C.16}$$

we find that $f_{XVI} \geqslant f' \geqslant f_{VI}$, and phase XVI does not occur at sufficiently small w_1, w_2, w_3, w_3', satisfying (1.20). Note that for $b = 0$ the equality signs hold, so that XVI and VI are degenerate in the absence of a magnetic field.

Finally, for phase XVI one has the condition $m_2^2 \geqslant 0$, or

$$\begin{aligned} &\frac{-(u_1 - u_2)}{8v - w_2 - 2w_3 - (a+1)w_3'} \\ &\quad \leqslant \frac{-2w_3(u_1 + u_2) - (4v + 4w_1 + w_2)(u_1 + u_2 - 2u_3)}{\{4w_3 + (a+1)w_3'\}\{4v + 4w_1 + w_2 + w_3 + 2\gamma v/(2v + \gamma)\}} \end{aligned} \tag{C.17}$$

and if (C.17) is not satisfied, phase VII is a more favourable phase.

Appendix D

In this appendix we show that the phases VII, VII′, VIII, VIII′, XV, XV′ and XVI do not occur under the conditions $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3' = w_3 = 2v$, and $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3 = 1.90v$, $w_3' = 1.94v$, assumed in figs. 6 and 7, respectively. We first consider the various phases in the case that $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3' = w_3 = 2v$.

Solutions VII, VII′. From (C.1) one finds that $f_{VII} < f_{A1}$ under the condition $y + 2.538y^2 \leqslant 0.269x$, which can be only satisfied in the region, where solution I or solution VI exists. From (1.14) and table IV, and using the condition $f_{VII} < f_{A1}$, we have

$$\begin{aligned} \frac{f_{VII}v}{(u_1 + u_2)^2} &= -\frac{13}{368} - \frac{1}{46}\frac{y}{x} - \frac{1}{46}\left(\frac{y}{x}\right)^2 + \frac{9}{184}\frac{y^2}{x} - \frac{1}{40}\frac{y^3}{x^2} - \frac{13}{368}\left(\frac{y^2}{x}\right)^2 \\ &\geqslant -\frac{13}{368} - \frac{1}{46}\frac{y}{x} - \frac{1}{46}\left(\frac{y}{x}\right)^2 . \end{aligned} \tag{D.1}$$

Comparing the right-hand side of (D.1) with (3.25) and (3.26) it is easy to show that $f_{\mathrm{VII}} > f_{\mathrm{I}}$ and also that $f_{\mathrm{VII}} > f_{\mathrm{VI}}$ under the condition $y/x \leqslant \frac{1}{20}$, which is the condition for VI to exist. Phase VII′ does not occur, since $f_{\mathrm{VII}'} > f_{\mathrm{VII}}$, as follows from (C.8).

Solution VIII. From (1.14) and table IV, we have, taking into account the last inequality of (1.15), i.e. $\frac{1}{9}x \geqslant \frac{8}{9}y + 2y^2$,

$$\frac{f_{\mathrm{VIII}}v}{(u_1+u_2)^2} = -\frac{1}{32} - \frac{1}{4}\left(\frac{y}{x}\right)^2 - \frac{1}{16}\frac{y^2}{x} - \frac{1}{2}\frac{y^3}{x^2} - \frac{9}{32}\left(\frac{y^2}{x}\right)^2$$

$$\geqslant -\frac{9}{256} + \frac{1}{64}\frac{y^2}{x} - \frac{27}{64}\left(\frac{y^2}{x}\right)^2 \geqslant -\frac{3}{80}\,, \tag{D.2}$$

implying that f_{VIII} is larger than f_{VI} at $b = 0$.

Solution XV. From the gap equation corresponding to Φ_{XV}, i.e. (C.11) with $\epsilon = -1$, one can solve $m_1^2 - m_2^2$, $m_1^2 + m_2^2 + 2m_3^2$ and $m_1^2 + m_2^2 + 4m_3^2$. The inequality $m_1^2 + m_2^2 - 4m_3^2 < m_1^2 + m_2^2 + 2m_3^2$ leads to the condition

$$\frac{-4(u_1+u_2+u_3)}{32v + 24w_1 + 3w_2 + 4w_3}$$

$$\geqslant \frac{-3v(u_1-u_2) - (5v - w_2 - w_3)(u_1 + u_2 - 2u_3)}{(w_3 - v)(5v - w_2 - w_3) - 3v^2}\,, \tag{D.3}$$

in which the equality sign describes the bifurcation to phase I. In the special case under consideration here (D.3) reduces to $y + \frac{27}{10}y^2 < \frac{1}{20}x$, so that phase XV can only occur in the region where VI exists. From (C.13) we have

$$\frac{f_{\mathrm{XV}}v}{(u_1+u_2)^2} = -\frac{3}{80} - \frac{1}{4}\left(\frac{y}{x}\right)^2 + \frac{1}{20}\frac{y^2}{x} - \frac{27}{20}\left(\frac{y^2}{x}\right)^2 - \frac{y^3}{x^2}\,, \tag{D.4}$$

and using $y + \frac{27}{10}y^2 < \frac{1}{20}x$ and (3.26) it follows that $f_{\mathrm{XV}} > f_{\mathrm{VI}}$.

Solution XVI. From the bifurcation with VII as given in table X we obtain $y - \frac{1}{10}y^2 < \frac{1}{20}x$, implying that XVI can occur only in the region where the BW phase X exists. Working out (C.15) we have

$$\frac{f_{\mathrm{XVI}}v}{(u_1+u_2)^2} = -\frac{3}{80} - \frac{1}{4}\left(\frac{y}{x}\right)^2 + \frac{1}{10}\frac{y^2}{x} - \frac{3}{20}\left(\frac{y^2}{x}\right)^2, \tag{D.5}$$

so that $f_{\mathrm{XVI}} > f_{\mathrm{BW}}$, as follows from (3.21).

Solutions VIII′ and XV′. The solutions XV′ and VIII′ can be inferred from (C.11) with $\epsilon = 1$ as the minima with $m_1 m_2 m_3 \neq 0$ and with $m_2 = 0$, respectively. From (C.11) for $\epsilon = 1$ it is clear that one only can have an absolute minimum of the Landau expansion under the condition $m_1^2 - m_2^2 \geq 0$, $m_1^2 + m_2^2 - 4m_3^2 \leq 0$. (In fact, when $m_1^2 - m_2^2$ and $m_1^2 + m_2^2 - 4m_3^2$ have the same sign, the right-hand side of (C.12) with $\epsilon = -1$ has a lower value than the right-hand side with $\epsilon = 1$, and if $m_1^2 - m_2^2 < 0$ and $m_1^2 + m_2^2 - 4m_3^2 > 0$ one can find a lower value of the right-hand side of (C.11) with $\epsilon = -1$ replacing m_1^2, m_2^2, m_3^2 by m_2^2, m_1^2, m_3^2, respectively.)

Assuming that $m_1^2 - m_2^2 > 0$, $m_1^2 + m_2^2 - 4m_3^2 < 0$, $m_1^2 + m_2^2 - m_3^2 > 0$ at the minimum of (C.11) with $\epsilon = 1$, one can introduce

$$\bar{m}_1^2 = \tfrac{4}{3}m_1^2 - \tfrac{2}{3}m_2^2 + \tfrac{8}{3}m_3^2\,, \quad \bar{m}_2^2 = \tfrac{4}{3}m_2^2 - \tfrac{2}{3}m_1^2 + \tfrac{8}{3}m_3^2\,,$$
$$\bar{m}_3^2 = \tfrac{1}{3}(m_1^2 + m_2^2 - m_3^2)\,, \tag{D.6}$$

so that $\bar{m}_1^2 - \bar{m}_2^2 = m_1^2 - m_2^2$, $\bar{m}_1^2 + \bar{m}_2^2 + 2\bar{m}_3^2 = m_1^2 + m_2^2 + 2m_3^2$, $\bar{m}_1^2 + \bar{m}_2^2 - 4\bar{m}_3^2 = 4m_3^2 - m_1^2 - m_2^2$, leading to

$$\begin{aligned}\Phi \geq\; & \tfrac{1}{6}(u_1 + u_2 + u_3)(\bar{m}_1^2 + \bar{m}_2^2 + 2\bar{m}_3^2) + \tfrac{1}{12}(u_1 + u_2 - 2u_3)(\bar{m}_1^2 + \bar{m}_2^2 - 4\bar{m}_3^2)\\ & + \tfrac{1}{4}(u_1 - u_2)(\bar{m}_1^2 - \bar{m}_2^2) + (\tfrac{4}{3}v + w_1 + \tfrac{1}{4}w_2 + \tfrac{1}{6}w_3)(\bar{m}_1^2 + \bar{m}_2^2 + 2\bar{m}_3^2)^2\\ & + \tfrac{1}{4}(5v - w_2 - w_3)(\bar{m}_1^2 - \bar{m}_2^2)^2 + \tfrac{1}{12}(w_3 - v)(\bar{m}_1^2 + \bar{m}_2^2 - 4\bar{m}_3^2)^2\\ & - \tfrac{1}{2}\epsilon v(\bar{m}_1^2 - \bar{m}_2^2)(\bar{m}_1^2 + \bar{m}_2^2 - 4\bar{m}_3^2)\,.\end{aligned} \tag{D.7}$$

Comparing (D.7) and (C.11) it is clear that the right-hand side of (D.7) for $\epsilon = 1$ cannot be smaller than the absolute minimum of (C.11) for $\epsilon = -1$, implying that solution XV′ is less favorable than solution XV in this case. Finally in the case that $m_3^2 \geq m_1^2 + m_2^2$, we have the following inequality for Φ with $\epsilon = 1$:

$$\begin{aligned}\Phi \geq\; & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (3v + w_1)m_1^4 + (6v + w_1)m_2^4\\ & +(4v + 4w_1 + w_2 + 2w_3)m_3^4 + (4v + 4w_1 + w_2)m_3^2(m_1^2 + m_2^2)\\ & +(2w_1 + w_2 + w_3 + 4v)m_1^2 m_2^2\,,\end{aligned} \tag{D.8}$$

implying that $\Phi \geq \Phi_{\text{XVI}}$ under the condition

$$(12v^2 - 2w_3 v - 2w_3' v - w_3 w_3')/(2v + w_3') \geq 0\,. \tag{D.9}$$

Hence, phase XV′ does not occur when (D.9) is satisfied. This holds also for some other solutions with $\epsilon = 1$ such as VIII′ and I′ that can be obtained via bifurcation with XV′, see also fig. 1 at the end of ref. 1. The argument does not apply to phase VII, since in the case $m_2 = 0$ the inequality (D.8) reduces to an equality in which the right-hand side is equal to Φ_{VII}.

In the case that $w_1 = \frac{1}{4}v$, $w_2 = -v$, $w_3 = 1.90v$, $w_3' = 1.94v$ considered in fig. 7, the phases XV′ and VIII′ cannot occur because of the same reason. The other phases can be ruled out by arguments which in part are similar to the ones used above. More specifically one can show that:

i) $f_{\mathrm{VII}} < f_{\mathrm{A1}}$, only in the region where phase I or phase VI exists. It is straightforward to show that $f_{\mathrm{VII}} > f_{\mathrm{I}}$, and also that $f_{\mathrm{VII}} > f_{\mathrm{VI}}$, when $y/x \leqslant 0.0673$ which is the condition of existence of phase VI and $y^2/x \leqslant 1$.

ii) Evaluating f_{VIII} and using the last inequality of (1.15), i.e. $\frac{1}{9}x \geqslant \frac{8}{9}y + 2y^2$, one has again $f_{\mathrm{VIII}} > f_{\mathrm{VI}}$ $(b = 0)$, so that phase VIII does not occur.

iii) Taking into account the bifurcation condition with phase I, cf. (D.3), we obtain $y + 2.879y^2 \leqslant 0.03536x$ as condition of existence of phase XV, so that XV can only exist where VI exists. Evaluating f_{XV} and f_{VI}, and using the condition $y + 2.879y^2 \leqslant 0.03536x$, it is straightforward to check that $f_{\mathrm{XV}} > f_{\mathrm{VI}}$.

iv) Using the bifurcation of XVI with phase VII, cf. (C.17), one has the condition $y \leqslant 0.0635x + 0.1336y^2$, implying that for $x < 5$ one must have $y/x \leqslant 0.0673$, so that phase XVI can exist only where phase VI exists. Then $f_{\mathrm{XVI}} > f_{\mathrm{VI}}$, because of (C.16) being valid in the case under consideration here.

Appendix E

In this appendix we show that the phases VIII, XV, XVI, I, VI and VII do not occur for $w_3, w_3' < 0$ at sufficiently small values of w_1, w_2, w_3, w_3'. First of all, for $w_3 < 0$ it is clear that (C.9) of appendix C cannot be satisfied, implying that VIII and XV do not occur. Secondly, from (C.14) it follows that Φ_{XVI} at fixed $m_1^2 - m_2^2$ can only have a minimum with $m_1^2 + m_2^2 \neq 0$, $m_1^2 + m_2^2 \neq m_1^2 - m_2^2$, $m_3^2 \neq 0$ under the condition

$$p \equiv \frac{2(4v + 4w_1 + w_2)}{\{(4v + 4w_1 + w_2 + 2w_3)(4v + 4w_1 + w_2 + 2w_3 + (a+1)w_3')\}^{1/2}} < 0\,, \tag{E.1}$$

which cannot be satisfied for negative and sufficiently small values of w_3, w_3'. Thus, phase XVI does not occur. Thirdly, phase I does not occur, since both terms in the right-hand side of (2.10) are negative for $w_3 < 0$. For phase VI we can use the condition $x_2 \equiv \boldsymbol{m}_2 \cdot \boldsymbol{m}_2 / m_2^2 \geqslant -1$. From eq. (A.21) of ref. 1 we then have

$$\frac{m_2^2}{m_1^2} > \frac{-2v\gamma}{4v^2-\gamma^2} > \frac{-\gamma}{2v}\,. \tag{E.2}$$

Inserting (E.2) into (2.11) we find

$$f_{\mathrm{III}} - f_{\mathrm{VI}} \leqslant m_1^4\left[\frac{v\gamma^2}{4v^2-\gamma^2}\left(1+\frac{\gamma}{2v}\right)^2 - \left(w_3 - \frac{2\gamma v}{2v+\gamma}\right)\frac{\gamma}{2v}\right]. \tag{E.3}$$

Using the inequality $w_3 < w_3'$, which means that phase III is more favourable than phase II, or equivalently that $-2w_3\gamma \leqslant -4\gamma^2$, we find

$$f_{\mathrm{III}} - f_{\mathrm{VI}} \leqslant \frac{m_1^4\gamma^2}{4v^2-\gamma^2}\{5\gamma^2 - 4v^2\}\,, \tag{E.4}$$

which is negative for $\gamma^2 < \frac{4}{5}v^2$. A similar argument for $f_{\mathrm{II}} - f_{\mathrm{VI}}$ can be applied in the case that $w_3' < w_3$, implying that phase VI does not occur for $w_3, w_3' < 0$ at sufficiently small w_1, w_2, w_3, w_3'.

From the explicit result (C.2) for f_{VII} together with (2.5) one finds

$$\frac{f_{\mathrm{VII}} - f_{\mathrm{III}}}{(u_1+u_2)^2} = \frac{-\left\{w_3\left(1+\frac{y}{x}\right) + (4v+4w_1+w_2)\left(\frac{1}{2}\frac{y}{x}+\frac{y^2}{x}\right)\right\}^2}{(4v+4w_1+w_2+2w_3)\{4(4v+4w_1+w_2+2w_3)(3v+w_1) - (4v+4w_1+w_2)^2\}}$$
$$+ \frac{\frac{1}{4}\left(\frac{y}{x}\right)^2}{4v-w_2-2w_3} + \frac{\frac{y^2}{x}\left(1-\frac{y^2}{x}\right)}{4v+4w_1+w_2+2w_3}\,, \tag{E.5}$$

in which x and y have been defined by (2.25).

From the condition $m_1^2 \geqslant 0$ for phase VII, i.e. the third inequality of (1.15), which by the way cannot be satisfied at $b=0$, we have

$$(4v+4w_1+w_2)\left(\frac{1}{2}\frac{y}{x}+\frac{y^2}{x}\right) + w_3\left(1+\frac{y}{x}\right) > 0\,. \tag{E.6}$$

From eq. (C.1), in which $A = \frac{2}{3}$ in the limit $w_1, w_2, w_3, w_3' \to 0$, one finds

$$\frac{1}{4}\left(\frac{y}{x}\right)^2 + \frac{y^2}{x}\left(1-\frac{y^2}{x}\right)\frac{4v-w_2-2w_3}{4v+4w_1+w_2+2w_3} \geqslant \left(\frac{1}{2}\frac{y}{x}+\frac{y^2}{x}\right)^2 \tag{E.7}$$

at sufficiently small w_1, w_2, w_3, w_3'. Inserting (E.6) and (E.7) into (E.5) we finally have $(2w_1 + w_2 + 2w_3 < 0)$

$$\frac{f_{\mathrm{VII}} - f_{\mathrm{III}}}{(u_1+u_2)^2} \geqslant \frac{w_3^2\left(1+\frac{y}{x}\right)^2}{(4v+4w_1+w_2+2w_3)\{4(4v+4w_1+w_2+2w_3)(3v+w_1) - (4v+4w_1+w_2)^2\}}$$

$$+\frac{\left(\frac{1}{2}\frac{y}{x}+\frac{y^2}{x}\right)^2}{4v-w_2-2w_3}$$

$$\left\{1-\frac{(4v+4w_1+w_2)^2(4v-w_2-2w_3)}{(4v+4w_1+w_2+2w_3)\{4(4v+4w_1+w_2+2w_3)(3v+w_1)-(4v+4w_1+w_2)^2\}}\right\},$$

which is positive at sufficiently small w_1, w_2, w_3, w_3', implying that phase VII does not occur.

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